HARMONIC TOTALLY REAL MAPS OF THE 3-SPHERE INTO THE COMPLEX PROJECTIVE SPACES

ISAMI KOGA AND YASUYUKI NAGATOMO

Abstract. Applying a generalized do Carmo-Wallach theory based on a generalization of Theorem of Tsunero Takahashi, we classify harmonic totally real maps of the 3-sphere into the complex projective spaces. This means that we employ differential geometry of vector bundles with connections and construct the moduli spaces of those maps explicitly.

1. Introduction

The purpose of the present paper is to classify harmonic totally real maps with constant energy density of the 3-sphere S^3 into the complex projective spaces $\mathbf{C}P^n$. This work is motivated by Li's result on isometric minimal totally real submanifold S^3 of $\mathbf{C}P^n$ [5], in which, he uses the Hopf fibration $S^{2n+1} \to \mathbf{C}P^n$ and a Theorem of Takahashi on isometric minimal immersion from S^3 into S^{2n+1}. Instead of the original one, we apply a generalization of Theorem of Takahashi based on differential geometry of vector bundles and connections and a generalization of do Carmo-Wallach theory [6] which are reviewed in §2. In §3, we describe the (complex-valued) function space $C^\infty(S^3)$ on S^3 as an $\mathrm{SU}(2) \times \mathrm{SU}(2)$-module in a standard way ([2] or [8]). Then the spectral decomposition of the Laplacian emerges and the set of Hermitian endomorphisms on each eigenspace $\mathcal{H}^k$ is decomposed into irreducible components in which the moduli space will be realized.

In §4, a generalization of Theorem of Takahashi relates maps under consideration to an eigenspace $\mathcal{H}^k$ of the Laplacian. Then using $\mathcal{H}^k$, we introduce the standard map into $\mathbf{P}(\mathcal{H}^{k^*})$ which is an $\mathrm{SU}(2) \times \mathrm{SU}(2)$-equivariant desired map (Proposition 4.2). Then a generalization of do Carmo-Wallach theory requires a representation theoretic argument for an explicit construction of the moduli spaces (Theorem 4.4). At this stage, our approach (Proposition 4.3) is similar to that in Toth-D'ambra [8]. Indeed, we have the direct relation of results of [8] on moduli spaces of harmonic maps of S^3 into S^n with constant energy densities. Since those maps induce harmonic totally real maps with constant energy density of S^3 into $\mathbf{C}P^n$ via the two-fold covering $S^n \to \mathbf{R}P^n$ and a totally geodesic and totally real embedding $\mathbf{R}P^n \to \mathbf{C}P^n$, our

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moduli spaces include those by Toth-D'ambra (Theorem 4.7). Following Li, those are called *absolutey real*. Li gave examples of absolutely real minimal submanifolds which are standard maps in our terminology. Furthermore, he also obtains a totally real submanifold of $\mathbf{C}P^{11}$ which is not absolutely real. We will show that the standard maps are unique $\mathrm{SU}(2) \times \mathrm{SU}(2)$-equivariant harmonic totally real maps with constant energy density of S^3 to $\mathbf{C}P^n$ up to image equivalence (Proposition 4.2). Finally, we will obtain harmonic totally real maps f_t with constant energy density parametrized by $t \in [0, 1]$. The map f_0 is the standard map and the others are non absolutely real. The map f_1 corresponds to a point in the boundary of our moduli space and is regarded as a map into $\mathbf{C}P^{11}$.

2. Preliminaries

2.1. **Vector bundles.** For a complex vector bundle $V \to M$, $\Gamma(V)$ denotes the space of (smooth) sections of $V \to M$. Then for each $x \in M$, we have a linear map $ev_x : \Gamma(V) \to V_x$ called the *evaluation map* defined by $t \mapsto t(x)$ for all $t \in \Gamma(V)$, $x \in M$. (see, for example, [1, p.298]). If a (finite-dimensional) subspace $W \subset \Gamma(V)$ is given, then the restriction of ev_x to W is also called the evaluation map which is denoted by the same symbol $ev_x : W \to V_x$.

Generically, $\underline{W} \to M$ will stand for a trivial (complex) vector bundle with fibre W over a base manifold M: $M \times W \to M$. Thus the evaluation map is considered as a bundle map $ev : \underline{W} \to V$.

We assume that a vector bundle $V \to M$ has a Hermitian metric h and a connection ∇ compatible with the metric h, for which we write $(V \to M, h, \nabla)$ or (V, h, ∇). The curvature form of ∇ is denoted by R^V. In this article, a vector bundle $V_1 \to M$ is called to be *isomorphic* to $V_2 \to M$ if there exists a bundle map $\phi : V_1 \to V_2$ such that ϕ is an isomorphism of vector bundles preserving the metrics and the connections. Then ϕ is called a *bundle isomorphism*.

2.2. **Geometry of Projective spaces.** Let $\mathbf{C}^{n+1}$ be a complex vector space of dimension $n+1$ and $\mathbf{P}^n = Gr_n(\mathbf{C}^{n+1})$ the complex projective space of hyperplanes in $\mathbf{C}^{n+1}$. Then we have an exact sequence $0 \to T^*(1) \to \underline{\mathbf{C}^{n+1}} \to \mathcal{O}(1) \to 0$ of holomorphic vector bundles over $\mathbf{P}^n$ where T^* is the holomorphic cotangent bundle and $\mathcal{O}(1)$ is the line bundle of degree 1. By using the natural projection $\underline{\mathbf{C}^{n+1}} \to \mathcal{O}(1)$, we can regard $\mathbf{C}^{n+1}$ as a subset of $\Gamma(\mathcal{O}(1))$. By fixing a Hermitian inner product on $\mathbf{C}^{n+1}$ the holomorphic cotangent bundle and $\mathcal{O}(1)$ inherit metrics, and can be given the Hermitian connections (see, for example, [4, p.11,Proposition 4.9]). When the curvature 2-form of the Hermitian connection on $\mathcal{O}(1)$ is denoted by $R^{\mathcal{O}(1)}$, the Kähler form ω on $\mathbf{P}^n$ is given as:

$$\omega = -\sqrt{-1}\, R^{\mathcal{O}(1)}.$$

Let $f : M \to \mathbf{P}^n$ be a map. If f satisfies $f^*\omega = 0$, then f is called a *totally real map*. The real projective space $\mathbf{R}P^n$ is realized as a totally geodesic and totally real submanifold of $\mathbf{P}^n$, which is the typical example. Following Li [5], when the image of a totally real map $f : M \to \mathbf{P}^n$ is included in a totally geodesic and totally real submanifold $\mathbf{R}P^n$, f is said to be *absolutely real*.

