The Topology of the Normalization of Complex Surface Germs

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Abstract

Let (X, p) be a reduced complex surface germ and let L_X be its link. If (X, p) is normal at p, David Mumford [10] shows that (X, p) is smooth if and only if L_X is simply connected. Moreover, if p is an isolated singular point, L_X is a three dimensional Waldhausen graph manifold. Then, the Plumbing Calculus of Walter Neumann [11] shows that the homeomorphism class of L_X determines a unique plumbing in normal form.

Here, we do not assume that X is normal at p, and so, the singular locus (Σ, p) of (X, p) can be one dimensional. We describe the topology of the singular link L_X and we show that the homeomorphism class of L_X (Theorem 4.1) determines the homeomorphism class of the normalization and consequently the plumbing of the minimal good resolution of (X, p) which provides the dual graph of the minimal good resolution of (X, p).

In Proposition 5.1, we obtain the following generalization of the above quoted theorem of Mumford:

Let $(X, 0)$ be an irreducible surface germ. If the link L_X of $(X, 0)$ is simply connected, then the normalization, $\nu : (X', p') \to (X, 0)$, is a homeomorphism and (X', p') is smooth at p'. In particular, L_X is a topological manifold and the normalization is the good minimal resolution of $(X, 0)$.

In this article, the tools to study an algebraic data (the normalization morphism) are mainly topological. [1]

1 Introduction

Let I be a reduced ideal in $\mathbb{C}\{z_1, \dots, z_n\}$ such that the quotient algebra $A_X = \mathbb{C}\{z_1, \dots, z_n\}/I$ is two-dimensional. The zero locus, at the origin 0 of $\mathbb{C}^n$, of a set of generators of I is an analytic surface germ embedded in $(\mathbb{C}^n, 0)$. Let $(X, 0)$ be its intersection with the compact ball B_ϵ^{2n} of radius a sufficiently small ϵ, centered at the origin in $\mathbb{C}^n$, and let L_X be its intersection with the boundary S_ϵ^{2n-1} of B_ϵ^{2n}. Let Σ be the set of the singular points of $(X, 0)$.

As I is reduced, Σ is empty when $(X, 0)$ is smooth, Σ is equal to the origin when 0 is an isolated singular point and Σ is a curve when the germ has a non-isolated singular locus (in particular $(X, 0)$ can be a reducible germ).

If Σ is a curve, $K_\Sigma = \Sigma \cap S_\epsilon^{2n-1}$ is the disjoint union of r one-dimensional circles (r being the number of irreducible components of Σ) embedded in L_X. We say that K_Σ is the link of Σ. By the conical structure theorem, for a sufficiently small ϵ, $(X, \Sigma, 0)$ is homeomorphic to the cone on the pair (L_X, K_Σ). This conical

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structure theorem, which dates back to K. Brauner [3] (1928), is proved by J. Milnor (Theorem 2.10 in [9]) for germs with isolated singular points, and a general proof is written in [2] (Theorem 9.3.6), by J. Bochnak, M. Coste and M-F. Roy.

On the other hand, thanks to A.Durfee (Proposition 3.5 in [5]), the homeomorphism class of $(X,0)$ depends only on the isomorphism class of the algebra A_X (i.e. is independent of a sufficiently small ϵ and of the choice of the embedding in $(\mathbb{C}^n,0)$). The analytic type of $(X,0)$ is given by the isomorphism class of A_X, and its topological type is given by the homeomorphism class of $(X,0)$.

Definition I *The link of* $(X,0)$ is the homeomorphism class of L_X. *The link of* $(X,\Sigma,0)$ is the homeomorphism class of the pair (L_X,K_Σ). If L_X is not a topological three dimensional manifold, we say that L_X is *topologically singular* .

