COHOMOLOGY AND K-THEORY OF GENERALIZED DOLD MANIFOLDS FIBRED BY COMPLEX FLAG MANIFOLDS

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Abstract. Let $\nu = (n_1, \ldots, n_s), s \geq 2$, be a sequence of positive integers and let $n = \sum_{1\leq j\leq s} n_j$. Let $\mathbb{C}G(\nu) = U(n)/(U(n_1) \times \cdots \times U(n_s))$ be the complex flag manifold. Denote by $P(m, \nu) = P(\mathbb{S}^m, \mathbb{C}G(\nu))$ the generalized Dold manifold $\mathbb{S}^m \times \mathbb{C}G(\nu)/\langle\theta\rangle$ where $\theta = \alpha \times \sigma$ with $\alpha : \mathbb{S}^m \to \mathbb{S}^m$ being the antipodal map and $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$, the complex conjugation. The manifold $P(m, \nu)$ has the structure of a smooth $\mathbb{C}G(\nu)$-bundle over the real projective space $\mathbb{R}P^m$. We determine the additive structure of $H^*(P(m, \nu); R)$ when $R = \mathbb{Z}$ and its ring structure when R is a commutative ring in which 2 is invertible. As an application, we determine the additive structure of $K(P(m, \nu))$ almost completely and also obtain partial results on its ring structure. The results for the singular homology are obtained for generalized Dold spaces $P(S, X) = S \times X/\langle\theta\rangle$, where $\theta = \alpha \times \sigma$, $\alpha : S \to S$ is a fixed point free involution and $\sigma : X \to X$ is an involution with $\mathrm{Fix}(\sigma) \neq \emptyset$, for a much wider class of spaces S and X.

1. Introduction

Let $\alpha : S \to S$ be a fixed point free involution and let $\sigma : X \to X$ be an involution with non-empty fixed point set. Let θ be the involution $\alpha \times \sigma$ on $S \times X$. Then $P(S, X) = S \times X/\langle\theta\rangle$ is a *generalized Dold space* considered in [19]. (See also [22], [23].) The special case when S is the sphere $\mathbb{S}^m$ with antipodal involution and $X = \mathbb{C}P^n$ where σ is given by complex conjugation on $\mathbb{C}^{n+1}$, corresponds to the classical Dold manifold [9], denoted by $P(m, n)$.

In [19], we described the $\mathbb{Z}_2$-cohomology algebra of a generalized Dold spaces $P(S, X)$ for a family of spaces which included, among others, the case $P(\mathbb{S}^n, \mathbb{C}G(\nu))$ with antipodal involution on the sphere $\mathbb{S}^m$ and the complex conjugation on the complex flag manifold $\mathbb{C}G(\nu)$ of type $\nu = (n_1, \ldots, n_s)$. (Thus σ is induced by the complex conjugation on $\mathbb{C}^n$, where $\nu = (n_1, \ldots, n_s)$.)

Our goal in this paper is to study the *integral* (co)homology and the complex K-theory of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$. We denote $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ by $P(m, \nu)$. We determine the integral cohomology *groups* of $P(m, \nu)$. In fact the additive structure of $H^*(P(S, X); \mathbb{Z})$ is obtained for a much wider class of spaces. We obtain a presentation of the cohomology ring of $P(m, \nu)$ with coefficients in a commutative ring R, having 2 as a unit. The complex K-ring of the classical Dold manifold $P(m, n)$ was determined by Fujii in [10], [11]. Note that if ω is a complex vector bundle over a finite CW complex B, one has the associated

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$\mathbb{C}G(\nu)$-bundle $\mathbb{C}G(\omega;\nu) \to B$. In this case, the complex K-ring of $\mathbb{C}G(\omega;\nu)$ is well-known. See [14, §3,Chapter IV] and also [2]. However, it is easily seen that $P(m,\nu)$ is not homeomorphic to $\mathbb{C}G(\omega;\nu)$ for any complex vector bundle ω over $\mathbb{R}P^m$. Indeed, the flag manifold bundle $\mathbb{C}G(\nu) \hookrightarrow \mathbb{C}G(\omega;\nu) \to \mathbb{R}P^m$ admits a cohomology extension of the fibre over $\mathbb{Z}$ and so, by the Leray-Hirsch theorem [24, §7, Chapter 5], $H^*(\mathbb{C}G(\omega;\nu);\mathbb{Z})$ is a free $H^*(\mathbb{R}P^m;\mathbb{Z})$-module of rank equal to $\mathrm{rank}_{\mathbb{Z}}(H^*(\mathbb{C}G(\nu);\mathbb{Z}))$. This implies, in particular, that the rank of $H^2(\mathbb{C}G(\omega;\nu);\mathbb{Z})$ is positive. However, $H^2(P(m,\nu);\mathbb{Z})$ has rank 0; see Theorem 3.8.

When $m \geq 1$, $P(m,\nu)$ is not a homogeneous space of $G = Spin(m+1)\times U(n)$, although $\mathbb{S}^m \times \mathbb{C}G(\nu) = G/H$ where $H \cong Spin(m)\times U(\nu)$. Consequently, the techniques available for studying the K-theory of homogeneous spaces are not directly applicable for $P(m,\nu)$, and the computation of the K-theory of $P(m,\nu)$ becomes a nontrivial and interesting problem.

Our approach to the computation of $K(P(m,\nu))$ is applicable for a larger class of manifolds, such as generalized Dold spaces fibred by a non-singular projective toric variety over $\mathbb{R}P^m$, but we confine our attention to $P(m,\nu)$.

We shall now state the main results of this paper in their weaker form so as to keep notations to a minimum.

Denote by ν_o the number of odd numbers in the sequence ν and let $\lfloor \nu/2 \rfloor := (n'_1,\ldots,n'_s)$ where $n'_j := \lfloor n_j/2 \rfloor$. As usual, $\binom{n}{\nu}$ denotes the multinomial coefficient $n!/(n_1!\cdots n_s!)$. We set $H^{\mathrm{ev}}(B;\mathbb{Z}) := \bigoplus_{q\geq 0} H^{2q}(B;\mathbb{Z})$ and $H^{\mathrm{odd}}(B;\mathbb{Z}) := \bigoplus_{q\geq 0} H^{2q+1}(B;\mathbb{Z})$.

Theorem A (See Theorem 3.11). Let $m \geq 1, \nu = (n_1,\ldots,n_s), s \geq 2, n := |\nu| = \sum_{1\leq j\leq s} n_j$. Then

$$H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}, \quad H^{\mathrm{odd}}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$$

where

$$b_e = \binom{n}{\nu},\ b_o = 0, \text{if } m \equiv 0 \pmod 2,$$

$$b_e = b_o = \ell_e = \begin{cases} \binom{n}{\nu}/2, & \text{if } \nu_o \geq 2 \\ \binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor} & \text{if } \nu_o \leq 1, \end{cases} \text{ if } m \equiv 1 \pmod 2,$$

and,

$$b'_e = \lfloor m/2 \rfloor \cdot \ell_e, b'_o = \lfloor (m-1)/2 \rfloor \cdot \ell_e.$$

A description of $H^*(P(m,\nu);R)$ as a quotient of a polynomial ring is given in Theorem 3.14, when R is any commutative ring in which 2 is invertible.

Next, we state our result on the additive structure of $K^*(P(m,\nu))$.

Theorem B (See Theorem 4.1). With notations as in Theorem A,
(i) $K^0(P(m,\nu)) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{\lfloor m/2 \rfloor} \oplus A_0$ where $o(A_0) = 2^t$ for some t, $0 \leq t \leq b'_e - \lfloor m/2 \rfloor$.
(ii) $K^1(P(m,\nu)) \cong \mathbb{Z}^{b_o} \oplus A_1$ where $o(A_1) = 2^t$ for some t, $0 \leq t \leq b'_o$.

We describe in §4, a subring $\mathcal{K}^0$ of $K^0(P(m,\nu))$ generated by the classes of certain vector bundles and show that as an abelian group, $K^0(P(m,\nu))/\mathcal{K}^0$ is finite. See Theorem 4.13.

Notation: Let $f : X \to Y$ be a continuous map. We shall use the same symbol $f^!$ for pull-back of a vector bundle over Y to X and also for the induced homomorphism $K^*(Y) \to K^*(X)$ in the complex K-ring. We denote by f^* and also by $H^m(f;R)$, the induced homomorphism in cohomology $H^m(Y;R) \to H^m(X;R)$.

2. Preliminaries

In this section, we shall obtain some basic results concerning the topology of generalized Dold spaces $P(S,X)$ which will be needed for our purposes. Some results (and notations) from [22] and [19] will also be recalled.

Let $(S,\alpha),(X,\sigma)$ be as in the Introduction. Then $\theta = \alpha \times \sigma : S \times X \to S \times X$ is a fixed point free involution. Set $Y := S/{\sim_\alpha}$ and $X_\mathbb{R} := \mathrm{Fix}(\sigma)$. Note that one has a fibre bundle with projection $p : P(S,X) \to Y$, defined as $[u,x] \mapsto [u] \in Y$, with fibre space X. A useful fact is that we have an inclusion $Y \times X_\mathbb{R} \hookrightarrow P(S,X)$ defined as $([u],x) \mapsto [u,x]$ and the projection $p|_{Y\times X_\mathbb{R}}$ is the first projection $pr_1 : Y \times Fix(\sigma) \to Y$. In particular, for any $x \in X_\mathbb{R}$, we have a cross-section $Y \to P(S,X)$ defined as $s_x([u]) = [u,x]$.

We generalize the construction of the vector bundle $\hat{\omega}$ over $P(S,X)$ for a σ conjugate vector bundle ω over X.

Suppose that ω is a real vector bundle over a space B with projection $p_\omega : E(\omega) \to B$. Let $f : B \to B$ be a fixed point free involution that is covered by an involutive bundle automorphism $\hat{f} : E(\omega) \to E(\omega)$. We obtain a vector bundle $\hat{\omega}$ over $\bar{B} := B/{\sim_f}$ with total space $E(\hat{\omega}) = E(\omega)/{\sim_{\hat{f}}}$. The bundle projection $E(\hat{\omega}) \to \bar{B}$ sends $[v] = \{v, \hat{f}(v)\}$ to $[b]$ for all $v \in E_b(\omega)$. If ω is a complex vector bundle and $\hat{f}$ is a complex vector bundle morphism, then $\hat{\omega}$ is a complex vector bundle over $\bar{B}$. We emphasize that the isomorphism class of $\hat{\omega}$ depends not only on ω, but also on $\hat{f}$.

Recall from [22] that a σ-conjugate vector bundle η over (X,σ) is a (real) vector bundle with a bundle involution $\hat{\sigma} : E(\eta) \to E(\eta)$ that covers $\sigma : X \to X$. Such a pair $(\eta,\hat{\sigma})$ leads to the construction of a real vector bundle $\hat{\eta}$ over $P(S,X)$ as follows: The bundle involution $\hat{\sigma}$ yields a fixed point free involution $\hat{\theta} : S \times E(\eta) \to S \times E(\eta)$ which is also a vector bundle morphism of $pr_2^!(\eta)$ that covers θ. It follows that $P(S,E(\eta)) \to P(S,X)$ is the projection of a vector bundle, denoted by $\hat{\eta}$.

A complex vector bundle ω is said to be a σ-*conjugate vector bundle* if $\hat{\sigma} : E(\omega) \to E(\omega)$ is a bundle involution that covers σ and is *conjugate complex linear* on the fibres of ω. Then the vector bundle $\hat{\omega}$ on $P(S,X)$ is isomorphic to $\hat{\bar{\omega}}$. Moreover, its complexification $\hat{\omega} \otimes_\mathbb{R} \mathbb{C}$ restricts to $\omega \oplus \bar{\omega}$ on any fibre of the X-bundle $p : P(S,X) \to Y$.

The construction of $\hat{\omega}$ behaves well with respect to Whitney sum, tensor products, taking exterior powers, etc., so long as all the bundles involved are σ-conjugate (complex) vector bundles, i.e., the respective bundle involutions cover the *same* $\sigma : X \to X$. (See [22].)

2.1. **Half-spin bundles over $\mathbb{S}^{2r}$.** Assume that $m = 2r$ is even. An explicit description of a complex vector bundle ξ over $\mathbb{S}^m$ such that $[\xi] - \mathrm{rank}(\xi)$ is a generator of $\tilde{K}(S^m) \cong \mathbb{Z}$ is well-known. See [3], [7]. For the sake of completeness, we give the details.

Explicitly, ξ may be taken to be associated to a half-spin representation of $\mathrm{Spin}(m)$ where $\mathbb{S}^m$ is viewed as the homogeneous space $\mathrm{Spin}(m+1)/\mathrm{Spin}(m)$. We shall explain this construction below, assuming familiarity with the basic properties of the spin groups, as given in [13].

We consider $\mathbb{R}^{m+1}$ as the standard $SO(m+1)$-representation. It becomes a representation of $\mathrm{Spin}(m+1)$ via the projection $\mathrm{Spin}(m+1) \to SO(m+1)$. Regard $\mathrm{Spin}(m) \subset \mathrm{Spin}(m+1)$ is the subgroup that stabilizes the vector $e_1 \in \mathbb{R}^{m+1}$. Consider bundles ξ^+, ξ^- associated to the half-spin (complex) representations Δ^+, Δ^- of the spin group $\mathrm{Spin}(m)$. Then $[\xi^+] - 2^{r-1}$ is a generator of $\tilde{K}(\mathbb{S}^m)$. See [3], [13, Theorem 13.3, Chapter 14]. (Note that the degrees of the representations $\Delta^{\pm}$ equal 2^{r-1}.) It turns out that $[\xi^+] + [\xi^-] = 2^r$ in $\tilde{K}(\mathbb{S}^m)$. In fact, the representation $\Delta^+ \oplus \Delta^-$ is the restriction of the spin representation Δ of $\mathrm{Spin}(m+1)$. As the bundle associated to $\Delta|_{\mathrm{Spin}(m)}$ is trivial, we obtain that $\xi^+ \oplus \xi^- = 2^r \epsilon_{\mathbb{C}}$, whence

$$[\xi^+] - 2^{r-1} = -([\xi^-] - 2^{r-1}). \tag{1}$$

Recall that $\alpha : \mathbb{S}^m \to \mathbb{S}^m$ denotes the antipodal map.

Theorem 2.1. *Let $m = 2r$ be even. The ring $K^0(\mathbb{S}^m)$ is isomorphic to the truncated polynomial ring $\mathbb{Z}[u_m]/\langle u_m^2 \rangle$ under an isomorphism that sends u_m to $[\xi^+] - 2^{r-1}$. In particular, $[\xi^{\pm}]^2 = 2^r[\xi^{\pm}] - 2^{2r-2}$, and, $[\xi^+][\xi^-] = 2^{2r-2}$. Moreover, $\alpha^! : K^0(\mathbb{S}^m) \to K^0(\mathbb{S}^m)$ sends $[\xi^+]$ to $[\xi^-]$.*

Proof. The description of the ring structure and the fact that u_m is represented by $[\xi^+] - 2^{r-1}$ are well-known. (See [3] and [13].) The asserted quadratic relation follows from $([\xi^+] - 2^{r-1})^2 = u_m^2 = 0$. Since $[\xi^+] + [\xi^-] = 2^r$, the same relation is satisfied by $[\xi^-]$. Also, $[\xi^+][\xi^-] = [\xi^+](2^r - [\xi^+]) = 2^{2r-2}$.

It remains to show that $\alpha^!(u_m) = -u_m$. This follows from the following facts: (i) $\alpha^* : H^m(\mathbb{S}^m; \mathbb{Q}) \to H^m(\mathbb{S}^m; \mathbb{Q})$ is $-id$, (ii) the Chern character $\mathrm{ch} : K(\mathbb{S}^m) \to H^*(\mathbb{S}^m; \mathbb{Q})$ is a monomorphism, and, (iii) $\alpha^* \circ \mathrm{ch} = \mathrm{ch} \circ \alpha^!$. □

We have not been able to find a natural and explicit bundle isomorphism $\xi^- \to \xi^+$ which covers the antipodal map $\alpha : \mathbb{S}^m \to \mathbb{S}^m$. For this reason, we work with the pull-back bundle $\alpha^!(\xi^+)$, which will suffice for our purposes.

Denote the pull-back bundle $\alpha^!(\xi^+)$ by η^-. When $m \geq 2$ is even, we have $2^{r-1} \geq m/2$ and any two vector bundles of rank 2^{r-1} which represent the same class in $K(\mathbb{S}^m)$ are isomorphic, using [13, Theorem 1.5, Chapter 8].

For future reference, we have the following remark.

Remark 2.2. From the above discussion, we have $[\eta^-] = [\xi^-] = 2^r - [\xi^+]$ in $K(\mathbb{S}^m)$ for all $m \equiv 0 \pmod 2$.

