Horizontal Δ-semimartingales on orthonormal frame bundles

Fumiya Okazaki

Abstract

In this article, we deal with stochastic horizontal lifts and anti-developments of semimartingales with jumps on complete and connected Riemannian manifolds. We prove two one-to-one correspondences among some classes of discontinuous semimartingales on Riemannian manifolds, orthonormal frame bundles and Euclidean spaces by using the stochastic differential geometry with jumps introduced by Cohen (1996). Both of these two results are extension of the one shown in Pontier-Estrade (1992). The first result is the correspondence in the case where jumps of semimartingales are regarded as initial velocities of geodesics which are not necessarily minimal. In the second result, we also established the correspondence in the situation where jumps of semimartingales are given by connection rules, but we impose the condition that the jumps of semimartingales are small. The latter result enables us to construct martingales with small jumps for a given connection rule on any compact manifold from local martingales on a Euclidean space through horizontal semimartingales on orthonormal frame bundles.

1 Introduction and main theorems

A stochastic parallel displacement of a frame along a diffusion was defined in [10,18]. This can be regarded as the horizontal lift of a diffusion on a manifold to a frame bundle. The horizontal lift of a continuous semimartingale on a manifold to more general principal bundles was considered in [24]. The horizontal lift of semimartingales is an extension of that of smooth curves on manifolds. Moreover, by employing horizontal lifts, we can regard continuous semimartingales on manifolds as developments of continuous semimartingales on tangent spaces above initial values, which are called anti-developments. Then we can describe those horizontal lifts as solutions of SDE's on the frame bundle driven by the anti-developments. This description was utilized in [3,4] in the study of differential families of continuous martingales on manifolds, which are naturally obtained from smooth harmonic maps.

MSC2020 Subject Classifications: 60H05, 60G44

Keywords and phrases: Stochastic differential equations on manifolds; Riemannian manifolds; Jump processes; Semimartingales; Orthonormal frame bundles

Horizontal lifts and anti-developments of discontinuous semimartingales were also considered in [23], which dealt with discontinuous semimartingales on manifolds whose jumps can be connected by unique minimal geodesics. The aim of this article is to extend the result in [23] so that we can construct anti-developments of discontinuous semimartingales in other situations.

To begin with, we recall an overview of the stochastic differential geometry for continuous semimartingales on manifolds referring to [9, 14]. Throughout this paper, we always assume that we are given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le\infty}, \mathbb{P})$ and the usual hypotheses for $\{\mathcal{F}_t\}_{0\le t\le\infty}$ hold. A càdlàg process X valued in a manifold M is called an M-valued semimartingale if $f(X)$ is an $\mathbb{R}$-valued semimartingale for all $f \in C^\infty(M)$. To begin with, let us recall basic facts about continuous semimartingales on manifolds. It is known that for an M-valued continuous semimartingale X, we can define the Stratonovich integral of 1-forms ϕ along X and the quadratic variation of 2-tensors ψ. They are denoted by $\displaystyle\int \phi(X) \circ dX$ and $\displaystyle\int \psi(X)\, d[X, X]$, respectively. Furthermore, given a torsion-free connection on M, we can define the Itô integral of 1-form ϕ denoted by $\displaystyle\int \phi(X)\, dX$ and the equation

$$\int \phi(X) \circ dX = \int \phi(X)\, dX + \frac{1}{2}\int \nabla\phi(X)\, d[X, X]$$

holds. When we apply the stochastic analysis to semimartingales on manifolds, it is helpful to consider lifts of semimartingales to fiber bundles on M. To see this, let (M, g) be a Riemannian manifold and $\pi : \mathcal{O}(M) \to M$ an orthonormal frame bundle, namely, we set

$$\begin{aligned}
\mathcal{O}_x(M) &:= \{u : \mathbb{R}^d \to T_xM \mid u \text{ is a linear isometric map}\},\\
\mathcal{O}(M) &:= \bigsqcup_{x\in M} \mathcal{O}_x(M),\\
\pi : \mathcal{O}(M) &\to M,\ \pi(u) = x,\ u \in \mathcal{O}_x(M).
\end{aligned}$$

Let $O(d)$ be an orthogonal group and $\mathfrak{o}(d)$ its Lie algebra. Then $O(d)$ acts on $\mathcal{O}(M)$ and the action is defined by

$$ua := u \circ a \in \mathcal{O}_x(M),\ x \in M,\ u \in \mathcal{O}_x(M),\ a \in O(d).$$

We define a map $R_a : \mathcal{O}(M) \to \mathcal{O}(M)$ by

$$R_au := ua,\ u \in \mathcal{O}(M)$$

for $a \in O(d)$. An $\mathfrak{o}(d)$-valued 1-form $\theta \in \Omega^1(\mathcal{O}(M); \mathfrak{o}(d))$ is called a connection form on $\mathcal{O}(M)$ if θ satisfies the following:

- For all $\mathcal{X} \in \mathfrak{o}(d)$, $\langle\theta, \mathcal{X}^\sharp\rangle = \mathcal{X}$;
- for all $a \in O(d)$, $R_a^*\theta = \mathrm{Ad}(a^{-1}) \circ \theta$,

where

$$\mathcal{X}^\sharp(u) = \left(\frac{d}{dt}\right)_{t=0} u \exp t\mathcal{X},\ u \in \mathcal{O}(M), \tag{1.1}$$

$R_a^*\theta$ is the pull-back of θ by R_a and $\mathrm{Ad}\colon O(d) \to \mathrm{End}(\mathfrak{o}(d))$ is the adjoint representation. For a connection form θ, set

$$H_u = \{\mathcal{A} \in T_u\mathcal{O}(M) \mid \langle \theta, \mathcal{A}\rangle = 0\},\ u \in \mathcal{O}(M). \tag{1.2}$$

This is called the horizontal subspace of $T_u\mathcal{O}(M)$. The restriction of π_{*u} to H_u denoted by $\pi_{*|H_u} : H_u \to T_{\pi u}M$ is a linear isomorphism for each $u \in \mathcal{O}(M)$ and for $\mathbf{X} \in T_{\pi u}M$, the vector

$$\tilde{\mathbf{X}} := (\pi_{*|H_u})^{-1}(\mathbf{X})$$

is called the horizontal lift of $\mathbf{X}$. Connection forms on $\mathcal{O}(M)$ and connections on M admit one-to-one correspondence through the relation

$$\nabla_{\mathbf{X}}\mathbf{Y}(\pi u) = u(\tilde{\mathbf{X}}(\pi^*\mathbf{Y})(u)),\ \mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M),\ u \in \mathcal{O}(M),$$

where the map $\pi^*\mathbf{Y}\colon \mathcal{O}(M) \to \mathbb{R}^d$ is defined by

$$\pi^*\mathbf{Y}(u) := u^{-1}\mathbf{Y}(\pi u),\ u \in \mathcal{O}(M).$$

A connection form on $\mathcal{O}(M)$ also determines the notion of horizontal semimartingales on $\mathcal{O}(M)$, which enables us to describe every semimartingale on M through SDE's. Indeed, it is known that given an M-valued continuous semimartingale X, an $\mathcal{O}(M)$-valued $\mathcal{F}_0$-measurable random variable u_0 such that $\pi u_0 = X_0$ and a connection ∇ on M, the $\mathcal{O}(M)$-valued continuous semimartingale U is uniquely determined which satisfies $U_0 = u_0$, $\pi U = X$ and

$$\int \theta \circ dU = 0, \tag{1.3}$$

where θ is the connection form θ corresponding to ∇. Furthermore, for an $\mathcal{O}(M)$-valued semimartingale U satisfying (1.3), the stochastic integral of the solder form $\mathfrak{s}$ along U yields a continuous semimartingale on a Euclidean space, which is called the anti-development of U or of $X := \pi U$. Here the solder form $\mathfrak{s} \in \Omega^1(\mathcal{O}(M); \mathbb{R}^d)$ is defined by

$$\mathfrak{s}_u(A) = u^{-1}\pi_* A,\ u \in \mathcal{O}(M),\ A \in T_u\mathcal{O}(M).$$

Conversely, we can construct an $\mathcal{O}(M)$-valued semimartingale U satisfying (1.3) from a continuous semimartingale W starting at 0 on Euclidean space. In fact, by the existence and uniqueness of solutions of stochastic differential equations (SDE's) on manifolds (e.g. [14]), there exists an $\mathcal{O}(M)$-valued continuous semimartingale U satisfying

$$F(U_t) - F(U_0) = \int_0^t \mathcal{L}_k F(U_s) \circ dW_s^k,\ F \in C^\infty(\mathcal{O}(M))$$

where $\mathcal{L}_k$ $(k = 1, \ldots, d)$ are the canonical horizontal vector fields on $\mathcal{O}(M)$ defined by

$$\mathcal{L}_k(u) = (\pi_{*|H_u})^{-1}(u\varepsilon_k),\ (k = 1, \ldots, d)$$

and we have used the Einstein summation convention. Furthermore, by projecting U onto the base space M, we obtain a continuous semimartingale on M. In [14], we can find the proof of the one-to-one correspondence using an embedding of the manifold.

1.1 Discontinuous semimartingales on manifolds

The purpose of this article is to establish the discontinuous version of the one-to-one correspondence mentioned above. However, stochastic analysis for discontinuous semimartingales on manifolds is complicated since we need to deal with jumps of processes on non-flat spaces. Thus in order to state our main results clearly, we recall some notions regarding discontinuous semimartingales on manifolds and stochastic analysis for them beforehand.

First, we let M be a d-dimensional C^∞ manifold. As mentioned above, càdlàg semimartingales on M can be defined without any other structures of M. However, in order to consider the stochastic integral of 1-forms along X, we need the direction of jumps of X which is supposed to be given by tangent vectors. In [22], maps from $M \times M$ to TM called connection rules are introduced and the Itô integral of 1-forms is defined through connection rules. A connection rule can determine the direction of jumps on a manifold, which is necessary for the definition of the stochastic integral.

Definition 1.1. *A mapping $\gamma : M \times M \to TM$ is a connection rule if it is measurable, C^2 on a neighborhood of the diagonal set of $M \times M$, and if it satisfies, for all $x, y \in M$,*

(i) $\gamma(x, y) \in T_xM$;

(ii) $\gamma(x, x) = 0$;

(iii) $d\gamma(x, \cdot)_x = \mathrm{id}_{T_xM}$.

Remark 1.2. Conditions (ii) and (iii) in Definition 1.1 are only relevant near the diagonal set. Thus we can choose the values of connection rules γ arbitrarily outside the diagonal set as long as γ is Borel measurable and satisfies condition (i).

Remark 1.3. As mentioned in [22], for each connection rule γ, there exists a unique torsion-free connection ∇ such that

$$f(y) - f(x) = \langle df(x), \gamma(x, y)\rangle + \frac{1}{2}\nabla df(x)(\gamma(x, y), \gamma(x, y)) + \mathrm{O}(d(x, y)^3) \ (y \to x)$$

for all $f \in C^\infty(M)$, where d is a distance compatible with the topology of M. Note that the correspondence is not one-to-one. For two connection rules γ_1 and γ_2, they induce the same connection ∇ if and only if

$$|\gamma_1(x, y) - \gamma_2(x, y)| = \mathrm{O}(d(x, y)^3) \ (y \to x) \tag{1.4}$$

for all $x \in M$.

Given a connection rule γ, we can determine the direction of jumps of a semimartingale X by $\gamma(X_{s-}, X_s)$ as one option. Moreover, for a given M-valued semimartingale X and a connection rule γ, we can define the stochastic integral of each TM-valued càdlàg process ϕ_t with $\phi_t \in T^*_{X_t}M$ by the limit in probability of the sum

$$\sum_{i=1}^{k_n} \langle \phi_{\tau^n_{i-1}\wedge t}, \gamma(X_{\tau^n_{i-1}\wedge t}, X_{\tau^n_i\wedge t})\rangle$$

as $n \to \infty$, where $\tau_0^n < \tau_1^n < \cdots < \tau_{k_n}^n$ be a certain random partition satisfying

$$\lim_{n\to\infty} \max\{|\tau_i^n - \tau_{i-1}^n| \,;\, i = 1, \ldots, k_n\} = 0$$

and

$$\lim_{n\to\infty} \tau_{k_n}^n = \infty.$$

We denote the limit by

$$\int \phi_{s-}\, \gamma dX.$$

More generally, $TM \times M$-valued processes called Δ-semimartingales have been introduced in [22], which is a pair of a càdlàg semimartingale and "directions of jumps".

Definition 1.4. *Let $Y = (\Delta X, X)$ be an adapted $TM \times M$-valued process. The process Y is called a Δ-semimartingale if it satisfies the following:*

(i) X is an M-valued semimartingale;

(ii) $\Delta X_s \in T_{X_{s-}}M$ for all $s > 0$;

(iii) $\Delta X_0 \in T_{X_0}M$, $\Delta X_0 = 0$;

(iv) for all connection rules γ and T^*M-valued càdlàg processes ϕ,

$$\sum_{0<s\le t} \langle \phi_{s-}, \Delta X_s - \gamma(X_{s-}, X_s)\rangle < \infty, \text{ for all } t > 0. \tag{1.5}$$

Remark 1.5. Although we denote the TM-valued part of a Δ-semimartingale by ΔX in Definition 1.4, it is not uniquely determined solely by the M-valued part X. Indeed, for a given M-valued semimartingale X and for any choice of a connection rule γ, the pair $(\gamma(X_-, X), X)$ forms a Δ-semimartingale. Therefore, if we consider the case where we are given a Δ-semimartingale $(\Delta X, X)$, it implies that we consider a specific TM-valued process ΔX which satisfies the conditions in Definition 1.4.

Remark 1.6. It is sufficient that condition (iv) in Definition 1.4 is satisfied for some connection rule since all the connection rules have the same order near the diagonal set by condition (iii) of Definition 1.1. Condition (1.5) means that the difference between the TM-valued part ΔX_s and $\gamma(X_{s-}, X_s)$ becomes small for $s \in [0, t]$ with ΔX_s small, and there are only a few $s \in [0, t]$ where ΔX_s is large.

Let $(\Delta X, X)$ be a Δ-semimartingale and ∇ a torsion-free connection. Then by Proposition 3.5 of [22], we can define the Itô integral along $(\Delta X, X)$ by

$$\int_0^t \phi_{s-}\, dX_s := \int_0^t \phi_{s-}\, \gamma dX_s + \sum_{0<s\le t} \langle \phi_{s-}, \Delta X_s - \gamma(X_{s-}, X_s)\rangle,$$

where ϕ is a T^*M-valued càdlàg process above X, i.e. $\phi_t \in T_{X_t}M$, and γ is a connection rule which induces ∇. In a similar way, for a Δ-semimartingale $(\Delta X, X)$, the quadratic variation of a $T^*M \otimes T^*M$-valued càdlàg process ψ above X is also defined and denoted by $\displaystyle\int \psi_{s-}\, d[X,X]_s$. Discontinuous martingales on manifolds can be introduced based on the Itô integral on manifolds.

Definition 1.7. *Let M be a d-dimensional manifold with a torsion-free connection ∇, and $(\Delta X, X)$ an M-valued Δ-semimartingale. We call $(\Delta X, X)$ a ∇-martingale if for all T^*M-valued càdlàg processes ϕ above X, the Itô integral $\displaystyle\int \phi_-\, dX$ is a local martingale.*

Note that the definition of martingales with jumps depends on the direction of jumps ΔX. If $(\gamma(X_-, X), X)$ is a ∇-martingale, we call X a γ-martingale.

Next we introduce the Stratonovich integral of 1-forms along Δ-semimartingales. The definition below is an extension of that of [23].

Definition 1.8. *Let ∇ be a torsion-free connection and $(\Delta X, X)$ an M-valued Δ-semimartingale. For $\alpha \in \Omega^1(M)$, we define*

$$\int_0^t \alpha \circ dX := \int_0^t \alpha(X_{s-})\, dX_s + \frac{1}{2}\int_0^t (\nabla\alpha)(X_{s-})\, d[X,X]^c_s.$$

This is called the Stratonovich integral of 1-form α along $(\Delta X, X)$. We also denote the integral by $\displaystyle\int_0^t \alpha(X_s) \circ dX_s$.

Remark 1.9. As we will see in Proposition 2.4 in Subsection 2.1 below, the notion of the Stratonovich integral introduced above does not depend on the choice of the torsion-free connection ∇.