Remark Let $\nu : (X',p') \to (X,0)$ be the normalization morphism of $(X,0)$. If $(X,0)$ is reducible, let $(\cup_{1\leq i\leq r}X_i,0)$ be its decomposition as a union of irreducible surface germs. Let $\nu_i : (X'_i,p_i) \to (X_i,0)$ be the normalization of the irreducible components of $(X,0)$. The morphisms ν_i induce the normalization morphism on the disjoint union $\coprod_{1\leq i\leq r}(X'_i,p_i)$.

By standard arguments (see A. Durfee [5]), we can associate to (X',p') and $(X,0)$ well defined links $L_{X'}$ and L_X such that $\nu(L_{X'}) = L_X$. Proposition 2.3.12, in [8], proves that L_X is a topological manifold if and only if ν restricted to $L_{X'}$ is a homeomorphism onto L_X. This result has already been stated by I. Luengo and A. Pichon (Proposition 2.1 in [7]). Here, we need a more detailed description of the topological singular locus of L_X. First of all, we will define the branches of Σ which provide the topological singular locus K_{Σ_+} of L_X.

Definition II

1) If $(\Sigma,0)$ is a one-dimensional germ, let σ be an irreducible component of Σ. Let $\sigma'_j, 1\leq j\leq n(\sigma)$, be the $n(\sigma)$ irreducible components of $\nu^{-1}(\sigma)$ and let d_j be the degree of ν restricted to σ'_j. The following number

$$k(\sigma) =: d_1 + \ldots + d_j + \ldots + d_{n(\sigma)}.$$

is *the total degree of* ν *above the curve* σ.

2) Let Σ_+ be the union of the irreducible components σ of Σ such that $k(\sigma) > 1$. In L_X, let K_{Σ_+} be the link of Σ_+. We choose a compact regular neighbourhood $N(K_{\Sigma_+})$ of K_{Σ_+}. Let $E(K_{\Sigma_+})$ be the closure of $L_X \setminus N(K_{\Sigma_+})$. By definition $E(K_{\Sigma_+})$ *is the (compact) exterior of* K_{Σ_+}.

For each irreducible component σ of Σ_+, we prove, by Lemma 3.2, that the integers $n(\sigma)$ and $d_j, 1\leq j\leq n(\sigma)$, only depend on the topology of L_X. In [8], Sections 2.3.3 and 2.3.4, one can find a description of the topology of $N(K_{\Sigma_+})$ which implies the following lemma 3.1. In order to be be self-contained, we begin Section 3 with a quick proof of it.

Lemma (3.1)
1. The restriction of ν to $\nu^{-1}(E(K_{\Sigma_+}))$ is an homeomorphism and $(L_X \setminus K_{\Sigma_+})$ is a topological manifold.
2. The link K_{Σ_+} is the topological singular locus of L_X.
3. The homeomorphism class of L_X determines the homeomorphism class of $N(K_{\Sigma_+})$ and $E(K_{\Sigma_+})$.
4. The number of connected components of $E(K_{\Sigma_+})$ is equal to the number of irreducible components of $(X,0)$.

Let $\nu : (X',p') \to (X,0)$ be the normalization morphism of $(X,0)$. Let Σ'_+ be the curve $\nu^{-1}(\Sigma_+)$.The definition of Σ_+ implies that ν restricted to $L_{X'}$ is not bijective exactly on the link $K_{\Sigma'_+} = L_{X'} \cap \Sigma'_+$.

Theorem (4.1) Let $(X, 0)$ be a reduced surface germ. The homeomorphism class of L_X determines the homeomorphism class of the pair of links $(L_{X'}, K_{\Sigma'_+})$.

Then, the plumbing calculus of W. Neumann [11] implies the following corollary.