We have a bundle isomorphism $\tilde{\alpha}^- : E(\eta^-) \subset \mathbb{S}^m \times E(\xi^+) \to E(\xi^+)$ given by the second projection that covers the antipodal map α. Explicitly, $\tilde{\alpha}^- : E(\eta^-) \to E(\xi^+)$ is the map that $(v, e) \in E_v(\eta^-)$ to $e \in E_{-v}(\xi^+)$. Then $\tilde{\alpha}^-$ is a bundle map that covers α. Similarly $\tilde{\alpha}^+ : E(\xi^+) \to E(\eta^-)$ is the bundle map that sends $e \in E_v(\xi^+)$ to $(-v, e) \in E_{-v}(\eta^-)$. Henceforth, we shall identify the fibre $E_v(\eta^-) = \{v\} \times E_{-v}(\xi^+)$ with $E_{-v}(\xi^+)$ so that the total space $E(\eta^-)$ is identified with $E(\xi^+)$. We have, for any $v \in \mathbb{S}^m$ and $e \in E_v(\xi^+)$, $\tilde{\alpha}^+(\tilde{\alpha}^-(e)) = e$, i.e., $\tilde{\alpha}^+ \circ \tilde{\alpha}^- = id_{\eta^-}$. Similarly $\tilde{\alpha}^- \circ \tilde{\alpha}^+ = id_{\xi^+}$.

Set $\tilde{\xi} := \xi^+ \oplus \eta^-$. A vector in the fibre $E_v(\tilde{\xi})$ over $v \in \mathbb{S}^m$ will be denoted by a triple $(v; e, e'), e \in E_v(\xi^+), e' \in E_{-v}(\xi^+) = E_v(\eta^-), v \in \mathbb{S}^m$. Define $\tilde{\alpha} : E(\xi^+ \oplus \eta^-) \to E(\xi^+ \oplus \eta^-)$ as $\tilde{\alpha}(v; e, e') = (-v; \tilde{\alpha}_v^-(e'), \tilde{\alpha}_v^+(e))$. Then $\tilde{\alpha}$ covers α and $\tilde{\alpha} \circ \tilde{\alpha} = id$.

We therefore obtain a complex vector bundle ξ^0 over $\mathbb{R}P^m$ where $E(\xi^0) := E(\tilde{\xi})/\langle \tilde{\alpha} \rangle$. A point of $E_{[v]}(\xi^0)$ is $[v; x, y] = \{(v; x, y), (-v; y, x)\}$ where $x \in E_v(\xi^+), y \in E_v(\eta^-) = E_{-v}(\xi^+)$.

Let us consider the bundle maps $\tilde{\beta}^+ : E(\eta^-) \subset \mathbb{S}^m \times E(\xi^+) \to E(\xi^+)$ defined by $(v, e) \mapsto ie \in E_{-v}(\xi^+)$ and similarly $\tilde{\beta}^- : E(\xi^+) \to E(\eta^-)$ which maps $e \in E_v(\xi^+)$ to $(-v, ie) \in E_{-v}(\eta^-)$. These two bundle maps cover α. It is easy to see that $\tilde{\beta}^+ \circ \tilde{\beta}^- = -id_{E(\eta^-)}$ and $\tilde{\beta}^- \circ \tilde{\beta}^+ = -id_{E(\xi^+)}$. Define $\tilde{\beta} : E(\tilde{\xi}) \to E(\tilde{\xi})$, $(v; e, e') \mapsto (-v; -\tilde{\beta}^-(e'), \tilde{\beta}^+(e))$ which is a bundle involution covering α. Now we can form a vector bundles ξ^1 over $\mathbb{R}P^m$ where $E(\xi^1) = E(\tilde{\xi})/\langle \tilde{\beta} \rangle$. A point of $E_{[v]}(\xi^1)$ is $[v; x, y] = \{(v; x, y), (-v; -iy, ix)\}$ where $x \in E_v(\xi^+), y \in E_v(\eta^-) = E_{-v}(\xi^+)$.

Remark 2.3. We note that ξ^0 and ξ^1 are isomorphic. An explicit isomorphism can be defined by $E(\xi^0) \ni [v; x, y] \mapsto [v; -ix, y] \in E(\xi^1)$.

3. Cohomology of certain generalized Dold manifolds

In [19], we determined the singular mod 2 cohomology of certain families of generalized Dold spaces. Here we consider the integral cohomology groups of $P(S, X)$. We assume that X has a CW structure with cells only in even dimensions and that each cell is stable by the involution σ. Examples of such spaces are complex flag manifolds and complex projective toric varieties where the involution is given by complex conjugation. In the case of complex flag manifolds the CW structure is given by Schubert varieties (see [6]) and, in the case of complex projective toric varieties, the required cell structure is obtained as orbit closures for the natural complex torus $\cong (\mathbb{C}^*)^n$ action on it. (See [12].)

Notation: Let B be a (locally finite) CW complex. We shall denote the closed cells of B by e, e_j, etc., and the corresponding open cells by $\mathring{e}, \mathring{e}_j$, etc. An orientation on a (closed) k-cell e in B is the choice of a generator of $H_k(e, \partial e; \mathbb{Z}) \cong H_k(\mathbb{S}^k; \mathbb{Z})$. An orientation on e is equivalent to an orientation of the open cell $\mathring{e} \cong \mathbb{R}^k$.

3.1. **Homology of** $P(S, X)$**.** In this section, we shall consider the homology of $P(S, X)$ where X is a connected locally finite CW complex that has only even dimensional cells. We assume that S is a connected locally finite CW complex. Further restrictions on S and X will be placed later as and when required. Some of the results here are established in the context of a double cover $p : \widetilde{B} \to B$ with appropriate hypotheses on $\widetilde{B}$.

Lemma 3.1. *Let X be a connected locally finite CW complex with only even dimensional cells. Suppose that $\sigma : X \to X$ is an involution that stabilizes each cell in X and that σ is orientation preserving on any $2k$-dimensional cell e if and only if k is even. Then σ_* acts by $(-1)^k$ on $H_{2k}(X; \mathbb{Z})$.* □

We omit the proof, which is straightforward.

Suppose that $p : \tilde{B} \to B$ is a double covering projection where B is a connected locally finite CW complex. We obtain a G-equivariant CW structure on $\tilde{B}$ by lifting the cell structure on B where $G \cong \mathbb{Z}_2$ is the deck transformation group. Denote by $\phi : \tilde{B} \to \tilde{B}$ the involution that generates G. For each (closed) cell e of B, we have a pair of cells e^+, e^- in $\tilde{B}$ such that $p(e^\pm) = e$ and $\phi(e^+) = e^-$ (as unoriented cells).

Let R be any commutative ring. It is always assumed that $0 \neq 1 \in R$. We denote by $(C_*(B; R), \partial)$ the cellular chain complex with R-coefficients. Recall that $C_r(B; R)$ is the free R-module with basis the set of all closed oriented r-cells of B modulo the relations $e + e' = 0$ where e' is the same underlying cell as e but with opposite orientation. (We denote by the same symbol e an oriented cell as well as the corresponding element in the cellular chain group $C_k(B; R)$.) It is clear that $C_r(B; R)$ is isomorphic to the free R-module with basis the set of *unoriented* r-cells of B.

For each oriented cell e in B, fix orientations on e^+ and e^- so that $p|_{e^\pm} : e^\pm \to e$ is orientation preserving; thus $p_*(e^\pm) = e$. Then $\phi|_{e^+} : e^+ \to e^-$ is orientation preserving since $p = p \circ \phi$.

Suppose that 2 is invertible in R. We set $\varepsilon^+ := (e^+ + e^-)/2$ and $\varepsilon^- := (e^+ - e^-)/2$ for each closed cell e of B. Then $C_r(\tilde{B}; R) = C_r^+(\tilde{B}; R) \oplus C_r^-(\tilde{B}; R)$ as a direct sum of R-modules where $C_r^+(\tilde{B}; R)$ is the free R-module with basis $\{\varepsilon_i^+\}_i$ as e_i varies over the set of r-cells of B. Similarly $\{\varepsilon_i^-\}_i$ is an R-basis for $C_r^-(\tilde{B}; R)$. It is readily seen that $C_r^+(\tilde{B}; R) = \mathrm{Fix}(\phi_*)$, the R-submodule of $C_r(\tilde{B}; R)$ element-wise fixed by ϕ_* and that ϕ_* acts as multiplication by -1 on $C_r^-(\tilde{B}; R)$. Clearly p_* maps $C_r^+(\tilde{B}; R)$ isomorphically onto $C_*(B; R)$ and vanishes identically on $C_r^-(\tilde{B}; R)$. Note that $\partial : C_*(\tilde{B}; R) \to C_*(\tilde{B}; R)$ maps $C_*^\pm(\tilde{B}; R)$ to itself since $\partial \circ \phi_* = \phi_* \circ \partial$. Hence $H_r(\tilde{B}; R)$ breaks up as a direct sum of two R-modules on which ϕ_* acts by 1 and (-1) respectively.

Lemma 3.2. *We keep the above notations. Let R be any commutative ring in which 2 is invertible. Then $p_* : H_*(\tilde{B}; R) \to H_*(B; R)$ maps $\mathrm{Fix}(\phi_*)$ isomorphically onto $H_*(B; R)$. Moreover $p_*[z] = 0$ if $\phi_*([z]) = -[z]$.*

Proof. Since $p_* = p_* \circ \phi_*$ in homology, if $\phi_*([z]) = -[z]$, then $p_*[z] = -p_*[z]$. Since 2 is invertible in R, we have $p_*([z]) = 0$.

Suppose that $\phi_*[z] = [z]$. Write $z = z^+ + z^-$ where $z^\pm \in C_r^\pm(\tilde{B}; R)$. Since $\partial z_i^\pm \in C_r^\pm(\tilde{B}; R)$ (as observed above), we have $\partial z^\pm = 0$ and so $[z] = [z^+] + [z^-]$. Since $[z] = \phi_*([z]) = [z^+] - [z^-]$, we have $2[z^-] = 0$ which implies that $[z^-] = 0$ as 2 is a unit in R. So we may (and do) assume that $z = z^+$.

Suppose that $p_*[z] = 0$. Write $p_*(z) = \partial(\sum a_j e_j)$ and set $\tilde{u} := \sum a_j \varepsilon_j^+ \in C_{r+1}^+(\tilde{B})$. Then $p_*(\partial(\tilde{u})) = \partial p_*(\tilde{u}) = p_*(z)$. Since $p_* : C_*^+(\tilde{B}; R) \to C_*(B; R)$ is a monomorphism, we have $z = \partial \tilde{u}$ and so $[z] = 0$. Thus $p_*|_{\mathrm{Fix}(\phi_*)}$ is a monomorphism.

Next suppose that $z = \sum a_j e_j$ is a cycle in B (with $a_j \in R$). Then $p_*(\tilde{z}) = z$ where $\tilde{z} = \sum a_j \varepsilon_j \in C_r^+(\tilde{B}; R)$. Since $p_*(\partial \tilde{z}) = \partial z = 0$ and since $p_* : C_*^+(\tilde{B}; R) \to C_*(B; R)$ is a monomorphism, we see that $\partial \tilde{z} = 0$. Clearly $p_*([\tilde{z}]) = [z]$ and so p_* is an epimorphism. □

As an immediate corollary, we obtain the following.

Proposition 3.3. *Let $[z] \in H_r(\tilde{B}; \mathbb{Z})$. If $p_*([z]) = 0$ and $\phi_*([z]) = [z]$, then $2^m[z] = 0$ for some $m \geq 1$. If $\phi_*([z]) = -[z]$, then $2p_*([z]) = 0$.* □

Example 3.4. *Let B be the $2n$-dunce hat, defined as the mapping cone of $f : \mathbb{S}^1 \to \mathbb{S}^1$ where $f(z) = z^{2n}$. Assume that $n > 1$. Then $\tilde{B}$ is a two dimensional complex. We shall regard the monomorphism $p_* : \pi_1(\tilde{B}) \cong \mathbb{Z}_n \to \mathbb{Z}_{2n} \cong \pi_1(B)$ as an inclusion. We have $H_k(\tilde{B}; \mathbb{Z}) = 0$ for $k \geq 2$. The deck transformation group induces id on $H_1(\tilde{B}; \mathbb{Z}) \cong \mathbb{Z}_n$ and $p_* : H_1(\tilde{B}; \mathbb{Z}) \to H_1(B; \mathbb{Z})$ corresponds to the inclusion $\mathbb{Z}_n \cong \pi_1(\tilde{B}) \xrightarrow{p_*} \pi_1(B) \cong \mathbb{Z}_{2n}$. We see that $H_*(p; R)$ is an isomorphism if 2 is a unit in R. Although B has a CW complex structure which is perfect mod 2, taking $n = 2^k$, $k \geq 1$, the 2-torsion elements in $H_1(B; \mathbb{Z})$ need not be of order 2.* □

Suppose that $\pi_1(\tilde{B}) \cong \mathbb{Z}^n$. We have an exact sequence of groups where $C_2 \cong \mathbb{Z}_2$:

$$1 \to \pi_1(\tilde{B}) \to \pi_1(B) \to C_2 \to 1. \tag{2}$$

If $\pi_1(B)$ is abelian, then either $\pi_1(B) \cong \mathbb{Z}^n \oplus \mathbb{Z}_2$ or $\pi_1(B) \cong \mathbb{Z}^n$ according as whether the exact sequence splits or not. Hence $H_1(B; \mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z}_2$ or $\mathbb{Z}^n$.

Suppose that $\pi_1(B)$ is not abelian. We will use additive notation for $\pi_1(\tilde{B})$. Conjugation by $y \in \pi_1(B) \backslash \pi_1(\tilde{B})$ defines an automorphism $A : \pi_1(\tilde{B}) \to \pi_1(\tilde{B})$. Fixing a basis $x_1, \ldots, x_n$, we obtain a matrix $A = (a_{ij}) \in GL(n, \mathbb{Z})$ such that $A^2 = I_n$ and $yxy^{-1} = Ax$ $\forall x \in \mathbb{Z}^n$. Since $\pi_1(B)$ is not abelian, $A \neq I_n$. We assume that y is chosen so that it has order 2 when the above exact sequence splits. The abelianization of $\pi_1(B)$, namely $H_1(B; \mathbb{Z})$, has the following presentation

$$H_1(B, \mathbb{Z}) = (\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n + \mathbb{Z}y)/H$$

where H is generated by the following elements $2y - \sum_{1 \leq i \leq n} c_i x_i, (A - I_n)x_j, 1 \leq j \leq n$, for some $c \in \mathbb{Z}^n$. If $c = 0$, then y generates a cyclic subgroup of order 2. Suppose that $c \neq 0$. Then the above exact sequence does not split by our hypothesis on y, and there exists a j so that c_j is odd. We may assume that all the non-zero c_i are 1. Relabeling the x_j, we may assume that $c_1 = 1$. Now replacing x_1 by $\sum c_i x_i$ in the basis $x_1, \ldots, x_n$ of $\pi_1(B)$ if necessary, we may (and do) assume that $2y = x_1$. Note that $yx_1y^{-1} = x_1$. It follows that the image of y is an element of infinite order in $H_1(B;\mathbb{Z})$ and that $H_1(B;\mathbb{Z})$ is generated by at most n elements.

Set $A_- := \{x \in \pi_1(\tilde{B}) \mid Ax = -x\} \subset \pi_1(B)$. Its image $\bar{A}_-$ in $H_1(B;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2^k$ where $k = \mathrm{rank}(A_-)$.

We claim that $H_1(B;\mathbb{Z})/\bar{A}_-$ is a free abelian group of rank $n-k$. To see this, let $A_+ = \{x \in \pi_1(\tilde{B}) \mid Ax = x\} \subset \pi_1(B)$. Since A is invertible and since $A_+ + A_-$ has the same rank n as $\pi_1(\tilde{B})$, it suffices to show that A_+ maps monomorphically into $H_1(B;\mathbb{Z})$. This is immediate from the above presentation of $H_1(B;\mathbb{Z})$ since $yxy^{-1} = Ax = x$. We summarise the result in the proposition below.

Proposition 3.5. *Let $p : \tilde{B} \to B$ be a double cover where B is connected and locally path connected. Suppose that $\pi_1(\tilde{B})$ is free abelian of finite rank, say n. Then $H_1(B;\mathbb{Z}) \cong \mathbb{Z}^l \oplus \mathbb{Z}_2^k$ for some $k, l \geq 0$. Moreover, $k + l = n$ except when the deck transformation group $\mathbb{Z}_2$ acts trivially on $\pi_1(\tilde{B})$, in which case $l = n, k = 1$.* □

We apply Lemma 3.2 to the special case of the double covering projection $\pi : S \times X \to P(S,X)$ satisfying further hypotheses. Denote by $H_p^{\pm}(S;R)$ the R-submodule of $H_p(S;R)$ on which α_* acts as ± 1 and denote its dimension by $b_p^{\pm}$. Thus the r-th Betti number $b_r(S)$ of S equals $b_r^+ + b_r^-$.