1.2 Main theorems

Again we let (M, g) be a Riemannian manifold. Throughout this article, for a smooth vector field $\mathbf{A}$ on M, we denote by

$$\mathrm{Exp}t\mathbf{A} \colon M \to M,\ x \mapsto \mathrm{Exp}_x t\mathbf{A}$$

the one-parameter transformation generated by $\mathbf{A}$. On the other hand, the exponential map defined through geodesics on M is denoted by $\exp\colon TM \to M$. Let $\{\varepsilon_i\}$ and $\{\mathcal{X}_\alpha\}$ be orthonormal bases on $\mathbb{R}^d$ and $\mathfrak{o}(d)$ with respect to the standard inner product, respectively. Then the connection form θ and the solder form $\mathfrak{s}$ can be written as

$$\theta = \theta^\alpha \mathcal{X}_\alpha,\ \mathfrak{s} = \mathfrak{s}^k \varepsilon_k.$$

Note that the set of vector fields $\{\mathcal{L}_k, \mathcal{X}^\sharp_\alpha\}^{k=1,\dots,d}_{\alpha=1,\dots,\frac{d(d-1)}{2}}$ is a basis of each tangent space on $\mathcal{O}(M)$ and $\{\mathfrak{s}^k, \theta^\alpha\}$ is its dual basis. We define the Riemannian metric $\tilde{g}$ on $\mathcal{O}(M)$ by

$$\tilde{g} := \sum_{\alpha=1}^{\frac{d(d-1)}{2}} \theta^\alpha \otimes \theta^\alpha + \sum_{k=1}^{d} \mathfrak{s}^k \otimes \mathfrak{s}^k \tag{1.6}$$

and denote the corresponding Levi-Civita connection by $\tilde{\nabla}$. Fundamental properties of $\tilde{g}$ are given in Appendix. By employing the connection on the orthonormal frame bundle, we can define the Itô integral of 1-forms and the quadratic variation of 2-tensors on $\mathcal{O}(M)$ with respect to $\tilde{\nabla}$. To state our results precisely, we introduce two more definitions.

Definition 1.10. *(1) Let $(\Delta U, U)$ be an $\mathcal{O}(M)$-valued Δ-semimartingale. Then $(\Delta U, U)$ is said to be horizontal if*

$$\int_0^t \theta(U_s) \circ dU_s = 0$$

for all $t \geq 0$.

(2) Let $(\Delta U, U)$ be a horizontal Δ-semimartingale on $\mathcal{O}(M)$. The anti-development of $(\Delta U, U)$ is defined by

$$W = \int \mathfrak{s}(U) \circ dU.$$

(3) Let $(\Delta U, U)$ be an $\mathcal{O}(M)$-valued horizontal Δ-semimartingale and $(\Delta X, X)$ an M-valued Δ-semimartingale. Then $(\Delta U, U)$ will be called a horizontal lift of $(\Delta X, X)$ if

$$\pi U = X, \ \pi_* \Delta U = \Delta X.$$

(4) Let W be an $\mathbb{R}^d$-valued semimartingale W with $W_0 = 0$ and U_0 an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable. Then by the existence and uniqueness of solutions of SDE's on manifold (see [6] or Example 2.9 of this article), there exists an $\mathcal{O}(M)$-valued semimartingale U such that for $F \in C^\infty(\mathcal{O}(M))$,

$$\begin{aligned} F(U_t) - F(U_0) &= \int_0^t \mathcal{L}_k F(U_{s-}) \circ dW_s^k \\ &\quad + \sum_{0<s\leq t} \{F(\mathrm{Exp}_{U_{s-}} \Delta W_s^k \mathcal{L}_k(U_{s-})) - F(U_{s-}) - \mathcal{L}_k F(U_{s-}) \Delta W_s^k\}. \end{aligned} \tag{1.7}$$

This process U is called a development of W.

In our first theorem below, we consider Δ-semimartingales satisfying

$$\exp_{X_{s-}} \Delta X_s = X_s \tag{1.8}$$

on (M, g). We also consider condition (1.8) for processes on $(\mathcal{O}(M), \tilde{g})$.

Theorem 1.11. *(1) Let W be an $\mathbb{R}^d$-valued semimartingale with $W_0 = 0$ and U a development of W. Suppose that U does not explode in finite time. If we set $\Delta U_s = \Delta W_s^k \mathcal{L}_k(U_{s-})$, then $(\Delta U, U)$ is a Δ-semimartingale satisfying (1.8) on $\mathcal{O}(M)$ and it holds that*

$$\int \theta(U_-) \circ dU = \int \theta(U_-) \, dU = 0,$$

$$\int \mathfrak{s}(U_-) \circ dU = \int \mathfrak{s}(U_-) \, dU = W.$$

In particular, the anti-development of $(\Delta U, U)$ is W.

(2) *Let $(\Delta U, U)$ be a horizontal Δ-semimartingale satisfying* (1.8) *on $\mathcal{O}(M)$ and W the anti-development of $(\Delta U, U)$. Then the development of W equals to $(\Delta U, U)$.*

(3) *Let $(\Delta U, U)$ be a horizontal Δ-semimartingale satisfying* (1.8) *on $\mathcal{O}(M)$. Set $X := \pi U$, $\Delta X := \pi_* \Delta U$. Then $(\Delta X, X)$ is a Δ-semimartingale satisfying* (1.8) *on M.*

(4) *Let $(\Delta X, X)$ be an M-valued Δ-semimartingale and u_0 an $\mathcal{O}_{X_0}(M)$-valued $\mathcal{F}_0$-measurable random variable satisfying* (1.8)*. Then there exists a unique horizontal lift $(\Delta U, U)$ of $(\Delta X, X)$ with $U_0 = u_0$ satisfying* (1.8)*. Furthermore, if we let $(\varepsilon_1, \ldots, \varepsilon_d)$ be a standard basis of $\mathbb{R}^d$ and $(\varepsilon^1, \ldots, \varepsilon^d)$ its dual basis, then it holds that*

$$W_t^i = \int_0^t U_{s-}\varepsilon^i \, dX_s,$$

where $U_{t-} : (\mathbb{R}^d)^* \to T^*_{X_{t-}}M$ is defined by

$$\langle U_{t-}a, v \rangle = \langle a, U_{t-}^{-1} v \rangle, \; a \in (\mathbb{R}^d)^*, \; v \in T^*_{X_{t-}}M,$$

and W is the anti-development of $(\Delta U, U)$.

This result is a generalization of the result of [23]. In [23], this kind of result was shown only in the case where jumps of semimartingales can be uniquely connected by minimal geodesics, but our result includes some cases where this assumption is not satisfied. As we will see later in Lemma 3.1, we can take a connection rule on any complete connected Riemannian manifold which provides the initial velocities of geodesics between two points even if minimal geodesics between two points are not uniquely determined. Thus we can construct a horizontal lift of every càdlàg semimartingale and establish the one-to-one correspondence for each chosen connection rule, which will be shown in Lemma 3.6.

Moreover, by replacing jumps with other geodesics, we can construct horizontal lifts of semimartingales of which jumps are described by geodesics that are not necessarily minimal. For example, let N be a Poisson process with intensity λ and X a semimartingale on the unit circle $\mathbb{S}^1$ given by $X_t = e^{\frac{\pi}{2} i (N_t - 3\lambda t)}$. Then each jump of X can be connected by the unique minimal geodesic $c_t(s) = X_{t-} e^{\frac{\pi}{2} is}$ ($s \in [0,1]$) and we can set the jump to $\Delta X_t = c_t'(0) \in T_{X_{t-}}\mathbb{S}^1$. On the other hand, we can also take a geodesic $\hat{c}_t(s) = X_{t-} e^{-\frac{3}{2}\pi is}$ ($s \in [0,1]$) connecting X_{t-} and X_t and set $\widehat{\Delta X}_t := \hat{c}_t'(0) \in T_{X_{t-}}\mathbb{S}^1$. The anti-developments of $(\Delta X, X)$ and $(\widehat{\Delta X}, X)$ are $\frac{\pi}{2}(N_t - 3\lambda t)$ and $\frac{3}{2}\pi(N_t - \lambda t)$, respectively and in particular, the latter one is a martingale. This kind of situation naturally happens if we consider SDE's on orthonormal frame bundles on general complete Riemannian manifolds which are in the form of (1.7). Let W be a semimartingale on $\mathbb{R}^d$ with W_0, U the solution of the SDE (1.7) and $X = \pi(U)$. In this case, it is natural that we set the jumps pf X to $\Delta X_t := \pi_*(\mathcal{L}_k(U_{t-}))\Delta W_t^k$ rather than velocities of minimal geodesics, especially when W is a semimartingale with special properties such as a local martingale, a Lévy process, etc. Moreover the horizontal lift and the anti-development are essentially unique for the determined jumps and equal to U and W, respectively due to Theorem 1.11.

Our next result includes cases of Δ-semimartingales whose jumps are described by connection rules which are not necessarily given by geodesics. For a connection rule γ on

M, we consider a Δ-semimartingale given in the form of $(\gamma(X_-, X), X)$. The aim here is to find an $\mathbb{R}^d$-valued semimartingale Z satisfying the following two requisites:

(a) Itô integral of 1-forms on M along the semimartingale X with respect to the connection rule γ can be described by the stochastic integral on $\mathbb{R}^d$ along Z;

(b) the semimartingale Z can reconstruct X through the SDE.

The motivation of this aim is derived from the fact that the notion of discontinuous martingales depends on connection rules. For instance, in the recent studies [19, 20], it has been shown that harmonic maps valued in Riemannian submanifolds of Euclidean spaces with respect to non-local Dirichlet forms can be characterized through martingales with respect to the connection rule which is not defined through the exponential map but the embedding. Typical examples of these kinds of harmonic maps are fractional harmonic maps introduced in [11, 12], which are critical points of the fractional Dirichlet energy. Our next result guarantees the one-to-one correspondence between Δ-semimartingales in the form of $(\gamma(X_-X), X)$ on M with small jumps and $\mathbb{R}^d$-valued semimartingales satisfying conditions (a) and (b) above for any compact Riemannian manifold M and connection rule M which induces the Levi-Civita connection. Thus our result might be instrumental in the study of the probabilistic description of fractional harmonic maps and associated martingales on manifolds. In Theorem 1.12 below, we denote geodesic balls on $\mathbb{R}^d$, M and T_xM with radius $r > 0$ by $B_r(z)$, $B_r^M(x)$ and $B_r^{T_xM}(v)$, respectively, where $z \in \mathbb{R}^d$, $x \in M$ and $v \in T_xM$.

Theorem 1.12. *Let (M, g) be a compact Riemannian manifold and $\gamma \in C^\infty(M \times M; TM)$ a connection rule which induces the Levi-Civita connection. Then there exist $\delta_0 = \delta_0(M, g, \gamma)$, $\delta = \delta(M, g, \gamma) > 0$ and $h \in C^\infty(\mathcal{O}(M) \times B_\delta(0); B_{\delta_0}(0))$ such that if we extend the map h to a map on $\mathcal{O}(M) \times \mathbb{R}^d$ by setting 0 on $\mathcal{O}(M) \times B_\delta(0)^c$, the following hold:*

(1) Let Z be a semimartingale on $\mathbb{R}^d$ with

$$\sup_{0 \le t < \infty} |\Delta Z_t| < \delta, \ \mathbb{P}\text{-}a.s. \tag{1.9}$$

and u_0 an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable. Then there exists a unique U satisfying $U_0 = u_0$ and

$$\begin{aligned} F(U_t) - F(U_0) = &\int_0^t \mathcal{L}_k F(U_{s-}) \circ dZ_s^k \\ &+ \sum_{0 < s \le t} \{F(\mathrm{Exp}_{U_{s-}}(h^k(U_{s-}, \Delta Z_s)\mathcal{L}_k)) - F(U_{s-}) - \mathcal{L}_k F(U_{s-})\Delta Z^k\} \end{aligned} \tag{1.10}$$

for all $F \in C^\infty(\mathcal{O}(M))$ and the process $X = \pi(U)$ satisfies

$$\int \phi_- \, \gamma dX = \int \langle U_-^{-1}\phi_-, \mathcal{L}_k(U_-)\rangle \, dZ^k \tag{1.11}$$

for any T^*M-valued càdlàg process ϕ above X.

(2) *Let X be an M-valued semimartingale satisfying*

$$X_t \in \left(\gamma_{X_{t-}}|_{B^M_{\delta_0}(X_{t-})}\right)^{-1}\left(B^{T_{X_{t-}}M}_{\delta}(0)\right) \text{ for all } t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where $\gamma_x := \gamma(x, \cdot)$ for $x \in M$. Then there exists a semimartingale Z on $\mathbb{R}^d$ such that if U is an $\mathcal{O}(M)$-valued semimartingale satisfying (1.10)*, then the pair $(h^k(U_-, \Delta Z)\mathcal{L}_k(U_-), U)$ is a horizontal lift of $(U_- h(U_-, \Delta Z), X)$ and X satisfies* (1.11).

SDE (1.10) is different from (1.7) in that the map h constructed from the connection rule γ appears in the jump part. The map h plays the role of replacing the jump $\Delta Z^k \mathcal{L}_k(U_-)$ with $h^k(U_-, \Delta Z)\mathcal{L}_k(U_-)$, which yields the desired relation (1.11). The precise construction of δ_0, δ and h appearing in Theorem 1.12 will be given in the proof. In the case where M is a sphere, we can write them explicitly. See Example 3.12. In particular, by taking Z in such a way that Z is a local martingale satisfying (1.9) in Theorem 1.12, we can construct martingales on compact Riemannian manifolds with respect to an arbitrary connection rule which induces the Levi-Civita connection from local martingales on Euclidean spaces.

Outline

We give an outline of the paper. First, we recall stochastic differential geometry with jumps developed in [6, 7]. We give proofs of our main results in Section 3. We summarize some facts and simple calculation regarding orthonormal frame bundles and Riemannian metrics on them in Appendix.

2 Preliminaries

2.1 Stochastic integrals along discontinuous semimartingales on manifolds

We have recalled some notions regarding stochastic analysis for discontinuous semimartingales on manifolds in Subsection 1.1. In this subsection, we will recall some basic properties of Stratonovich integral defined in Subsection 1.1. Throughout this subsection, we let M be a C^∞ manifold and fix a torsion-free connection ∇ on M.

Proposition 2.1. *For any Δ-semimartingale $(\Delta X, X)$, $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$,*

$$\int f\alpha \circ dX = \int f(X) \circ d\left(\int \alpha \circ dX\right). \tag{2.1}$$

In (2.1), the right-hand side is the Stratonovich integral of $\mathbb{R}$-valued semimartingale $f(X)$ along $\mathbb{R}$-valued semimartingale $\displaystyle\int \alpha \circ dX$; namely, for $\mathbb{R}$-valued semimartingales Y and Z,

$$\int Y \circ dZ := \int Y_-\, dZ + \frac{1}{2}[Y, Z]^c.$$

Proof. We begin with the left-hand side of (2.1):

$$\begin{aligned}
\int_0^t (f\alpha)(X_s) \circ dX_s &= \int_0^t f\alpha(X_{s-})\, dX_s + \frac{1}{2}\int_0^t \nabla(f\alpha)(X_{s-})\, d[X,X]^c_s \\
&= \int_0^t f(X_{s-})\, d\left(\int_0^s \alpha\, dX\right) + \frac{1}{2}\int_0^t f(X_{s-})\, d\left(\int_0^s (\nabla\alpha)(X_-)\, d[X,X]^c\right) \\
&\quad + \frac{1}{2}\int_0^t \alpha \otimes df(X_{s-})\, d[X,X]^c_s,
\end{aligned}$$

where we have used stochastic calculus rules for the Ito integral from Proposition 3.5 (b) and Proposition 3.6 (b) in [22]. On the other hand, by the straightforward calculation, the right-hand side of (2.1) can be written down as

$$\begin{aligned}
&\int_0^t f(X_s) \circ d\left(\int_0^s \alpha \circ dX\right) \\
&\quad = \int_0^t f(X_{s-})\, d\left(\int_0^s \alpha \circ dX\right) + \frac{1}{2}\left[f(X), \int_0^\cdot \alpha \circ dX\right]^c_t \\
&\quad = \int_0^t f(X_{s-})\, d\left(\int_0^s \alpha\, dX\right) + \frac{1}{2}\int_0^t f(X_{s-})\, d\left(\int_0^s \nabla\alpha(X_-)\, d[X,X]^c\right) \\
&\quad\quad + \frac{1}{2}\left[f(X), \int_0^\cdot \alpha\, dX\right]^c_t.
\end{aligned}$$

Furthermore, by the claim below Proposition 3.6 in [22], we have

$$\begin{aligned}
\left[f(X), \int_0^\cdot \alpha\, dX\right]^c_t &= \left[\int_0^\cdot df(X_-)\, dX, \int_0^\cdot \alpha\, dX\right]^c_t \\
&= \int_0^t df \otimes \alpha(X_-)\, d[X,X]^c.
\end{aligned}$$

Therefore we obtain

$$\int f\alpha \circ dX = \int f(X) \circ d\left(\int \alpha \circ dX\right),$$

and this is precisely the assertion of the proposition. □

Proposition 2.2. *Let $(\Delta X, X)$ be a Δ-semimaritngale. Then for α, $\beta \in \Omega^1(M)$,*

$$\left[\int \alpha(X) \circ dX, \int \beta(X) \circ dX\right] = \int \alpha \otimes \beta(X_-)\, d[X,X].$$

Proof. By Definition 1.8 and the claim below Proposition 3.6 in [22], we have

$$\begin{aligned}\left[\int \alpha(X)\circ dX, \int \beta(X)\circ dX\right] &= \left[\int \alpha(X_-)\, dX + \frac{1}{2}\nabla\alpha(X_-)\, d[X,X]^c,\right.\\ &\qquad \left.\int \beta(X_-)\, dX + \frac{1}{2}\int \nabla\beta(X_-)\, d[X,X]^c\right]\\ &= \left[\int \alpha(X_-)\, dX, \int \beta(X_-)\, dX\right]\\ &= \int \alpha\otimes\beta(X_-)\, d[X,X].\end{aligned}$$

This is our claim. □

Since the stochastic integral along a Δ-semimartingale has a càdlàg modification, we can consider the stochastic integral on a random interval.