2.3. **Evaluation homomorphism and induced maps.** Suppose that $L \to M$ is a line bundle and consider a subspace W of $\Gamma(L)$. The line bundle $L \to M$ is said to be *globally generated by* W if the evaluation is surjective. Under this hypothesis, there is a map $f : M \to \mathbf{P}(W^*)$, defined by $f(x) := \mathrm{Ker}\, ev_x, = \{t \in W : t(x) = 0\}$. The map f is called the *induced map by* $(L \to M, W)$, or simply by W if $L \to M$ is already specified.

2.4. **Maps satisfying the gauge condition.** Let $f : M \to \mathbf{P}(W^*)$ be a smooth map. The map $f : M \to \mathbf{P}(W^*)$ is said to be *full* if the induced linear map $W \to \Gamma(f^*\mathcal{O}(1))$ is a monomorphism. When the line bundle $\mathcal{O}(1) \to \mathbf{P}(W^*)$ is equipped with a Hermitian metric $h^{\mathcal{O}(1)}$ and a connection $\nabla^{\mathcal{O}(1)}$, these are pulled back to a metric $f^*h^{\mathcal{O}(1)}$ and a connection $\nabla^{f^*\mathcal{O}(1)}$ on the pull-back bundle $f^*\mathcal{O}(1) \to M$.

We fix a complex line bundle (L, h, ∇) over a manifold M. We will say that $f : M \to \mathbf{P}(W^*)$ *satisfies the gauge condition for* (L, h, ∇) if there exists a bundle isomorphism $(L, h, \nabla) \cong (f^*\mathcal{O}(1),\ f^*h^{\mathcal{O}(1)}, \nabla^{f^*\mathcal{O}(1)})$.

2.5. **A generalization of do Carmo-Wallach theory.** First of all, we introduce a generalization of Theorem of Takahashi [6] specialized to the case where the target is the projective space.

Theorem 2.1. *Let $f : M \to Gr_n(\mathbf{C}^{n+1}) = \mathbf{P}^n$ be a full map with constant energy density from a Riemannian manifold M. Then f is harmonic if and only if $\mathbf{C}^{n+1}$ is a subspace of an eigenspace of the Laplacian Δ acting on sections of $f^*\mathcal{O}(1) \to M$ defined by the induced connection $\nabla^{f^*\mathcal{O}(1)}$ with $e(f)/2$ as the eigenvalue, where $e(f)$ is the energy density of f.*

Next, we develop a generalization of do Carmo-Wallach theory. Let $Gr_p(\mathbf{C}^n)$ denote a Grassmannian of p-planes in $\mathbf{C}^n$. Suppose that f_1 and $f_2 : M \to Gr_p(\mathbf{C}^n)$ are smooth maps. Then f_1 is called *image equivalent* to f_2, if there exists an isometry $\psi \in \mathrm{SU}(n)$ of $Gr_p(\mathbf{C}^n)$ such that $f_2 = \psi \circ f_1$.

Let G be a compact Lie group and W a unitary representation of G with an invariant Hermitian inner product $(,)_W$. The set of Hermitian endomorphisms of W is denoted by $\mathrm{H}(W)$. Then G naturally acts on $\mathrm{H}(W)$. If we equip $\mathrm{H}(W)$ with an inner product $(\cdot, \cdot)_H$ such that $(A, B) = \mathrm{trace}\, AB$, for $A, B \in \mathrm{H}(W)$, then $(\cdot, \cdot)_H$ is G-invariant. We define a Hermitian endomorphism $H(u,v)$ for u, $v \in W$ as:

$$H(u,v) := \frac{1}{2}\left\{u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\right\}.$$

If U and V are (complex) subspaces of W, we denote by $H(U,V)$ a real subspace of $\mathrm{H}(W)$ spanned by $H(u,v)$ where $u \in U$ and $v \in V$. In a similar fashion, $GH(U,V)$ denotes the subspace of $\mathrm{H}(W)$ spanned by $gH(u,v)$, where $g \in G$, and so $GH(U,V)$ is a G-submodule of $\mathrm{H}(W)$.

Let G/K be a compact reductive Riemannian homogeneous space with K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. On the principal fiber bundle $G \to G/K$ with the fiber K, the canonical connection is defined as taking the horizontal subspace as $L_g\mathfrak{m}$, where $g \in G$ and L_g means the left translation by g.

For an irreducible unitary K-module V_0, the vector bundle $V = G \times_K V_0$ is called an *irreducible homogeneous vector bundle*. Then the canonical connection induces the covariant derivative on $V \to G/K$ which is also said the canonical connection denoted by ∇^0.

Let $W \subset \Gamma(V)$ be a G-invariant subspace of $\Gamma(V)$ with the evaluation map $ev : \underline{W} \to V$ and a G-invariant Hermitian inner product $(\cdot,\cdot)_W$. Then we can realize V_0 as a subspace of W by Frobenius reciprocity and the adjoint map $ev^* : V \to \underline{W}$ of ev.

Denote by U_0 the orthogonal complement of V_0 in W. Then, the map $f_0 : M \to Gr_p(W)$ is defined as:

$$f_0([g]) = gU_0 \subset W, \quad \text{for all } [g] \in G/K, \quad g \in G,$$

which is called the *standard map*. It is obviously an G-equivariant map.

In the sequel, $Q \to Gr_p(\mathbf{C}^n)$ denotes the universal quotient bundle [1, p.292]. A Hermitian inner product on $\mathbf{C}^n$ induces the Hermitian metric h_Q on $Q \to Gr_p(\mathbf{C}^n)$. Since $Q \to Gr_p(\mathbf{C}^n)$ is a holomorphic vector bundle, we have the connection ∇^Q compatible with h_Q and holomorphic vector bundle structure as the Hermitian connection.