Proposition 3.6. *We keep the above notations. Suppose that S is a connected locally finite CW complex equipped with a fixed-point free involution α and that $H_*(S;\mathbb{Z})$ is free abelian. Assume that X is a connected locally finite CW complex with cells only in even dimensions. Assume that the involution $\sigma : X \to X$ satisfies the hypothesis of Lemma 3.1. Let R be a ring where 2 is invertible. Then (i) $\pi_* : H_*(S \times X; R) \to H_*(P(S,X); R)$ is a monomorphism when restricted to $\mathrm{Fix}(\alpha_* \otimes \sigma_*)$. In fact, we have an isomorphism*

$$H_r(P(S,X);R) \cong \bigoplus_{p+4t=r} \left(H_p^+(S;R) \otimes H_{4t}(X;R) \oplus H_{p-2}^-(S;R) \otimes H_{4t+2}(X;R)\right) \tag{3}$$

In particular,

$$\dim H_r(P(S,X);\mathbb{Q}) = \sum_{p+4q=r} (b_p^+ b_{4q}(X) + b_{p-2}^- b_{4q+2}(X)). \tag{4}$$

(ii) Suppose that $\pi_1(S)$ is abelian. Any torsion element of $H_r(P(S,X);\mathbb{Z})$ has order a power of 2.

Proof. (i). We shall apply Lemma 3.2. In view of our hypotheses and the Künneth theorem, we have $H_r(S \times X;\mathbb{Z}) = \bigoplus_{p+2q=r} H_p(S;\mathbb{Z}) \otimes H_{2q}(X;\mathbb{Z})$. Recall that $\pi = \pi \circ \theta$ where $\theta = \alpha \times \sigma$. So $\pi_* = \pi_* \circ \theta_*$ and so $\theta_* = \alpha_* \otimes \sigma_*$. Since σ_* acts on $H_{2q}(X;\mathbb{Z})$ by $(-1)^q$ we obtain that $H_r(P(S,X);R) \cong \mathrm{Fix}(H_r(\theta;R)) = \bigoplus_{p+4t=r}(H_p^+(S;R) \otimes H_{4t}(X;R)) \oplus (H_{p-2}^-(S;R) \otimes H_{4t+2}(X;R))$. Taking $R = \mathbb{Q}$ we obtain the asserted formula for the r-th Betti number of $P(S,X)$.

(ii) We only need to show that there is no odd torsion element in $H_r(P(S,X);\mathbb{Z})$. Our hypothesis on $\pi_1(S)$ and the freeness of $H_1(S;\mathbb{Z})$ implies that $\pi_1(S) \cong \mathbb{Z}^k$ for some

$k \geq 0$. If $k = 0$, then $\pi_1(P(S,X)) \cong \mathbb{Z}_2$. In general, $\pi_1(P(S,X))$ has an index 2 subgroup $H := \pi_*(\pi_1(S \times X)) \cong \mathbb{Z}^k$. Then by Proposition 3.5, $H_1(P(S,X);\mathbb{Z}) \cong \mathbb{Z}^l \oplus \mathbb{Z}_2^t$ for some $l, t \geq 0$ and so has no odd torsion.

Since $H_*(S \times X;\mathbb{Z})$ has no torsion and since π_* maps $\mathrm{Fix}(H_*(\theta;\mathbb{Q}))$ isomorphically onto $H_*(P(S,X);\mathbb{Q})$, π induces a monomorphism $\mathrm{Fix}(H_*(\theta;\mathbb{Z})) \to H_*(P(S,X);\mathbb{Z})$. Suppose that $H_r(P(S,X);\mathbb{Z})$ has an element of order an odd prime p. We assume that r is the least positive integer for which this happens. So $r > 1$ and $\mathrm{Tor}(H_{r-1}(P(S,X);\mathbb{Z}), \mathbb{Z}_p) = 0$. By the universal coefficient theorem, $H_r(P(S,X);\mathbb{Z}_p) \cong H_r(P(S,X);\mathbb{Z}) \otimes \mathbb{Z}_p$. Since the r-th Betti number of $P(S,X)$ equals $\dim H_r(P(S,X);\mathbb{Z}_p)$ by (i), every non-zero element in $H_r(P(S,X);\mathbb{Z}_p)$ has to be the reduction mod p of an element of $H_r(P(S,X);\mathbb{Z})$ of *infinite* order. This implies that $H_r(P(S,X);\mathbb{Z})$ has no element of order p, contrary to our choice of r. This completes the proof. □

Remark 3.7. (i) The cohomology version of the above proposition is valid and is equivalent to it by the universal coefficient theorem.
(ii) Suppose that S, X as in the above theorem are compact connected orientable manifolds (without boundary). Suppose that $\dim S = m, \dim X = 2d$. Then σ is orientation preserving if and only if d is even. It follows that $P(S,X)$ is orientable if and only if α is orientation preserving and d is even, or, α orientation reversing and d is odd. This follows from computation of $H_{m+2d}(P(S,X);\mathbb{Q})$. This is also seen from the fact that θ is orientation preserving if and only if both α, σ are simultaneously either orientation preserving or reversing.

3.2. **The generalized Dold manifold** $P(m,\nu)$.

We shall now consider the specific classes of manifolds $P(S,X)$ where
(i) S is the sphere $\mathbb{S}^m$ and $\alpha : \mathbb{S}^m \to \mathbb{S}^m$ is the antipodal map;
(ii) X is a complex flag manifold and $\sigma : X \to X$ is the complex conjugation (induced by the complex conjugation $^{-}: \mathbb{C}^n \to \mathbb{C}^n$).

Recall that $\mathbb{C}G_{n,k} \cong U(n)/(U(k) \times U(n-k))$ is the space all k-dimensional complex vector subspaces of $\mathbb{C}^n$. Let $\sigma : \mathbb{C}G_{n,k} \to \mathbb{C}G_{n,k}$ be the involution that sends $L \in \mathbb{C}G_{n,k}$ to $\bar{L} = \{\bar{v} \in \mathbb{C}^n \mid v \in L\}$.

We put the standard hermitian inner product on $\mathbb{C}^n$. The complex flag manifold $\mathbb{C}G(\nu)$ of type $\nu := (n_1 \cdots, n_s)$ is the space of all flags $\mathbf{L} := (L_1, \cdots, L_s)$ where: (i) L_j is a vector subspace of $\mathbb{C}^n$ of dimension $n = |\nu| := \sum_{1 \leq j \leq s} n_j$, and, (ii) where $L_i \perp L_j$ if $i \neq j$; thus $L_1 + \cdots + L_s = \mathbb{C}^n$. The complex flag manifold $\mathbb{C}G_\nu$ is identified with the homogeneous space $U(n)/U(\nu)$ where $U(\nu) := (U(n_1) \times \cdots \times U(n_s)) \subset U(n)$ is the subgroup of block diagonal matrices with block sizes $n_1, \cdots, n_s$. When $s = 2$, $\mathbb{C}G(\nu)$ is the Grassmann manifold $\mathbb{C}G_{n,n_1}$.

3.3. **Schubert cell structure on** $\mathbb{C}G(\nu)$.

One has the Schubert cell structure for the complex Grassmann manifold $\mathbb{C}G(\nu)$ (cf. [20, Chapter 6]) which has a generalization to the case of complex flag manifolds given by the Bruhat decomposition (see [8], [4]).The closed cells, the Schubert varieties in $\mathbb{C}G(\nu)$, are in bijection with the set of T-fixed points of $\mathbb{C}G(\nu)$, where $T \subset U(n)$ is the diagonal subgroup, which is a maximal torus of $U(n)$. These T-fixed points are in bijection with the coset space S_n/S_ν where $S_\nu := S_{n_1} \times \cdots \times S_{n_s}$. Here S_n is the permutation group on $[n] = \{1, 2, \cdots, n\}$. In the case of $\mathbb{C}G_{n,k}$, $S_n/(S_k \times S_{n-k})$ is identified with the set $I(n;k)$ of all strictly increasing sequences $\mathbf{i} = (i_1, i_2, \ldots, i_k), 1 \leq i_p \leq n$. The Schubert variety $X(\mathbf{i}) \subset \mathbb{C}G_{n,k}$ is defined as

$$X(\mathbf{i}) := \{L \in \mathbb{C}G_{n,k} \mid \dim_{\mathbb{C}}(L \cap \mathbb{C}^{i_p}) \geq p \ \forall p \leq k\}.$$

The corresponding T-fixed point is $E_{\mathbf{i}}$, the $\mathbb{C}$-span of the standard basis vectors $e_{i_p}, 1 \leq p \leq k$. $X(\mathbf{i})$ has the structure of a complex projective variety. One has the following formula for the dimension of $X(\mathbf{i})$ as a complex variety:

$$\dim_{\mathbb{C}} X(\mathbf{i}) = \ell(\mathbf{i}) := \sum_{1 \leq p \leq k} (i_p - p).$$

The corresponding (open) Schubert cell is $\mathring{X}(\mathbf{i}) = \{L \in \mathbb{C}G_{n,k} \mid \dim(L \cap \mathbb{C}^{i_p}) = p, \dim(L \cap \mathbb{C}^{i_p - 1}) = p - 1 \ \forall p\} \cong \mathbb{C}^{\ell(\mathbf{i})}$. We shall denote by $I_q(n;k) \subset I(n;k)$ the set of all $\mathbf{i} \in I(n;k)$ with $\ell(\mathbf{i}) = q$.

In the case of $\mathbb{C}G(\nu)$, we identify S_n/S_ν with the subset $I(\nu) \subset S_n$ defined as follows. Let $\mathbf{i} = (i_1, \cdots, i_n)$ be a permutation of $[n]$ (in one-line notation). Let $m_0 := 0, m_p = n_1 + \cdots + n_p$ if $1 \leq p \leq s$. Let $\mathbf{i}_p$ be the sequence $\mathbf{i}_p = i_{m_{p-1}+1}, i_{m_{p-1}+2}, \ldots, i_{m_p}, 0 \leq p < s$. We shall refer to $\mathbf{i}_p$ as the pth block of $\mathbf{i}$. Note that $\mathbf{i}_p$ has length n_p and that $\mathbf{i}$ equals $(\mathbf{i}_1, \ldots, \mathbf{i}_s)$. By definition,

$$I(\nu) := \{\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_s) \in S_n \mid \mathbf{i}_p \in I(n; n_p) \ \forall 1 \leq p \leq s\}.$$

With respect to the generating set $\mathcal{S} \subset S_n$ consisting of transpositions $s_i = (i, i+1), 1 \leq i < n$, one has the length function $\ell = \ell_{\mathcal{S}}$ on the Coxeter group $(S_n, \mathcal{S})$. The set $I(\nu)$ is the set of minimal length representatives of elements of S_n/S_ν. See [16, §12.8]. We have the formula:

$$\ell(\mathbf{i}) = |\{(t,r) | 1 \leq t < r \leq n; i_r < i_t\}| = \sum_{1 \leq t < n} |\{r \mid t < r \leq n, i_r < i_t\}|.$$

When $s = 2$, we have $n = n_1 + n_2$ and the set $I(\nu) \subset S_n$ is in bijection with the set $I(n;k)$ where $k = n_1$. The above notation is consistent with the notation $\ell(\mathbf{i})$ for $\mathbf{i} \in I(n;k)$. The formula for $\ell(\mathbf{i}), \mathbf{i} \in I(\nu)$, for $s \geq 3$ can be proved by induction on s: If $\mathbf{i} \in I(\nu)$, write $\nu' = (n_1, \ldots, n_{s-2}, n_{s-1} + n_s)$ and let $\mathbf{i}' = (\mathbf{i}_1, \cdots, \mathbf{i}_{s-2}, \mathbf{i}'_{s-1}) \in I(\nu')$ where $\mathbf{i}'_{s-1} \in I(n; n_{s-1} + n_s)$ is the sequence $\mathbf{i}_{s-1}, \mathbf{i}_s$ rearranged in the increasing order. Then, it is easily seen that

$$\ell(\mathbf{i}) = \ell(\mathbf{i}') + \ell(i'_{s-1}).$$

As in the case of the Grassmannian Schubert varieties, the set of all Schubert varieties (with reference to the standard complete flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$) in $\mathbb{C}G(\nu)$ and the set of T-fixed points of $\mathbb{C}G(\nu)$ are in bijective correspondence with $I(\nu)$. An element $\mathbf{i} \in I(\nu)$ corresponds to the T-fixed point $(E_{\mathbf{i}_1}, E_{\mathbf{i}_2}, \ldots, E_{\mathbf{i}_s}) \in \mathbb{C}G(\nu)$ and the corresponding Schubert variety is denoted $X(\mathbf{i})$. An element $V = (V_1, \ldots, V_s) \in \mathbb{C}G(\nu)$ belongs to $X(\mathbf{i})$ if and only if $V_1 + \cdots + V_p$ belongs to $X(\mathbf{j}_p) \subset \mathbb{C}G_{n,m_p}$ for all $p < s$ where $\mathbf{j}_p \in I(n; m_p)$ is the sequence $(\mathbf{i}_1, \ldots, \mathbf{i}_p)$ rearranged in increasing order. The Schubert variety $X(\mathbf{i}) \subset \mathbb{C}G(\nu)$ has (complex) dimension the length of $\mathbf{i}$. That is, $\dim_{\mathbb{C}} X(\mathbf{i}) = \ell(\mathbf{i})$. See [17, §3.3], [8, §1.2]. The dimension formula may be established for $s \geq 3$ by induction using the Grassmann bundle $\mathbb{C}G_{n,k} \hookrightarrow \mathbb{C}G(\nu) \to \mathbb{C}G(\nu')$ (with ν' as above) where $(V_1, \ldots, V_s) \in \mathbb{C}G(\nu)$ projects to $(V_1, \ldots, V_{s-2}, V_{s-1} + V_s) \in \mathbb{C}G(\nu')$.

The Schubert cells have natural orientations arising from the fact that the complex vector spaces have canonical orientations. In fact, the open cell $\mathring{X}(\mathbf{i})$ is isomorphic to the affine space $\mathbb{C}^{\ell(\mathbf{i})}$. See [8, §1.2]. We note that the conjugation map $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$ that sends $\mathbf{L} = (L_1, \ldots, L_s)$ to $\bar{\mathbf{L}} = (\bar{L}_1, \ldots, \bar{L}_s)$ stabilizes each Schubert cell. In fact, under the identification $\mathring{X}(\mathbf{i}) \cong \mathbb{C}^{\ell(\mathbf{i})}$, the σ corresponds to the complex conjugation on $\mathbb{C}^{\ell(\mathbf{i})}$. In particular, σ is orientation preserving (resp. reversing) on $X(\mathbf{i})$ when $\ell(\mathbf{i})$ is even (resp. odd).

Since the real dimension of each Schubert cell is even, it follows that the differential of the cellular chain complex of $\mathbb{C}G(\nu)$ vanishes identically. Consequently, $H_*(\mathbb{C}G(\nu); \mathbb{Z}) \cong C_*(\mathbb{C}G(\nu); \mathbb{Z})$ and $H^*(\mathbb{C}G(\nu); \mathbb{Z}) \cong C^*(\mathbb{C}G(\nu); \mathbb{Z})$. Thus $\mathbb{C}G(\nu)$ satisfies the hypotheses of Lemma 3.1. The rank of $H^{2q}(\mathbb{C}G(\nu); \mathbb{Z})$ equals the cardinality of $I_q(\nu) := \{\mathbf{i} \in I(\nu) \mid \ell(i) = q\}$.

The fixed set $\mathrm{Fix}(\sigma) \subset \mathbb{C}G(\nu)$ is identified with the real flag manifold $\mathbb{R}G(\nu) = O(n)/(O(n_1) \times \cdots \times O(n_s))$ via the embedding defined by the complexification $V \mapsto V \otimes \mathbb{C} \subset \mathbb{C}^n$ where V is a real vector space $V \subset \mathbb{R}^n$.

3.4. **A cell structure on $P(m, \nu)$.** We put the standard $\mathbb{Z}_2$-equivariant cell structure $\{C_k^{\pm}\}_{0 \leq k \leq m}$ on the sphere $\mathbb{S}^m$. Here the equivariance is with respect to the cyclic group generated by the antipodal map $\alpha : \mathbb{S}^m \to \mathbb{S}^m$. The cell C_k^+ (resp. C_k^-) is the upper (resp. lower) hemisphere of $\mathbb{S}^k \subset \mathbb{S}^m$ where $\mathbb{S}^k \subset \mathbb{S}^m$ is the unit sphere in $\mathbb{R}^{k+1}$ spanned by $e_j, 1 \leq j \leq k+1$. The projection $p : \mathbb{S}^m \to \mathbb{R}P^m$ is cellular with respect to the standard cell structure on $\mathbb{R}P^m$. The unique closed k-cell in $\mathbb{R}P^m$ is $\mathbb{R}P^k$ corresponding the inclusion $\mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{m+1}$.

We put the standard orientation on C_j^+ for each j and put the orientation on C_j^- induced on it via α. The cells C_j in $\mathbb{R}P^m$ are given the orientation induced from C_j^+ via $p|_{C_j^+} : C_j^+ \to C_j$.