Definition 2.3. *Let S,T be stopping times with $S<T$ and $(\Delta X, X)$ a Δ-semimartingale. For a T^*M-valued càdlàg process ϕ above X, we define*

$$\begin{aligned}\int_{(S,T]} \phi_{s-}\, dX_s &:= \int_0^T \phi_{s-}\, dX_s - \int_0^S \phi_{s-}\, dX_s,\\ \int_{\{T\}} \phi_{s-}\, dX_s &:= \langle \phi_{T-}, \Delta X_T\rangle,\\ \int_{(S,T)} \phi_{s-}\, dX_s &:= \int_{(S,T]} \phi_{s-}\, dX_s - \int_{\{T\}} \phi_{s-}\, dX_s.\end{aligned}$$

We define the quadratic variation and the Stratonovich integral on $(S,T]$, $\{T\}$, (S,T) in the same way.

Proposition 2.4. *Let $(\Delta X, X)$ be a Δ-semimartingale and $(U; x^1,\dots,x^d)$ a local coordinate neighborhood. Let $\alpha\in\Omega^1(M)$ and b a 2-tensor field with*

$$\alpha = \alpha_i dx^i,\ \ b = b_{ij} dx^i\otimes dx^j \text{ on } U.$$

Let S, T be stopping times such that $S<T$ and $X_s\in U$ for $s\in(S,T)$. Then

$$\int_{(S,T)} \alpha(X_s)\circ dX_s = \int_{(S,T)} \alpha_i(X_s)\circ dX_s^i + \sum_{S<s<T} \alpha_i(X_{s-})\left(\xi_s^i - \Delta X_s^i\right), \tag{2.2}$$

$$\begin{aligned}\int_{(S,T)} b(X_{s-})\, d[X,X]_s = &\int_{(S,T)} b_{ij}(X_{s-})\, d[X^i,X^j]_s\\ &+ \sum_{S<s<T} b_{ij}(X_{s-})\left(\xi_s^i\xi_s^j - \Delta X_s^i, \Delta X_s^i\right),\end{aligned} \tag{2.3}$$

where $X^i = x^i(X)$, $\Delta X_s^i = X_s^i - X_{s-}^i$ and $\xi_s^i = \langle dx^i(X_{s-}), \Delta X_s\rangle$ on U. (Note that ΔX is not equal to $\Delta X^i\dfrac{\partial}{\partial x^i}$) In particular, the Stratonovich integral and the quadratic variation of Δ-semimartingales are independent of the connection ∇.

Proof. We begin with the left-hand side of (2.2):

$$\int_{(S,T)} \alpha(X) \circ dX = \int_{(S,T)} \alpha_i(X_-)\, d\left(\int dx^i dX\right) + \frac{1}{2}\int_{(S,T)} \left(\frac{\partial \alpha_i}{\partial x^j} - \Gamma^k_{ij}\alpha_k\right)(X_-) d\left(\int dx^i \otimes dx^j\ d[X,X]^c\right),$$

where Γ^k_{ij} is the Christoffel symbol for the connection ∇. On the other hand,

$$\begin{aligned}
&\int_{(S,T)} \alpha_i(X) \circ d\left(\int dx^i \circ dX\right) \\
&= \int_{(S,T)} \alpha_i(X) \circ d\left(\int dx^i dX\right) + \frac{1}{2}\int_{(S,T)} \alpha_i(X) \circ d\left(\int \nabla dx^i d[X,X]^c\right) \\
&= \int_{(S,T)} \alpha_i(X_-)\, d\left(\int dx^i\ dX\right) + \frac{1}{2}\int_{(S,T)} \left(\frac{\partial \alpha_k}{\partial x^j} - \Gamma^i_{jk}\alpha_i\right)(X_-)\ dx^j \otimes dx^k\ d[X,X]^c.
\end{aligned}$$

Therefore

$$\int_{(S,T)} \alpha(X) \circ dX = \int_{(S,T)} \alpha_i(X) \circ d\left(\int dx^i \circ dX\right).$$

Thus we have

$$\begin{aligned}
\int_{(S,T)} \alpha(X) \circ dX &= \int_{(S,T)} \alpha_i(X) \circ d\left(X^i - X^i_0 - \sum_{0<s\le\cdot}(X^i_s - X^i_{s-} - \xi^i_s)\right) \\
&= \int_{(S,T)} \alpha_i(X) \circ dX^i + \sum_{S<s<T} \alpha_i(X_{s-})\left(\xi^i_s - \Delta X^i_s\right).
\end{aligned}$$

(2.3) follows in the same way. □

2.2 Second-order stochastic differential geometry with jumps

Next, we recall the theory of second-order stochastic differential geometry with jumps. In [6, 7], S. Cohen formulated the stochastic integral of order 2 along càdlàg semimartingales valued in manifolds and the stochastic defferential equation. In this section we summarize results on them. We refer to [1, 6, 7, 13]. Several works done in [6, 7] have also been summarized in [17]. A linear map $L : C^2(M) \to \mathbb{R}$ is called a second-order differential operator without constant at $x \in M$ if for a local coordinate (x^i) including x, there exist $a^{ij} \in \mathbb{R}$, $b^k \in \mathbb{R}$ $(i, j, k = 1, \ldots, d)$ such that L is denoted by

$$Lf(x) = \sum_{i,j=1}^d a^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{k=1}^d b^k \frac{\partial f}{\partial x^k}(x),\ \ f \in C^2(M).$$

This definition does not depend on local coordinates. Denote the vector space of all second-order differential operators without constants at $x \in M$ by $\mathbb{T}_x M$. The space

$$\mathbb{T}M = \bigsqcup_{x\in M} \mathbb{T}_x M$$

is called the second-order tangent bundle on M. Let $\mathbb{T}_x^*M$ be the dual space of $\mathbb{T}_xM$ for each $x \in M$. The space

$$\mathbb{T}^*M = \bigsqcup_{x\in M} \mathbb{T}_x^*M$$

is called the second-order cotangent bundle on M.

Definition 2.5. *A Borel measurable function $f : M \to \mathbb{R}$ is called a form of order 2 specified in x if f is twice differentiable at x and $f(x) = 0$. Define*

$$\begin{aligned}\overset{\triangle}{\mathbb{T}^*}{}_xM &:= \{f : M \to \mathbb{R} \mid f \text{ is a form of order 2 specified in } x\},\\ \overset{\triangle}{\mathbb{T}^*}M &:= \bigsqcup_{x\in M} \overset{\triangle}{\mathbb{T}^*}{}_xM,\end{aligned}$$

and $\mathcal{G}_x : \overset{\triangle}{\mathbb{T}^*}{}_xM \to \mathbb{T}_x^*M$ by

$$\mathcal{G}_xf(L) := Lf(x),\ \ f \in \overset{\triangle}{\mathbb{T}^*}{}_xM,\ \ L \in \mathbb{T}M.$$

By Theorem 1 of [6], for an M-valued semimartingale X and a $\overset{\triangle}{\mathbb{T}^*}M$-valued predictable locally bounded process Θ above X, we can define the stochastic integral $\displaystyle\int \Theta\, \overset{\triangle}{d}X$. Note that the stochastic integral defined in Theorem 1 of [6] can recover the Itô integral with respect to connection rules. Let γ be a connection rule on M, X an M-valued semimartingale and ϕ a T^*M-valued càdlàg process. We set

$$\left(\overset{\triangle}{\gamma}_{X_{s-}}\phi_{s-}\right)(y) = \langle \phi_{s-}, \gamma(X_{s-}, y)\rangle,\ \ y \in M. \tag{2.4}$$

Then $\overset{\triangle}{\gamma}_{X_{s-}}\phi_{s-} \in \overset{\triangle}{\mathbb{T}^*}{}_{X_{s-}}M$ and

$$\int \overset{\triangle}{\gamma}_{X_{s-}}\phi_{s-}\, \overset{\triangle}{d}X_s = \int \phi_{s-}\, \gamma dX_s.$$

Next we recall the theory of SDE's on manifolds with jumps formulated in [6, 7].

Definition 2.6. *Let M and N be manifolds. Suppose that C is a closed submanifold of $M \times N$ such that the projection p_1 from C to M is onto and a submersion. A measurable map $\varphi : C \times M \to N$ is called a constraint coefficient from $C \times M$ to N if*

(i) for each $(x, y) \in C$, $\varphi(x, y, x) = y$,

(ii) φ is C^3 in a neighborhood of $\{(z,\ p_1(z)) | z \in C\}$,

(iii) for all $x \in M$ and $z \in C$, $(x,\ \varphi(z, x)) \in C$.

In the case $C = M \times N$, we refer to ϕ as just a coefficient of SDE's.

Definition 2.7. *Let M and N be manifolds, C a closed submanifold of $M \times N$, and $\phi : C \times M \to N$ a constraint coefficient from $C \times M$ to N. Fix an M-valued semimartingale X and an N-valued $\mathcal{F}_0$-measurable random variable y_0 with $(X_0, y_0) \in C$. A pair of a positive predictable stopping time η and an N-valued semimartingale Y on $[0, \eta)$ is called a solution of the SDE*

$$\begin{cases} \overset{\triangle}{d}Y = \phi(Y, \overset{\triangle}{d}X), \\ Y_0 = y_0, \end{cases} \tag{2.5}$$

if $Y_0 = y_0$, $(X, Y) \in C$ and for all $\overset{\triangle}{\mathbb{T}}{}^*N$-valued locally bounded predictable processes Θ with $\Theta_t \in \overset{\triangle}{\mathbb{T}}{}^*{}_{Y_t}N$ on $[0, \eta)$,

$$\int \Theta \, \overset{\triangle}{d}Y = \int \phi^* \Theta \, \overset{\triangle}{d}X,$$

where

$$\phi^*\Theta_t(z) := \Theta_t(\phi(X_{t-}, Y_{t-}, z)).$$

Theorem 2 of [6] and Theorem 1 of [8] guarantee that equation (2.5) admits a unique solution.

Remark 2.8. Let $\iota\colon M \to \mathbb{R}^d$ be an embedding. Then by Remark 7 and Proposition 4 of [6], (Y, η) is a solution of the SDE (2.5) if and only if for any $f \in C^\infty(N)$ and $t < \eta$,

$$\begin{aligned} f(Y_t) - f(Y_0) = &\int_0^t \frac{\partial f \circ \Phi_s}{\partial z^i}(X_{s-}) \, dX_s^i + \frac{1}{2} \int_0^t \frac{\partial^2 f \circ \Phi_s}{\partial z^i \partial z^j}(X_{s-}) \, d[X^i, X^j]_s^c \\ &+ \sum_{0 < s \le t} \left\{ f(\Phi_s(X_s)) - f(Y_{s-}) - \frac{\partial f \circ \Phi_s}{\partial z^i}(X_{s-}) \Delta X_s^i \right\}, \end{aligned}$$

where $\Phi_t\colon \mathbb{R}^d \to N$ is an extension of $\phi(X_{t-}, Y_{t-}, \cdot)$ to a function on $\mathbb{R}^d$ which is C^2 at $z = X_{t-}$, $(z^1, \ldots, z^d)$ is the canonical coordinate on $\mathbb{R}^d$ and

$$\iota(X_t) = (X_t^1, \ldots, X_t^d).$$

Example 2.9. Let $\mathbf{A}_i$ $(i = 1, \ldots, r)$ be complete vector fields on M. We suppose that any $\mathbb{R}$-linear combination of $\{\mathbf{A}_i\}_{i=1,\ldots,r}$ is also complete. Let $h\colon M \times \mathbb{R}^r \to \mathbb{R}^r$ be a function which is C^∞ on the neighborhood of $M \times \{0\}$ and

$$h(x, 0) = 0$$

for all $x \in M$. Define $\phi\colon \mathbb{R}^r \times M \times \mathbb{R}^r \to M$ by

$$\phi(z, x, w) := \operatorname{Exp}_x \left(\sum_{k=1}^r h^k(x, w - z) \mathbf{A}_k \right),$$

where $h(x, z) = (h^1(x, z), \ldots, h^r(x, z))$. Then ϕ is a coefficient of SDE's. Let W be a d-dimensional semimartingale with $W_0 = 0$. Then the SDE

$$\overset{\triangle}{d}X = \phi(X, \overset{\triangle}{d}W) \tag{2.6}$$

admits a unique solution (X,η). By Remark 2.8, this means that for all $f \in C^\infty(M)$, X satisfies that

$$\begin{aligned}
f(X_t)-f(X_0) = &\int_0^t \mathbf{A}_k f(X_{s-}) \frac{\partial h_s^k}{\partial w^i}(W_{s-})\, dW_s^i \\
&+ \frac{1}{2}\int_0^t \left\{ \mathbf{A}_k \mathbf{A}_l f(X_{s-}) \frac{\partial h_s^k}{\partial w^i}(W_{s-}) \frac{\partial h_s^l}{\partial w^j}(W_{s-}) \right. \\
&\left. + \mathbf{A}_k f(X_{s-}) \frac{\partial^2 h_s^k}{\partial w^i \partial w^j}(W_{s-}) \right\} d[W^i, W^j]_s^c \\
&+ \sum_{0<s\le t} \left\{ f(\mathrm{Exp}_{X_{s-}}(h_s^k(W_s)\mathbf{A}_k)) - f(X_{s-}) - \mathbf{A}_k f(X_{s-}) \frac{\partial h_s^k}{\partial w^i}(W_{s-}) \Delta W_s^i \right\},
\end{aligned}$$

where

$$h_t^k(w) = h^k(X_{t-}, w - W_{t-}).$$

Lemma 2.10 below states that we can describe the Stratonovich integral and the quadratic variation along the solution of (2.6) by the integral along the driving semimartingale on a Euclidean space.

Lemma 2.10. *Set $\Delta X_t := \mathbf{A}_k(X_{t-}) h^k(X_{t-}, \Delta W_t)$ under the conditions stated in Example 2.9. Suppose that $(\Delta X, X)$ is a Δ-semimartingale and h satisfies*

$$d_0 h(x,\cdot) = \mathrm{id}\colon \mathbb{R}^r \to \mathbb{R}^r$$

for all $x \in M$, where the left-hand side is the derivative of the map $h(x,\cdot)\colon \mathbb{R}^r \to \mathbb{R}^r$ at the origin. Then for $\alpha \in \Omega^1(M)$ and $b \in \Gamma(T^*M \otimes T^*M)$,

$$\begin{aligned}
\int \alpha(X) \circ dX = &\int \langle \alpha, \mathbf{A}_k\rangle(X) \circ dW^k \\
&+ \frac{1}{2}\int \frac{\partial^2 h_s^k}{\partial w^i \partial w^j}(W_{s-}) \langle \alpha, \mathbf{A}_l\rangle\, d[W^k, W^l]^c \\
&+ \sum_{0<s\le t} \langle \alpha(X_{s-}), R_s^k \mathbf{A}_k(X_{s-})\rangle, \\
\int b(X)\, d[X,X] = &\int b(X_-)(\mathbf{A}_k(X_-), \mathbf{A}_l(X_-))\, d[W^k, W^l] \\
&+ \sum_{0<s\le \cdot} \Delta W_s^k R_s^l b(X_{s-})(\mathbf{A}_k, \mathbf{A}_l) \\
&+ \sum_{0<s\le \cdot} R_s^k \Delta W_s^l b(X_{s-})(\mathbf{A}_k, \mathbf{A}_l) \\
&+ \sum_{0<s\le \cdot} R_s^k R_s^l b(X_{s-})(\mathbf{A}_k, \mathbf{A}_l),
\end{aligned}$$

where we set

$$R_t^k := h^k(X_{t-}, \Delta W_t) - \Delta W_t^i.$$

Proof. To begin with, we take $f \in C^\infty(M)$ and $\alpha = df$. Then by Itô's formula,

$$f(X_t) - f(X_0) = \int_0^t df(X) \circ dX + \sum_{0<s\le t} \{f(X_s) - f(X_{s-}) - \langle df(X_{s-}), \Delta X_s\rangle\}$$