Corollary 1.1 *The homeomorphism class of L_X determines the dual graph of the minimal good resolution of the germ $(X, 0)$ and the dual graph of the minimal good resolution of the pair (X', Σ'_+).*

Let K_σ be the link associated to an irreducible component σ of Σ_+. Point 2 of Lemma 3.2 proves that a compact regular neighbourhood $N(K_\sigma)$ of K_σ in L_X is a singular pinched solid torus as defined below (Definition III). Such a description of $N(K_\sigma)$ is stated in Section 2 of [7] and detailed in Section 2.3.4 of [8]. Here, we need to refine the description of $N(K_\sigma)$. In particular, we introduce, for the first time, the definition of meridian curves (see 2.2 and 2.5) on each connected component of the boundary of $N(K_\sigma)$. It is the key point in the proof of Theorem 4.1 which can be summarized as follows.

If L_X is not a topological manifold, let $\mathcal{K}$ be its topological singular locus. By Statement 2 of Lemma 3.1, we know that $\mathcal{K} = K_{\Sigma_+}$ is a disjoint union of circles. Let $N(\mathcal{K})$ be a regular compact neighbourhood of $\mathcal{K}$ and let $E(\mathcal{K})$ be the closure of $(L_X \setminus N(\mathcal{K}))$. Let τ be one of the n connected components of the boundary of $E(\mathcal{K})$. By 2.2, τ is given with a meridian curve m_τ. We glue a solid torus T_τ on $E(\mathcal{K})$ with the Dehn filling construction associated to m_τ, which is detailed in the proof of 4.1. Let L be the result of such Dehn fillings performed on each connected component of the boundary of $E(\mathcal{K})$. In the proof of 4.1, we obtain a continuous map $\nu_{top} : L \to L_X$ such that ν_{top} restricted to $E(\mathcal{K})$ is equal to the identity and $\nu_{top}^{-1}(\mathcal{K})$ is the disjoint union K_{top} of the n cores of the n solid tori glued to $E(\mathcal{K})$. The topology of the pair (L, K_{top}) only depends on the topology of L_X.
On the other hand, $\tau' = \nu^{-1}(\tau)$ is the boundary of a solid torus T' in $L_{X'}$. Lemma 3.2 and Lemma 2.1 imply that $\nu^{-1}(m_\tau)$ is a meridian curve of T'. By unicity of the Dehn filling construction, there exists an orientation preserving homeomorphism $f : L_{X'} \to L$ such that $f(K_{\Sigma'_+}) = K_{top}$ and $\nu = \nu_{top} \circ f$.

Definition III

1. A *d-curling* $\mathcal{C}_d$ is a topological space homeomorphic to the following quotient of a solid torus $S \times D$:
$$\mathcal{C}_d = S \times D/(u, 0) \sim (u', 0) \Leftrightarrow u^d = (u')^d.$$
The associated quotient morphism $q : (S \times D) \to \mathcal{C}_d$ is a *d-curling morphism.* By definition, $l_0 = q(S \times \{0\})$ is the *core of* $\mathcal{C}_d$. Moreover, $\mathcal{C}_d$ is given with the following orientations: the oriented circle S and the oriented disc D induce an orientation on the circles $l_z = q(S \times \{z\}),\ z \in D$, and on the topological discs $q(\{u\} \times D),\ u \in S$.

2. Let $q_j : (S \times D) \to \mathcal{C}_{d_j}, 1 \leq j \leq n$, be n disjoint d_j-curling morphisms. Let $l_{0_j} = q_j(S \times \{0\})$ be the core of $\mathcal{C}_{d_j}$. Let $\gamma_j : S \to l_{0_j}$ be n orientation preserving diffeomorphisms. A *singular pinched solid torus of sheets* $\mathcal{C}_{d_j}, 1 \leq j \leq n$, is a topological space orientation preserving homeomorphic to the quotient of the disjoint union of the d_j-curlings, $(\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})$, by the identification of their cores. More precisely, for all $x \in S$, the equivalence relation is defined by: $\gamma_j(x) \sim \gamma_l(x), 1 \leq j \leq n, 1 \leq l \leq n$. The induced quotient morphism, $\Gamma : (\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j}) \to ((\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/\sim)$, is the *identification morphism.* By definition, $\tilde{l}_0 = \Gamma(l_{0_j})$ does not depend on $j, 1 \leq j \leq n$, it is the *core* of the singular pinched solid torus $((\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/\sim)$.