A cell decomposition of $\mathbb{S}^m \times \mathbb{C}G(\nu)$ which is equivariant with respect to $\theta = \alpha \times \sigma$ is obtained by taking the product cells $X^{\pm}(j, \mathbf{i}) := C_j^{\pm} \times X(\mathbf{i})$ as $(j, \mathbf{i})$ varies in $I(m, \nu) := \{j \mid 0 \leq j \leq m\} \times I(\nu)\}$. The cell $C_j^{\pm} \times X(\mathbf{i})$ is given the product orientation.

We shall denote the image of the oriented cell $X^+(j, \mathbf{i})$ under the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m, \nu)$ by $X(j, \mathbf{i})$ and put the induced orientation on it. The deck transformation group of π is generated by $\theta = \alpha \times \sigma$.

We note that $\theta|_{X^+(j,\mathbf{i})} : X^+(j, \mathbf{i}) \to X^-(j, \mathbf{i})$ is orientation preserving if and only if $\sigma|_{X^+(\mathbf{i})} : X^+(\mathbf{i}) \to X^-(\mathbf{i})$ is, if and only if $\ell(\mathbf{i})$ is even.

Since the boundary map of $C_*(\mathbb{C}G(\nu);\mathbb{Z})$ vanishes, it is readily seen that the boundary map of $C_*(\mathbb{S}^m \times \mathbb{C}G(\nu))$ is determined by $\partial : C_*(\mathbb{S}^m;\mathbb{Z})$ and we obtain, for all $(j,\mathbf{i}) \in I(m,\nu)$, that

$$\partial(X^{\pm}(j,\mathbf{i})) = X^{\pm}(j-1,\mathbf{i}) + (-1)^j X^{\mp}(j-1,\mathbf{i}), \tag{5}$$

and

$$\theta_*(X^{\pm}(j,\mathbf{i})) = (-1)^{\ell(\mathbf{i})} X^{\mp}(j,\mathbf{i}). \tag{6}$$

Now $p_*(X^+(j,\mathbf{i})) = X(j,\mathbf{i})$ and,

$$p_*(X^-(j,\mathbf{i})) = p_*(\theta_*((-1)^{\ell(\mathbf{i})}X^+(j,\mathbf{i}))) = p_*((-1)^{\ell(\mathbf{i})}X^+(j,\mathbf{i})) = (-1)^{\ell(\mathbf{i})}X(j,\mathbf{i}).$$

It follows from Equation (5) that

$$\partial X(j,\mathbf{i}) = p_*\partial(X^+(j,\mathbf{i})) = (1+(-1)^{j+\ell(\mathbf{i})})X(j-1,\mathbf{i}) \ \forall (j,\mathbf{i}) \in I(m,\nu). \tag{7}$$

3.5. **(Co)homology of $P(m,\nu)$.** We now turn to the determination of the integral homology of $P(m,\nu)$. It follows from Equation (7) that, for $0 < j \leq m$, $X(j,\mathbf{i})$ is a cycle in $P(m,\nu)$ if and only if $j+\ell(\mathbf{i})$ is odd. When $j+\ell(\mathbf{i})$ is odd and $j<m$, we have $\partial(X(j+1,\mathbf{i})) = 2X(j,\mathbf{i})$ and so the homology class $[X(j,\mathbf{i})]$ has order 2.

It is evident from Equation (7) that $X(0,\mathbf{i})$ is a cycle for all $\mathbf{i} \in I(\nu)$. When $\ell(\mathbf{i})$ is even, $[X(0,\mathbf{i})]$ is of infinite order. If $\ell(\mathbf{i})$ is odd, $X(0,\mathbf{i})$ is of order 2. Similarly, if m is odd and $\ell(\mathbf{i})$ is even, $[X(m,\mathbf{i})]$ is of infinite order. If both m and $\ell(\mathbf{i})$ are even, then $X(m,\mathbf{i})$ is not a cycle. If m is even and $\ell(\mathbf{i})$ is odd, then $[X(m,\mathbf{i})]$ is of infinite order.

Let $q \geq 0$. Define the sets

$$\begin{array}{rcl}
I_e(\nu) & := & \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \pmod 2\},\\
I_o(\nu) & := & \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 1 \pmod 2\} = I(\nu) \setminus I_e(\nu),\\
\mathcal{B}_{2q} & := & \{[X(0,\mathbf{i})] \mid q = \ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\} \cup \{[X(m,\mathbf{i})] \mid 2q = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_o(\nu)\},\\
\mathcal{B}_{2q+1} & := & \{[X(m,\mathbf{i})] \mid 2q+1 = m+2\ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\},\\
\mathcal{B}'_q & := & \{[X(j,\mathbf{i})] \mid q = j+2\ell(\mathbf{i}), j+\ell(\mathbf{i}) \equiv 1 \pmod 2, 0 \leq j < m, \ \mathbf{i} \in I(\nu)\}.
\end{array}$$

We note that if m is even, then $\mathcal{B}_{2q+1} = \emptyset$, and if m is odd, then $\{[X(m,\mathbf{i})] \mid 2q = m+2\ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\} = \emptyset$.

Observe that the cycle group $Z_q(P(m,\nu);\mathbb{Z})$ is the free abelian group generated by $\{X(j,\mathbf{i})\}$ where $[X(j,\mathbf{i})] \in \mathcal{B}_q \cup \mathcal{B}'_q$ for any $q \geq 0$. In particular $H_q(P(m,\nu);\mathbb{Z})$ is generated by $\mathcal{B}_q \cup \mathcal{B}'_q$.

We have the following result where we leave out the known cases namely, $m=0$ as $P(0,\nu) \cong \mathbb{C}G(\nu)$ and $\nu = (1)$ in which case $P(m,\nu) = \mathbb{S}^m$.

Theorem 3.8. *We keep the above notations. Suppose that $m \geq 1$ and $n = |\nu| \geq 2$. Then there is no odd torsion in $H_*(P(m,\nu);\mathbb{Z})$ and every 2-torsion element is of order 2. The set $\mathcal{B}_q$ is a basis for $H_q(P(m,\nu);\mathbb{Z})/$torsion and $\mathcal{B}'_q$ is a $\mathbb{Z}_2$-basis for the torsion subgroup of $H_q(P(m,\nu);\mathbb{Z})$.*

Proof. We first show that $\mathcal{B}_q$ is $\mathbb{Z}$-linearly independent. Suppose that $Z = \sum a_{j,\mathbf{i}} X(j, \mathbf{i})$ with $a_{j,\mathbf{i}} \in \mathbb{Z}$, is a boundary where the sum is over $(j, \mathbf{i}) \in I(m, \nu)$ such that $[X(j, \mathbf{i})] \in \mathcal{B}_q$, i.e. $Z = \partial C$ for some $C \in C_{q+1}(P(m, \nu))$. Suppose that some $a_{j,\mathbf{i}} \neq 0$. Then there has to be a cell $X(k, \mathbf{l})$ such that $X(j, \mathbf{i})$ occurs with non-zero coefficient in the expression of $\partial X(k, \mathbf{l})$ for some $X(k, \mathbf{l})$. However, Equation (7) shows that the only possibility is $k = j + 1 \leq m, \mathbf{l} = \mathbf{i}$ and that $\partial X(j + 1, \mathbf{i}) = (1 + (-1)^{j+1+\ell(\mathbf{i})})X(j, \mathbf{i})$. As $[X(j, \mathbf{i})] \in \mathcal{B}_q$, we have $\partial X(j + 1, \mathbf{i}) = 0$. This shows that there is no such C and so the $\mathcal{B}_q$ is $\mathbb{Z}$-linearly independent.

Note that $\mathcal{B}_q \subset \mathrm{Fix}(\theta_*)$. In view of this one can also prove the linear independence of $\mathcal{B}_q$ using Proposition 3.3.

Since $\mathcal{B}_q \cup \mathcal{B}'_q$ generates $H_q(P(m, \nu); \mathbb{Z})$ as observed above, it follows that $\mathcal{B}_q$ generates the abelian group $H_q(P(m, \nu); \mathbb{Z})/\text{torsion}$. Therefore $\mathcal{B}_q$ is a $\mathbb{Z}$-basis for $H_q(P(m, \nu); \mathbb{Z})$ mod torsion. It is easily seen that $\mathcal{B}'_q$ is $\mathbb{Z}_2$-linearly independent.

Any torsion element of $H_q(P(m, \nu); \mathbb{Z})$ is of order 2, in view of the fact that $H_q(P(m, \nu); \mathbb{Z})$ is generated by $\mathcal{B}_q \cup \mathcal{B}'_q$. Indeed, $\mathcal{B}_q$ generates a free abelian subgroup of $H_q(P(m, \nu); \mathbb{Z})$ and $\mathcal{B}'_q$ generates an elementary abelian 2-group contained in $H_q(P(m, \nu); \mathbb{Z})$. Hence the torsion subgroup equals the subgroup generated by $\mathcal{B}'_q$ and our assertion follows. □

The cohomology groups of $P(m, \nu)$ can be read off from the above theorem using the universal coefficient theorem. Thus $H^q(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^t \oplus \mathbb{Z}_2^p$ where the values of t and p depend on m, ν can be explicitly determined in terms of $|\mathcal{B}_q|$ and $|\mathcal{B}'_{q-1}|$. As a warm up, we compute $H^2(P(m, \nu); \mathbb{Z})$. Note that $\mathcal{B}_2 = \emptyset$ and $\mathcal{B}'_1 = \{[X(1, \mathbf{i}_0)]\}$ where $\mathbf{i}_0 = (1, 2 \ldots, n)$ is the identity permutation (which is the unique element of length 0) if $m > 1$ and $\mathcal{B}'_1$ is empty if $m = 1$. So the rank of $H_2(P(m, \nu); \mathbb{Z})$ is zero. Also $H_1(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}_2$ if $m > 1$ and is isomorphic to $\mathbb{Z}$ if $m = 1$.

Proposition 3.9. *Let $m \geq 1$. With the above notations,*

$$H^2(P(m, \nu); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } m > 1 \\ 0 & \text{if } m = 1. \end{cases}$$

Also the Picard group $Pic(P(m, \nu)) \cong \mathbb{Z}_2$, generated by $\xi_{\mathbb{C}}$ if $m > 1$ and is trivial otherwise.

Proof. The first assertion follows from the universal coefficient theorem. The second assertion follows from the fact that the first Chern class map yields an isomorphism of the Picard group onto $H^2(P(m, \nu); \mathbb{Z})$. □

We now turn to the determination of the rank and the torsion subgroup of the integral cohomology groups of $P(m, \nu)$ explicitly in terms of m, ν.

Write $H^{\mathrm{ev}}(P(m, \nu); \mathbb{Z}) = \bigoplus_{q \geq 0} H^{2q}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}$ and $H^{\mathrm{odd}}(P(m, \nu); \mathbb{Z}) = \bigoplus_{q \geq 1} H^{2q-1}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$. We shall obtain formulas for the numbers b_e, b'_e, b_o, b'_o in terms of m, ν.

Set $\mathcal{B} := \bigcup_{q\geq 0}\mathcal{B}_q, \mathcal{B}_e := \bigcup_{q\geq 0}\mathcal{B}_{2q}$ and let $\mathcal{B}_o = \mathcal{B}\setminus\mathcal{B}_e$. Likewise $\mathcal{B}' := \bigcup_{q\geq 1}\mathcal{B}'_q$ and $\mathcal{B}'_e := \{[X(j,\mathbf{i})]\in\mathcal{B}' \mid j\equiv 0 \pmod 2\}$, and $\mathcal{B}'_o = \mathcal{B}'\setminus\mathcal{B}'_e$. Also, set $\beta_e := |\mathcal{B}'_e|, \beta_o := |\mathcal{B}'_o|$.

By the universal coefficient theorem, $b_e = |\mathcal{B}_e|, b_o = |\mathcal{B}_o|$, whereas $b'_e = \beta_o, b'_o = \beta_e$.

Set $\ell_e := |I_e(\nu)|$ and $\ell_o = |I_o(\nu)|$. Thus $\ell_e + \ell_o = \binom{n}{\nu} = \chi(\mathbb{C}G(\nu))$. We shall determine the values of ℓ_e, ℓ_o.

The Poincaré polynomial $P_t(\nu)$ of $\mathbb{C}G(\nu)$ is the q-Gaussian multinomial coefficient $Q_q(\nu) := \binom{n}{\nu}_q = [n]_q/([n_1]_q\cdots[n_s]_q)$ where $q := t^2$ and $[k]_q = \prod_{1\leq j\leq k}(1-q^j)$. The right hand side of Q_q is also the Poincaré polynomial $P_q(\mathbb{R}G(\nu);\mathbb{Z}_2)$ of the *real* flag manifold $\mathbb{R}G(\nu)$ with $\mathbb{Z}_2$-coefficients. See [4, §26], [5]. Therefore

$$\ell_e - \ell_o = \chi(\mathbb{R}G(\nu)). \tag{8}$$

We conclude that

$$\ell_e = (\chi(\mathbb{C}G(\nu)) + \chi(\mathbb{R}G(\nu)))/2;\quad \ell_o = (\chi(\mathbb{C}G(\nu)) - \chi(\mathbb{R}G(\nu)))/2. \tag{9}$$

If $\chi(\mathbb{R}G(\nu)) = 0$, then $\ell_e = \ell_o = \binom{n}{\nu}/2$.

The Euler-Poincaré characteristic of $\mathbb{R}G(\nu)$ is can be computed using standard arguments and the classical result that $\chi(G/T) = |W(G,T)|$, the order of the Weyl group $W(G,T)$ when G is a compact connected Lie group and T is a maximal torus. We include a proof for the sake of completeness as we could not find an explicit reference for the result.

Lemma 3.10. *Let $\nu = (n_1,\ldots,n_s), s\geq 2$. Set $\nu_o := \{1\leq j\leq s \mid n_j\equiv 1 \mod 2\}$.*

Then $\chi(\mathbb{R}G(\nu)) = 0$ if $\nu_o\geq 2$. Suppose that $\nu_o\leq 1$. Set $n'_j := \lfloor n_j/2\rfloor, 1\leq j\leq s$, and $\lfloor\nu/2\rfloor := (n'_1, n'_2,\ldots,n'_s)$. Then $\chi(\mathbb{R}G(\nu)) = \binom{\lfloor n/2\rfloor}{\lfloor\nu/2\rfloor}$.

Proof. Without loss of generality, assume that n_j is even for all $j > \nu_o$. When $\nu_o\geq 2$, $\mathbb{R}G(\nu)$ is fibred by the real Grassmann manifold $\mathbb{R}G(n_1,n_2)$ over $\mathbb{R}G(n_1+n_2, n_3,\ldots,n_s)$. Since $\dim\mathbb{R}G(n_1,n_2) = n_1n_2$ is odd, $\chi(\mathbb{R}G(\nu)) = 0$ and so the vanishing of $\chi(\mathbb{R}G(\nu))$ follows from the multiplicative property of the Euler-Poincaré characteristics for fibrations. See [24, Theorem 1,§3, Chapter 9]

Let $\nu_o\leq 1$. Consider first the Euler-Poincaré characteristic of the *oriented* flag manifold $\widetilde{\mathbb{R}G}(\nu) := SO(n)/SO(\nu)$ where $SO(\nu) := SO(n_1)\times\cdots\times SO(n_s)$. As $\nu_o\leq 1$, $SO(\nu)$ has the same rank as $SO(n)$. We take $T = (SO(2))^{\lfloor n/2\rfloor}\subset SO(\nu)$ to be the standard maximal torus of $SO(n)$ which sits in $SO(n)$ as 2×2 block diagonal with top diagonal entry being 1 if n is odd. Using the fibration $SO(n)/T\to SO(n)/SO(\nu)$ with fibre $SO(\nu)/T$, we obtain that $\chi(SO(n)/T) = \chi(SO(\nu)/T)).\chi(SO(n)/SO(\nu))$. It is a classical result that when G is a compact connected Lie group and $T\subset G$ is a maximal torus of G, then $\chi(G/T) = |W(G,T)|$, the order of the Weyl group of G with respect to T. See [21, Part-II]. It is known [13] that $W(SO(n),T)$ has order $2^{\lfloor(n-1)/2\rfloor}\cdot\lfloor n/2\rfloor!$. This yields that $\chi(\widetilde{\mathbb{R}G}(\nu)) = 2^{s-1}\binom{\lfloor n/2\rfloor}{\lfloor\nu/2\rfloor}$. The value of $\chi(\mathbb{R}G(\nu))$ is then determined to be $\binom{\lfloor n/2\rfloor}{\lfloor\nu/2\rfloor}$ using the covering projection $\widetilde{\mathbb{R}G}(\nu)\to\mathbb{R}G(\nu)$, which has degree 2^{s-1}. $\square$

With notations as in the above lemma, if $\nu_o \leq 1$, then $\ell_e = (\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$ and $\ell_o = (\binom{n}{\nu} - \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$.

We tabulate below the values of ℓ_e and ℓ_o.