Combining Theorem 5.12 with Theorem 5.30 in [6], we obtain

Theorem 2.2. *Let G/K be a compact reductive Riemannian homogeneous space with K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Fix a rank q irreducible homogeneous vector bundle $(V = G \times_K V_0, h, \nabla^0)$ with an invariant metric h and the canonical connection ∇^0. Since ∇^0 is G-invariant, W can be regarded as $\mathfrak{g}$-representation $\varrho : \mathfrak{g} \to \mathrm{End}\,(W)$. We can realize V_0 as a subspace of W by Frobenius reciprocity and $(\cdot,\cdot)_W$.*

If $f : G/K \to (Gr_p(\mathbf{C}^n), (\cdot,\cdot))$ $(n = p + q)$ is a full harmonic map with the gauge condition for (V, h, ∇^0) and we fix a bundle isomorphism between (V, h, ∇^0) and $(f^*Q, f^*h_Q, \nabla^{f^*Q})$, then there exist an eigenspace $W \subset \Gamma(V)$ of the Laplacian with the L^2 Hermitian inner product $(\cdot,\cdot)_W$, a unique linear injection $\iota : \mathbf{C}^n \to W$ and a positive semi-definite Hermitian endomorphism T on W such that

(a) *T satisfies*

(2.1) $$\left(T^2 - Id_W, GH(V_0,V_0)\right)_H = 0,\ \left(T^2, GH(\varrho(\mathfrak{m})V_0,V_0)\right)_H = 0,$$

(b) $\iota(\mathbf{C}^n) = \operatorname{Ker} T^{\perp}$ *and* $(\iota^* T\iota\cdot, \iota^* T\iota\cdot) = \iota^*(\cdot,\cdot)_W$.
(c) $f : G/K \to (Gr_p(\mathbf{C}^n), (\cdot,\cdot))$ *is written as :*

(2.2) $$f([g]) = (\iota^* T\iota)^{-1}\left(f_0([g]) \cap KerT^{\perp}\right),$$

which is called the map induced by a triple $(V, \mathbf{C}^n, \iota(\iota^* T\iota))$.

Conversely, if a positive semi-definite Hermitian endomorphism T on W satisfies condition (a) *and* $\mathbf{C}^n := (\operatorname{Ker} T)^{\perp}$ *globally generates* $V \to M$, *then the map defined by* (2.2) *is a full harmonic map into* $(Gr_p(\mathbf{C}^n), \iota^*(\cdot,\cdot)_W)$ *with the gauge condition for* (V, h, ∇^0), *where* $\iota : \mathbf{C}^n \to W$ *is the inclusion.*

Let $f_i : M \to (Gr_p(\mathbf{C}^n), (\cdot,\cdot)_W)$ *be the maps induced by those triples* $(V, \mathbf{C}^n, \iota(\iota^* T_i\iota))$ *such that* $\iota(\mathbf{C}^n)^{\perp} = \operatorname{Ker} T_1 = \operatorname{Ker} T_2$, *where* $\iota : \mathbf{C}^n \to W$ *is the inclusion. Then,* f_1 *and* f_2 *are gauge equivalent if and only if* $T_1 = T_2$.

Remark. In the above Theorem, we adopt *gauge equivalence* of maps to classify harmonic maps. When the target is the projective space, the gauge equivalence can be replaced by *image equivalence* [6] and we do not need the definition of gauge equivalence in this article.

Remark. If $evev^* = Id_V$ or equivalently, the standard map is a full harmonic map with the gauge condition for (V, h, ∇^0), then we can show that $\operatorname{trace} T^2 = \operatorname{trace} Id_W$ [6]. Hence $C = T^2 - Id_W$ is orthogonal to Id_W from the definition of the inner product on $\mathrm{H}(W)$.

3. 3-sphere

Let S^3 be the 3-dimensional sphere with the standard metric. The corresponding symmetric pair is denoted by $(\mathrm{SU}_+(2) \times \mathrm{SU}_-(2), \Delta)$, which is abbreviated to $(\mathrm{SU}_+ \times \mathrm{SU}_-, \Delta)$, where Δ is a diagonal subgroup of $\mathrm{SU}_+(2) \times \mathrm{SU}_-(2)$. When we denote by $S^k\mathbf{C}^2$ the k-th symmetric power of the standard representation $\mathbf{C}^2$ of $\mathrm{SU}(2)$, $S^k\mathbf{C}^2$ inherits an invariant Hermitian inner product h and a real or quaternion structure denoted by $\tau = j^k$, where j is a quaternion structure on $\mathbf{C}^2$. The irreducible representation of $\mathrm{SU}_\pm(2)$ and Δ are denoted by $S^k_\pm$ and S^k_Δ, respectively and the induced invariant structures are indicated by adding $\pm$ and Δ to the symbols, for instance, $\tau_\pm$ and h_Δ.

By Clebsch-Gordan, $S^{k_1}_+ \otimes S^{k_2}_-$ irreducibly decomposes as Δ-module:

$$S^{k_1}_+ \otimes S^{k_2}_- = S^{k_1+k_2}_\Delta \oplus S^{k_1+k_2-2}_\Delta \oplus \cdots \oplus S^{|k_1-k_2|-2}_\Delta \oplus S^{|k_1-k_2|}_\Delta.$$

Hence $S^{k_1}_+ \otimes S^{k_2}_-$ is a class one representation of $(\mathrm{SU}_+ \times \mathrm{SU}_-, \Delta)$ if and only if $k_1 = k_2$. Thus $\mathcal{H}^k := S^k_+ \otimes S^k_-$ is a class one representation of $(\mathrm{SU}_+ \times \mathrm{SU}_-, \Delta)$:

$$\mathcal{H}^k = S^k_+ \otimes S^k_- = S^{2k}_\Delta \oplus S^{2k-2}_\Delta \oplus \cdots \oplus S^2_\Delta \oplus S^0_\Delta.$$

Since every class one representation appears in $C^\infty(S^3)$ with multiplicity one, we can conclude that

$$C^\infty(S^3) = \sum_{k=0}^{\infty} S_+^k \otimes S_-^k = \sum_{k=0}^{\infty} \mathcal{H}^k.$$

When we denote by $V \to S^3$ a trivial bundle $(\mathrm{SU}_+ \times \mathrm{SU}_-) \times_\Delta S_\Delta^0$, the evaluation map $ev : \mathcal{H}^k \to V$ is written as $ev_{[g]}(w) = ([g], \pi_0(gw))$, where $g \in \mathrm{SU}_+ \times \mathrm{SU}_-$, $w \in \mathcal{H}^k$ and $\pi_0 : \mathcal{H}^k \to S_\Delta^0$ is the orthogonal projection. We can see that $\mathcal{H}^k$ is the eigenspace of the Laplacian with eigenvalue $k(k+2)$ and the L^2 Hermitian inner product $(\cdot,\cdot)_k$ on $\mathcal{H}^k$ is an $\mathrm{SU}_+ \times \mathrm{SU}_-$-invariant inner product. Since $\mathcal{H}^k$ is irreducible as an $\mathrm{SU}_+ \times \mathrm{SU}_-$-module, we can suppose that $(\cdot,\cdot)_k$ is induced by an invariant inner product $h_\pm$ on $S_\pm^k$ and the restriction of $(\cdot,\cdot)_k$ to S_Δ^0 is h_Δ. Thus $ev : \mathcal{H}^k \to V$ satisfies $ev ev^* = Id_V$.