Note that

$$\mathbf{A}_k(X_{s-})\frac{\partial h_s^k}{\partial w^i}(W_{s-}) = \mathbf{A}_i(X_{s-}).$$

by the assumption for h. Then in view of Example 2.9, we have

$$\begin{aligned}
\int_0^t df(X) \circ dX &= f(X_t) - f(X_0) - \sum_{0<s\le t} \{f(X_s) - f(X_{s-}) - \langle df(X_{s-}), \Delta X_s\rangle\} \\
&= \int_0^t \mathbf{A}_k f(X_{s-})\frac{\partial h_s^k}{\partial w^i}(W_{s-})\, dW_s^i \\
&\quad + \frac{1}{2}\int_0^t \Bigg\{\mathbf{A}_k\mathbf{A}_l f(X_{s-})\frac{\partial h_s^k}{\partial w^i}(W_{s-})\frac{\partial h_s^l}{\partial w^j}(W_{s-}) \\
&\quad + \mathbf{A}_k f(X_{s-})\frac{\partial^2 h_s^k}{\partial w^i \partial w^j}(W_{s-})\Bigg\}\, d[W^i, W^j]_s^c \\
&\quad + \sum_{0<s\le t} \langle df(X_{s-}), \Delta X_s - \mathbf{A}_k(X_{s-})\frac{\partial h_t^k}{\partial w^k}(W_{s-})\Delta W_s^i\rangle \\
&= \int_0^t \langle df, \mathbf{A}_i\rangle(X_{s-})\, dW_s^i + \frac{1}{2}\int_0^t \mathbf{A}_i\mathbf{A}_j f(X_{s-})\, d[W^i, W^j]_s^c \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 h_s^k}{\partial w^i \partial w^j}(W_{s-})\langle df, \mathbf{A}_k\rangle(X_{s-})\, d[W^i, W^j]_s^c \\
&\quad + \sum_{0<s\le t} \left\langle df(X_{s-}), R_s^k \mathbf{A}_k(X_{s-})\right\rangle.
\end{aligned}$$

Moreover, by substituting f to $\mathbf{A}_i f$ in the equality in Example 2.9, we have

$$\begin{aligned}
\int_0^t \langle df, \mathbf{A}_i\rangle(X) \circ dW_s^i &= \int_0^t \langle df, \mathbf{A}_i\rangle(X_{s-})\, dW_s^i + \frac{1}{2}[\mathbf{A}_i f(X), W^j]_s^c \\
&= \int_0^t \langle df, \mathbf{A}_i\rangle(X_{s-})\, dW_s^i + \frac{1}{2}\int_0^t \mathbf{A}_i\mathbf{A}_j f(X_{s-})\, d[W^i, W^j]_s^c.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_0^t df(X) \circ dX &= \int_0^t \langle df, \mathbf{A}_k\rangle(X) \circ dW_s^k \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 h_s^k}{\partial w^i \partial w^j}(W_{s-})\langle df, \mathbf{A}_k\rangle(X_{s-})\, d[W^i, W^j]_s^c \\
&\quad + \sum_{0<s\le t} \left\langle df(X_{s-}), R_s^k \mathbf{A}_k(X_{s-})\right\rangle. \qquad (2.7)
\end{aligned}$$

Next, we take another $g \in C^\infty(M)$. Then by (2.7),

$$\begin{aligned}\int df \otimes dg(X)\, d[X,X] &= \left[\int df(X) \circ dX, \int dg(X) \circ dX\right] \\ &= \int df \otimes dg(\mathbf{A}_k, \mathbf{A}_l)(X_-)\, d[W^k, W^l] \\ &\quad + \sum_{0<s\le\cdot} \Delta W_s^k R_s^l df \otimes dg(\mathbf{A}_k, \mathbf{A}_l)(X_{s-}) \\ &\quad + \sum_{0<s\le\cdot} R_s^k \Delta W_s^l df \otimes dg(\mathbf{A}_k, \mathbf{A}_l)(X_{s-}) \\ &\quad + \sum_{0<s\le\cdot} R_s^k R_s^l df \otimes dg(\mathbf{A}_k, \mathbf{A}_l)(X_{s-}). \end{aligned} \tag{2.8}$$

Thus we obtain the desired equality for $\alpha = df$ and $b = df \otimes dg$. Without loss of generality, we can suppose that M is isometrically embedded in a higher dimensional Euclidean space by $\iota\colon M \to \mathbb{R}^d$. Note that any $\alpha \in \Omega^1(M)$ and $b \in \Gamma(T^*M \otimes T^*M)$ can be expressed as

$$\alpha = \alpha_i d\iota^i, \tag{2.9}$$

$$b = b_{ij} d\iota^i \otimes d\iota^j, \tag{2.10}$$

where $\alpha_i = \langle \alpha, \nabla \iota^i \rangle$, $b_{ij} = b(\nabla\iota^i, \nabla\iota^j)$. These expressions yield the desired equality for general α and b. $\square$

3 Proofs of Theorems 1.11 and 1.12

Let (M, g) be a complete and connected Riemannian manifold and $\mathcal{O}(M)$ an orthonormal frame bundle on M. We use the notation defined in the previous sections. In this section, we prove Theorems 1.11 and 1.12.

3.1 Proofs of Theorem 1.11 (1)–(3)

Proof of Theorem 1.11 (2). By Itô's formula, the horizontal Δ-semimartingale $(\Delta U, U)$ satisfies

$$\begin{aligned} F(U_t) - F(U_0) &= \int_0^t dF(U_s) \circ dU_s \\ &\quad + \sum_{0<s\le t} \{F(U_s) - F(U_{s-}) - \langle dF(U_{s-}), \Delta U_s\rangle\}. \end{aligned} \tag{3.1}$$

Since dF can be written as $dF = \mathcal{L}_k F \mathfrak{s}^k + \mathcal{X}_\alpha^\sharp F \theta^\alpha$, by Proposition 2.1, equation (3.1) becomes

$$\begin{aligned} F(U_t) - F(U_0) &= \int_0^t \mathcal{L}_k F(U_s) \circ dW_s^k \\ &\quad + \sum_{0<s\le t} \{F(U_s) - F(U_{s-}) - \mathcal{L}_k F(U_{s-}) \Delta W_s^k\}, \end{aligned}$$

where we used $W = \int \mathfrak{s}(U) \circ dU$, $\Delta W = \langle \mathfrak{s}(U_-), \Delta U \rangle$. This implies that U is the stochastic development of W. □

We denote by C_g the set of connection rules γ which satisfy the following: for all $x, y \in M$, $\exp_x t\gamma(x,y)$, $t \in [0,1]$ is a minimal geodesic connecting x and y. If M is a strongly convex Riemannian manifold, $\gamma \in C_g$ can be written as

$$\gamma(x,y) = \exp_x^{-1} y. \tag{3.2}$$

In general, as we will show in the Proposition 3.1 below, as long as we assume that (M,g) is a complete connected Riemannian manifold, we can take a connection rule γ such that $\gamma(x,y)$ is an initial velocity of a minimal geodesic connecting x and y for all $x, y \in M$ even though the cut locus is not empty. We use the notion $\|\cdot\|$ as the norm with respect to the Riemannian metric.

Proposition 3.1. *Let (M,g) be a complete and connected Riemannian manifold. Then $C_g \neq \varnothing$.*

Proof. Let $\pi_{TM} : TM \to M$ be a canonical projection associated with the tangent bundle and let

$$UM := \{\mathbf{A} \in TM \mid |\mathbf{A}| = 1\}.$$

Define $t : UM \to [0,\infty]$ by

$$t(\mathbf{A}) := \sup\{t \geq 0 \mid d(\pi_{TM}\mathbf{A}, \exp(t\mathbf{A})) = t\},$$

where d is the Riemannian distance. Set

$$D_x := \{t\mathbf{A} \in TM \mid \mathbf{A} \in U_x M,\ t \in [0, t(u)]\},$$

$$D := \bigsqcup_{x \in M} D_x.$$

Then D is a closed subset of TM. Define $F : TM \to M \times M$ by

$$F(\mathbf{A}) := (\pi_{TM}\mathbf{A}, \exp(\mathbf{A})).$$

Then F is a continuous map because the solution of the geodesic equations depends continuously on the initial value. We further set

$$\Phi(x,y) := \{\mathbf{A} \in T_x M \mid \exp_x \mathbf{A} = y,\ \|\mathbf{A}\| = d(x,y)\}.$$

and

$$\Phi^{-1}(B) := \{(x,y) \mid \Phi(x,y) \cap B \neq \varnothing\}$$

for $B \subset TM$. Then for any compact set $C \subset TM$, we have

$$\Phi^{-1}(C) = F(C \cap D).$$

Since F is continuous, this is a compact subset in $M \times M$. In particular, $\Phi^{-1}(C)$ is a Borel subset. Therefore by measurable section theorem ([21], Theorem 5.2), there exists a map $\gamma : M \times M \to TM$ such that $\gamma(x,y) \in \Phi(x,y)$ and $\gamma^{-1}(C)$ is a Borel set for any compact subset $C \subset TM$. Then γ is Borel measurable. Moreover, since γ can be written in the form of (3.2) whenever possible, it satisfies conditions (i)–(iii) in Definition 1.1. □

Lemma 3.2. *Let $(\Delta X, X)$ be a $TM \times M$-valued process satisfying (i), (ii), (iii) in Definition 1.4 and* (1.8)*. We further assume that for every $t > 0$, ΔX satisfies*

$$\sum_{0<s\le t} |\Delta X_s|^2 < \infty \ a.s.$$

Then for all $\gamma \in C_g$ and $t > 0$, the number of $s \in [0,t]$ with

$$\Delta X_s \neq \gamma(X_{s-}, X_s)$$

is finite almost surely. Furthermore, $(\Delta X, X)$ is a Δ-semimartingale, namely, the process ΔX also satisfies (iv) in Definition 1.4.

Proof. Fix $t \geq 0$, $\omega \in \Omega$, and a connection rule $\gamma \in C_g$. Let $r : M \to [0,\infty]$ be the injectivity radius. Then r is positive and continuous on M. Since the path $\overline{X(\omega,[0,t])}$ is compact, r admits the minimum value r_0 on the set. Since $\sum_{0<s\le t} |\Delta X_s(\omega)|^2 < \infty$, the number of $s \in [0,t]$ with $|\Delta X_s(\omega)| \geq r_0$ is finite. For s with $|\Delta X_s| < r_0$, we have $\Delta X_s(\omega) = \gamma(X_{s-}(\omega), X_s(\omega))$ since both of them are the initial velocity of the minimal geodesic connecting $X_{s-}(\omega)$ and $X_s(\omega)$. Therefore the number of $s \in [0,t]$ with $\Delta X_s(\omega) \neq \gamma(X_{s-}(\omega), X_s(\omega))$ is finite. Furthermore, for all T^*M-valued càdlàg processes ϕ above X, condition (1.5) is satisfied and hence $(\Delta X, X)$ is a Δ-semimartingale. □

Proofs of Theorem 1.11 (1) and (3). By virtue of Theorem 1.11 (2), any horizontal Δ-semimartingale $(\Delta U, U)$ can be described as the development of the anti-development W of $(\Delta U, U)$. Thus we deal with (1) and (3) in Theorem 1.11 simultaneously. Since $\mathrm{Exp}_{U_{s-}} \Delta W_s^k \mathcal{L}_k = U_s$, ΔU is the initial velocity of the geodesic from U_{s-} to U_s with regard to $\tilde{g}$ and ΔX_s is the initial velocity of the geodesic from X_{s-} to X_s. Since $[W,W]_t(\omega) < \infty$, $\sum_{0<s\le t} |\Delta W_s(\omega)|^2 < \infty$ for any fixed $t \geq 0$. Hence it holds that $\sum_{0<s\le t} |\Delta X_s(\omega)|^2 < \infty$ and $\sum_{0<s\le t} |\Delta U_s(\omega)|^2 < \infty$ because

$$|\Delta W| = |\Delta U| = |\Delta X|.$$

Therefore by Lemma 3.2, $(\Delta X, X)$ is a Δ-semimartingale. This completes the proof of Theorem 1.11 (3). By Lemma 2.10 and Proposition A.1 in Appendix,

$$\int \theta(U) \circ dU = \int \langle \theta, \mathcal{L}_k \rangle \circ dW^k = 0,$$
$$\int \tilde{\nabla}\theta(U_-)\, d[U,U]^c = \int \tilde{\nabla}\theta(\mathcal{L}_k(U_-), \mathcal{L}_l(U_-))\, d[W^k, W^l]^c = 0.$$

Thus we obtain

$$\int \theta(U_-)\, dU = 0.$$

Similarly, it holds that

$$\int \mathfrak{s}^k(U) \circ dU = \int \langle \mathfrak{s}^k, \mathcal{L}_l \rangle \circ dW^l = \int \delta^k_l \circ dW^l = W^k,$$
$$\int \tilde{\nabla}\mathfrak{s}(U_-)\, d[U,U]^c = \int \tilde{\nabla}\mathfrak{s}(\mathcal{L}_k(U_-), \mathcal{L}_l(U_-))\, d[W^k, W^l]^c = 0.$$

Hence we obtain

$$\int \mathfrak{s}(U_-)\, dU = W.$$

This completes the proof of Theorem 1.11 (1). □

3.2 Proof of Theorem 1.11 (4)

In this subsection, we divide the proof of Theorem 1.11 (4) into Theorems 3.7, 3.10, and 3.11. Let $\gamma \in C_g$. We set

$$C := \{(x,u) \in M \times \mathcal{O}(M) \mid \pi u = x\}$$

and define a map $\varphi : C \times M \to \mathcal{O}(M)$ by

$$\varphi(x,u,y) := \tilde{\eta}_{x,y,u}(1),\ x, y \in M,\ u \in \mathcal{O}_x(M), \tag{3.3}$$

where $\eta_{x,y}(t)$ is the geodesic on M with $\eta_{x,y}(0) = x$, $\eta'_{x,y}(0) = \gamma(x,y)$, and $\tilde{\eta}_{x,y,u}$ is a horizontal lift of $\eta_{x,y}$ whose initial value is u. Since γ is measurable on $M \times M$ and differentiable on the diagonal set of $M \times M$, the map φ is a constraint coefficient of SDE's. We define $D_{\mathcal{O}(M)}, C_{\mathcal{O}(M)} \subset \mathcal{O}(M) \times \mathcal{O}(M)$ by

$$\begin{aligned} D_{\mathcal{O}(M)} = \{(u,v) \in \mathcal{O}(M) \times \mathcal{O}(M) \mid\ & u \text{ and } v \text{ can be connected} \\ & \text{by a unique minimal geodesic with respect to } \tilde{g}\}, \\ C_{\mathcal{O}(M)} = \mathcal{O}(M) \times \mathcal{O}(M) \backslash & D_{\mathcal{O}(M)}. \end{aligned}$$

Then every pair (u,v) of elements in $\mathcal{O}(M)$ can be classified into $D_{\mathcal{O}(M)}$ and $C_{\mathcal{O}(M)}$ in terms of the proximity between u and v. We define the connection rule $\tilde{\gamma}$ on $\mathcal{O}(M)$ by

$$\tilde{\gamma}(u,v) = \begin{cases} \text{The initial velocity of minimal geodesic from } u \text{ to } v, & (u,v) \in D_{\mathcal{O}(M)}, \\ (\pi_*|_{H_u})^{-1}\gamma(\pi u, \pi v), & (u,v) \in C_{\mathcal{O}(M)}, \end{cases}$$

where H_u is the horizontal subspace of $T_u\mathcal{O}(M)$ defined by (1.2). We can easily check that the map $\tilde{\gamma}$ is actually a connection rule on $\mathcal{O}(M)$. Indeed, the conditions (i), (ii), (iii) can be easily checked since $\tilde{\gamma}$ is defined through the exponential map near the diagonal set of $\mathcal{O}(M) \times \mathcal{O}(M)$ and the Borel-measurability of $\tilde{\gamma}$ on the whole $\mathcal{O}(M) \times \mathcal{O}(M)$ is derived from the measurability of γ.

Remark 3.3. Even if u and v in $\mathcal{O}(M)$ can be connected by a horizontal geodesic, the tangent vector $\tilde{\gamma}(u,v)$ is not necessarily horizontal. However, if one of the minimal geodesics between u and v is horizontal, then $\tilde{\gamma}(u,v)$ is horizontal and it coincides the horizontal lift of the initial velocity of a minimal geodesic connecting πu and πv.

For later use, we start with three lemmas.