Section 2 contains also the presentation of the following example which is a typical illustration of a d-curling.

Example I (detailed in 2.4) Let $X = \{(x,y,z) \in \mathbb{C}^3 \textit{ where } z^d - xy^d = 0 \textit{ and } d > 1\}$. The normalization of $(X,0)$ is smooth, i.e. the morphism $\nu : (\mathbb{C}^2,0) \to (X,0)$ defined by $(u,v) \mapsto (u^d, v, uv)$ is a normalization morphism. The singular locus of $(X,0)$ is the line $l_x = \{(x,0,0) \in \mathbb{C}^3, x \in \mathbb{C}\}$. We choose the union of the two solid tori $T = \{(u,v) \in \mathbb{C}^2,\ |u| = 1,\ |v| \leq 1\}$ and $T' = \{(u,v) \in \mathbb{C}^2,\ |u| \leq 1,\ |v| = 1\}$ to represent the link of the normalization of $(X,0)$. We have $n(l_x) = 1$ because $\nu^{-1}(l_x)$ is the line $\{(u,0) \in \mathbb{C}^2, u \in \mathbb{C}\}$ and $d_1 = d$. Moreover $N(K_{l_x}) = \nu(T)$ is a tubular neighbourhood of K_{l_x} and ν restricted to T is a d-curling morphism. In this example L_X is not simply connected. In fact: $H_1(L_X, \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$.

In [10], D. Mumford proves that a normal surface germ which has a simply connected link is a smooth germ of surface. However, there exist surface germs with one dimensional singular locus and simply connected links. Obvious examples are obtained as follows.

Let $f(x,y)$ be an irreducible element of $\mathbb{C}\{x,y\}$ of multiplicity $m > 1$ at 0.
Let $Z = \{(x,y,z) \in \mathbb{C}^3 \textit{ such that } f(x,y) = 0\}$. The singular locus of $(Z,0)$ is the line $l_z = \{(0,0,z),\ z \in \mathbb{C}\}$. A normalization morphism $\nu : (\mathbb{C}^2,0) \to (Z,0)$ can be given by a Puiseux expansion of $f(x,y)$. So, the link L_Z is the sphere S^3. Lê 's conjecture states that this family is the only family of singular irreducible surface germs with one dimensional singular locus and links homeomorphic to the 3-dimensional sphere (see [6] for detailed statements, see [7] and [1] for partial results).

In Section 5, we prove the following proposition which is a kind of generalization of Mumford's theorem for non-normal surface germs:

Proposition (5.1) Let $(X,0)$ be an irreducible surface germ. If the link L_X of $(X,0)$ is simply connected, then the normalization $\nu : (X',p') \to (X,0)$ is a homeomorphism and (X',p') is smooth at p'. In particular, L_X is a topological manifold and the normalization is the good minimal resolution of $(X,0)$.

Just below, we present an example of a reducible germ $(Y,0)$ with a simply connected and topologically singular link. It implies that the hypothesis of Proposition 5.1 is necessary.

Example II There exist reducible surface germs with simply connected link for which the normalization is not a homeomorphism. For example let $Y = \{(x,y,z) \in \mathbb{C}^3 \textit{ where } xy = 0\}$.
In $\mathbb{C}^3$, $(Y,0)$ is the union of two planes and $\Sigma = \Sigma_+ = l_z = \{(0,0,z),\ z \in \mathbb{C}\}$. Moreover, the normalization is the obvious quotient $\nu : (\mathbb{C}^2,0) \coprod (\mathbb{C}^2,0) \to (Y,0)$ given by $\nu(x,z) = (x,o,z)$ and $\nu(y,z) = (0,y,z)$. Let $K_i \subset S^3_i, i = 1,2$, be two copies of S^3 given with a trivial knot K_i. The link L_Y of $(Y,0)$ is the quotient of the disjoint union $S^3_1 \coprod S^3_2$, by the identification point by point of the trivial knots K_1 and K_2. So L_Y is simply connected. On the other hand, a regular neighbourhood, $N(K_{l_z})$, of the link of the singular locus l_z, is a singular pinched solid torus of sheets two solid tori as defined in Definition III.