By the real or quaternion structure $\tau = j^k$ on $S^k\mathbf{C}^2$, $w \mapsto h(\cdot, \tau(w))$ gives $S^k\mathbf{C}^2 \cong S^k\mathbf{C}^{2*}$. Thus we can induce the real structure σ on $\mathcal{H}^k$ and regard $\mathrm{End}\,\mathcal{H}^k$ with $\mathcal{H}^k \otimes \mathcal{H}^k$. From Clebsch-Gordan, $\mathrm{End}\,\mathcal{H}^k$ has an $\mathrm{SU}_+ \times \mathrm{SU}_-$-irreducible decomposition:

$$\begin{aligned}
&\mathrm{End}\,\mathcal{H}^k = S_+^k \otimes S_-^k \otimes S_+^k \otimes S_-^k \\
&\cong \left(S_+^{2k} \oplus S_+^{2k-2} \oplus \cdots \oplus S_+^2 \oplus S_+^0\right) \otimes \left(S_-^{2k} \oplus S_-^{2k-2} \oplus \cdots \oplus S_-^2 \oplus S_-^0\right) \\
&= \bigoplus_{i,j=0}^{k} S_+^{2k-2i} \otimes S_-^{2k-2j} = \bigoplus_{i,j=0}^{k} S_+^{2i} \otimes S_-^{2j}.
\end{aligned}$$

When $S^2\mathcal{H}^k$ (resp. $\wedge^2\mathcal{H}^k$) denotes the space of symmetric (resp. skew-symmetric) product of degree 2 on $\mathcal{H}^k$, $\mathrm{End}\,\mathcal{H}^k$ has another decomposition: $\mathrm{End}\,\mathcal{H}^k = S^2\mathcal{H}^k \oplus \wedge^2\mathcal{H}^k$. Since

$$S^2(S^k\mathbf{C}^2) = \bigoplus_{i=0}^{[\frac{k}{2}]} S^{2k-4i}, \quad \wedge^2(S^k\mathbf{C}^2) = \begin{cases} \bigoplus_{i=0}^{[\frac{k}{2}]-1} S^{2k-4i-2}, & k : \text{even} \\ \bigoplus_{i=0}^{[\frac{k}{2}]} S^{2k-4i-2}, & k : \text{odd} \end{cases}$$

where $[\frac{k}{2}]$ is the greatest integer which does not exceed $\frac{k}{2}$, we get

$$S^2\mathcal{H}^k = \bigoplus_{l,m=0,|l-m|\equiv 0 (\mathrm{mod} 2)}^{k} S_+^{2l} \otimes S_-^{2m},$$

$$\wedge^2\mathcal{H}^k = \bigoplus_{l,m=0,|l-m|\equiv 1 (\mathrm{mod} 2)}^{k} S_+^{2l} \otimes S_-^{2m}. \tag{3.1}$$

The corresponding Lie algebra homomorphisms into $\mathcal{H}^k$ (resp. $S^2\mathcal{H}^k$ and $\wedge^2\mathcal{H}^k$) are denoted by $\varrho_{\mathcal{H}^k}$, (resp. $\varrho_{S^2\mathcal{H}^k}$ and $\varrho_{\wedge^2\mathcal{H}^k}$).

4. Moduli spaces

We can specialize Theorem 2.2 to the case where the domain manifold is the 3-sphere $\mathrm{SU}_+ \times \mathrm{SU}_-/\Delta$ and $f : S^3 \to \mathbf{P}^n = \mathbf{P}(\mathbf{C}^{n+1*})$ is a harmonic totally real map with a constant energy density. The canonical complement of the pair $(\mathrm{SU}_+ \times \mathrm{SU}_-, \Delta)$ is denoted by $\mathfrak{m}$ [7].

Since the Kähler form on $\mathbf{P}^n$ is the curvature form of $\mathcal{O}(1) \to \mathbf{P}^n$ and the pull-back of the Kähler form vanishes from f being a totally real map, the pull-back bundle of $\mathcal{O}(1) \to \mathbf{P}^n$ is a flat bundle on S^3. It also has a Hermitian metric compatible with the connection. Since S^3 is simply-connected, $f^*\mathcal{O}(1) \to S^3$ is a trivial bundle with a product connection and the space of sections of $f^*\mathcal{O}(1)$ is identified with $C^\infty(S^3)$. Thus f satisfies the gauge condition for $(V \to S^3, h_\Delta, \nabla^0)$, where h_Δ is now recognized as the fiber metric on $V \to S^3$ and $f^*\mathcal{O}(1) \to S^3$ has a preferred trivialization. Notice that a trivial line bundle $V \to S^3$ is regarded as an irreducible homogeneous vector bundle and the product connection as the canonical one ∇^0. Since the energy density is constant, Theorem 2.1 yields that $\mathbf{C}^{n+1}$ is the eigenspace of the Laplacian.

Definition 4.1. Let $f : S^3 \to \mathbf{P}^n = \mathbf{P}(\mathbf{C}^{n+1*})$ be a harmonic totally real map with a constant energy density. If $\mathbf{C}^{n+1}$ is a subspace of the eigenspace with $k(k+2)$ as the eigenvalue of the Laplacian, then f is said to be of *degree k*.

Let $\mathcal{H}^k$ be a class one representation of $(\mathrm{SU}_+ \times \mathrm{SU}_-, \Delta)$. We abbreviate $S^0_\Delta \subset \mathcal{H}^k$ to V_0. From the identification $S^k\mathbf{C}^2 \cong S^k\mathbf{C}^{2*}$ by $w \mapsto h(\cdot, \tau(w))$, $\mathcal{H}^k$ can be regarded as $\mathrm{Hom}(S^k_-, S^k_+) \cong S^k_+ \otimes S^k_-{}^*$. Since $\mathrm{Hom}(S^k_-, S^k_+)$ is $\mathrm{End}(S^k_\Delta, S^k_\Delta)$ as Δ-module, V_0 is identified with the subspace of $\mathcal{H}^k$ generated by $Id_{S^k_\Delta}$. When we denote by $v_0 \in \mathcal{H}^k$ the basis of V_0 corresponding to $Id_{S^k_\Delta}$ and by $w^\pm_k, w^\pm_{k-2}, \cdots, w^\pm_{-k+2}, w^\pm_{-k}$ a unitary basis of $S^k_\pm$ with weight $k-2i$ $(i = 0, 1, \cdots, k)$, it follows from the definition of $v_0 \in \mathcal{H}^k = S^k_+ \otimes S^k_-{}^*$ that

$$v_0 = \sum_{i=0}^{k} w^+_{k-2i} \otimes {w^-_{k-2i}}^*,$$

where ${w^-_k}^*, {w^-_{k-2}}^*, \cdots, {w^-_{-k+2}}^*, {w^-_{-k}}^*$ is the dual basis of $S^k_-{}^*$. From the identification, we can see that

$$w_{k-2i} \mapsto (-1)^i w^*_{-k+2i}, \quad \text{or,} \quad w^*_{k-2i} \mapsto (-1)^{k-i} w_{-k+2i}.$$