Lemma 3.4. *For $u, v \in \mathcal{O}(M)$ and $a \in O(d)$, it holds that*

$$R_{a*}\tilde{\gamma}(u,v) = \tilde{\gamma}(ua, va).$$

Proof. It holds that $(u,v) \in D_{\mathcal{O}(M)} \Leftrightarrow (ua, va) \in D_{\mathcal{O}(M)}$ for $u, v \in \mathcal{O}(M)$ and $a \in O(d)$ by Proposition A.6 in Appendix. First let $(u,v) \in D_{\mathcal{O}(M)}$ and set $\tau(t) = \exp_u t\tilde{\gamma}(u,v)$. Then $R_a\tau(t)$ is a unique minimal geodesic from ua to va by Proposition A.6 in Appendix again. Therefore

$$R_{a*}\tilde{\gamma}(u,v) = R_{a*}\frac{d\tau}{dt}(0) = \frac{d}{dt}R_a\tau(0) = \tilde{\gamma}(ua, va).$$

Next we suppose $(u,v) \in C_{\mathcal{O}(M)}$. Then $\tilde{\gamma}(u,v)$ is the horizontal lift of $\gamma(\pi u, \pi v)$ at u. Thus $R_{a*}\tilde{\gamma}(u,v)$ is the horizontal lift of $\gamma(\pi u, \pi v)$ at ua and equals $\tilde{\gamma}(ua, va)$. $\square$

Lemma 3.5. *Let $(\Delta U, U)$ be an $\mathcal{O}(M)$-valued Δ-semimartingale satisfying* (1.8)*. Then $(\Delta U, U)$ is horizontal if and only if it holds that*

$$\int \theta(U_-)\, dU = 0. \tag{3.4}$$

Proof. By (1) and (3) of Theorem 1.11, if $(\Delta U, U)$ is horizontal, then (3.4) holds. Conversely, suppose that (3.4) holds. For any stopping times S, T with $S \le T$, it holds that

$$\tilde{\nabla}\theta(U_S)(\tilde{\gamma}(U_S, U_T), \tilde{\gamma}(U_S, U_T)) = 0 \tag{3.5}$$

by Proposition A.1 in Appendix. Similarly, it holds that for $s \ge 0$,

$$\begin{aligned} \tilde{\nabla}\theta(U_{s-})(\tilde{\gamma}(U_{s-}, U_s), \tilde{\gamma}(U_{s-}, U_s)) = 0, \\ \tilde{\nabla}\theta(U_{s-})(\Delta U_s, \Delta U_s) = 0. \end{aligned} \tag{3.6}$$

Equations (3.5) and (3.6) imply

$$\int \tilde{\nabla}\theta(U_-)\, d[U,U] = 0.$$

In particular, we have

$$\int \theta(U) \circ dU = \int \theta(U_-)\, dU + \frac{1}{2}\int \nabla\theta(U_-)\, d[U,U]^c = 0.$$

Therefore, $(\Delta U, U)$ is horizontal. $\square$

Lemma 3.6. *Let X be an M-valued semimartingale. Fix an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable u_0 such that $u_0 \in \mathcal{O}_{X_0}(M)$. Let $\varphi\colon C \times M \to \mathcal{O}(M)$ be a map defined in* (3.3)*. Let U be a semimartingale valued in $\mathcal{O}(M)$ solving the following SDE*

$$\begin{cases} \overset{\triangle}{d}U = \varphi(U, \overset{\triangle}{d}X), \\ U_0 = u_0. \end{cases} \tag{3.7}$$

Then U does not explode in finite time with probability one and

$$\int \theta \circ \tilde{\gamma} dU = \int \theta \ \tilde{\gamma} dU = 0. \tag{3.8}$$

In particular, the Δ-semimrtingale $(\tilde{\gamma}(U_-, U), U)$ is a horizontal lift of $(\gamma(X_-, X), X)$. Furthermore, it holds that

$$\int \mathfrak{s} \circ \tilde{\gamma} dU = \int \mathfrak{s} \ \tilde{\gamma} dU, \tag{3.9}$$

$$\int \mathfrak{s}^k(U_-) \, \tilde{\gamma} dU = \int U_- \varepsilon^k \ \gamma dX, \ k = 1, \ldots, d. \tag{3.10}$$

Proof. Since the map φ satisfies

$$\pi(\varphi(x, u, y)) = y \text{ for } x, y \in M, \ u \in \mathcal{O}(M)$$

and U is the solution of (3.7), it holds that

$$\pi(U_t) = X_t$$

by Remark 2.8. Moreover, it holds that

$$\varphi(X_{t-}, U_{t_-}, X_t) = U_t.$$

This implies that U_{t-} is connected to U_t by the horizontal lift of $\exp_{X_{t-}} t\gamma(X_{t-}, X_t)$, which is one of the minimal geodesics on $(\mathcal{O}(M), \tilde{g})$ between U_{t-} and U_t by Proposition A.3. Therefore $\tilde{\gamma}(U_{t-}, U_t)$ is horizontal by the definition of $\tilde{\gamma}$. Let ζ be an explosion time of U and assume

$$\mathbb{P}(\zeta < \infty) > 0.$$

Then for $\omega \in \{\zeta < \infty\}$, $\{U_t(\omega)\}_{0 \leq t < \zeta(\omega)}$ is not relatively compact. On the other hand, since X does not explode in finite time,

$$A(\omega) := \{X_t(\omega) \mid 0 \leq t \leq \zeta(\omega)\}$$

is relatively compact in M. Now it holds that $\{U_t(\omega)\}_{0 \leq t < \zeta(\omega)} \subset \pi^{-1}(\overline{A(\omega)})$ and the right-hand side is compact because $O(d)$ is compact. This is a contradiction. Therefore $\zeta = \infty$, $\mathbb{P}$-a.s. Next we will show the second claim. By taking a $\overset{\triangle}{\mathbb{T}}\mathcal{O}(M)$-valued càdlàg process $\theta_{U_{t-}}(\tilde{\gamma}(U_{t-}, \cdot))$, we have

$$\begin{aligned}
\int \theta(U_-) \ \tilde{\gamma} dU &= \int \theta_{U_-}(\tilde{\gamma}(U_-, \cdot)) \ \overset{\triangle}{d} U \\
&= \int \theta_{U_-}(\tilde{\gamma}(U_-, \phi(X_-, U_-, \cdot))) \ \overset{\triangle}{d} X \\
&= 0.
\end{aligned}$$

Therefore by Lemma 3.5, equation (3.8) holds and consequently, $(\tilde{\gamma}(U_-, U), U)$ is a horizontal lift of $(\gamma(X_-, X), X)$. Therefore (3.9) can be obtained by Theorem 1.11 (1). Finally we will show (3.10). We begin with the left-hand side of the claimed equation:

$$\begin{aligned}\int \mathfrak{s}^k(U_-)\ \tilde{\gamma} dU &= \int \pi^*(U_-\varepsilon^k)\ \tilde{\gamma} dU \\ &= \int \pi^*(U_-\varepsilon^k)(\tilde{\gamma}(U_-, \cdot))\ \overset{\triangle}{d} U \\ &= \int \pi^*(U_-\varepsilon^k)(\tilde{\gamma}(U_-, \phi(X_-, U_-, \cdot)))\ \overset{\triangle}{d} X \\ &= \int U_-\varepsilon^k(\gamma(X_-, \cdot))\ \overset{\triangle}{d} X \\ &= \int (U_-\varepsilon^k)\ \gamma dX.\end{aligned}$$

Therefore we obtain (3.10) and this completes the proof. □

Now Lemma 3.6 guarantees the existence of horizontal lift of Δ-semimartingales of the form $(\gamma(X_-, X), X)$ with $\gamma \in C_g$. Next we show the existence of the horizontal lift of any Δ-semimartingale with (1.8) in Theorem 3.7 below. In the proof of Theorem 3.7, we first take the horizontal lift $(\tilde{\gamma}(U_-^\gamma, U^\gamma), U^\gamma)$ of $(\gamma(X_-, X), X)$ for some $\gamma \in C_g$ and then we construct a suitable $O(d)$-valued process a_t and set $U = U^\gamma a$ in order that the jumps $\tilde{\gamma}(U_-^\gamma, U^\gamma)$ are replaced into the horizontal lifts of ΔX.

Theorem 3.7. *Let $(\Delta X, X)$ be an M-valued Δ-semimartingale satisfying* (1.8) *and u_0 an $\mathcal{O}_{X_0}(M)$-valued $\mathcal{F}_0$-measurable random variable. Then there exists a horizontal lift $(\Delta U, U)$ of $(\Delta X, X)$ with $U_0 = u_0$ satisfying* (1.8).

Proof. Fix a connection rule $\gamma \in C_g$. Let U^γ be the horizontal lift of $(\gamma(X_-, X), X)$. If an $\mathcal{O}(M)$-valued semimartingale U satisfies $\pi U = X$, there exists an $O(d)$-valued process a_s such that $U = U^\gamma a$. We will specify the process a_s. For each $s \geq 0$, Set

$$\begin{aligned} c_s^1(t) &:= \exp t\gamma(X_{s-}, X_s),\ t \in [0, 1], \\ c_s^2(t) &:= \exp t\Delta X_s,\ t \in [0, 1], \\ (c_s^1)^{-1}(t) &:= c_s^1(1-t),\ t \in [0, 1], \end{aligned}$$

and

$$c_s^2 \cdot \left(c_s^1\right)^{-1} = \begin{cases} (c_s^1)^{-1}(2t),\ t \in [0, \frac{1}{2}], \\ c_s^2(2t-1),\ t \in [\frac{1}{2}, 1]. \end{cases}$$

Denote by $\widetilde{c_s^2 \cdot (c_s^1)^{-1}}$ the horizontal lift of $c_s^2 \cdot (c_s^1)^{-1}$ starting at U_s^γ. Then there exists a unique element $b_s \in O(d)$ satisfying

$$\widetilde{c_s^2 \cdot (c_s^1)^{-1}}(1) = U_s^\gamma b_s. \tag{3.11}$$

Since b_s equals the unit element e in $O(d)$ for $s \geq 0$ with $\gamma(X_{s-}, X_s) = \Delta X_s$, b_s equals the unit element except for finite number of $s \in [0, t]$ for fixed $t \geq 0$ by Lemma 3.2. Let $0 < T_1 < T_2 < \cdots$ be a sequence of stopping times which exhausts the time s with $\gamma(X_{s-}, X_s) \neq \Delta X_s$. We define $O(d)$-valued processes δ_s, a_s as follows:

$$\begin{aligned} &a_s = \delta_s = e, \;\; s \in [0, T_1), \\ &\delta_s = \left(a_{T_{i-1}}\right)^{-1} b_{T_i} a_{T_{i-1}}, \;\; s \in [T_i, T_{i+1}), \\ &a_s = a_{T_{i-1}} \delta_{T_i}, \;\; s \in [T_i, T_{i+1}). \end{aligned} \tag{3.12}$$

Then a_t satisfies

$$a_t = \prod_{0 < s \leq t} b_{t-s} \tag{3.13}$$

for each t. We set

$$U_s := U_s^\gamma a_s,$$

and

$$\Delta U_s := (\text{the horizontal lift of } \Delta X_s \text{ at } U_{s-}).$$

Obviously we have $\pi U = X$ and $\pi_* \Delta U = \Delta X$. Let us prove that $(\Delta U, U)$ is a Δ-semimartingale satisfying (1.8). Denote the horizontal lift of ΔX_s at U_{s-}^γ by $\Delta U_s'$. Then

$$\Delta U_s = R_{a_{s-}*} \Delta U_s'.$$

By the definition of b_s, it holds that

$$\exp_{U_{s-}^\gamma} \Delta U_s' = U_{s-}^\gamma b_s.$$

Therefore we obtain

$$\begin{aligned} \exp_{U_{s-}} \Delta U_s &= \exp_{U_{s-} a_{s-}} \left(R_{a_{s-}*} \Delta U_s'\right) \\ &= \left(\exp_{U_{s-}^\gamma} \Delta U_s'\right) a_{s-} \\ &= U_s^\gamma b_s a_{s-} \\ &= U_s^\gamma a_s \\ &= U_s. \end{aligned}$$

At the second equality, we used Proposition A.3 (2) in Appendix and at the third equality, we used (3.13). Thus $(\Delta U, U)$ satisfies (1.8) and consequently it is a Δ-semimartingale by Lemma 3.2. Next we prove that $(\Delta U, U)$ is horizontal. It suffices to show that

$$\int \theta(U_-) \, dU = 0, \tag{3.14}$$

by Lemma 3.5. For each i, it is obvious that

$$\langle \theta(U_{T_i -}), \Delta U_{T_i} \rangle = 0 \tag{3.15}$$

by the definition of ΔU. For $r, s \in (T_i, T_{i+1})$ with $r < s$, by Lemma 3.4,

$$\begin{aligned}\langle \theta(U_r), \tilde{\gamma}(U_r, U_s) \rangle &= \langle \theta(U_r^\gamma a_{T_i}), \tilde{\gamma}(U_r^\gamma a_{T_i}, U_s^\gamma a_{T_i}) \rangle \\ &= \langle \theta(R_{a_{T_i}} U_r^\gamma), R_{a_{T_i}*} \tilde{\gamma}(U_r^\gamma, U_s^\gamma) \rangle \\ &= \mathrm{Ad}(a_{T_i}) \langle \theta(U_r^\gamma), \tilde{\gamma}(U_r^\gamma, U_s^\gamma) \rangle.\end{aligned}$$

Therefore it holds that

$$\int_{(T_i, T_{i+1})} \theta(U_{s-})\, dU_s = \mathrm{Ad}(a_{T_i}) \int_{(T_i, T_{i+1})} \theta(U_s)\, \tilde{\gamma} dU_s = 0 \tag{3.16}$$

for each i. Combining (3.15) and (3.16), we obtain (3.14). Therefore we can deduce that $(\Delta U, U)$ is a horizontal lift of $(\Delta X, X)$. □

Next we show the uniqueness of the horizontal lift. Let $(\Delta U, U)$ be a horizontal lift of $(\Delta X, X)$ satisfying $U_0 = u_0$ and $\exp_{U_{s-}} \Delta U_s = U_s$. Let $\gamma \in C_g$ and denote the horizontal lift of $(\gamma(X_-, X), X)$ with an initial value u_0 by U^γ again. Then there exists an $O(d)$-valued adapted process a_t satisfying $U = U^\gamma a$ and such a_t is unique since the action of $O(d)$ to each fiber of $\mathcal{O}(M)$ is free. Note that it holds that

$$\Delta U_s = (\text{the horizontal lift of } \Delta X_s \text{ at } U_{s-})$$

by the definition of horizontal lifts. We will show that the process a_t equals the process which has been constructed in the proof of Theorem 3.7. We denote U^γ by V to simplify the notation.

Lemma 3.8. *It holds that*

$$a_{s-}(\omega) \neq a_s(\omega) \Rightarrow \gamma(X_{s-}(\omega), X_s(\omega)) \neq \Delta X_s(\omega)$$

for $s \geq 0$, $\omega \in \Omega$.

Proof. If s and ω satisfy

$$\gamma(X_{s-}(\omega), X_s(\omega)) = \Delta X_s(\omega),$$

then each of $\Delta U_s(\omega)$ and $\Delta V_s(\omega)$ is the horizontal lift of $\gamma(X_{s-}(\omega), X_s(\omega))$ at U_{s-} and V_{s-}, respectively. Therefore we have

$$\exp t\Delta U_s(\omega) = (\exp t\Delta V_s(\omega))\, a_{s-}(\omega),\ \ t \in [0, 1]$$

by Proposition A.2 in Appendix. In particular, $U_s(\omega) = V_s(\omega) a_{s-}(\omega)$. On the other hand, the process a_s satisfies $U_s(\omega) = V_s(\omega) a_s(\omega)$. Thus we can deduce $a_s(\omega) = a_{s-}(\omega)$. □

Combining Lemma 3.2 and Lemma 3.8, for any fixed $t \geq 0$ and $\omega \in \Omega$, the number of $s \in [0, t]$ with $a_s(\omega) \neq a_{s-}(\omega)$ is finite. Let $T_1 < T_2 < \cdots$ be a sequence of stopping times which exhausts the jumps of a_t. Then we have $\Delta U_s = \tilde{\gamma}(U_{s-}, U_s)$, $s \in (T_i, T_{i+1})$. Next we will show that a_s is constant on each $[T_i, T_{i+1})$. This can be shown in the same way as Theorem 3.2 in [23].

Lemma 3.9. *Suppose that j is the canonical 1-form, which is a 1-form on $O(d)$ valued in $\mathfrak{o}(d)$ defined by*

$$j(fA) = A,\ \ f \in O(d),\ \ A \in \mathfrak{o}(d).$$

Then it holds that

$$\int_{(T_i,T_{i+1})} j \circ da = 0 \tag{3.17}$$

and consequently, a_t is constant on (T_i, T_{i+1}) for each i.