ν	ℓ_e	ℓ_o
$\nu_o \geq 2$	$\binom{n}{\nu}/2$	$\binom{n}{\nu}/2$
$\nu_o \leq 1$	$(\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$	$(\binom{n}{\nu} - \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$

Table 1: The values of ℓ_e and ℓ_o.

From the definition of $\mathcal{B}'_e$ we see that if $\mathbf{i} \in I_o(\nu)$, then $[X(j, \mathbf{i})] \in \mathcal{B}'_e$ for $\lfloor (m+1)/2 \rfloor$ distinct values of j, namely, those which satisfy $j \equiv 0 \pmod 2, 0 \leq j < m$. Moreover, if $[X(j, \mathbf{i})] \in \mathcal{B}'_e$, then $\mathbf{i} \in I_o(\nu)$. Therefore $\beta_e = \lfloor (m+1)/2 \rfloor \cdot \ell_o$. Similarly $\beta_o = \lfloor m/2 \rfloor \cdot \ell_e$.

The values of b_e, b_o depend on ν and the parity of m. We proceed to determine them.

We have $b_e + b_o = \dim_{\mathbb{Q}}(H^*(P(m, \nu); \mathbb{Q}))$ and $b_e - b_o = \chi(P(m, \nu))$.

Case 1: $\underline{m \text{ is even.}}$ Then $H^q(P(m, \nu); \mathbb{Q}) = 0$ if q is odd and so $b_o = 0$. It follows that $b_e = \chi(P(m, \nu)) = (1/2)\chi(\mathbb{S}^m \times \mathbb{C}G(\nu)) = \binom{n}{\nu} = n!/(n_1! \cdots n_s!)$.

Case 2: $\underline{m \text{ is odd.}}$ We have and $b_e - b_o = \chi(P(m, \nu)) = 0$. Therefore $b_e = b_o$. Also $b_e = |\{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \mod 2\}| = \ell_e$.

We summarise the above results in the theorem below.

Theorem 3.11. *Let $m \geq 1$ and $\nu = (n_1, \ldots, n_s)$ where $s \geq 2$. Let $\nu_o := |\{1 \leq j \leq s \mid n_j \equiv 1 \pmod 2\}|$. Then $H^{\mathrm{ev}}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}$ and $H^{\mathrm{odd}}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$ where b_e, b_o, b'_e, b'_o are as follows:*
(i) When m is odd,

$$b_e = b_o = \ell_e = \begin{cases} \binom{n}{\nu}/2, & \text{if } \nu_o \geq 2 \\ (\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2, & \text{if } \nu_o \leq 1. \end{cases}$$

When m is even, $b_e = \binom{n}{\nu}$ and $b_o = 0$.
(ii) $b'_e = \beta_o = \lfloor m/2 \rfloor \cdot \ell_e$, $\quad b'_o = \beta_e = \lfloor (m+1)/2 \rfloor \cdot \ell_o$.

In particular, $H^{ev}(P(1, \nu); \mathbb{Z})$ has no torsion. □

The following remark gives some partial information about the ring structure of the integral cohomology of $P(m, \nu)$.

Remark 3.12. (i) It is readily seen that $\mathbb{C}G(\nu)$ equals the Schubert variety $X(\mathbf{w}_0)$ that corresponds to the permutation $w_0 = (n, n-1, \ldots, 2, 1) \in S_n$ (in the one-line notation). So σ is orientation preserving if and only if $d = \dim_{\mathbb{C}} \mathbb{C}G(\nu) = \sum_{1 \leq i < j \leq s} n_i n_j$ is even, if and only if $\binom{\nu_o}{2}$ is even. It follows that $P(m, \nu)$ is orientable if and only if $m + \binom{\nu_o}{2}$ is odd. (See Remark 3.7.) When $P(m, \nu)$ is orientable, the natural orientation on it corresponds to the fundamental class $[X(m, \mathbf{w}_0)]$.

(ii) Since the bundle $p : P(m,\nu) \to \mathbb{R}P^m$ admits a cross-section $s : \mathbb{R}P^m \to P(m,\nu)$, $p^* : H^*(\mathbb{R}P^m;\mathbb{Z}) \to H^*(P(m,\nu);\mathbb{Z})$ is a monomorphism of rings and its image is a direct summand.

(iii) Let $\iota : \mathbb{C}G(\nu) \hookrightarrow P(m,\nu)$ denote the fibre-inclusion of the bundle $p : P(m,\nu) \to \mathbb{R}P^m$ over a point, say $[e_1] \in \mathbb{R}P^m$. Since $\{[X^+(0,\mathbf{i})] \mid \mathbf{i} \in I_e(\nu)\}$ is a $\mathbb{Z}$-basis for $\bigoplus_{q\geq 0} H_{4q}(\mathbb{C}G(\nu);\mathbb{Z}) \subset H_*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Z})$ and since $\iota_*([X(\mathbf{i})]) = \pi_*[X^+(0,\mathbf{i})] = [X(0,\mathbf{i})]$, it follows that $\iota_*(\bigoplus H_{4q}(\mathbb{C}G(\nu);\mathbb{Z})) \cong \mathbb{Z}^{\ell_e}$ is a direct summand of $H_{4q}(P(m,\nu);\mathbb{Z})$. Therefore

$$\iota^* : \bigoplus_{q\geq 0} H^{4q}(P(m,\nu);\mathbb{Z}) \to \bigoplus_{q\geq 0} H^{4q}(\mathbb{C}G(\nu);\mathbb{Z}) \tag{10}$$

is a surjective ring homomorphism. Since $\bigoplus_{q\geq 0} H^{4q}(\mathbb{C}G(\nu);\mathbb{Z})$ is torsion free, the kernal of ι^* contains the torsion subgroup of $\bigoplus_{q\geq 0} H^{4q}(P(m,\nu);\mathbb{Z})$.

(iv) Let $\{x(m,\mathbf{i})\}$ in $Hom(H_*(P(m,\nu);\mathbb{Z}),\mathbb{Z})$ be dual to $[X(m,\mathbf{i})]$ with respect to the basis $\mathcal{B} = \bigcup_{q\geq 0} \mathcal{B}_q$ of $H_*(P(m,\nu);\mathbb{Z})/\text{torsion}$. We have

$$\begin{aligned} \langle \iota^*(x(m,\mathbf{i})), [X(\mathbf{j})]\rangle &= \langle x(m,\mathbf{i}), \iota_*([X(\mathbf{j})])\rangle \\ &= \langle x(m,\mathbf{i}), [X(0,\mathbf{j})]\rangle \\ &= 0, \end{aligned}$$

for all $[X(\mathbf{j})] \in H_*(\mathbb{C}G(\nu);\mathbb{Z})$. Thus the kernel $\iota^* : H^*(P(m,\nu);\mathbb{Z}) \to H^*(\mathbb{C}G(\nu);\mathbb{Z})$ contains $x(m,\mathbf{i})$.

(v) The cup-product $x(m,\mathbf{i}) \smile x(m,\mathbf{j})$ is a torsion element. This is obvious when any one of the factors is a torsion element. Suppose that both are of infinite order. Now $\pi^*(x(m,\mathbf{i})), \pi^*(x(m,\mathbf{j})) \in H^m(\mathbb{S}^m;\mathbb{Z}) \otimes H^*(\mathbb{C}G(\nu);\mathbb{Z})$. Hence $x(m,\mathbf{i}) \smile x(m,\mathbf{j}) \in \ker \pi^*$. By Proposition 3.6, $\ker(\pi^*)$ consists only of torsion elements, and so our assertion follows.

3.6. **The ring structure of $H^*(P(m,\nu);R)$.** Let R be a commutative ring in which 2 is a unit. From Theorem 3.8 we know that as an R-module, $H^q(P(m,\nu);R)$ is free of rank $|\mathcal{B}_q|$. We now turn to the ring structure of $H^*(P(m,\nu);R)$. We shall first describe it as a subring of $H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);R)$ via the homomorphism π^* induced by the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$. Note that since 2 is a unit in R, π^* is a monomorphism; see Proposition 3.6. Then, in Theorem 3.14 we shall describe $H^*(P(m,\nu);R)$ as a quotient of a polynomial algebra.

It will be convenient to use Chern classes of the canonical bundles over $\mathbb{C}G(\nu)$ as ring generators of $H^*(\mathbb{C}G(\nu);\mathbb{Z})$.

Let $1 \leq j \leq s$ and let γ_j denote the complex n_j-plane bundle over $\mathbb{C}G(\nu)$ whose fibre over $\mathbf{L} = (L_1,\ldots,L_s)$ is L_j. Then

$$\bigoplus_{1\leq j\leq s} \gamma_j = n\epsilon_{\mathbb{C}}.$$

Denote by $c(\omega,t)$ the total Chern polynomial $1 + c_1(\omega)t + c_2(\omega)t^2 + \cdots + c_r(\omega)t^r$ in the indeterminate t, where $c_j(\omega)$ is the jth Chern class of ω and r is the rank of ω. The following relation holds in $H^*(\mathbb{C}G(\nu);\mathbb{Z})$, in view of the above vector bundle isomorphism, where

$$\sum_{0\le r\le n} f_p t^p := \prod_{1\le j\le s} c(\gamma_j, t) = 1. \tag{11}$$

Recall that

$$H^*(\mathbb{C}G(\nu);\mathbb{Z}) = \mathbb{Z}[c_{r,j}; 1\le r\le n_j, 1\le j\le s]/\langle f_1,\cdots,f_n\rangle$$

where the f_j are defined by the equation

$$\sum_{0\le r\le n} f_r t^r = \prod_{1\le j\le s}(1+c_{1,j}t+\cdots+c_{n_j,j}t^{n_j}) \tag{12}$$

under an isomorphism that maps $c_{r,j}$ to $c_r(\gamma_j)$. See [4, Proposition 31.1].

Denote by $u_m \in H^m(\mathbb{S}^m;\mathbb{Z})$ the positive generator (with respect to the standard orientation on the sphere).

We now describe, using Proposition 3.8, the subring fixed by θ^* in $H^*(\mathbb{S}^m\times\mathbb{C}G(\nu);R)\cong R[u_m]\otimes R[c_{r,j}; 1\le r\le n_j, 1\le j\le s]/\langle u_m^2,\ f_p, 1\le p\le n\rangle$. Since π^* defines an isomorphism of $H^*(P(m,\nu);R)\cong \operatorname{Fix}(\theta^*)$, we have the following result. We omit the proof, which involves a straightforward verification.

Theorem 3.13. *Suppose that 2 is a unit in R. Then $H^*(P(m,\nu))$ is isomorphic to the subalgebra $\operatorname{Fix}(\theta^*)\subset H^*(\mathbb{S}^m\times\mathbb{C}G(\nu);R)$ which is generated by the following elements:*
Case (1): <u>m is even.</u>

$$\begin{aligned} u_m c_{2p-1}(\gamma_r),&\quad 1\le 2p-1\le n_r,\ 1\le r\le s;\\ c_{2j}(\gamma_r),&\quad 1\le j\le n_r,\ 1\le r\le s;\ \textit{and}\\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j),&\quad 1\le 2p-1\le n_i,\ 1\le 2q-1\le n_j,\ 1\le i\le j\le s.\end{aligned}$$

Case (2): <u>m is odd.</u>

$$\begin{aligned} u_m;\ \ c_{2j}(\gamma_r),&\quad 1\le j\le n_r,\ 1\le r\le s;\ \textit{and}\\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j),&\quad 1\le 2p-1\le n_i,\ 1\le 2q-1\le n_j,\ 1\le i\le j\le s.\end{aligned}$$

□

Next, we shall present $H^*(P(m,\nu);R)$ as a quotient of a polynomial algebra over R. We introduce commuting 'variables' $u_m, c_{2p,i}, c'_{2p-1,i,}, c''_{2p-1,2q-1,i,j}$, $1\le 2p-1\le n_i, 1\le 2q-1\le n_j, 1\le i\le j\le s$. Their degrees are assigned as follows: $|u_m| = m, |c_{2p,i}| = 4p, |c'_{2p-1,i}| = m+4p-2, |c''_{2p-1,2q-1,i,j}| = 4(p+q-1)$.

It will be tacitly assumed that $c''_{2p-1,2q-1,i,j}$ is the same as the variable $c''_{2q-1,2p-1,j,i}$.

Recall the polynomials $f_r, 1\le r\le n$, defined in Equation (12). When $r = 2r'$ is even, one may express f_r as a polynomial in $c_{2p,i}$ and $c_{2p'-1,i'}c_{2q'-1,j'}$, where p,p',q' vary in appropriate ranges. Now, replacing $c_{2p'-1,i'}c_{2q'-1,j'}$ by $c''_{2p'-1,2q'-1,i',j'}$, one obtains a polynomial F_r in the variables $c_{2p,i}$ and $c''_{2p'-1,2q'-1,i',j'}$.

When m is even, the same procedure applied to $c_{2t-1,l}f_r, 2p-1\le n_l$, for r odd yields the polynomials $F_{r,2t-1,l}$ in the variables $c_{2q,j}, c'_{2p-1,i}, c''_{2p'-1,2q'-1,i',j'}$, $1\le 2p'-1, 2p-1\le n_{i'}, 1\le 2q'-1, 2q\le n_j, 1\le i,j\le s$ replacing $u_m c_{2p-1,i}$ and $c_{2p'-1,i'}c_{2q'-1,j'}$ by $c'_{2p-1,i}$ and

$c''_{2p'-1,2q'-1,i',j'}$, respectively. Similarly, assuming that r is odd, $u_m f_r$ can be expressed as a polynomial in the variables $c_{2q,j}, c'_{2p-1,i}, c''_{2p'-1,2q'-1,i',j'}$ in several ways. We choose one such expression and denote the resulting polynomial by $F'_{r,m}$.

Theorem 3.14. *We keep the above notations. Let R be a commutative ring in which 2 is a unit. Then $H^*(P(m,\nu);R)$ is isomorphic, as a graded R-algebra to $\mathcal{R}/\mathcal{J}$ where $\mathcal{R}$ is a polynomial algebra. The algebra $\mathcal{R}$ and the ideal $\mathcal{J}\subset\mathcal{R}$, whose descriptions depend on the parity of m, are defined below.*

Case (1). *Suppose that $m\equiv 0 \pmod 2$. Then*

$$\mathcal{R} := R[c_{2t,i};\ c'_{2p-1,i};\ c''_{2p-1,2q-1,i,j};\ 1\le 2t, 2p-1\le n_i,\ 1\le 2q-1\le n_j,\ 1\le i,j\le s],$$

with $|c_{2t,i}| = 4t, |c'_{2p-1,i}| = m+4p-2, |c''_{2p-1,2q-1,i,j}| = 4p+4q-4$. The ideal $\mathcal{J}\subset\mathcal{R}$ is the ideal generated by the following elements:
(i) $c'_{2p-1,i}c'_{2q-1,j};\quad c'_{2p-1,i}c''_{2q-1,2q'-1,j,j'} - c'_{2q'-1,j'}c''_{2p-1,2q-1,i,j}$;
(ii) $c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'}$;
(iii) $F_{2r},\ F'_{2k-1,m},\ F_{2k-1,2t-1,l},\ 1\le 2k-1, 2r\le n,\ 1\le 2t-1\le n_l,\ 1\le l\le s$;
where $2p-1\le n_i,\ 2p'-1\le n_{i'},\ 2q-1\le n_j,\ 2q'-1\le n_{j'},\ 1\le i,j,i',j'\le s$.
The isomorphism $\eta:\mathcal{R}/\mathcal{J}\to H^*(P(m,\nu);R)$ is established by

$$c_{2p,i}\mapsto c_{2p}(\gamma_i),\ \ c'_{2p-1,i}\mapsto u_m c_{2p-1}(\gamma_i),\ \ c''_{2p-1,2q-1,i,j}\mapsto c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j).$$

Case (2). *Suppose that $m\equiv 1 \pmod 2$. Then*

$$\mathcal{R} := R[u_m;\ c_{2p,i}, 1\le 2p\le n_i;\ c''_{2p-1,2q-1,i,j},\ 2p-1\le n_i,\ 2q-1\le n_j,\ 1\le i,j\le s],$$

with $|u_m| = m, |c_{2p,i}| = 4p, |c''_{2p-1,2q-1,i.j}| = 4p+4q-4$. The ideal $\mathcal{J}\subset\mathcal{R}$ is generated by the following elements:
(i) $u_m^2;\quad c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'}$;
(ii) $F_{2r},\ F_{2k-1,2t-1,l},\ 1\le 2t-1\le n_l,\ 1\le l\le s$;
where $2p-1\le n_i,\ 2p'-1\le n_{i'},\ 2q-1\le n_j,\ 2q'-1\le n_{j'},\ 1\le i,j,i',j'\le s$.
An isomorphism $\eta:\mathcal{R}/\mathcal{J}\to H^*(P(m,\nu);R)$ is obtained as

$$u_m\mapsto u_m,\ \ c_{2p,i}\mapsto c_{2p}(\gamma_i),\ \ c''_{2p-1,2q-1,i,j}\mapsto c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j).$$

Proof. It is clear that η is a well-defined R-algebra homomorphism. Also image of η equals $\mathrm{Fix}(\theta^*)$. Since 2 is a unit in R, we have $H^*(P(m,\nu);R)\cong\mathrm{Fix}(\theta^*)\subset H^*(\mathbb{S}^m\times\mathbb{C}G(\nu);R)$ and so, to complete the proof, we only need to show that it is a monomorphism of R-modules.