Conventions and notations

The boundary of a pseudo-manifold W will be denoted by $b(W)$.

A *disc* (resp. an *open disc*) will always be an oriented topological manifold orientation preserving homeomorphic to $D = \{z \in \mathbb{C}, |z| \leq 1\}$ (resp. to $\dot{D} = \{z \in \mathbb{C}, |z| < 1\}$). A *circle* will always be an oriented topological manifold orientation preserving homeomorphic to $S = \{z \in \mathbb{C}, |z| = 1\}$.

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2 The topology of d-curlings

In this section, we study in details the topological properties of d-curlings (defined Section 1, definition III) because they are the key to make the proofs of the two original statements of this paper self-contained. In particular, we need well defined, up to isotopy, meridian curves on the boundary of d-curlings since the proof of Theorem 4.1 is based on Dehn fillings associated to these meridian curves. In this section, we suppose that $d > 1$ to avoid the trivial case $d = 1$.

Definition IV

1. A *d-pinched disc*, $d(D)$, is orientation preserving homeomorphic to the quotient of the disjoint union of d oriented and ordered discs $D_i, 1 \leq i \leq d$, with origin 0_i, by the relation $0_i \sim 0_j$ for all $i, 1 \leq i \leq d$, and $j, 1 \leq j \leq d$. So, all the origins $0_i,\ 1 \leq i \leq d$, are identified in a unique point $\tilde{0}$. By definition $\tilde{0}$ is the *origin of* $d(D)$. The class $\tilde{D}_i$ of each disc D_i in $d(D)$ is *an irreducible component* of $d(D)$.

2. Let $\mathcal{C}_d$ be a d-curling given with a d-curling morphism $q : (S \times D) \to \mathcal{C}_d$. An oriented simple closed curve m, on the boundary of $\mathcal{C}_d$, is a *meridian curve of* $\mathcal{C}_d$ if the class of m, in $\pi_1(b(\mathcal{C}_d))$, is equal to the class of the oriented boundary m_u of $q(\{u\} \times D), u \in S$. Let m be a meridian curve of $\mathcal{C}_d$. An oriented simple closed curve l on the boundary of $\mathcal{C}_d$ is a *parallel curve of* $\mathcal{C}_d$ if $m \cap l = +1$. The *core* of $\mathcal{C}_d$ is $l_0 = q(S \times \{0\})$.

We have to detail the following lemma because it is a key point in the proof of Theorem (4.1).

Lemma 2.1 *Let $\mathcal{C}_d$ be a d-curling given with a d-curling morphism $q : (S \times D) \to \mathcal{C}_d$. Let m_u be oriented boundary of $q(\{u\} \times D), u \in S$. The kernel of the homomorphism $i_1 : \pi_1(b(\mathcal{C}_d)) \to \pi_1(\mathcal{C}_d)$, induced by the inclusion $b(\mathcal{C}_d) \subset \mathcal{C}_d$, is infinite cyclic generated by the class m^1 of the oriented simple closed curve m_u. Let $l_v = q(S \times \{v\}),\ v \in S$ and let l^1 be its homotopy class. Then, (m^1, l^1) is a basis of $\pi_1(b(\mathcal{C}_d))$ and the class l_0^1 of the core l_0 is a generator of $\pi_1(\mathcal{C}_d)$. Moreover $i_1(l^1) = d.l_0^1$ and the cokernel of i_1 is isomorphic to $\mathbb{Z}/d.\mathbb{Z}$..*