Thus, when $\mathcal{H}^k$ is considered as $S^k_+ \otimes S^k_-$, v_0 is written as:

$$v_0 = \sum_{i=0}^{k} (-1)^{k-i} w^+_{k-2i} \otimes w^-_{-k+2i}.$$

We consider $\mathrm{H}(\mathcal{H}^k)$ the set of Hermitian endomorphisms of $\mathcal{H}^k$, when applying Theorem 2.2. Then, we need to specify $GH(V_0, V_0)$

and $GH(\mathfrak{m}V_0, V_0)$ in $\mathrm{H}(\mathcal{H}^k)$. To do this, we complexify $GH(V_0, V_0)$ and $GH(\mathfrak{m}V_0, V_0)$ and specify them in $\mathrm{End}\,\mathcal{H}^k$. The complexified spaces are denoted by the same symbols.

Since V_0 is a trivial representation and $\mathfrak{m}V_0 = S^2_\Delta$, $H(V_0, V_0)$ is also trivial and $H(\mathfrak{m}V_0, V_0)$ consists of vectors of weight ± 2 as Δ-modules. It follows from Frobenius reciprocity and Clebsch-Gordan that

$$
\begin{aligned}
(4.1) \qquad & GH(V_0, V_0),\ GH(\mathfrak{m}V_0, V_0) \\
& \subset \bigoplus_{l,m=0,|l-m|=0}^{k} S_+^{2l} \otimes S_-^{2m} \ \oplus \bigoplus_{l,m=0,|l-m|=2}^{k} S_+^{2l} \otimes S_-^{2m} \\
& = \bigoplus_{l=0}^{k} S_+^{2l} \otimes S_-^{2l} \ \oplus \ \bigoplus_{l=1}^{k} S_+^{2l} \otimes S_-^{2l-2} \oplus S_+^{2l-2} \otimes S_-^{2l}.
\end{aligned}
$$

Proposition 4.2. *The standard map $f_0 : S^3 \to \mathbf{P}({\mathcal{H}^k}^*)$ induced by a pair $(S^3 \times V_0, \mathcal{H}^k)$ is a a harmonic totally real map of degree k with a constant energy density.*

Proof. Since the standard map is $\mathrm{SU}_+ \times \mathrm{SU}_-$-invariant, f_0 has a constant energy density. In Theorem 2.2, the standard map f_0 corresponds to Id_W. For $W = \mathcal{H}^k$, (4.1) assures that Id_W satisfies (2.1). Theorem 2.2 yields that it is a harmonic map and the curvature of the pull-back connection is flat. Thus f_0 is a totally real map. Since $\mathcal{H}^k$ is the eigenspace with eigenvalue $k(k+2)$, f_0 is of degree k. $\square$

We now see the decomposition $\mathrm{End}\,\mathcal{H}^k = S^2\mathcal{H}^k \oplus \wedge^2\mathcal{H}^k$ in detail. Let $\mathcal{H}^k_{\mathbf{R}}$ denote the real subspace of $\mathcal{H}^k$ invariant by $\sigma = \tau_+ \otimes \tau_-$ on $\mathcal{H}^k$. Then the real Grassmannian of hyperplanes in $\mathcal{H}^k_{\mathbf{R}}$ is a totally geodesic and totally real submanifold $i : \mathbf{RP}({\mathcal{H}^k_{\mathbf{R}}}^*) \to \mathbf{P}({\mathcal{H}^k}^*)$.

If $C \in S^2\mathcal{H}^k$ (resp. $C \in \wedge^2\mathcal{H}^k$) is a Hermitian endomorphism of $\mathcal{H}^k$, then, by the real structure σ on $\mathcal{H}^k$, we have that $\sigma C \sigma = C$ (resp. $\sigma C \sigma = -C$). For any $w \in \mathcal{H}^k_{\mathbf{R}}$, when $C \in H(\mathcal{H}^k) \cap S^2\mathcal{H}^k$,

$$
(4.2) \qquad \sigma(Cw) = \sigma(\sigma C \sigma \sigma(w)) = Cw,
$$

and when $C \in H(\mathcal{H}^k) \cap \wedge^2\mathcal{H}^k$,

$$
(4.3) \qquad \sigma(Cw) = \sigma(-\sigma C \sigma \sigma(w)) = -Cw.
$$

Suppose that $C \in H(\mathcal{H}^k) \cap S^2\mathcal{H}^k$. For (4.2), C preserves $\mathcal{H}^k_{\mathbf{R}}$ and defines a symmetric endomorphism on $\mathcal{H}^k_{\mathbf{R}}$. Then Toth-D'ambra classification [8] yields that C corresponds to a full harmonic map with constant energy density of S^3 into the sphere. Since they adopt an (image) equivalence relation by the orthogonal group, the target can be replaced by the real projective space $\mathbf{RP}({\mathcal{H}^k_{\mathbf{R}}}^*)$.

Proposition 4.3. *We have that*

$$
\begin{aligned}
&GH(V_0, V_0) \oplus GH(\mathfrak{m}V_0, V_0) \\
=&\bigoplus_{l=0}^{k} S_+^{2l} \otimes S_-^{2l} \ \oplus \ \bigoplus_{l=1}^{k} S_+^{2l} \otimes S_-^{2l-2} \oplus S_+^{2l-2} \otimes S_-^{2l}.
\end{aligned}
$$

Proof. It follows from do Carmo-Wallach [2] and Toth-D'ambra [8] that

$$GH(V_0, V_0) \cap S^2\mathcal{H}^k = \bigoplus_{l=0}^{k} S_+^{2l} \otimes S_-^{2l}.$$

We use induction on k to show the result. Since $\mathcal{H}^1 = S_+^1 \otimes S_-^1$,

$$\wedge^2\mathcal{H}^1 = S_+^2 \oplus S_-^2.$$

As $\mathfrak{m}V_0 = S_\Delta^2$, the definition of $GH(\mathfrak{m}V_0, V_0)$ yields the result.