Let $\widetilde{\mathcal{R}}_0$ be the polynomial algebra $\widetilde{\mathcal{R}}_0 = R[c_{q,i}\mid 1\le q\le n_i; 1\le i\le s]$ and let $\mathcal{R}_0$ be the R-subalgebra of $\widetilde{\mathcal{R}}_0$ generated by the $c_{2q,i},\ \ c_{2p-1,2q-1,i,j} := c_{2p-1,i}c_{2q-1,j}$. The algebra $\widetilde{\mathcal{R}}_0$ is graded by $|c_{q,i}| = 2q\ \forall i$. Let $\tilde\sigma:\widetilde{\mathcal{R}}_0\to\widetilde{\mathcal{R}}_0$ be the R-algebra involution defined by $c_{q,i}\mapsto(-1)^q c_{q,i}$. Then $\mathcal{R}_0 = \bigoplus_{t\ge 0}\widetilde{\mathcal{R}}^{4t} = \mathrm{Fix}(\tilde\sigma)$. The cohomology algebra $H^*(\mathbb{C}G(\nu);R)$ is isomorphic to $\widetilde{\mathcal{R}}_0/\widetilde{\mathcal{J}}_0$ where $\widetilde{\mathcal{J}}_0$ is generated by $f_1,\cdots,f_n$ defined as in Equation (11). The isomorphism is obtained by sending $c_{q,i}$ to $c_q(\gamma_i)\in H^{2q}(\mathbb{C}G(\nu);R)$.

The ideal $\widetilde{\mathcal{J}}_0$ is graded and we have that $\widetilde{\mathcal{J}}_0 \cap \mathcal{R}_0 = \oplus_{t\geq 0}\widetilde{\mathcal{J}}_0^{4t}$. So $\mathcal{J}_0 := \mathcal{R}_0 \cap \widetilde{\mathcal{J}}_0 \subset \mathcal{R}_0$ is generated as an ideal by the following elements:

$$f_{2r},\ c_{2p-1,i}f_{2r-1} \in \mathrm{Fix}(\tilde{\sigma}), 1 \leq 2p-1 \leq n_i, 1 \leq i \leq s.$$

We may express each of these elements as polynomials in the chosen generators of $\mathcal{R}_0$. The non-uniqueness of such expressions for a given ideal generator arises from the equality $c_{2p-1,2q-1,i,j}c_{2p'-1,2q'-1,i',j'} = c_{2p-1,2p'-1,i,i'}c_{2q-1,2q'-1,j,j'}$.

Consider the polynomial algebra R_0 over R generated by the indeterminates $c_{2p,i}, 1 \leq 2p \leq n_i, c''_{2p-1,2q-1,i,j}, 1 \leq 2p-1 \leq n_i, 1 \leq 2q-1 \leq n_j; 1 \leq i, j \leq s$. It is understood that $c''_{2p-1,2q-1,i,j} = c''_{2q-1,2p-1,j,i}$. Let J be the ideal of R_0 generated by the following elements

$$c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'},$$

where $1 \leq 2p-1 \leq n_i, 1 \leq 2q-1 \leq n_j, 1 \leq 2p'-1 \leq n_{i'}, 1 \leq 2q-1 \leq n_j, 1 \leq 2q'-1 \leq j', 1 \leq i, i', j, j' \leq s$. Then the quotient algebra R_0/J is isomorphic to $\mathcal{R}_0 \subset \widetilde{\mathcal{R}}_0$ via the R-algebra homomorphism

$$c_{2p,i} \mapsto c_{2p,i},\ c''_{2p-1,2q-1,i,j} \mapsto c_{2p-1,2q-1,i,j}.$$

Let $J_0 \subset R_0$ be the ideal generated by $J \cup \{F_{2r}, F_{2r-1,2t-1,l} \mid 1 \leq 2k-1, 2r \leq n, 1 \leq 2t-1 \leq n_l, 1 \leq l \leq s\}$. Then

$$R_0/J_0 \cong \mathcal{R}_0/\mathcal{J}_0 \cong \bigoplus_{t\geq 0} H^{4t}(\mathbb{C}G(\nu); R) \cong \mathrm{Fix}(\sigma^*). \tag{13}$$

The ring $\mathcal{R}/\mathcal{J}$ is an $\mathcal{R}_0/\mathcal{J}_0$-module whose (definition and) structure depends on the parity of m.

First, consider the case when m is odd. In this case, $\mathcal{R}/\mathcal{J}$ is a free $\mathcal{R}_0/\mathcal{J}_0$-module with basis $\{1, u_m\}$. Also, $H^*(\mathbb{S}^m; R) = R[u_m]/\langle u_m^2\rangle$ is fixed by α^*.

Since $u_m^2 \in \mathcal{J}$, we have the following isomorphisms of R-algebras

$$\begin{aligned} \mathcal{R}/\mathcal{J} &\cong (\mathcal{R}_0/\mathcal{J}_0)[u_m]/\langle u_m^2\rangle \\ &\cong R[u_m]/\langle u_m^2\rangle \otimes_R (\mathcal{R}_0/\mathcal{J}_0) \\ &\cong H^*(\mathbb{S}^m; R) \otimes \mathrm{Fix}(\sigma^*) \\ &\cong \mathrm{Fix}(\theta^*) \cong H^*(P(m,\nu); R). \end{aligned}$$

This completes the proof when m is odd.

Now let m be even. We have $\alpha^*(u_m) = -u_m$ and

$$\mathrm{Fix}(\theta^*) = \bigoplus_{t\geq 0} H^{4t}(\mathbb{C}G(\nu); R) \oplus \left(\bigoplus_{t\geq 0} Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R)\right). \tag{14}$$

Since each $H^{4t+2}(\mathbb{C}G(\nu); R)$ is a free R-module for $t \geq 0$, so is $Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R) \cong H^{4t+2}(\mathbb{C}G(\nu); R)$.

On the other hand, $\mathcal{R}/\mathcal{J}$ is generated, as a $\mathcal{R}_0/\mathcal{J}_0$-algebra, by the $c'_{2p-1,i}$. Since $c'_{2p-1,i}c'_{2q-1,j} = 0$, we see that $\mathcal{R}/\mathcal{J}$ is generated as an $\mathcal{R}_0/\mathcal{J}_0$-module by $\{1, c'_{2p-1,i}\}$. Further, in view of the relation $c'_{2p-1,i}c''_{2q-1,2q'-1,j,j'} = c'_{2q'-1,j'}c''_{2p-1,2q-1,i,j}$ in $\mathcal{R}/\mathcal{J}$, if $x \in \mathcal{R}/\mathcal{J}$

is a monomial in the generators $c''_{2p-1,2q-1,i,j}$ and if r such that $1 \leq 2r-1 \leq n_l, 1 \leq l \leq s$, then $c'_{2r-1,l}x$ is *uniquely* expressible as $c'_{2r_0-1,l_0}y$ with y a monomial in the generators $c''_{2p-1,2q-1,i,j}$ and (r_0, l_0) is the *least* in lexicographic order among such expressions. This shows that, as R-submodules of $\mathcal{R}/\mathcal{J}$,

$$\sum(\mathcal{R}_0/\mathcal{J}_0)c'_{2p-1,i} \cong \bigoplus_{t\geq 0} Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R).$$

Since $c'_{2p-1,i}c'_{2q-1,j} = 0$ and since $u_m^2 = 0$, the above is an R-algebra isomorphism. It follows that $\mathcal{R}/\mathcal{J} \cong \mathcal{R}_0/\mathcal{J}_0 \oplus \big(\bigoplus_{t\geq 0}(Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R))\big)$ is an R-algebra isomorphism. Thus, in view of Equation (14), we have the R-algebra isomorphism $\mathcal{R}/\mathcal{J} \cong \mathrm{Fix}(\theta^*) \cong H^*(P(m,\nu); R)$ defined by η. □

4. K-theory of $P(m, \nu)$

Our goal is to determine the ring structure of the complex K-theoretic ring $K^*(P(m,\nu))$ for any $m \geq 1$ and $\nu = (n_1, \ldots, n_s), s \geq 2$. However, we have been greeted with limited success in our efforts. However, our results are complete up to a 'finite ambiguity' in a sense that will be made precise later.

We now proceed to describe our results on (a) the additive structure of $K^*(P(m,\nu))$, (b) the ring structure of $K^*(P(m,\nu)) \otimes \mathbb{Q}$, and, (c) a subring of $K^0(P(m,\nu))$ generated by the classes of certain canonical (complex) vector bundles on $P(m,\nu)$.

4.1. **The additive structure of** $K^*(P(m,\nu))$**.** Recall, from [3], the Atiyah-Hirzebruch spectral sequence $(E_r^{p,q}, d_r)$, which abuts to $K^*(B)$ for a finite CW complex B. The E_2-page is defined as $E_2^{p,q} = H^p(B; K^q(\{*\}))$ where $\{*\}$ denotes a one-point space. The differential is of bidegree $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ has bidegree $(r, 1-r)$. As such $d_r = 0$ if r is even, since $K^q(\{*\}) = 0$ for q odd.

Let ω be a complex vector bundle over B. Recall that the Chern character $\mathrm{ch}(\omega) \in H^*(B;\mathbb{Q})$ is defined as $\mathrm{ch}(\omega) = \sum_{k\geq 0}(\sum_{1\leq j\leq r} x_i^k/k!)$ where $x_j, 1 \leq j \leq r$ are the 'Chern roots' of ω. Thus, the x_j are defined by a formal factorization of the Chern polynomial:

$$c(\omega, t) = \sum_{0\leq p\leq r} c_p(\omega)t^p = \prod_{1\leq j\leq r}(1 + x_j t).$$

Atiyah and Hirzebruch [3] proved that the Chern character $\mathrm{ch} : K^0(B)\otimes\mathbb{Q} \longrightarrow H^{\mathrm{ev}}(B;\mathbb{Q}) = \bigoplus_{q\geq 0} H^{2q}(B;\mathbb{Q})$ defined by $[\omega] \mapsto \mathrm{ch}(\omega)$ is an isomorphism of rings. In particular, the rank of $K^0(B)$ equals that of $H^{\mathrm{ev}}(B;\mathbb{Z})$. It follows that if $H^*(B;\mathbb{Z})$ has no p-torsion for a prime p, neither does $K^0(X)$. The same result applied to the suspension $S(B)$ of B yields an isomorphism $K^1(B) \otimes \mathbb{Q} \to H^{\mathrm{odd}}(B;\mathbb{Q}) = \bigoplus_{q\geq 1} H^{2q-1}(B;\mathbb{Q})$.

Taking B to be $P(m,\nu)$ we obtain the following. Denote by $\xi_{\mathbb{C}}$ the complexification of the Hopf line bundle ξ over $\mathbb{R}P^m$. Clearly $\xi_{\mathbb{C}} \otimes \xi_{\mathbb{C}} \cong \epsilon_{\mathbb{C}}$ and so $([\xi_{\mathbb{C}}] - 1)^2 = -2([\xi_{\mathbb{C}}] - 1)$. Adams [1] showed that $K^0(\mathbb{R}P^m) \cong \mathbb{Z}[y]/\langle y^2 + 2y, y^{r+1}\rangle = \mathbb{Z} \oplus \mathbb{Z}_{2^r}y$ where $y = [\xi_{\mathbb{C}}] - 1$ and $r = \lfloor m/2 \rfloor$. We shall denote by the same symbol $\xi_{\mathbb{C}}$ the pull-back $p^!(\xi_{\mathbb{C}})$ on $P(m,\nu)$.

Theorem 4.1. *Let $m \geq 1$, $\nu = (n_1, \ldots, n_s)$, $s \geq 2$. Let b_e, b_o, b'_e, b'_o be as in Theorem 3.11. Then:*
(i) $K^0(P(m,\nu)) \cong \mathbb{Z}^{b_e} \oplus A_0$ where A_0 is a finite abelian group of order 2^k for some k, with $0 \leq k \leq b'_e$. The group A_0 contains a summand $\mathbb{Z}_{2^{\lfloor m/2 \rfloor}}$, generated by $y = [\xi_{\mathbb{C}}] - 1$.
(ii) $K^1(P(m,\nu)) \cong \mathbb{Z}^{b_o} \oplus A_1$, where A_1 is a finite abelian group of order 2^t for some $0 \leq t \leq b'_o$.
In particular, $K^1(P(m,\nu))$ is a torsion group if m is even, and $K^0(P(1,\nu))$ is a torsion-free group

Proof. Since $\mathrm{ch} : K^*(P(m,\nu)) \otimes \mathbb{Q} \to H^*(P(m,\nu);\mathbb{Q})$ is an isomorphism and since $K^*(P(m,\nu))$ is finitely generated group, it follows that the ranks of $K^0(P(m,\nu))$ and $K^1(P(m,\nu))$ are the same as those of $H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z})$ and $H^{\mathrm{odd}}(P(m,\nu);\mathbb{Z})$ respectively. This shows that the ranks of $K^0(P(m,\nu)), K^1(P(m,\nu))$ as asserted.

Since the Atiyah-Hirzebruch spectral sequence converges to $K^*(P(m,\nu))$, $K^0(P(m,\nu))$ has a filtration so that the associated graded group is isomorphic to $\bigoplus_{p\geq 0} E_\infty^{p,-p}$. Since the torsion subgroup of $\bigoplus_{p\geq 0} E_2^{2p,-2p} = H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z})$ is a 2-group of order $2^{b'_e}$, we obtain that the torsion subgroup of $K^0(P(m,\nu))$ is a 2-group whose orders are bounded above by $2^{b'_e}$. Similar argument yields that the torsion subgroup of $K^1(P(m,\nu))$ is a 2-group of order at most $2^{b'_o}$.

To see that $K^0(P(m,\nu))$ contains a summand isomorphic to $\mathbb{Z}_{2^{\lfloor m/2 \rfloor}}$, we note that $p : P(m,\nu) \to \mathbb{R}P^m$ admits a cross-section, namely $s([v]) = [v, \mathbf{E}]$ where $\mathbf{E} = (\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_s})$ in which the standard basis vectors $e_{n_1+\cdots+n_{j-1}+1}, \ldots, e_{n_1+\cdots+n_j}$ span the j-th coordinate $\mathbb{C}^{n_j}$. It follows that the composition $s^! \circ p^!$ is the identity map of $K^0(\mathbb{R}P^m)$. Hence $p^!$ is a monomorphism whose image is a summand of $K^0(P(m,\nu)$. By the work of Adams recalled above $\widetilde{K}^0(\mathbb{R}P^m) = \mathbb{Z}_{2^{\lfloor m/2 \rfloor}} y$. This completes the proof. $\square$

We pause for an example.

Example 4.2. (i) In the case of classical Dold manifolds, $\nu = (1, n-1)$ and $P(m,\nu) = P(m, n-1)$. The group $K^*(P(m,n-1))$ had been completely determined by Fujii [10, Theorem 3.14]. The table below summarises Fujii's results (in our notations) yielding the rank and the order of the torsion groups of $K^0(P(m,n-1)), K^1(P(m,n-1))$. Our results are consistent with Fujii's work but we have not been able to determine the groups A_0, A_1.

$P(m,n-1)$		K^0		K^1	
m	$n-1$	b_e	$o(A_0)$	b_o	$o(A_1)$
$2r$	$2t$	$2t+1$	2^r	0	0
$2r+1$	$2t$	$t+1$	2^r	$t+1$	0
$2r$	$2t+1$	$2t+2$	2^r	0	2^r
$2r+1$	$2t+1$	$t+1$	2^r	$t+1$	2^{r+1}

Table 2: The values of $b_e, b_o, o(A_0), o(A_1)$ for $P(m,n-1)$.

(ii) When $\mathbb{C}G(\nu)$ is the complete flag manifold, that is, when $\nu = (1, \dots, 1)$, $\nu_o = n$. We leave out the trivial case when $n = 1$ and assume $n \geq 2$. Now the values of b_e, b'_e, b_o, b'_o for $P(m, (1, \dots, 1))$ in Theorem 4.1 are described in the following table.

$P(m,\nu)$	K^0		K^1	
m	b_e	b'_e	b_o	b'_o
$2r$	$n!/2$	$rn!/2$	$n!/2$	$rn!/2$
$2r+1$	$n!$	$rn!/2$	0	$(r+1)n!/2$

Table 3: The values of b_e, b'_e, b_o, b'_o for $P(m, (1, \dots, 1))$.

4.2. **The ring structure of $K^0(P(m,\nu))$.** Recall that $\gamma_j, 1 \leq j \leq s$, denotes the canonical complex n_j-plane bundle over $\mathbb{C}G(\nu)$. It is a σ-conjugate bundle, so we obtain the real vector bundle $\hat{\gamma}_j$ over $P(m,\nu)$. Denote by $\hat{\gamma}_{j\mathbb{C}}$ the complexification $\hat{\gamma}_j \otimes_{\mathbb{R}} \mathbb{C}$.