Proof
Let us choose $u \in S$. The kernel of the homomorphism $j_1 : \pi_1(S \times S) \to \pi_1(S \times D)$ induced by the inclusion $(S \times S) \subset (S \times D)$ is infinite cyclic generated by the class of the closed simple curve

$$m = (\{u\} \times S) = b(\{u\} \times D).$$

Moreover, m is oriented as the boundary of the oriented disc $(\{u\} \times D)$. This defines a unique generator $\tilde{m}$ of the kernel of j_1. So, any closed simple curve on the boundary of $(S \times D)$ which generates the kernel of j_1, can be oriented to be isotopic to m. By definition it is a meridian curve of $(S \times D)$. The d-curling $\mathcal{C}_d$ is defined by the d-curling morphism $q : (S \times D) \to \mathcal{C}_d$. But, q restricted to $(S \times S)$ is the identity. So, $m_u \cap l_v = +1$ and (m^1, l^1) is a basis of $\pi_1(b(\mathcal{C}_d))$. Moreover $q(\{u\} \times D)$ is a topological disc in $\mathcal{C}_d$.
So, the kernel of the homomorphism $i_1 : \pi_1(b(\mathcal{C}_d)) \to \pi_1(\mathcal{C}_d)$, induced by the inclusion $b(\mathcal{C}_d) \subset \mathcal{C}_d$, is infinite cyclic generated by the class m^1 of the oriented simple closed curve $m_u = q(m)$. As for the solid torus, any simple closed curve in $b(\mathcal{C}_d)$ which generates the kernel of i_1 can be oriented to be isotopic to $q(m)$.

But, l_0 is a deformation retract of $\mathcal{C}_d$ and d is the degree of such a retraction restricted to l_v. It implies that $i_1(l^1) = d.l_0^1$.

End of proof

Corollary 2.2 *1. On the boundary of a d-curling, there exists a unique, up to isotopy, oriented simple meridian curve which generates the kernel of* $i_1 : \pi_1(b(\mathcal{C}_d)) \to \pi_1(\mathcal{C}_d)$.

2. *For any* $v \in S$, $l_v = q(S \times \{v\})$ *is a parallel curve of* $\mathcal{C}_d$. *But, contrary to meridians, parallels are not unique up to isotopy.*

3. *The integer d is a topological invariant of a d-curling.*

We gather together the topological properties of a d-curling $\mathcal{C}_d$ in the following Lemma.

Lemma 2.3 *1. Let* $q : (S \times D) \to \mathcal{C}_d$ *be a d-curling morphism. Let* $\pi_d : (S \times D) \to (S \times D)$ *be the ramified covering of degree d defined, for all* $(u, z) \in (S \times D)$, *by* $\pi_d(u, z) = (u^d, z)$. *But,* π_d *induces a unique topological morphism* $\bar{\pi}_d : \mathcal{C}_d \to (S \times D)$ *such that* $\pi_d = \bar{\pi}_d \circ q$. *By construction, for all* $t \in S$, $\mathcal{D}_t = \bar{\pi}_d^{-1}(\{t\} \times D)$ *is a d-pinched disc. If* $u^d = t$, *the origin of* $\mathcal{D}_t$ *is* $q(u, 0)$.

2. *The d-curling* $\mathcal{C}_d$ *is homeomorphic to the mapping torus of an orientation preserving homeomorphism of the d-pinched disc* $\mathcal{D}_t$, *which induces a cyclic permutation of the d irreducible components of* $\mathcal{D}_t$.