From (3.1), $\wedge^2\mathcal{H}^k$ is decomposed as $\mathrm{SU}_+ \times \mathrm{SU}_-$-module:

$$\wedge^2\mathcal{H}^k = \bigoplus_{j=0}^{[\frac{k}{2}]} \left(S_+^{2k} \otimes S_-^{2k-4j-2} \oplus S_+^{2k-4j-2} \otimes S_-^{2k}\right) \oplus \wedge^2\mathcal{H}^{k-1}.$$

We would like to claim that

$$
\begin{aligned}
&GH(\mathfrak{m}V_0, V_0) \cap \wedge^2\mathcal{H}^k \\
=&S_+^{2k} \otimes S_-^{2k-2} \oplus S_+^{2k-2} \otimes S_-^{2k} \oplus \left(GH(\mathfrak{m}V_0, V_0) \cap \wedge^2\mathcal{H}^{k-1}\right).
\end{aligned}
$$

Since, for $Z, W \in \mathfrak{sl}(2, \mathbf{C})$ and a non-negative integer p,

$$\varrho_{\mathcal{H}^k}(Z, W)^p v_0 = \sum_{i=0}^{k} (-1)^{k-i} \sum_{r=0}^{p} \binom{p}{r} \left(\varrho(Z)^{p-r} w_{k-2i}^+\right) \otimes \left(\varrho(W)^r w_{-k+2i}^-\right),$$

we get

$$
\begin{aligned}
&\varrho_{\mathcal{H}^k}(Z, -Z)^{k-1} v_0 \\
&= \sum_{i=0}^{k-1} (-1)^{k-i} \sum_{r=0}^{k-1} \binom{k-1}{r} \left(\varrho(Z)^{k-1-r} w_{k-2i}^+\right) \otimes \left(\varrho(-Z)^r w_{-k+2i}^-\right).
\end{aligned} \tag{4.4}
$$

We pick up an element $Z \in \mathfrak{m}^C \subset \mathfrak{sl}(2, \mathbf{C})$, where $\mathfrak{m}^C$ denotes the complexified space of $\mathfrak{m}$, in such a way that

$$\varrho(Z) w_{k-2i} = \sqrt{(k-i)(i+1)}\, w_{k-2i-2}, \tag{4.5}$$

and put $v_1 = \varrho_{\mathcal{H}^k}(Z, -Z) v_0$. Since v_0 is σ invariant by definition and $\varrho_{\mathcal{H}^k}$ commutes with σ, v_1 is also σ invariant. This yields that $v_0 \wedge v_1 \in GH(\mathfrak{m}V_0, V_0) \cap \wedge^2\mathcal{H}^k$.

It follows from (4.5) that

$$\varrho(Z)^q w_{k-2i} = \sqrt{\frac{(k-i)!}{(k-i-q)!} \frac{(i+q)!}{i!}}\, w_{k-2(i+q)}, \tag{4.6}$$

and

$$\varrho(Z)^q w_{-k+2i} = \sqrt{\frac{(k-i+q)!}{(k-i)!}\frac{i!}{(i-q)!}}\, w_{-k+2(i-q)}. \tag{4.7}$$

Since w_{k-2i} and $w_{-k+2i} \in S^k\mathbf{C}^2$, we see from (4.6) and (4.7) that

$$\varrho(Z)^q w_{k-2i} = 0 \iff k-2i-2q < -k \iff q > k-i,$$

$$\varrho(Z)^q w_{-k+2i} = 0 \iff -k+2i-2q < -k \iff q > i.$$

Consequently,

$$\varrho(Z)^{k-1-r} w_{k-2i} = 0 \iff k-1-r > k-i \iff r < i-1.$$

Thus only the terms in the range $i-1 \leqq r \leqq i$ in (4.4) remains:

$$\begin{aligned}
&\varrho_{\mathcal{H}^k}(Z,-Z)^{k-1}v_0 \\
&= \sum_{i=1}^{k}(-1)^{k-i}\left\{\binom{k-1}{i-1}\left(\varrho(Z)^{k-i}w^+_{k-2i}\right)\otimes\left(\varrho(-Z)^{i-1}w^-_{-k+2i}\right)\right\} \\
&+ \sum_{i=0}^{k-1}(-1)^{k-i}\left\{\binom{k-1}{i}\left(\varrho(Z)^{k-i-1}w^+_{k-2i}\right)\otimes\left(\varrho(-Z)^{i}w^-_{-k+2i}\right)\right\} \\
&= \sum_{i=1}^{k}(-1)^{k-1}\left\{\binom{k-1}{i-1}\left(\varrho(Z)^{k-i}w^+_{k-2i}\right)\otimes\left(\varrho(Z)^{i-1}w^-_{-k+2i}\right)\right\} \\
&+ \sum_{i=0}^{k-1}(-1)^{k}\left\{\binom{k-1}{i}\left(\varrho(Z)^{k-i-1}w^+_{k-2i}\right)\otimes\left(\varrho(Z)^{i}w^-_{-k+2i}\right)\right\}.
\end{aligned} \tag{4.8}$$

It follows from (4.6) and (4.7) that

$$\begin{aligned}
\varrho(Z)^{k-i-1}w^+_{k-2i} &= \sqrt{\frac{(k-i)!(k-1)!}{i!}}\, w^+_{-k+2}, \\
\varrho(Z)^{k-i}w^+_{k-2i} &= \sqrt{\frac{(k-i)!k!}{i!}}\, w^+_{-k}, \\
\varrho(Z)^{i-1}w^-_{-k+2i} &= \sqrt{\frac{(k-1)!i!}{(k-i)!}}\, w^-_{-k+2}, \\
\varrho(Z)^{i}w^-_{-k+2i} &= \sqrt{\frac{k!i!}{(k-i)!}}\, w^-_{-k}.
\end{aligned}$$

Hence (4.8) reduces to:

$$
\begin{aligned}
(4.9)\qquad & \varrho_{\mathcal{H}^k}(Z,-Z)^{k-1}v_0 \\
=&(-1)^{k-1}\sqrt{k}(k-1)!\left\{\sum_{i=1}^{k}\binom{k-1}{i-1}\right\}w^+_{-k}\otimes w^-_{-k+2} \\
+&(-1)^{k}\sqrt{k}(k-1)!\left\{\sum_{i=0}^{k-1}\binom{k-1}{i}\right\}w^+_{-k+2}\otimes w^-_{-k} \\
=&(-1)^{k-1}\sqrt{k}(k-1)!2^{k-1}\left(w^+_{-k}\otimes w^-_{-k+2}-w^+_{-k+2}\otimes w^-_{-k}\right).
\end{aligned}
$$