Proof. Define $\Phi : \mathcal{O}(M) \times O(d) \to \mathcal{O}(M)$ and $\Phi_u : O(d) \to \mathcal{O}(M)$ by

$$\begin{aligned} \Phi(u, g) &= u \cdot g, \\ \Phi_u(g) &= u \cdot g. \end{aligned}$$

for $u \in \mathcal{O}(M)$. Then

$$U_t = \Phi(V_t, a_t).$$

Let $(\mathcal{U}; u^\alpha)$ be a local coordinate of $\mathcal{O}(M)$. Suppose that V lives in the coordinate neighborhood $\mathcal{U}$ on the random interval $[\sigma, \tau) \subset [T_i, T_{i+1})$, where σ and τ are stopping times. We denote $V^\alpha = u^\alpha(V)$, and $U^\alpha = u^\alpha(U)$ on $[\sigma, \tau)$. Let (a^k) be a coordinate of $O(d)$. Then the connection form θ can be expressed as $\theta = \theta_\alpha du^\alpha$ with $\theta_\alpha \in C^\infty(\mathcal{U}; \mathfrak{o}(d))$. By Itô's formula, it holds that

$$\begin{aligned} U_t^\alpha - U_\sigma^\alpha = &\int_{(\sigma,t]} \left\{ \frac{\partial \Phi^\alpha}{\partial u^\beta}(V_{s-}, a_s) \circ dV_s^\beta + \frac{\partial \Phi^\alpha}{\partial a^k}(V_{s-}, a_s) \circ da_s^k \right\} \\ &+ \sum_{\sigma < s \le t} \left\{ \Phi^\alpha(V_s, a_s) - \Phi^\alpha(V_{s-}, a_s) - \frac{\partial \Phi^\alpha}{\partial u^\beta}(V_{s-}, a_s) \Delta V_s^\beta \right\} \end{aligned}$$

for $t \in (\sigma, \tau)$. Therefore by Proposition 2.4,

$$\begin{aligned} \int_{(\sigma,t]} \theta \circ dU &= \int_{(\sigma,t]} \theta_\alpha \circ dU^\alpha + \sum_{\sigma < s \le t} \left\langle \theta, \Delta U - \Delta U^\alpha \frac{\partial}{\partial u^\alpha} \right\rangle \\ &= \int_{(\sigma,t]} \theta_\alpha \left\{ \frac{\partial \Phi^\alpha}{\partial u^\beta}(V_{s-}, a_s) \circ dV_s^\beta + \frac{\partial \Phi^\alpha}{\partial a^k}(V_{s-}, a_s) \circ da_s^k \right\} \\ &\quad + \sum_{\sigma < s \le t} \theta_\alpha \left\{ \Phi^\alpha(V_s, a_s) - \Phi^\alpha(V_{s-}, a_s) - \frac{\partial \Phi^\alpha}{\partial u^\beta}(V_{s-}, a_s) \Delta V_s^\beta \right\} \\ &\quad + \sum_{\sigma < s \le t} (\langle \theta, \Delta U \rangle - \theta_\alpha \Delta U^\alpha), \end{aligned} \tag{3.18}$$

Note that it holds that

$$\begin{aligned} \theta_\alpha \frac{\partial \Phi^\alpha}{\partial u^\beta}(u, a) &= \mathrm{Ad}(a^{-1}) \theta_\beta(u), \\ \theta_\alpha \frac{\partial \Phi^\alpha}{\partial a^k}(u, a) &= \left\langle d\left(\Phi_u^* \theta\right), \frac{\partial}{\partial a^k} \right\rangle = \left\langle j, \frac{\partial}{\partial a^k} \right\rangle. \end{aligned}$$

In addition, it holds that

$$\begin{aligned}\langle\theta(U_{s-}),\Delta U_s\rangle =&\langle\theta(V_{s-}a_s),R_{a_s*}\Delta V_s\rangle\\ =&\langle R_{a_s}^*\theta(V_{s-}),\Delta V_s\rangle\\ =&\langle\mathrm{Ad}(a_s^{-1})\theta(V_{s-}),\Delta V_s\rangle.\end{aligned}$$

Therefore (3.18) can be rewritten as

$$\begin{aligned}&\int_{(\sigma,t]}\theta\circ dU\\ &\quad=\int_{(\sigma,t]}\left\{\mathrm{Ad}(a^{-1})\theta_\beta(V)\circ dV^\beta+\left\langle j,\frac{\partial}{\partial a^k}\right\rangle(a)\circ da^k\right\}\\ &\qquad+\sum_{\sigma<s\le t}\left(\langle\theta(U_{s-}),\Delta U_s\rangle-\mathrm{Ad}(a_s^{-1})\theta_\beta\Delta V_s^\beta\right)\\ &\quad=\int_{(\sigma,t]}\mathrm{Ad}(a^{-1})\,d\left(\int\theta_\beta(V)\circ dV^\beta+\sum_{0<s\le\cdot}\left(\theta_\beta(V_{s-})\langle du^\beta,\Delta V_s\rangle-\theta_\beta(V_{s-})\Delta V_s^\beta\right)\right)\\ &\qquad+\int_{(\sigma,t]}j\circ da\\ &\quad=\int_{(\sigma,t]}\mathrm{Ad}(a^{-1})\,d\left(\int\theta\circ dV\right)+\int_{(\sigma,t]}j\circ da,\end{aligned}$$

where we applied Proposition 2.4 at the third equality. Therefore, we have

$$\int_{(\sigma,t]}j\circ da=0.$$

Thus (3.17) follows and this implies that a_s is constant on (T_i,T_{i+1}) for each i. □

By Lemma 3.9, we obtain the following uniqueness result.

Theorem 3.10. *Let $(\Delta X,X)$ be a Δ-semimartingale satisfying (1.8). Then the horizontal lift of $(\Delta X,X)$ is uniquely determined.*

Proof. Let $(\Delta\widetilde{U},\widetilde{U})$ be an arbitrary horizontal lift of $(\Delta X,X)$. We denote the horizontal lift of $(\Delta X,X)$ constructed in Theorem 3.7 by $(\Delta U,U)$. Then it suffices to show that $\widetilde{U}=U$. We fix $\gamma\in C_g$ and let $(\tilde{\gamma}(U_-^\gamma,U^\gamma),U^\gamma)$ be the horizontal lift of $(\gamma(X_-,X),X)$. Then there exists a unique $O(d)$-valued process $\tilde{a}$ satisfying $\widetilde{U}=U^\gamma\tilde{a}$. We show that $\tilde{a}_t$ equals the process a_t defined in (3.12). Let b_t be an $O(d)$-valued process defined through (3.11). Denote the horizontal lift of ΔX_s at U_s^γ by $\widetilde{\Delta X_s}$. Then for $s\ge 0$,

$$\begin{aligned}U_s^\gamma\tilde{a}_s&=\widetilde{U}_s\\ &=\exp_{\widetilde{U}_{s-}}\Delta\widetilde{U}_s\\ &=\exp_{U_{s-}^\gamma\tilde{a}_{s-}}\left(R_{\tilde{a}_{s-}*}\widetilde{\Delta X_s}\right)\\ &=\left(\exp_{U_{s-}^\gamma}\widetilde{\Delta X_s}\right)\tilde{a}_{s-}\\ &=U_s^\gamma b_s\tilde{a}_{s-},\end{aligned}$$

where we have applied (1.8) in the second equality and the fact that $R_{\tilde{a}_{s-}*}\widetilde{\Delta X_s}$ is the horizontal lift of ΔX at $\widetilde{U}_{s-}$ in the third equality. Let $T_1 < T_s < \cdots$ be a sequence of stopping times which exhausts the jumps of $\tilde{a}$. Then by Lemma 3.9, $\tilde{a}_t$ is constant on (T_i, T_{i+1}) for each i. Therefore, it holds that

$$\tilde{a}_t = \prod_{0 \leq s < t} b_{t-s}, \ t \geq 0,$$

which equals the one constructed in (3.12). Thus we have $\widetilde{U} = U$. □

We end the proof of Theorem 1.11 (4) with the following theorem.

Theorem 3.11. *Let $(\Delta X, X)$ be a Δ-semimartingale satisfying (1.8) and $(\Delta U, U)$ its horizontal lift. Then it holds that*

$$W_t^i = \int_0^t U_{s-}\varepsilon^i \, dX_s,$$

where W is the anti-development of $(\Delta U, U)$.

Proof. Fix $\gamma \in C_g$. Then U can be written as $U = U^\gamma a$, where a is the $O(d)$-valued process constructed in (3.12). Let $0 = T_0 < T_1 < \cdots$ be a sequence of stopping times which exhausts jumps of a_t. Then Since

$$\int_{(0,t]} U_{s-}\varepsilon^i \ dX = \sum_{m=1}^{\infty} \int_{(T_m \wedge t, T_{m+1} \wedge t]} U_{s-}\varepsilon^i \ dX,$$

it suffices to show

$$\int_{(T_m \wedge t, T_{m+1} \wedge t)} U_{s-}\varepsilon^i \ dX = \int_{(T_m \wedge t, T_{m+1} \wedge t)} \mathfrak{s}^i(U_{s-}) \ dU_s, \tag{3.19}$$

$$\langle U_{T_m \wedge t-}\varepsilon^i, \Delta X_{T_m \wedge t} \rangle = \langle \mathfrak{s}^i(U_{T_m \wedge t-}), \Delta U_{T_m \wedge t} \rangle \tag{3.20}$$

for each m. Equation (3.20) can be easily obtained by the definition of the solder form. Thus we will show (3.19). Set $c_j^i(s) = a_i^j(s)$. Since a_t is constant and $\Delta X = \gamma(X_-, X)$ on (T_m, T_{m+1}) for each m, we have

$$\begin{aligned}
\int_{(T_m \wedge t, T_{m+1} \wedge t)} U_{s-}\varepsilon^i \ dX_s &= \int_{(T_m \wedge t, T_{m+1} \wedge t)} U_{s-}\varepsilon^i \ \gamma dX_s \\
&= \int_{(T_m \wedge t, T_{m+1} \wedge t)} (U_{s-}^\gamma a) \, \varepsilon^i \ \gamma dX_s \\
&= c_j^i \int_{(T_m \wedge t, T_{m+1} \wedge t)} U_{s-}^\gamma \varepsilon^j \ \gamma dX_s \\
&= c_j^i \int_{(T_m \wedge t, T_{m+1} \wedge t)} \mathfrak{s}^j(U_{s-}^\gamma) \ \tilde{\gamma} dU_s^\gamma,
\end{aligned} \tag{3.21}$$

by (3.10), where we write $c^i_j = c^i_j(T_m \wedge t)$, $a = a(T_m \wedge t)$ to simplify the notation. Furthermore, for stopping times S, T with $T_i < S \le T < T_{i+1}$, it holds that

$$\begin{aligned} c^i_j \langle \mathfrak{s}^j(U^\gamma_S), \tilde\gamma(U^\gamma_S, U^\gamma_T)\rangle =& c^i_j \langle \mathfrak{s}^j(U^\gamma_S), R_{a^{-1}*} R_{a*} \tilde\gamma(U^\gamma_S, U^\gamma_T)\rangle \\ =& \langle \mathfrak{s}^i(U_S), \tilde\gamma(U_S, U_T)\rangle \end{aligned}$$

by Lemma 3.4. Therefore it holds that

$$c^i_j \int_{(T_m\wedge t, T_{m+1}\wedge t)} \mathfrak{s}^j(U^\gamma_{s-})\ \tilde\gamma dU^\gamma_s = \int_{(T_m\wedge t, T_{m+1}\wedge t)} \mathfrak{s}^i(U_{s-})\ dU_s. \tag{3.22}$$

Combining (3.21) and (3.22), we obtain (3.19) and the assertion follows. □

Combining Theorems 3.7, 3.10 and 3.11, we complete the proof of Theorem 1.11 (4).

3.3 Proof of Theorem 1.12

We end this section with the proof of Theorem 1.12 and its example. The idea of the proof of Theorem 1.12 is to construct a coefficient of SDE from a given connection rule γ, which was also considered in section 10.2 of [16]. In the proof of Theorem 1.12 below, we give a concrete construction of the coefficient of SDE through the orthonormal frame bundle.

Proof of Theorem 1.12. Let γ be an arbitrary connection rule which induces Levi-Civita connection. We divide the proof into several steps as follows.

- **Step 1.** We construct $\delta_0 > 0$ in such a way that for all $x \in M$, the map γ is a diffeomorphism on $B^M_{\delta_0}(x) \times B^M_{\delta_0}(x)$ into its image.
- **Step 2.** We construct $\delta > 0$ in such a way that for all $x \in M$, it holds that

$$\left(\gamma_x|_{B_{\delta_0}(x)}\right)^{-1} \left(B^{T_xM}_\delta(0)\right) \subset B^M_{\delta_0}(x). \tag{3.23}$$

- **Step 3.** We define h using δ_0 and δ constructed in the previous steps.
- **Step 4.** We prove (1) of Theorem 1.12.
- **Step 5.** We prove (2) of Theorem 1.12.

Step 1. We fix $\gamma^g \in C_g$. By the assumption for γ and the compactness of M, there exists a neighborhood $\mathcal{U}$ of the diagonal set $\mathrm{diag}(M)$ of M such that

$$|\gamma(x,y) - \gamma^g(x,y)| = O(d(x,y)^3) \tag{3.24}$$

on $\mathcal{U}$. Moreover, it is well known that the derivative of the map $TM \ni u \mapsto (\pi_{TM}u, \exp u) \in M \times M$ at the point $0 \in T_xM$ equals

$$\begin{bmatrix} \mathrm{id}_{T_xM} & \mathrm{id}_{T_xM} \\ \mathrm{id}_{T_xM} & 0 \end{bmatrix}.$$

Thus by (3.24), for any $x \in M$,

$$\gamma_{*(x,x)} = \begin{bmatrix} \mathrm{id}_{T_xM} & \mathrm{id}_{T_xM} \\ 0 & -\mathrm{id}_{T_xM} \end{bmatrix}.$$

Thus there exists $r > 0$ such that γ is a diffeomorphism on $B_r^M(x) \times B_r^M(x)$ into its image, where $B_r^M(x)$ is the geodesic ball on M centered at x with radius r. Let

$$R_x := \sup\{r > 0 \mid \gamma \text{ is a diffeomorphism on } B_r^M(x) \times B_r^M(x) \text{ into its image}\}$$

We will show that the function $x \mapsto R_x$ is lower semi-continuous. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to a point $x \in M$. Take $\varepsilon \in (0, R_x)$. Then for any sufficiently large n, $B^M_{R_x-\varepsilon}(x_n) \subset B^M_{R_x}(x)$. Thus γ is a diffeomorphism on $B^M_{R_x-\varepsilon}(x_n) \times B^M_{R_x-\varepsilon}(x_n)$. This means that $R_x - \varepsilon \le R_{x_n}$. Since $\varepsilon > 0$ is taken arbitrarily, we have $R_x \le \liminf_{n\to\infty} R_{x_n}$ and this means that $x \mapsto R_x$ is lower semi-continuous. In particular, R_x attains its minimum $R_0 > 0$ on M. Let r_0 be the injectivity radius of M. We set $\delta_0 := R_0 \wedge r_0$.

Step 2. Next, we set

$$\delta_x := \sup\{\delta > 0 \mid B_\delta^{T_xM}(0) \subset \gamma_x(B^M_{\delta_0}(x))\},$$

Then the map $x \mapsto \delta_x$ can also be shown to be lower semi-continuous as follows. Assume that there exist a point $x \in M$ and a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = x, \ \liminf_{n\to\infty} \delta_{x_n} < \delta_x.$$

By taking a proper subsequence, we suppose that $\lim_{n\to\infty} \delta_{x_n}$ exists and satisfies $\lim_{n\to\infty} \delta_{x_n} < \delta_x$. Let $\varepsilon \in (0, \delta_x - \lim_{n\to\infty} \delta_{x_n})$. Then there exists $N_\varepsilon \in \mathbb{N}$ such that $\delta_{x_n} \le \delta_x - \varepsilon$ for any $n \ge N_\varepsilon$. Then for each n, we take $v_n \in B^{T_{x_n}M}_{\delta_x - \frac{\varepsilon}{2}}(0) \cap \gamma_{x_n}(B^M_{\delta_0}(x_n))^c$. We set

$$U^q M := \{v \in TM \mid |v| \le q\}$$

for $q > 0$. Since $U^{\delta_x - \frac{\varepsilon}{2}}M$ is compact, there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that v_{n_k} converges to $v \in U_x^{\delta_x-\varepsilon}M$ with respect to a proper metric which is compatible to the topology on TM. Since γ_x is a diffeomorphism on $B^M_{\delta_0}(x)$, we can take $\varepsilon' > 0$ such that $B^{T_xM}_{\delta_x - \frac{\varepsilon}{2}}(0) \subset \gamma_x(B^M_{\delta_0-\varepsilon'}(x))$. This implies that $\gamma(B^M_{\delta_0-\varepsilon'}(x), B^M_{\delta_0-\varepsilon'}(x))$ is an open neighborhood of v in TM. Therefore, there exists $K_1 \in \mathbb{N}$ such that $v_{n_k} \in \gamma(B^M_{\delta_0-\varepsilon'}(x), B^M_{\delta_0-\varepsilon'}(x))$ for all $k \ge K_1$. On the other hand, there exists $K_2 \in \mathbb{N}$ such that $B^M_{\delta_0-\varepsilon'}(x) \subset B^M_{\delta_0}(x_{n_k})$ for all $k \ge K_2$. Thus for $k \ge K_1 \vee K_2$, $v_{n_k} \in \gamma_{x_{n_k}}(B^M_{\delta_0}(x_{n_k}))$. This contradicts to the choice of v_n. Therefore, we obtain the lower semi-continuity of δ_x and consequently, δ_x also attains its minimum on M. Let $\delta := \min_{x\in M} \delta_x > 0$. Then γ satisfies (3.23) and γ_x is a diffeomorphism on $\left(\gamma_x|_{B_{\delta_0}(x)}\right)^{-1}\left(B_\delta^{T_xM}(0)\right)$.