Let $f : \mathbb{C}G(\nu) \to P(m,\nu)$ be the inclusion of the fibre over the base point $[e_1] \in \mathbb{R}P^m$. Thus $f(\mathbf{L}) = [e_1, \mathbf{L}]$. Since $f^!(\hat{\gamma}_j) = \gamma_j$ as *real* vector bundles, it follows that $f^!(\hat{\gamma}_{j\mathbb{C}}) = \gamma_j \otimes \mathbb{C} \cong \gamma_j \oplus \bar{\gamma}_j$ as complex vector bundles. Note that complexification commutes with Whitney sums, tensor products, and exterior products. So, we have $f^!(\Lambda^k(\hat{\gamma}_{j\mathbb{C}})) = \Lambda^k(\gamma_j \oplus \bar{\gamma}_j) \cong \bigoplus_{p+q=k} \Lambda^p(\gamma_j) \otimes \Lambda^q(\bar{\gamma}_j) \cong \bigoplus_{p+q=k} \Lambda^p(\gamma_j) \otimes \overline{\Lambda^q(\gamma_j)}$ for all $k \geq 0, 1 \leq j \leq s$.

We recall the following description of the complex K-ring of $\mathbb{C}G(\nu)$. See [14, Theorem 3.6, Chapter-IV].

Theorem 4.3. *The ring $K^0(\mathbb{C}G(\nu))$ is isomorphic to the polynomial ring generated by $\lambda_{p,j},\ 1 \leq p \leq n_j,\ 1 \leq j \leq s$, modulo the ideal generated by the set $\{h_p - \binom{n}{p} \mid 1 \leq p \leq n\}$, where $h_p = h_p(\lambda_{q,j})$ is the coefficient of t^p in the following polynomial in the variable t:*

$$\sum_{0 \leq p \leq n} h_p t^p = \prod_{1 \leq j \leq s} \Big(\sum_{0 \leq r \leq n_j} \lambda_{r,j} t^r \Big).$$

Here $h_0 = 1, \lambda_{0,j} = 1\ \forall j$. The isomorphism is defined by sending $\lambda_{p,j}$ to $[\Lambda^p(\gamma_j)]$. Moreover, $K^1(\mathbb{C}G(\nu)) = 0$. □

We shall refer to the generators $\lambda_{p,j}$ as the *canonical generators* of $K^0(\mathbb{C}G(\nu))$.

Remark 4.4. For any compact connected Lie group G and a closed subgroup $H \subset G$, the α-construction yields a λ-ring homomorphism $RH \to K^0(G/H)$ defined by that sends $[V] \in RH$ to the class of the associated vector bundle with projection $G \times_H V \to G/H$. (See [3], [13].) The bundle γ_j is associated to the representation $\lambda_{1,j} : S(U(\nu)) := S(U(n_1) \times \cdots \times U(n_s)) \to U(n_j)$, the projection to the jth coordinate, via the so-called 'α-construction'. Taking the pth exterior power, we obtain that $\Lambda^p(\gamma_j)$ is associated to $\Lambda^p(\lambda_{1,j}) = \lambda_{p,j}$ in the representation ring $RS(U(\nu))$. Using the known description of the representation of the unitary group (see [13, Chapter 14]), it is readily seen that $RS(U(\nu)) = \mathbb{Z}[\lambda_{p,j}, \lambda^{-1}_{n_j,j} \mid 1 \leq j \leq n_j]/\langle \prod_{1 \leq j \leq s} \lambda_{n_j,j} - 1 \rangle$. Note that $\lambda^{-1}_{n_j,j} = \prod_{i \neq j} \lambda_{n_i,i}$ in $RS(U(\nu))$. Denote by λ_1 the standard representation of $SU(n)$ on $\mathbb{C}^n$. We denote by

$\rho([V]) \in RS(U(\nu))$ the restriction of the class $[V]$ of a $S(U(\nu))$ representation V. Thus $\rho(\lambda_1) = \sum_{1\leq j\leq s} \lambda_{1,j}$. Set $\Lambda_t([V]) := \sum_{1\leq j\leq r}[\Lambda^q(V)]t^q$. Then $\Lambda_t(\lambda_1) = \prod_{1\leq j\leq s} \Lambda_t(\lambda_{1,j})$.

If a complex representation of $S(U(\nu))$ arises as the extension of a representation of $SU(n)$, then the associated vector bundle is trivial. It follows that $\rho(h_p) = \binom{n}{p}$. So ρ defines a ring homomorphism $\bar{\rho} : RS(U(\nu))/\mathcal{I} \to K^0(\mathbb{C}G(\nu))$ where $\mathcal{I} \subset RS(U(\nu))$ is the ideal in $\langle h_p - \binom{n}{p} \mid 1 \leq p \leq n-1\rangle$. The above theorem is equivalent to the assertion that $\bar{\rho}$ is an isomorphism.

As a corollary, we obtain the following.

Lemma 4.5. *$K(\mathbb{C}G(\nu))$ is generated as an abelian group by the classes of σ-conjugate complex vector bundles.*

Proof. We need only observe that if ξ and η are σ-conjugate complex vector bundles so are $\xi \otimes \eta, \Lambda^q(\xi), q \geq 0$, and $\xi \otimes \eta$. (See [22, Example 2.2(iv), Lemma 2.3(ii)].) Since the canonical bundles $\gamma_j, 1 \leq j \leq s$, are all σ-conjugate bundles, the lemma follows from the above theorem. □

Since $\pi = \pi \circ \theta$, we have $\theta^! \circ \pi^! = (\pi \circ \theta)^! = \pi^!$. So, for any complex vector bundle ω over $P(m, \nu)$, $\pi^!(\omega)$ is a complex vector bundle over $\mathbb{S}^m \times \mathbb{C}G(\nu)$ such that $\theta^!(\pi^!(\omega)) = \pi^!(\omega)$. This implies that the image of $\pi^! : K^0(P(m, \nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ is contained in the fixed subring $\mathrm{Fix}(\theta^!)$.

Since $K^1(\mathbb{C}G(\nu)) = 0$, we have isomorphisms given by the exterior tensor products of vector bundles. See [2, Corollary 2.7.15.], [14, Prop. 3.24, Chapter IV].

$$K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) = K^0(\mathbb{S}^m) \otimes K^0(\mathbb{C}G(\nu)), \tag{15}$$

and,

$$K^1(\mathbb{S}^m \times \mathbb{C}G(\nu)) = K^1(\mathbb{S}^m) \otimes K^0(\mathbb{C}G(\nu)). \tag{16}$$

Also, if m is even $K^1(\mathbb{S}^m) = 0$ and if m is odd, $\widetilde{K}(\mathbb{S}^m) = 0$.

We shall identify $y \in K^0(\mathbb{C}G(\nu))$ with $pr_2^!(y) = 1 \otimes y \in K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ and $x \in K^0(\mathbb{S}^m)$ with $x \otimes 1 = pr_1^!(x)$ where pr_i denotes the projection to the ith factor of $\mathbb{S}^m \times \mathbb{C}G(\nu)$. Thus $x \otimes y$ is identified with $xy \in K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$. A similar notation holds for the $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$-module $K^1(\mathbb{S}^m \times \mathbb{C}G(\nu))$.

Let $m = 2r$ be even. By Theorem 2.1, the element $[\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-]$ generates $\ker(\mathrm{rank} : K^0(\mathbb{S}^m) \to \mathbb{Z}) \cong \mathbb{Z}$.

Definition 4.6. Let $m \geq 1$ and let $r = \lfloor m/2 \rfloor$. We set

$$\delta_m := \begin{cases} [\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-] & \text{if } m \equiv 0 \pmod 2 \\ 0 & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

We have $\theta^!(\delta_m) = \alpha^!(\delta_m) = -\delta_m$. (This follows from the naturality of the Chern character.)

For any $x \in K^0(\mathbb{C}G(\nu))$, $\theta^!(x) = \theta^!(1 \otimes x) = \sigma^!(x) = \bar{x}$.

Lemma 4.7. *Let $m \geq 1$ be arbitrary. Let $\mathcal{K}$ be the subring of $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ generated by the elements (i) $x + \bar{x}$ and, (ii) $\delta_m(x - \bar{x})$ when m is even, as x varies in $K^0(\mathbb{C}G(\nu))$. Then the subring $\mathrm{Fix}(\theta^!) \subset K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ contains $\mathcal{K}$. Moreover, if $z \in \mathrm{Fix}(\theta^!)$, then $2z \in \mathcal{K}$. Thus*

$$2\mathrm{Fix}(\theta^!) \subset \mathcal{K} \subset \mathrm{Fix}(\theta^!).$$

Proof. It is evident that, for any $x \in K^0(\mathbb{C}G(\nu))$, $x + \bar{x} \in \mathrm{Fix}(\theta^!)$.

When m is even, $\theta^!(\delta_m(x - \bar{x})) = \alpha^!(\delta_m)(\bar{x} - x) = -\delta_m(\bar{x} - x) = \delta_m(x - \bar{x})$ and so $\delta_m(x - \bar{x}) \in \mathrm{Fix}(\theta^!)$.

This shows that $\mathcal{K} \subset \mathrm{Fix}(\theta^!)$ for any $m \geq 1$.

Let $z \in \mathrm{Fix}(\theta^!)$ where we assume that $\mathrm{rank}(z) = 0$. Write $z = z_0 + \delta_m z_1$ where $z_i \in \widetilde{K}^0(\mathbb{C}G(\nu))$ (recall that $\delta_m = 0$ if m is odd). Then $z = \theta^!(z) = \bar{z}_0 - \delta_m z_1$ and so $2z = z + \theta^!(z) = z_0 + \bar{z}_0 + \delta_m(z_1 - \bar{z}_1) \in \mathcal{K}$. □

Remark 4.8. (i). If $y \in K^0(\mathbb{C}G(\nu))$, then, $y^2 + \bar{y}^2, (y + \bar{y})^2 \in \mathcal{K}$. Therefore $2y\bar{y} = (y + \bar{y})^2 - y^2 - \bar{y}^2 \in \mathcal{K}$.
(ii) Since $\bar{\lambda}_{n_j,j} = \lambda_{n_j,j}^{-1} = \prod_{i \neq j} \lambda_{n_i,i}$, we may write any $x \in K^0(\mathbb{C}G(\nu))$ as $x = P(\lambda_{p,j})$ as a polynomial in the canonical generators $\lambda_{p,j}, 1 \leq p \leq n_j, 1 \leq j \leq s$, of $K^0(\mathbb{C}G(\nu))$. In particular, the exponents of $\lambda_{n_j,j}, 1 \leq j \leq s$, that occur in $P(\lambda_{p,j})$ is non-negative for each j. Furthermore, we may assume that, in each monomial, at least one of the $\lambda_{n_j,j}$ does not occur. This is possible since $\prod_{1 \leq j \leq s} \lambda_{n_j,j} = 1$.

Recall from §2.1 that, when $m \equiv 0 \pmod 2$, the complex vector bundles ξ^+, ξ^- over $\mathbb{S}^m$ are associated to the half-spin representations, and the bundle η^- is defined as the pull-back $\alpha^!(\xi^+)$. Also we obtained bundle maps $\tilde{\alpha}^- : \eta^- \to \xi^+$ and $\tilde{\alpha}^+ : \xi^+ : \xi^+ \to \eta^-$ both covering α such that $\tilde{\alpha}^- \circ \tilde{\alpha}^+$ and $\tilde{\alpha}^+ \circ \tilde{\alpha}^-$ are the identity isomorphisms id_{ξ^+} and id_{η^-} respectively.

Let $\tilde{\sigma} : E(\omega) \to E(\omega)$ be a σ-conjugate bundle involution that covers $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$. We regard it as a complex linear isomorphism $\omega \to \bar{\omega}$ covering σ. We have a complex vector bundle isomorphism $\tilde{\theta}^+ : E(\xi^+ \otimes \omega) \to E(\eta^- \otimes \bar{\omega})$ that covers θ. Explicitly, $\tilde{\theta}^+(e^+ \otimes u) = \tilde{\alpha}^+(e^+) \otimes \tilde{\sigma}(u)$ and $\tilde{\theta}^-(e^- \otimes v) = \tilde{\alpha}^-(e^-) \otimes \tilde{\sigma}(v)$. The bundle isomorphism $\tilde{\theta}^- : E(\eta^- \otimes \bar{\omega}) \to E(\xi^+ \otimes \omega)$ covering θ is defined analogously. We have

$$\tilde{\theta}^+ \circ \tilde{\theta}^- = id_{\eta^- \otimes \bar{\omega}}, \text{ and } \tilde{\theta}^- \circ \tilde{\theta}^+ = id_{\xi^+ \otimes \omega}. \tag{17}$$

Set $\tilde{\xi}(\omega) := \xi^+ \otimes \omega \oplus \eta^- \otimes \bar{\omega}$ for any σ-conjugate vector bundle ω.

Lemma 4.9. *With the above notations, we have:*
(i) For any $m \geq 1$, we have $\pi^!(\hat{\omega}_{\mathbb{C}}) \cong \omega \oplus \bar{\omega}$.
(ii) Let $m \equiv 0 \pmod 2$. Then there exists a complex vector bundle $\xi^0(\omega)$ over $P(m, \nu)$ such that $\pi^!(\xi^0(\omega)) \cong \tilde{\xi}(\omega)$.

Proof. We have identified ω with the bundle $pr_2^!(\omega)$ on $\mathbb{S}^m \times \mathbb{C}G(\nu)$ in the statement and will do so in the proof as well.

(i) We shall prove the stronger statement that $\pi^!(\hat{\omega}) \cong \omega_{\mathbb{R}}$, the real vector bundle underlying ω. Let $\hat{\sigma}$ be the σ-conjugate vector bundle morphism of ω that covers $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$. Recall that the total space $E(\hat{\omega}) = P(\mathbb{S}^m, E(\omega))$ of $\hat{\omega}$ is obtained as the quotient of $\mathbb{S}^m \times E(\omega)$ where (v, e) is identified to $(-v, \hat{\sigma}(e))$. The vector bundle projection $p_{\hat{\omega}} : E(\hat{\omega}) \to P(m, \nu)$ maps $[v, e]$ to $[v, p_\omega(e)]$. We have a commuting diagram where the horizontal maps are quotient maps and the vertical maps are bundle projections:

$$\begin{array}{ccc} \mathbb{S}^m \times E(\omega) & \xrightarrow{\hat{\pi}} & E(\hat{\omega}) \\ {\scriptstyle id\times p_\omega}\downarrow & & \downarrow{\scriptstyle p_{\hat{\omega}}} \\ \mathbb{S}^m \times \mathbb{C}G(\nu) & \xrightarrow{\pi} & P(m,\nu) \end{array} \tag{18}$$

Since $\hat{\pi}$ is an $\mathbb{R}$-linear isomorphism on each fibre $v \times E_x(\omega)$, it follows that $\pi^!(\hat{\omega}) \cong \omega_{\mathbb{R}}$.

(ii) We have a bundle involution $\tilde{\theta} : E(\tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ that covers θ defined as:

$$\tilde{\theta}(e^+ \otimes u, e^- \otimes v) = (\tilde{\theta}^-(e^- \otimes v), \tilde{\theta}^+(e^+ \otimes u)),$$

for $e^+ \otimes u \in E_b(\xi^+ \otimes \omega), e^- \otimes v \in E_b(\eta^- \otimes \bar{\omega})$, $b \in \mathbb{S}^m \times \mathbb{C}G(\nu)$. Note that $\tilde{\theta}$ is fixed point free. So we obtain a vector bundle $\xi^0(\omega)$ over $P(m, \nu)$, whose total space is $E(\tilde{\xi}(\omega))/\langle\tilde{\theta}\rangle$ and we have the following commuting diagram in which the double covering projection $\tilde{\pi}$ is a bundle map:

$$\begin{array}{ccc} E(\tilde{\xi}(\omega)) & \xrightarrow{\tilde{\pi}} & E(\xi^0(\omega)) \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{S}^m \times \mathbb{C}G(\nu) & \xrightarrow{\pi} & P(m,\nu) \end{array} \tag{19}$$

Therefore $\pi^!(\xi^0(\omega)) = \tilde{\xi}(\omega)$. □

It is not clear whether there is a *unique* bundle (up to isomorphism) $\xi^0(\omega)$ such that $\pi^!(\xi^0(\omega)) \cong \tilde{\xi}(\omega)$. (However, $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$; see Lemma 4.12(ii) below.) In the sequel, $\xi^0(\omega)$ will always denote the bundle as constructed in the above proof. The following lemma implies, that the class of any other such bundle differs from $[\xi^0(\omega)]$ by a torsion element in $K^0(P(m, \nu))$.