It follows from $\varrho(Z)w_{-k+2}=\sqrt{k}w_{-k}$ that

$$
\begin{aligned}
(4.10)\quad \varrho_{\mathcal{H}^k}(Z,-Z)^{k}v_0 =&(-1)^{k-1}k!2^{k-1}\left(-w^+_{-k}\otimes w^-_{-k}-w^+_{-k}\otimes w^-_{-k}\right) \\
=&(-1)^{k}k!2^{k}w^+_{-k}\otimes w^-_{-k}
\end{aligned}
$$

and $\varrho_{\mathcal{H}^k}(Z,-Z)^{p}v_0=0$, when $p>k$. Then we have that

$$
\begin{aligned}
&\varrho_{\wedge^2\mathcal{H}^k}(Z,-Z)^{2k-2}\left(v_0\wedge v_1\right) \\
=&4(k-1)\binom{2k-1}{k}\varrho_{\mathcal{H}^k}(Z,-Z)^{k-1}v_0\wedge \varrho_{\mathcal{H}^k}(Z,-Z)^{k}v_0.
\end{aligned}
$$

From (4.9) and (4.10), considering the weights, we can conclude that

$$
\varrho_{\wedge^2\mathcal{H}^k}(Z,-Z)^{2k-2}\left(v_0\wedge v_1\right)\in S^{2k}_+\otimes S^{2k-2}_-\oplus S^{2k-2}_+\otimes S^{2k}_-,
$$

and $S^{2k}_+\otimes S^{2k-2}_-\oplus S^{2k-2}_+\otimes S^{2k}_-$ is a subset of $GH(\mathfrak{m}V_0,V_0)$. Combining this with (4.1), we obtain the result. □

Theorem 2.2 with Propositions 4.2 and 4.3 yields the main result. To state the main theorem, we adopt the convention that $S^{2k}_+\otimes S^{2l}_-$ stands for the real vector space invariant by the real structure $\tau_+\otimes\tau_-$. For $T\in \mathrm{H}(\mathcal{H}^k)$, we write $T>0$ to indicate that T is positive definite.

Theorem 4.4. *If $f:S^3\to \mathbf{P}(\mathbf{C}^{n*})$ is a full harmonic totally real map of degree k with a constant energy density, then $n\le (k+1)^2$ and $\mathbf{C}^n$ is considered as a subspace of $\mathcal{H}^k$.*

Let $\mathcal{M}_k$ be the moduli space of full harmonic totally real maps of degree k with a constant energy density of S^3 into $\mathbf{P}(\mathcal{H}^{k*})$ modulo image equivalence of maps. Then $\mathcal{M}_k$ is identified with a subset of $\bigoplus_{l,m=0,|l-m|\geqq 2}^{k} S^{2l}_+\otimes S^{2m}_-$:

$$
\mathcal{M}_k=\left\{C\in\bigoplus_{l,m=0,|l-m|\geqq 2}^{k}S^{2l}_+\otimes S^{2m}_-\;\middle|\; Id_{\mathcal{H}^k}+C>0\right\},
$$

where $\bigoplus_{l,m=0,|l-m|\geqq 2}^{k} S^{2l}_+\otimes S^{2m}_-$ is regarded as a subspace of $\mathrm{H}(\mathcal{H}^k)$.

Let $\overline{\mathcal{M}_k}$ be the closure of the moduli $\mathcal{M}_k$ by topology induced from the inner product. Every boundary point of $\overline{\mathcal{M}_k}$ distinguishes a subspace $\mathbf{C}^n$ of $\mathcal{H}^k$ and describes one of those maps into $\mathbf{P}(\mathbf{C}^{n*})$ which can be *regarded as totally geodesic submanifold of* $\mathbf{P}(\mathcal{H}^{k^*})$. *The Hermitian inner product on* $\mathcal{H}^k$ *determines the orthogonal decomposition of* $\mathcal{H}^k$: $\mathcal{H}^k = \mathbf{C}^n \oplus \mathbf{C}^{n^\perp}$. *Then the totally geodesic submanifold* $\mathbf{P}(\mathbf{C}^{n*})$ *can be obtained as the common zero set of sections of* $\mathcal{O}(1) \to \mathbf{P}(\mathcal{H}^{k^*})$ *which belongs to* $\mathbf{C}^{n^\perp}$.

Finally, $C \in \overline{\mathcal{M}_k}$ *corresponds to* $(Id_{\mathcal{H}^k} + C)^{-\frac{1}{2}} f_0$, *under the convention that the inverse of* $Id_{\mathcal{H}^k} + C$ *is taken on* Ker $(Id_{\mathcal{H}^k} + C)^\perp$.

Corollary 4.5. *Let* $f : S^3 \to \mathbf{P}(\mathbf{C}^{n*})$ *be an* $\mathrm{SU}_+ \times \mathrm{SU}_-$*-equivariant full harmonic totally real map. Then* f *is the standard map up to image equivalence.*

Proof. If f is of degree k, Theorem 4.4 implies that f corresponds to $C \in \overline{\mathcal{M}_k}$. Since f is $\mathrm{SU}_+ \times \mathrm{SU}_-$-equivariant, C is also an $\mathrm{SU}_+ \times \mathrm{SU}_-$-equivariant linear endomorphism on $\mathcal{H}^k$. Schur's lemma yields that C is proportional to the identity. However, C is orthogonal to the identity (see the remark after Theorem 2.2). We thus deduce that $C = 0$. Then the result follows form Proposition 4.2. □

In a similar vein, we can characterize $\mathrm{SU}_\pm$-equivariant harmonic totally real maps, a few examples of which are given in [3]:

Corollary 4.6. *A full harmonic totally real map* $f : S^3 \to \mathbf{P}(\mathbf{C}^{n*})$ *of degree* k *is* SU_+ *(resp.* SU_-*)-equivariant if and only if the Hermitian transform* C *corresponding to* f *is in* $\bigoplus_{m=2}^k S_+^0 \otimes S_-^{2m}$ *(resp.* $\bigoplus_{l=2}^k S_+^{2l} \otimes S_-^0$*).*

We define the subset $\mathcal{AM}_k$ of the moduli space $\mathcal{M}_k$ by:

$$\mathcal{AM}_k = \mathcal{M}_k \cap S^2\mathcal{H}^k.$$

Theorem 4.7. *The moduli space of absolutely real full harmonic totally real maps of degree* k *with constant energy density of* S^3 *into* $\mathbf{P}(\mathcal{H}^{k^*})$ *is identified with* $\mathcal{AM}_k$.