Step 3. The map $\exp_x$ is a diffeomorphism on $\exp_x^{-1}\left(\left(\gamma_x|_{B_{\delta_0}(x)}\right)^{-1}\left(B_\delta^{T_xM}(0)\right)\right)$ since this set is included in $B_{\delta_0}^{T_xM}(0)$ and $\delta_0 \leq r_0$. Thus we can define

$$b_x := \gamma(x, \exp_x(\cdot))\colon\ \exp_x^{-1}\left(\left(\gamma_x|_{B_{\delta_0}(x)}\right)^{-1}\left(B_\delta^{T_xM}(0)\right)\right) \to B_\delta^{T_xM}(0)$$

and b_x is a diffeomorphism. Define

$$a(u,z) := b_{\pi u}^{-1}(uz)$$

for $u \in \mathcal{O}(M)$ and $z \in B_\delta(0)$. Then a satisfies

$$\gamma(\pi u, \exp_{\pi u} a(u,z)) = uz$$

for all $u \in \mathcal{O}(M)$ and $z \in B_\delta(0)$ and $a(u,0) = 0$. Moreover, if we let $h(u,z) = u^{-1}a(u,z) \in B_{\delta_0}(0)$ for $(u,z) \in \mathcal{O}(M) \times B_\delta(0)$, then h is differentiable on $\mathcal{O}(M) \times B_\delta(0)$ and satisfies

$$d_0 h(u, \cdot) = \mathrm{id}_{\mathbb{R}^d}.$$

Next, we show that

$$\frac{\partial^2 h(u,\cdot)}{\partial z^i \partial z^j}(0) = 0 \tag{3.25}$$

for any $u \in \mathcal{O}(M)$. Fix $x \in M$ and we write $G_1(y) := \gamma(x,y)$ and $G_2(y) := \gamma^g(x,y)$. Then $a = G_2 \circ G_1^{-1}$ on $B_\delta^{T_xM}(0)$. Let $(\xi^1, \dots, \xi^d)$ be a coordinate on T_xM associated with an orthonormal basis and $(y^1, \dots, y^d)$ be a normal coordinate with $y^i(x) = 0$ for $i = 1, \dots, d$. Then we regard G_1 and G_2 as functions of $(y^1, \dots, y^d)$. By Taylor's theorem, we have

$$G_1(y) - G_2(y) = \frac{1}{2}\left(\frac{\partial^2 G_1}{\partial y^i \partial y^j}(0) - \frac{\partial^2 G_1}{\partial y^i \partial y^j}(0)\right) y^i y^j + \mathrm{o}(|y|^3)$$

for any y near x. Since both γ and γ^g induce Levi-Civita connection, this implies $\mathrm{Hess}G_1^i(0) = \mathrm{Hess}G_2^i(0)$ for each $i = 1, \dots, d$. Therefore, we have

$$\frac{\partial^2 G_2 \circ G_1^{-1}}{\partial \xi^i \partial \xi^j}(0) = 0.$$

This immediately yields (3.25).

Step 4. *Proof of (1) of Theorem 1.12.*
We set $\varphi\colon \mathbb{R}^d \times \mathcal{O}(M) \times \mathbb{R}^d \to \mathcal{O}(M)$ by

$$\varphi(w,u,z) := \mathrm{Exp}_u(h^k(u, z-w)\mathcal{L}_k).$$

Then obviously, the map φ is a constraint coefficient from $\mathbb{R}^d \times \mathcal{O}(M) \times \mathbb{R}^d$ to $\mathcal{O}(M)$. Therefore, for a given $\mathcal{F}_0$-measurable random variable U_0 and a semimartingale Z, we obtain the unique solution U of the SDE (1.10). We set

$$\Delta U_t := h^k(U_{t-}, \Delta Z_t)\mathcal{L}_k(U_{t-}).$$

Then $(\Delta U, U)$ is a horizontal Δ-semimartingale on $\mathcal{O}(M)$. In fact, by Lemma 2.10 and (3.25), we have

$$\begin{aligned}\int \theta(U) \circ dU &= \int \langle \theta, \mathcal{L}_k \rangle (U_-) \circ dZ^k + \sum_{0<s\leq \cdot} R^k_s \langle \theta, \mathcal{L}_k \rangle (U_{s-}) \\ &= 0,\end{aligned}$$

where

$$R^k_t := h^k(U_{t-}, \Delta Z_t) - \Delta Z^k_t.$$

Let $X = \pi U$ and $\Delta X = \pi_* \Delta U$. Then the pair $(\Delta X, X)$ is a Δ-semimartingale satisfying (1.8) and

$$\begin{aligned}\gamma(X_{t-}, X_t) &= \gamma(\pi U_{t-}, \exp_{\pi U_{t-}} h^k(U_{t-}, \Delta Z_t) U_{t-} \varepsilon_k) \\ &= \gamma(\pi U_{t-}, \exp_{\pi U_{t-}} a(U_{t-}, \Delta Z_t)) \\ &= U_{t-} \Delta Z_t.\end{aligned}$$

We set

$$W_t := Z_t + \sum_{0<s\leq t} U^{-1}_{s-} (\gamma^g(X_{s-}, X_s) - \gamma(X_{s-}, X_s)).$$

Then $\Delta W_t = U^{-1}_t \gamma^g(X_{t-}, X_t)$ and it satisfies

$$\exp_{X_{t-}} U_{t-} \Delta W_t = X_t = \exp_{X_{t-}} a(U_{t-}, \Delta Z_t).$$

Thus we have

$$U_{t-} \Delta W_t = a(U_{t-}, \Delta Z_t).$$

Therefore, for all $F \in C^\infty(\mathcal{O}(M))$, it holds that

$$\begin{aligned}F(U_t) - F(U_0) &= \int_0^t \mathcal{L}_k F(U_{s-}) \circ dZ^k_s \\ &\quad + \sum_{0<s\leq t} \{F(\mathrm{Exp}_{U_{s-}}(h^k(U_{s-}, \Delta Z_s)\mathcal{L}_k)) - F(U_{s-}) - \mathcal{L}_k F(U_{s-}) \Delta Z^k\} \\ &= \int_0^t \mathcal{L}_k F(U_{s-}) \circ dW^k_s \\ &\quad + \sum_{0<s\leq t} \{F(\mathrm{Exp}_{U_{s-}}(\Delta W^k_t \mathcal{L}_k)) - F(U_{s-}) - \mathcal{L}_k F(U_{s-}) \Delta W^k\}.\end{aligned}$$

This implies that W is an anti-development of $(\Delta X, X)$ with respect to the horizontal lift $(\Delta U, U)$. Therefore, we have

$$\begin{aligned}\int \phi_- \, \gamma dX &= \int \phi_- \, \gamma^g dX + \sum_{0<s\leq \cdot} \langle \phi_{s-}, \gamma(X_{s-}, X_s) - \gamma^g(X_{s-}, X_s) \rangle \\ &= \int \langle U^{-1}_- \phi_-, dW \rangle + \sum_{0<s\leq \cdot} \langle U^{-1}_{s-} \phi_{s-}, \Delta Z_s - \Delta W_s \rangle \\ &= \int \langle U^{-1}_- \phi_-, dZ \rangle\end{aligned}$$

for any T^*M-valued càdlàg process ϕ above X.

Step 5. *Proof of (2) of Theorem 1.12.*
Let X be an M-valued semimartingale satisfying

$$X_t \in \left(\gamma_{X_{t-}}|_{B^M_{\delta_0}(X_{t-})}\right)^{-1}\left(B^{T_{X_{t-}}M}_{\delta}(0)\right) \text{ for all } t \geq 0, \ \mathbb{P}\text{-a.s.}$$

Let W be an anti-development of $(\gamma^g(X_-, X), X)$ and V a horizontal lift of $(\gamma^g(X_-, X), X)$ with an initial value U_0. We set

$$Z_t := W_t + \sum_{0<s\leq \cdot} V_{s-}^{-1}(\gamma(X_{s-}, X_s) - \gamma^g(X_{s-}, X_s)).$$

Then by the assumption for X,

$$\begin{aligned}|\Delta W_t| &= |\gamma^g(X_{t-}, X_t) < \delta_0,\\ |\Delta Z_t| &= |\gamma(X_{t-}, X_t)| < \delta.\end{aligned}$$

Thus we have

$$\begin{aligned}\gamma(X_{s-}, \exp_{X_{s-}} a(V_{s-}, \Delta Z_s)) &= V_{s-}\Delta Z_s\\ &= \gamma(X_{s-}, X_s)\\ &= \gamma(X_{s-}, \exp_{X_{s-}} V_{s-}\Delta W_s)\end{aligned}$$

and consequently,

$$a(V_{s-}, \Delta Z_s) = V_{s-}\Delta W_s.$$

Therefore, V satisfies

$$\begin{aligned}F(V_t) - F(V_0) &= \int_0^t \mathcal{L}_k F(V_{s-}) \circ dW_s^k\\ &\quad + \sum_{0<s\leq t}\{F(\mathrm{Exp}_{V_{s-}}(\Delta W_s^k \mathcal{L}_k)) - F(V_{s-}) - \mathcal{L}_k F(V_{s-})\Delta W_s^k\}\\ &= \int_0^t \mathcal{L}_k F(V_{s-}) \circ dZ_s^k\\ &\quad + \sum_{0<s\leq t}\{F(\mathrm{Exp}_{V_{s-}}(h^k(V_{s-}, \Delta Z_s)\mathcal{L}_k)) - F(V_{s-}) - \mathcal{L}_k F(V_{s-})\Delta Z_s^k\}\end{aligned}$$

for all $F \in C^\infty(\mathcal{O}(M))$. This implies that V solves (1.10) and we have $V = U$ by the uniqueness of the solution of the SDE. Therefore, $(h^k(U_-, \Delta Z)\mathcal{L}(U_-), U)$ is the horizontal lift of X with the initial value U_0. In particular, the semimartingale $(U_- h(U_-, \Delta Z), X$ satisfies (1.11) by the claim of (1). □

Example 3.12. We consider the case

$$M = \mathbb{S}^d := \{(x^1, \ldots, x^{d+1}) \in \mathbb{R}^{d+1} \mid (x^1)^2 + \cdots + (x^{d+1})^2 = 1\}.$$

Let γ be a connection rule on $\mathbb{S}^d$ given by

$$\gamma(x,y) := \Pi_x(y-x),$$

where $\Pi_x\colon \mathbb{R}^{d+1} \to T_x\mathbb{S}^d$ is the orthonormal projection. Let g be the Riemannian metric on $\mathbb{S}^d$ associated with the embedding into $\mathbb{R}^{d+1}$ and fix $\gamma_g \in C_g$. Then it holds that

$$\gamma(x,y) = \frac{\sin d(x,y)}{d(x,y)}\gamma^g(x,y).$$

Therefore, we can easily check that $\delta_0 = \frac{\pi}{2}$, $\delta = 1$ and

$$a(u,z) = \frac{\arcsin|z|}{|z|}uz.$$

We define $f\colon [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{\arcsin t - t}{t}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Then for a local martingale Z on $\mathbb{R}^d$ with Z_0 and $\sup\limits_{0\le t<\infty} |\Delta Z_t| < 1$, if we set

$$W = Z + \sum_{0<s\le\cdot} f(|\Delta Z_s|)\Delta Z_s,$$

the development of W is an $\mathbb{S}^d$-valued γ-martingale.

A Appendix: Riemannian metric on $\mathcal{O}(M)$

We summarize some notions regarding the Riemannian metric on orthonormal frame bundles defined by (1.6). Fundamental properties of orthonormal frame bundles mentioned in this appendix are based on [15]. Let (M,g) be a Riemannian manifold and $\pi\colon \mathcal{O}(M) \to M$ the orthonormal frame bundle on M.

Let $\tilde{g}$ be a Riemannian metric $\tilde{g}$ on $\mathcal{O}(M)$ defined by (1.6). We denote the Levi-Civita connection on $T\mathcal{O}(M)$ corresponding to $\tilde{g}$ by $\tilde{\nabla}$. The Riemannian metric $\tilde{g}$, the Levi-Civita connection $\tilde{\nabla}$ and geodesics on $\mathcal{O}(M)$ have been considered in [23]. Covariant derivatives with respect to connections on soldered principal fiber bundles have been calculated in [5] under a more general situation. We can write covariant derivatives of vector fields $\{\mathcal{L}_k, \mathcal{X}^\sharp_\alpha\}^{k=1,\dots,d}_{\alpha=1,\dots,\frac{d(d-1)}{2}}$ as

$$\tilde{\nabla}\mathcal{L}_j = \sum_\beta \omega^\beta_j\ \mathcal{X}^\sharp_\beta + \sum_i \omega^i_j\ \mathcal{L}_i,$$
$$\tilde{\nabla}\mathcal{X}^\sharp_\alpha = \sum_\beta \omega^\beta_\alpha\ \mathcal{X}^\sharp_\beta + \sum_i \omega^i_\alpha\ \mathcal{L}_i,$$

where $\omega^\alpha_\beta, \omega^\alpha_j, \omega^i_j$ are 1-forms on $\mathcal{O}(M)$ defined by

$$\begin{aligned}
\omega^\alpha_\beta &= \langle \theta^\alpha, \tilde{\nabla} \mathcal{X}^\sharp_\beta \rangle, \\
\omega^\alpha_i &= \langle \theta^\alpha, \tilde{\nabla} \mathcal{L}_i \rangle, \\
\omega^i_\alpha &= \langle \mathfrak{s}^i, \tilde{\nabla} \mathcal{X}^\sharp_\alpha \rangle, \\
\omega^i_j &= \langle \mathfrak{s}^i, \tilde{\nabla} \mathcal{L}_j \rangle.
\end{aligned}$$

Here, note that we divide the global frame on the tangent bundle $T\mathcal{O}(M)$ into two groups; $\{\mathcal{L}_i\}_{i=1,\dots,d}$ and $\{\mathcal{X}^\sharp_\alpha\}_{\alpha=1,\dots,\frac{d(d-1)}{2}}$. That is why we distinguish the Greek index $\alpha, \beta, \gamma, \dots$ and the Latin index $i, j, k, \dots$ accordingly. Since $\tilde{\nabla}$ is torsion-free, it holds that

$$\omega^\alpha_j = -\omega^j_\alpha.$$

By setting

$$\begin{aligned}
\omega^\alpha_{\beta\gamma} &= \langle \theta^\alpha, \tilde{\nabla}_{\mathcal{X}^\sharp_\gamma} \mathcal{X}^\sharp_\beta \rangle, \ \omega^\alpha_{\beta k} = \langle \theta^\alpha, \tilde{\nabla}_{\mathcal{L}_k} \mathcal{X}^\sharp_\beta \rangle, \\
\omega^\alpha_{j\gamma} &= \langle \theta^\alpha, \tilde{\nabla}_{\mathcal{X}^\sharp_\gamma} \mathcal{L}_j \rangle, \ \omega^\alpha_{jk} = \langle \theta^\alpha, \tilde{\nabla}_{\mathcal{L}_k} \mathcal{L}_j \rangle, \\
\omega^i_{j\gamma} &= \langle \mathfrak{s}^i, \tilde{\nabla}_{\mathcal{X}^\sharp_\gamma} \mathcal{L}_j \rangle, \ \omega^i_{jk} = \langle \mathfrak{s}^i, \tilde{\nabla}_{\mathcal{L}_k} \mathcal{L}_j \rangle,
\end{aligned}$$

we can write

$$\begin{aligned}
\omega^\alpha_\beta &= \sum_\gamma \omega^\alpha_{\beta\gamma} \ \theta^\gamma + \sum_k \omega^\alpha_{\beta k} \ \mathfrak{s}^k, \\
\omega^\alpha_j &= \sum_\gamma \omega^\alpha_{j\gamma} \ \theta^\gamma + \sum_k \omega^\alpha_{jk} \ \mathfrak{s}^k, \\
\omega^i_j &= \sum_\gamma \omega^i_{j\gamma} \ \theta^\gamma + \sum_k \omega^i_{jk} \ \mathfrak{s}^k.
\end{aligned}$$

In view of calculations in [5, p. 897], it holds that

$$\begin{aligned}
\omega^\alpha_\beta &= \frac{1}{2} \sum_\gamma c^\alpha_{\beta\gamma} \ \theta^\gamma, \\
\omega^\alpha_j &= -\frac{1}{2} \sum_k \Omega^\alpha_{jk} \ \mathfrak{s}^k, \\
\omega^i_j &= \sum_\gamma \left\{ (\mathcal{X}^\sharp_\gamma)^{ij} - \frac{1}{2} \Omega^\gamma_{ij} \right\} \theta^\gamma,
\end{aligned} \tag{A.1}$$

where the coefficients

$$c^\gamma_{\alpha\beta}, \ \alpha, \beta, \gamma = 1, \dots, \frac{d(d-1)}{2}$$

are the structure constants with respect to an orthonormal basis $\{\mathcal{X}_\alpha\}_{\alpha=1,\dots,\frac{d(d-1)}{2}}$ of $\mathfrak{o}(d)$ defined through

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_{\mathfrak{o}(d)} = c^\gamma_{\alpha\beta} \ \mathcal{X}_\gamma,$$

and

$$\Omega^\alpha_{ij},\ i,j=1,\dots,d,\ \alpha=1,\dots,\frac{d(d-1)}{2}$$

are the components of the curvature form Ω^θ defined through

$$\begin{aligned}\Omega^\theta =& d\theta+\frac{1}{2}[\theta,\theta]\\ =&\frac{1}{2}\sum_{i,j,\alpha}\Omega^\alpha_{ij}\ \mathcal{X}^\sharp_\alpha\ \mathfrak{s}^i\wedge\mathfrak{s}^j\ (\Omega^\alpha_{ij}=-\Omega^\alpha_{ji}).\end{aligned}$$

Since the standard inner product of $\mathfrak{o}(d)$ is $O(d)$-invariant, $\{c^\gamma_{\alpha\beta}\}$ is totally anti-symmetric in α,β,γ. The following Propositions A.1 and A.2 can be easily obtained by (A.1) above.