Lemma 4.10. *Let $m \geq 1$ be arbitrary. With notations as in Lemma 4.7,* $\ker \pi^! : K^0(P(m, \nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ *is precisely the torsion subgroup of* $K^0(P(m, \nu))$ *and* $Im(\pi^!)$ *contains* $\mathcal{K}$.

Proof. By the naturality of the Chern character, we have the following commutative diagram:

$$
\begin{tikzcd}
K^0(P(m,\nu)) \arrow[r, "\pi^!"] \arrow[d, "\mathrm{ch}"'] & K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) \arrow[d, "\mathrm{ch}"] \\
H^*(P(m,\nu);\mathbb{Q}) \arrow[r, "\pi^*"] & H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q})
\end{tikzcd} \tag{20}
$$

Since $\mathrm{ch} : K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) \to H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q})$ is a monomorphism, and since $\pi^* : H^*(P(m,\nu);\mathbb{Q}) \to H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q})$ is a monomorphism, $\ker \pi^!$ equals the kernel of $\mathrm{ch} : K^0(P(m,\nu)) \to H^*(P(m,\nu);\mathbb{Q})$, which is precisely the torsion subgroup of $K^0(P(m,\nu))$.

It remains to show that $\mathcal{K} \subset Im(\pi^!)$. Recall that, by definition, $\delta_m = 0$ when m is odd. Since $\mathcal{K}$ is generated as a subring by $x + \bar{x}, \delta_m(x - \bar{x})$ where x varies in $K^0(\mathbb{C}G(\nu))$, we need only to show that $x + \bar{x}$ and $\delta_m(x - \bar{x})$ are in $Im(\pi^!)$.

By Lemma 4.5, any $x \in \widetilde{K}(\mathbb{C}G(\nu))$ may be expressed $[\omega] - d$ where ω is a σ-conjugate complex vector bundle and $d = \mathrm{rank}(\omega)$. Thus $\theta^!(x) = \sigma^!(x) = [\bar{\omega}] - d = \bar{x}$. Consider the complex vector bundle $\hat{\omega}_{\mathbb{C}} = \hat{\omega} \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of $\hat{\omega}$. We have $\pi^!([\hat{\omega}_{\mathbb{C}}] - 2d) = [\omega] + [\bar{\omega}] - 2d = x + \bar{x}$. Thus $x + \bar{x} \in Im(\pi^!)$. This proves that $\mathcal{K} \subset Im(\pi^!)$ when m is odd.

Let $m \equiv 0 \pmod 2$. Recall from Remark 2.2 that $[\xi^-] = [\eta^-]$ in $K(\mathbb{S}^m)$ and so $\delta_m = [\xi^+] - 2^{r-1} = -([\eta^-] - 2^{r-1})$ in $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$. Therefore $[\tilde{\xi}(\omega)] = [\xi^+][\omega] + [\eta^-][\bar{\omega}] = \delta_m([\omega] - [\bar{\omega}]) + 2^{r-1}([\omega] + [\bar{\omega}])$. By Lemma 4.9, $[\tilde{\xi}(\omega)] \in Im(\pi^!)$. It follows that $\delta_m(x - \bar{x}) = \delta_m \cdot ([\omega] - [\bar{\omega}]) \in Im(\pi^!)$ since $[\omega] + [\bar{\omega}] \in Im(\pi^!)$. Hence $\mathcal{K} \subset Im(\pi^!)$. $\square$

Recall that $\xi_{\mathbb{C}}$ denotes the complexification of the Hopf line bundle ξ over $\mathbb{R}P^m$. We shall denote by the same symbol ξ (resp. $\xi_{\mathbb{C}}$) the bundle $p^!(\xi)$ (resp.$p^!(\xi_{\mathbb{C}})$) over $P(m,\nu)$.

Definition 4.11. Let $m \geq 1$. With notations as above, define $\mathcal{K}^0 \subset K^0(P(m,\nu))$ to be the subring generated by the following elements:
(i) $[\xi_{\mathbb{C}}]$, $[\hat{\omega}_{\mathbb{C}}]$, and, (ii) when m is even, $[\xi^0(\omega)]$,
where ω varies over σ-conjugate complex vector bundles over $\mathbb{C}G(\nu)$.

We have the following proposition which gives partial information on the multiplicative structure of $K^0(P(m,\nu))$.

Proposition 4.12. *Let $m \geq 1$. The following relations hold in $K^0(P(m,\nu))$ for any σ-conjugate vector bundles ω_1, ω_2 and ω over $\mathbb{C}G(\nu)$.*
(i) $([\xi_{\mathbb{C}}] - 1)[\hat{\omega}_{\mathbb{C}}] = 0$, and, The (additive) order of $([\xi_{\mathbb{C}}] - 1)$ equals $2^{\lfloor m/2 \rfloor}$.
Furthermore, when $m = 2r$ is even, we have
(ii) $([\xi_{\mathbb{C}}] - 1)[\xi^0(\omega)] = 0$, $[\xi^0(\omega_1)] + [\xi^0(\omega_2)] = [\xi^0(\omega_1 \oplus \omega_2)]$;
(iii) $[\xi^0(\omega)] + [\xi^0(\bar{\omega})] \equiv 2^r[\hat{\omega}_{\mathbb{C}}]$ modulo torsion;
(iv) $[\xi^0(\omega_1)][\xi^0(\omega_2)] \equiv 2^r[\xi^0(\omega_1 \otimes \omega_2)] - 2^{2r-2}([\omega_1] - [\bar{\omega}_1])([\omega_2] - [\bar{\omega}_2])$ modulo torsion.

Proof. (i) The assertion that $([\xi_{\mathbb{C}}] - 1)[\hat{\omega}_{\mathbb{C}}] = 0$ holds since $\hat{\omega} \otimes \xi \cong \hat{\omega}$. (See [22, Lemma 2.7].) That $2^{\lfloor m/2 \rfloor}$ equals the additive order of $([\xi_{\mathbb{C}}] - 1) = 0$ follows readily from the work of Adams [1] in view of the fact that the projection $p : P(m,\nu) \to \mathbb{R}P^m$ induces a monomorphism $p^! : K^0(\mathbb{R}P^m) \to K^0(P(m,\nu))$ since p admits a cross-section.

(ii) The second assertion of (ii) is readily seen to be valid. We shall construct a bundle isomorphism $\tilde{\lambda} : \epsilon_{\mathbb{C}} \otimes \tilde{\xi}(\omega) \to \tilde{\xi}(\omega)$ that covers $\theta : \mathbb{S}^m \times \mathbb{C}G(\nu) \to \mathbb{S}^m \times \mathbb{C}G(\nu)$ and such that it descends to yield a bundle isomorphism $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$ on $P(m,\nu)$. This readily implies the first assertion of (ii).

We have $E(\xi_{\mathbb{C}}) = \mathbb{S}^m \times \mathbb{C}G(\nu) \times \mathbb{C}/\langle\mu\rangle$ where $\rho : \epsilon_{\mathbb{C}} \to \epsilon_{\mathbb{C}}$ is the involutive bundle isomorphism that covers θ defined as $\rho(b;t) = (\theta(b);-t)\ \forall b \in \mathbb{S}^m \times \mathbb{C}G(\nu), t \in \mathbb{C}$. We shall denote by $[b;t] \in E_{[b]}(\xi_{\mathbb{C}})$ the image of $(b;t) \in E_b(\epsilon_{\mathbb{C}})$ under bundle map $\epsilon_{\mathbb{C}} \to \xi_{\mathbb{C}}$ covering the projection $\mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$. Thus $[b;t] = [\theta(b);-t]$.

The total space $E(\xi^0(\omega))$ of $\xi^0(\omega)$ was described in the course of the proof of Lemma 4.9 as $E(\tilde{\xi})/\langle\tilde{\theta}\rangle$. We shall denote by $(b;x,y) \in E_b(\tilde{\xi}(\omega))$ the element $(x,y) \in E_b(\xi^+ \otimes \omega) \oplus E_b(\eta^- \otimes \bar{\omega})$ and by $[b;x,y]$ its image in $E_{[b]}(\xi^0(\omega))$ under the projection $E(\tilde{\xi}(\omega)) \to E(\xi^0(\omega))$. Thus $[b;x,y] = [\theta(b);\tilde{\theta}^-(y),\tilde{\theta}^+(x)]$.

The total space of $\xi_{\mathbb{C}} \otimes \xi^0(\omega)$ has the following description: The fibre over $[b] \in P(m,\nu)$ is the vector space $E_{[b]}(\xi_{\mathbb{C}}) \otimes E_{[b]}(\xi^0(\omega))$. Let $t \in \mathbb{C}, x \in E_b(\xi^+ \otimes \omega), y \in E_b(\eta^- \otimes \bar{\omega})$ so that $[b;t] \in E_{[b]}(\xi_{\mathbb{C}}), [b;x,y] \in E_{[b]}(\xi^0(\omega))$. Then the vector $[b;t] \otimes [b;x,y] \in E_{[b]}(\xi_{\mathbb{C}} \otimes \xi^0(\omega)) = E_{[b]}(\xi_{\mathbb{C}}) \otimes E_{[b]}(\xi^0(\omega))$ will be denoted $[(b;t) \otimes (x,y)]$. Note that

$$[(b;t) \otimes (x,y)] = [(\theta(b);-t) \otimes (\tilde{\theta}^-(y),\tilde{\theta}^+(x))]. \tag{21}$$

Consider the bundle map $\tilde{\lambda} : E(\epsilon_{\mathbb{C}} \otimes \tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ defined as follows: For $b \in \mathbb{S}^m \times \mathbb{C}G(\nu), t \in \mathbb{C}, x \in E_b(\xi^+ \otimes \omega), y \in E_b(\eta^- \otimes \bar{\omega})$,

$$\tilde{\lambda}((b;t) \otimes (b;x,y)) = (b;-tx,ty). \tag{22}$$

Then $\tilde{\lambda}$ is a complex vector bundle isomorphism that covers θ. Moreover,

$$\begin{aligned}
\tilde{\lambda} \circ (\rho \otimes \tilde{\theta})((b;t) \otimes (b;x,y)) &= \tilde{\lambda}((\theta(b);-t) \otimes (\theta(b);\tilde{\theta}^-(y),\tilde{\theta}^+(x))) \\
&= (\theta(b);t\tilde{\theta}^-(y),-t\tilde{\theta}^+(x)) \\
&= \tilde{\theta}(b;-tx,ty) \\
&= \tilde{\theta}(\tilde{\lambda}((b;t) \otimes (b;x,y)))
\end{aligned}$$

Therefore $\tilde{\lambda}$ yields a well-defined bundle isomorphism $\lambda : \xi_{\mathbb{C}} \otimes \xi^0(\omega) \to \xi^0(\omega)$, covering the identity map of $P(m,\nu)$, defined as $[(b;t) \otimes (x,y)] \mapsto [b;-tx,ty]$. This shows that $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$.

(iii) In view of the fact that $\ker(\pi^!)$ equals the torsion subgroup of $K^0(P(m,\nu))$, it suffices to show that $\pi^!([\xi^0(\omega)]+[\xi^0(\bar{\omega})]) = 2^r\pi^!([\hat{\omega}_{\mathbb{C}}])$. Since $\pi^!(\hat{\omega}_{\mathbb{C}}) = ([\omega]+[\bar{\omega}])$ by Lemma 4.9 and since $\pi^!(\xi^0(\omega)) = \tilde{\xi}(\omega)$, we need only to show that $[\tilde{\xi}(\omega)] + [\tilde{\xi}(\bar{\omega})] = 2^r([\omega]+[\bar{\omega}])$. It is clear that $\tilde{\xi}(\omega) \oplus \tilde{\xi}(\bar{\omega}) = (\xi^+ \oplus \eta^-) \otimes (\omega \oplus \bar{\omega}) \cong 2^r(\omega \oplus \bar{\omega})$ since we have a complex vector bundle isomorphism $\xi^+ \oplus \eta^- \cong \xi^+ \oplus \xi^- \cong 2^r\epsilon_{\mathbb{C}}$ on $\mathbb{S}^m$.

(iv) We have

$$\begin{array}{rcl}\pi^!(\xi^0(\omega_1)\otimes\xi^0(\omega_2)) &=& \tilde{\xi}(\omega_1)\otimes\tilde{\xi}(\omega_2)\\ &=& (\tilde{\xi}^+\otimes\omega_1+\eta^-\otimes\bar{\omega}_1)\otimes(\tilde{\xi}^+\otimes\omega_2+\eta^-\otimes\bar{\omega}_2)\\ &=& (\xi^+)^2\otimes(\omega_1\otimes\omega_2)+(\eta^-)^2\otimes(\bar{\omega}_1\otimes\bar{\omega}_2)\\ && +\xi^+\eta^-\cdot(\omega_1\otimes\bar{\omega}_2\oplus\bar{\omega}_1\otimes\omega_2).\end{array}$$

Now by Theorem 2.1 and Remark 2.2, $[\xi^+]^2=2^r[\xi^+]-2^{2r-2},[\eta^-]^2=2^r[\eta^-]-2^{2r-2}$ and $[\xi^+\eta^-]=2^{2r-2}$. Substituting in the above equation and using $\tilde{\xi}(\omega)=\xi^+\otimes\omega\oplus\eta^-\otimes\bar{\omega}$ (with $\omega=\omega_1\otimes\omega_2$) we obtain:

$$\begin{array}{rcl}\pi^!([\xi^0(\omega_1)\otimes\xi^0(\omega_2)]) &=& 2^r([\tilde{\xi}(\omega_1\otimes\omega_2)])-2^{2r-2}([\omega_1\otimes\omega_2]+[\bar{\omega}_1\otimes\bar{\omega}_2])\\ && +2^{2r-2}([\omega_1\otimes\bar{\omega}_2]+[\bar{\omega}_1\otimes\omega_2])\\ &=& 2^r\pi^!([\xi^0(\omega_1\otimes\omega_2)]-2^{2r-2}([\omega_1]-[\bar{\omega}_1])([\omega_2]-[\bar{\omega}_2]).\end{array}$$

Since $\ker(\pi^!:K(P(m,\nu))\to K(\mathbb{S}^m\times\mathbb{C}G(\nu)))$ consists only of torsion elements, our assertion follows. □

Theorem 4.13. *Let $m\geq 1$ and let $r=\lfloor m/2\rfloor$. The homomorphism $\pi^!:K^0(P(m,\nu))\to K^0(\mathbb{S}^m\times\mathbb{C}G(\nu))$ defines a surjective ring homomorphism $\mathcal{K}^0\to\mathcal{K}$, again denoted $\pi^!$, whose kernel equals the torsion ideal $\mathcal{T}^0\subset\mathcal{K}^0$. The quotient group $K^0(P(m,\nu))/\mathcal{K}^0$ is a finite abelian 2-group.*

Proof. We have $\pi^!([\hat{\omega}_\mathbb{C}])=[\omega]+[\bar{\omega}]\in\mathcal{K}$, and, when m is even, $\pi^!([\xi^0(\omega)])=[\tilde{\xi}(\omega)]=[\xi^+][\omega]+[\eta^-][\omega]=[\xi^+]([\omega-\bar{\omega}])+2^r[\bar{\omega}]=\delta_m([\omega]-[\bar{\omega}])+2^{r-1}\pi^!([\hat{\omega}_\mathbb{C}])$. Therefore $\delta_m([\omega]-[\bar{\omega}])$ is in $\pi^!(\mathcal{K}^0)$. This shows that the homomorphism $\pi^!:\mathcal{K}^0\to\mathcal{K}$ is surjective.

Since, by Lemma 4.10, the kernel of $\pi^!:K^0(P(m,\nu))\to K^0(\mathbb{S}^m\times\mathbb{C}G(\nu))$ consists only of torsion elements, it follows that the same is true of $\mathcal{K}^0\to\mathcal{K}$. By the same lemma, $\mathrm{rank}(\mathcal{K})=\mathrm{rank}(K^0(P(m,\nu)))$, and since $\mathcal{K}^0\subset K^0(P(m,\nu))$, it follows from the surjectivity of $\mathcal{K}^0\to\mathcal{K}$ that $\mathrm{rank}(\mathcal{K}^0)=\mathrm{rank}(K^0(P(m,\nu))$. Therefore $K^0(P(m,\nu))/\mathcal{K}^0$ is a torsion group. Since $H^*(P(m,\nu);\mathbb{Z})$ has no odd torsion, it follows (for example by using the Atiyah-Hirzebruch spectral sequence) that $K^0(P(m,\nu))$ has no odd torsion. This proves the last assertion of the theorem. □

Remark 4.14. It is clear that $\mathbb{Z}_{2^r}y$ is contained in the torsion subgroup of $K^0(P(m,\nu))$. We have not been able to determine the torsion part of $K^0(P(m,\nu))$. This seems to us to be a rather difficult, albeit an interesting problem.

Note added on 29th June, 2026: Very recently, we came across the paper of Kavitha and Renukadevi [15], in which the cohomology groups of $P(m,X)$ are computed when X is a complex Grassmannian $\mathbb{C}G_{n,k}$ or a complete flag manifold $\mathbb{C}G(1,1,\ldots,1)$.

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