Proof. In [8], Toth-D'ambra constructs the moduli space $\mathbf{M}_k$ of full harmonic maps from S^3 into the sphere of degree k with constant energy density:

$$\mathbf{M}_k := \left\{ C \in \bigoplus_{l,m=0,|l-m|=2i,i\geqq 1}^{k} S_+^{2l} \otimes S_-^{2m} \,\middle|\, Id_{\mathcal{H}^k_{\mathbf{R}}} + C > 0 \right\},$$

where $\bigoplus_{l,m=0,|l-m|=2i,i\geqq 1}^{k} S_+^{2l} \otimes S_-^{2m}$ is considered as a subset of symmetric endomorphisms on $\mathcal{H}^k_{\mathbf{R}}$. When $\mathcal{H}^k_{\mathbf{R}}$ is complexified, C defines a Hermitian endomorphism of $\mathcal{H}^k$ and belongs to $\mathcal{AM}_k$.

Suppose that $C \in \mathcal{AM}_k$. Since C preserves $\mathcal{H}^k_{\mathbf{R}}$ by (4.2), it defines a symmetric endomorphism on $\mathcal{H}^k_{\mathbf{R}}$. Thus we can identify $\mathbf{M}_k$ with $\mathcal{AM}_k$.

To see the correspondence of maps according to $\mathbf{M}_k \cong \mathcal{AM}_k$, we take a subspace $U_0^{\mathbf{R}} = U_0 \cap \mathcal{H}_{\mathbf{R}}^k$ to define the standard map $f_0^{\mathbf{R}}$ of S^3 into $\mathbf{RP}({\mathcal{H}_{\mathbf{R}}^k}^*)$ as $f_0^{\mathbf{R}}([g]) = gU_0^{\mathbf{R}}$, where $g \in \mathrm{SU}_+ \times \mathrm{SU}_-$. Then, by [8], $C \in \mathbf{M}_k$ corresponds to $(Id_{\mathcal{H}_{\mathbf{R}}^k} + C)^{-\frac{1}{2}} f_0^{\mathbf{R}}$. Our identification between $\mathbf{M}_k$ and $\mathcal{AM}_k$ yields $(Id_{\mathcal{H}^k} + C)^{-\frac{1}{2}} f_0 = i \circ (Id_{\mathcal{H}_{\mathbf{R}}^k} + C)^{-\frac{1}{2}} f_0^{\mathbf{R}}$.

Suppose that f is an absolutely real full harmonic totally real map with constant energy density. By image equivalence relation, f may be supposed to be a map into $\mathbf{RP}({\mathcal{H}_{\mathbf{R}}^k}^*)$. From Theorem 4.4 and our identification, there exists $C \in \mathbf{M}_k$ such that $f = i \circ (Id_{\mathcal{H}_{\mathbf{R}}^k} + C)^{-\frac{1}{2}} f_0^{\mathbf{R}}$. □

Example. Let $\mathcal{M}_2$ be the moduli space of full harmonic totally real maps of degree 2 with a constant energy density of S^3 into $\mathbf{P}(\mathcal{H}^{2*})$ modulo image equivalence:

$$\mathcal{M}_2 = \left\{ C \in S_+^4 \otimes S_-^0 \oplus S_+^0 \otimes S_-^4 \,\middle|\, Id_{\mathcal{H}^2} + C > 0 \right\}.$$

From Theorem 4.7, all maps are absolutely real and from Corollary 4.6, those are SU_+ or SU_--equivariant ones.

Li obtains an isometric minimal totally real immersion of S^3 into $\mathbf{P}^{11}$ which is not absolutely real [5]. This immersion is of degree 3. We also give such an example.

Example. Let $\mathcal{M}_3$ be the moduli space of those maps of degree 3 of S^3 into $\mathbf{P}(\mathcal{H}^{3*})$ modulo image equivalence.

$$\mathcal{M}_3 = \{ C \in S_+^6 \otimes S_-^2 \oplus S_+^2 \otimes S_-^6 \oplus S_+^4 \otimes S_-^0 \oplus S_+^0 \otimes S_-^4 \oplus S_+^6 \otimes S_-^0 \oplus S_+^0 \otimes S_-^6 \,|\, Id_{\mathcal{H}^3} + C > 0 \}.$$

The unitary basis of $\mathcal{H}^3$ is denoted by $w_{3-2l}^+ \otimes w_{3-2m}^-$, $(l, m = 0, \cdots, 3)$. If $C = i(w_6^+ - w_{-6}^+) \in S_+^6 \otimes S_-^0$, then C satisfies $\sigma C \sigma = -C$. From (4.3), we see that $C \in \mathrm{H}(\mathcal{H}^3)$ and it is written as:

$$C = i \left(\begin{array}{c|c|c|c} O & O & O & Id_4 \\ \hline O & O & O & O \\ \hline O & O & O & O \\ \hline -Id_4 & O & O & O \end{array} \right).$$

Hence $Id_{\mathcal{H}^3} + tC$ has 1 and $1 \pm t$ as its eigenvalues and each eigenspace is of dimension 4, which is equivalent to S_-^3 as SU_--module. If $t \in (0, 1]$, then $Id_{\mathcal{H}^3} + tC$ induces a full harmonic totally real map with constant energy density, which is not absolutely real. When $t = 1$, $Id_{\mathcal{H}^3} + C$ has a kernel of dimension 4. By Theorem 4.4, it induces a full map into $\mathbf{P}(\mathbf{C}^{12*})$, where $\mathbf{C}^{12} \subset \mathcal{H}_3$ is the orthogonal complement of $\mathrm{Ker}\,(Id_{\mathcal{H}^3} + C)$. Corollary 4.6 yields that those maps are SU_--equivariant.

In a similar way, using $w_4^+ + w_{-4}^+ \in S_+^4 \otimes S_-^0$, we obtain one parameter family f_t ($t \in [0,1]$) of harmonic totally real map with constant energy density, which is absolutely real and f_1 is a full map into $\mathbf{P}(\mathbf{R}^{8^*})$, where $\mathbf{R}^8$ is a subspace of $\mathrm{SU}_+ \times \mathrm{SU}_-$- invariant real subspace $\mathbf{R}^{16}$ of $\mathcal{H}^3$. From Corollary 4.6, we can see that f_t are SU_--equivariant.

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(IK) Faculty of Contemporary Business, Kyushu International University, Hirano, Yahatahigashi-ku, Kitakyushu-shi, Fukuoka 805-8512, JAPAN

(YN) Department of Mathematics, Meiji University, Higashi-Mita, Tama-ku, Kawasaki-shi, Kanagawa 214-8571, JAPAN

Email address: IK: i-koga@cb.kiu.ac.jp

Email address: YN: yasunaga@meiji.ac.jp