Proposition A.1. *For any $u\in\mathcal{O}(M)$ and $\mathcal{A}\in T_u\mathcal{O}(M)$, it holds that*

$$\tilde{\nabla}\theta(\mathcal{A},\mathcal{A})=0.$$

Furthermore, if $\mathcal{A}$ is horizontal, then

$$\tilde{\nabla}\mathfrak{s}(\mathcal{A},\mathcal{A})=0.$$

Proof. Any tangent vector $\mathcal{A}$ can be denoted by

$$\mathcal{A}=a^i\mathcal{L}_i(u)+b^\alpha X^\sharp_\alpha,\ a^i,b^\alpha\in\mathbb{R},\ i=1,\dots,d,\ \alpha=1,\dots,\frac{d(d-1)}{2}.$$

Therefore we can write

$$\begin{aligned}\tilde{\nabla}\theta^\alpha(\mathcal{A},\mathcal{A})=&a^ka^l(\tilde{\nabla}\theta^\alpha)(\mathcal{L}_k,\mathcal{L}_l)+a^kb^\gamma\tilde{\nabla}\theta^\alpha(\mathcal{L}_k,\mathcal{X}^\sharp_\gamma)\\ &+b^\beta a^l\tilde{\nabla}\theta^\alpha(\mathcal{X}^\sharp_\beta,\mathcal{L}_l)+b^\beta b^\gamma(\tilde{\nabla}\theta^\alpha)(\mathcal{X}^\sharp_\beta,\mathcal{X}^\sharp_\gamma).\end{aligned}$$

By using (A.1), it holds that

$$\begin{aligned}&a^ka^l(\tilde{\nabla}\theta^\alpha)(\mathcal{L}_k,\mathcal{L}_l)=a^ka^l\langle\omega^\alpha_l,\mathcal{L}_k\rangle=\frac{a^ka^l}{2}\Omega^\alpha_{lk},\\ &\tilde{\nabla}\theta^\alpha(\mathcal{L}_k,\mathcal{X}^\sharp_\gamma)=-\langle\omega^\alpha_\gamma,\mathcal{L}_k\rangle=0,\\ &\tilde{\nabla}\theta^\alpha(\mathcal{X}^\sharp_\beta,\mathcal{L}_l)=-\langle\omega^\alpha_l,\mathcal{X}^\sharp_\beta\rangle=0,\\ &b^\beta b^\gamma(\tilde{\nabla}\theta^\alpha)(\mathcal{X}^\sharp_\beta,\mathcal{X}^\sharp_\gamma)=b^\beta b^\gamma\langle\omega^\alpha_\gamma,\mathcal{X}^\sharp_\beta\rangle=-b^\beta b^\gamma c^\alpha_{\beta\gamma}.\end{aligned}$$

Since $\Omega^\alpha_{lk}=-\Omega^\alpha_{kl}$, we obtain

$$a^ka^l(\tilde{\nabla}\theta^\alpha)(\mathcal{L}_k,\mathcal{L}_l)=0.$$

Similarly, we obtain

$$b^\beta b^\gamma(\tilde{\nabla}\theta^\alpha)(\mathcal{X}^\sharp_\beta,\mathcal{X}^\sharp_\gamma)=0.$$

Therefore we deduce $\tilde{\nabla}\theta(\mathcal{A},\mathcal{A})=0$. Next suppose $\mathcal{A}$ is horizontal. Then we obtain

$$\tilde{\nabla}\mathfrak{s}^j(\mathcal{A},\mathcal{A})=a^ka^l\tilde{\nabla}\mathfrak{s}^j(\mathcal{L}_k,\mathcal{L}_l)=-a^ka^l\langle\omega^j_l\ \mathcal{L}_k\rangle=0.$$

This proves the proposition. □

Proposition A.2. *(1) Integral curves of the horizontal vector field $a^k\mathcal{L}_k$ for $a_k \in \mathbb{R}^k,\ k = 1,\dots,d$, are geodesics with respect to $\tilde{\nabla}$.*

(2) Integral curves of the vertical vector field $b^\alpha \mathcal{X}^\sharp_\alpha$ for $b^\alpha \in \mathbb{R}^d,\ \alpha = 1,\dots,\frac{d(d-1)}{2}$, are geodesics with respect to $\tilde{\nabla}$.

Proof. Let $u(t)$ be a curve on $\mathcal{O}(M)$ satisfying

$$\frac{du}{dt}(t) = a^k\mathcal{L}_k(u(t)).$$

Then it holds that

$$\begin{aligned}\tilde{\nabla}_{\frac{d}{dt}}\frac{du}{dt} &= a^k a^l(\tilde{\nabla}_{\mathcal{L}_l}\mathcal{L}_k)(u(t))\\ &= a^k a^l(\omega^\alpha_{kl}\mathcal{X}^\sharp_\alpha)\\ &= -\frac{1}{2}(a^k a^l\Omega^\alpha_{kl})\mathcal{X}^\sharp_\alpha\\ &= 0\end{aligned}$$

by the relation $\Omega^\alpha_{kl} = -\Omega^\alpha_{lk}$. Next let us denote the integral curve of $b^\alpha\mathcal{X}^\sharp_\alpha$ by $v(t)$. Then

$$\begin{aligned}\tilde{\nabla}_{\frac{d}{dt}}\frac{dv}{dt} &= b^\beta b^\gamma(\tilde{\nabla}_{\mathcal{X}^\sharp_\beta}\mathcal{X}^\sharp_\gamma)(v(t))\\ &= \frac{b^\beta b^\gamma}{2}c^\lambda_{\beta\gamma}\ \mathcal{X}^\sharp_\lambda(v(t))\\ &= 0.\end{aligned}$$

This completes the proof. □

The next proposition proved in [23] provides the relation between geodesics on $\mathcal{O}(M)$ and those on M.

Proposition A.3 ([23], Proposition 1.9)**.** *(1) Let x and y be two points in M and c a minimal geodesic from x to y. Let us denote the parallel transport along c by $P_c : T_xM \to T_yM$. Let $u \in \mathcal{O}_x(M)$, $v \in \mathcal{O}_y(M)$ with $v = P_c \circ u$. Then minimal geodesics from u to v with respect to $\tilde{g}$ are horizontal and the horizontal lift of c starting at u is one of minimal geodesics from u to v. Furthermore, if minimal geodesics from x to y on M are unique, then minimal geodesics from u to v are also unique.*

(2) Let τ be a geodesic on $\mathcal{O}(M)$ with respect to $\tilde{g}$. Suppose that $\tau'(0)$ is horizontal. Then τ is a horizontal curve and $\pi \circ \tau$ is a geodesic on M.

We prepare a simple lemma and propositions for the use in Section 3.

Lemma A.4. *Let $u \in \mathcal{O}(M)$, $\mathcal{A}, \mathcal{B} \in T_u\mathcal{O}(M)$ and $a \in O(d)$. Then*

$$\tilde{g}(R_{a*}\mathcal{A}, R_{a*}\mathcal{B}) = \tilde{g}(\mathcal{A}, \mathcal{B}).$$

Proof. By definition, it holds that

$$\tilde{g}(R_{a*}\mathcal{A}, R_{a*}\mathcal{B}) = \langle\langle\theta(ua), R_{a*}A\rangle, \langle\theta(ua), R_{a*}B\rangle\rangle_{\mathfrak{o}(d)} \\ + \langle\langle\mathfrak{s}(ua), R_{a*}\mathcal{A}\rangle, \langle\mathfrak{s}(ua), R_{a*}\mathcal{B}\rangle\rangle_{\mathbb{R}^d}.$$

Since $\langle\theta(ua), R_{a*}\mathcal{A}\rangle = \mathrm{Ad}(a^{-1})\langle\theta(u), \mathcal{A}\rangle$ and the metric on $\mathfrak{o}(d)$ is Ad-invariant,

$$\langle\langle\theta(ua), R_{a*}A\rangle, \langle\theta(ua), R_{a*}\mathcal{B}\rangle\rangle_{\mathfrak{o}(d)} = \langle\langle\theta(u), \mathcal{A}\rangle, \langle\theta(u), \mathcal{B}\rangle\rangle_{\mathfrak{o}(d)}.$$

On the other hand,

$$\begin{aligned}\langle\mathfrak{s}(ua), R_{a*}\mathcal{A}\rangle &= (ua)^{-1}\pi_* R_{a*}\mathcal{A} \\ &= a^{-1} \circ u^{-1}(\pi \circ R_a)_*\mathcal{A} \\ &= a^{-1} \circ u^{-1}(\pi_*\mathcal{A}) \\ &= a^{-1}\langle\mathfrak{s}(u), \mathcal{A}\rangle.\end{aligned}$$

Since a is isometric,

$$\langle\langle\mathfrak{s}(ua), R_{a*}\mathcal{A}\rangle, \langle\mathfrak{s}(ua), R_{a*}\mathcal{B}\rangle\rangle_{\mathbb{R}^d} = \langle\langle\mathfrak{s}(u), \mathcal{A}\rangle, \langle\mathfrak{s}(u), \mathcal{B}\rangle\rangle_{\mathbb{R}^d}.$$

Therefore $\tilde{g}(R_{a*}\mathcal{A}, R_{a*}\mathcal{B}) = \tilde{g}(\mathcal{A}, \mathcal{B})$. □

Proposition A.5. *Let $u, v \in \mathcal{O}(M)$. Then for all $a \in O(d)$,*

$$d_{\mathcal{O}(M)}(ua, va) = d_{\mathcal{O}(M)}(u, v),$$

where $d_{\mathcal{O}(M)}$ is the Riemannian distance with respect to $\tilde{g}$.

Proof. For any $\varepsilon > 0$, there exists a curve $\tau_\varepsilon : [0,1] \to \mathcal{O}(M)$ with $\tau_\epsilon(0) = u$, $\tau_\epsilon(1) = v$ satisfying $\int_0^t \left|\frac{d\tau_\varepsilon}{dt}\right| dt \le d_{\mathcal{O}(M)}(u, v) + \varepsilon$. Then by Lemma A.4,

$$\begin{aligned}\int_0^1 \left|\frac{d}{dt}R_a\tau_\varepsilon\right| dt &= \int_0^1 \left|R_{a*}\frac{d\tau_\varepsilon}{dt}\right| dt \\ &= \int_0^1 \left|\frac{d\tau_\varepsilon}{dt}\right| dt \\ &\le d_{\mathcal{O}(M)}(u, v) + \varepsilon.\end{aligned}$$

Thus

$$d_{\mathcal{O}(M)}(ua, va) \le d_{\mathcal{O}(M)}(u, v) + \varepsilon.$$

Since ε is arbitrary,

$$d_{\mathcal{O}(M)}(ua, va) \le d_{\mathcal{O}(M)}(u, v).$$

This inequality holds for all $u, v \in \mathcal{O}$ and $a \in O(d)$. Therefore, we have

$$d_{\mathcal{O}(M)}(ua, va) = d_{\mathcal{O}(M)}(u, v).$$

□

Proposition A.6. *Let $u, v \in \mathcal{O}(M)$ and $\tau(t)$ ($t \in [0,1]$) a minimal geodesic from u to v. Then for all $a \in O(d)$, $R_a\tau$ is a minimal geodesic from ua to va.*

Proof. By Lemma A.4 and Proposition A.5, we obtain

$$\begin{aligned}\int_0^1 \left|\frac{d}{dt} R_a\tau\right| dt &= \int \left|R_{a*}\frac{d\tau}{dt}\right| dt \\ &= \int_0^1 \left|\frac{d\tau}{dt}\right| dt \\ &= d_{\mathcal{O}(M)}(u,v) \\ &= d_{\mathcal{O}(M)}(ua, va).\end{aligned}$$

Therefore $R_a\tau$ is a minimal geodesic. □

Acknowledgements

The author is grateful to Professor Hariya for his careful reading of the previous version of the manuscript and for helpful comments. The author also thanks a referee for his/her constructive comments and corrections which have led to significant improvement of this paper.
This work was supported by JSPS KAKENHI Grant Number 24K22827.

References

[1] S. Albeverio, F. C. De Vecchi, P. Morando, and S. Ugolini, *Weak symmetries of stochastic differential equations driven by semimartingales with jumps*, Electron. J. Probab. **25** (2020), 1–34.

[2] D. Applebaum, *A horizontal Lévy process on the bundle of orthonormal frames over a complete riemannian manifold*, Séminaire de probabilités XXIX. Lecture Notes in Math. **1613** (1995), 166–180.

[3] M. Arnaudon and A. Thalmaier, *Stability of stochastic differential equations in manifolds*, Séminaire de probabilités **32** (1998), 188–214.

[4] ———, *Complete lifts of connections and stochastic Jacobi fields*, Journal de Mathématiques Pures et Appliquées **77** (1998), 283–315.

[5] C. Bär and D. Bleecker, *Connections on Solderd Principal Bundles*, Acta Physica Polonica B **29** (1998), 891–903.

[6] S. Cohen, *Géométrie différentielle stochastique avec sauts 1*, Stochastics: An International Journal of Probability and Stochastic Processes **56** (1996), 179–203.

[7] ———, *Géométrie différentielle stochastique avec sauts 2: discrétisation et applications des EDS avec sauts*, Stochastics: An International Journal of Probability and Stochastic Processes **56** (1996), 205–225.

[8] ———, *Some Markov properties of stochastic differential equations with jumps*, Séminaire de probabilités **29** (1995), 181–193.

[9] M. Emery, *Stochastic Calculus in Manifolds*, Springer-Verlag Berlin Heidelberg, 1989.

[10] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes, 2nd edition*, North-Holland Mathematical Library, vol. 24, North-Holland, 1989.

[11] F. Da Lio and T. Rivière, *Three-term commutator estimates and the regularity of $\frac{1}{2}$-harmonic maps into spheres*, Anal. PDE **4** (2011), no. 1, 149–190.

[12] ______, *Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps*, Adv. Math. **227** (2011), 1300-1348.

[13] F. C. De Vecchi, *Lie symmetry analysis and geometrical methods for finite and infinite dimensional stochastic differential equations*, Ph.D. thesis, Università degli Studi di Milano (2018).

[14] E. P. Hsu, *Stochastic Analysis on Manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, 2002.

[15] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I*, Wiley Classics Library, Interscience Publisher, 1963.

[16] L. Maillard-Teyssier, *Calcul Stochastique Covariant à Sauts & Calcul Stochastique à Sauts Covariants*, Thèse de doctorat - Université de Versailles Saint Quentin en Yvelines (2003).

[17] ______, *Stochastic Covariant Calculus with Jumps and Stochastic Calculus with Covariant Jumps*, in: Séminaire de Probabilités XXXIX, In Memoriam Paul-André Meyer, in: Lecture Notes in Math., vol. 1874 (2006), 381–417.

[18] P. Malliavin, *Formules de la Moyenne, Calcul de Perturbations et Théorèmes d'Annulation pour les Formes Harmoniques*, Journal of Functional Analysis **17** (1974), 274–291.

[19] F. Okazaki, *Convergence of martingales with jumps on submanifolds of Euclidean spaces and its applications to harmonic maps*, Journal of Theoretical Probability **37** (2024), no. 2, 1168–1198.

[20] ______, *Probabilistic characterization of weakly harmonic maps for non-local Dirichlet forms*, Potential Analysis **62** (2024), no. 1, 1–25.

[21] T. Parthasarathy, *Selection Theorems and their Applications*, Lecture Notes in Mathematics, Springer Berlin, Heidelberg, 1972.

[22] J. Picard, *Calcul stochastique avec sauts sur une variété*, Séminaire de Probabilités XXV **1485** (1991).

[23] M. Pontier and A. Estrade, *Relèvement horizontal d'une semimartingale càdlàg*, Séminaire de Probabilités de Strasbourg **26** (1992), 127–145.

[24] I. Shigekawa, *On stochastic horizontal lifts*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **59** (1982), 211-221.

Affiliation: Graduate School of Engineering, The University of Tokyo, Tokyo, Japan

Email address: fumiya-okazaki@g.ecc.u-tokyo.ac.jp