3. *The mapping torus of an orientation preserving homeomorphism of a d-pinched disc, which induces a cyclic permutation of its d irreducible components, is always orientation preserving homeomorphic to a d-curling.*

Proof

1. If $u^d = t$, $\mathcal{D}_t$ is the union of $q(\{\sigma u\} \times D)$ where $\sigma^d = 1$.

2. The circles $(S \times \{z\})$, $z \in D$, equip the solid torus $T = (S \times D)$ with a trivial fibration in oriented circles. If we choose $t \in S$, the first return map along these circles induces the identity on the disc $(\{t\} \times D)$. Using $\bar{\pi}_d^{-1}$, we can lift these fibration by circles on $\mathcal{C}_d$. Let h be the automorphism of $\mathcal{D}_t$ defined by the first return map along these circles. So, h is an orientation preserving homeomorphism of $\mathcal{D}_t$ which induces a cyclic permutation of the d irreducible components of $\mathcal{D}_t$. Obviously h keeps the origin of $\mathcal{D}_t$ fixed.

3. When $d > 1$, a homeomorphism of a d-pinched disc keeps always the origin fixed. There is, up to isotopy, a unique orientation preserving homeomorphism of a d-pinched disc, which induces a cyclic permutation of its d irreducible components. By point 2, a d-curling is such a mapping torus.

End of proof

Let us detail Example I (Section1). Let $X = \{(x, y, z) \in \mathbb{C}^3 \text{ where } z^d - xy^d = 0\}$. The normalization of $(X, 0)$ is smooth i.e. $\nu : (\mathbb{C}^2, 0) \to (X, 0)$ is given by $(u, v) \mapsto (u^d, v, uv)$. Here $B = \nu(D \times D)$ is a good semi-analytic neighbourhood of $(X, 0)$ in the sense of A. Durfee [5]. So, $L_X = X \cap \nu((S \times D) \cup (D \times S))$ can represent the link of $(X, 0)$. As detailed in the following lemma, the link L_X is a basic example of a link which contains a d-curling.

Lemma 2.4 *Let* $T = \{(u, v) \in (S \times D) \subset \mathbb{C}^2\}$. *Let* $\pi_x : \nu(T) \to S$ *be the projection* $(x, y, z) \mapsto x$ *restricted to* $\nu(T)$. *Here the singular locus of* $(X, 0)$ *is the line* $l_x = \{(x, 0, 0) \in \mathbb{C}^3, x \in \mathbb{C}\}$.
Then, $N(K_{l_x}) = L_X \cap (\pi_x^{-1}(S)) = \nu(T)$ *is a d-curling. The link* L_X *is the union of* $N(K_{l_x})$ *with the solid torus* $\nu(D \times S)$. *Moreover,* $H_1(L_X, \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}/d.\mathbb{Z}$.

Proof
Let $q : T \to \mathcal{C}_d$ be the d-curling morphism (definition III, Section 1). There exists a well defined homeomorphism $f : \mathcal{C}_d \to N(K_{l_x})$ which satisfies $f(q(u, v)) = (u^d, v, uv)$. So, $N(K_{l_x})$ is a d-curling and K_{l_x} is its core. Moreover, f restricted to the core l_0 of $\mathcal{C}_d$ is a homeomorphism onto K_{l_x}.

Let us take $s = e^{2i\pi/d}$. The intersection $\mathcal{D}_1 = N(K_{l_x}) \cap \{x = 1\} = \{(1, y, z) \in \mathbb{C}^3 \ where \ z^d - y^d = 0\}$ is a plane curve germ at $(1, 0, 0)$ with d irreducible components given by $\nu(s^k \times D), \ 1 \leq k \leq d$.
On the torus $\tau = b(N(K_{l_x})) = \nu(S \times S)$, $m = \nu(\{1\} \times S)$ is a meridian curve of $N(K_{l_x})$ and $l_1 = \nu(S \times \{1\})$ is a parallel. Moreover, $N(K_{l_x})$ is saturated by the foliation in oriented circles $l_v = \nu(S \times \{v\})$ which cuts $\mathcal{D}_1$ transversally at the d points $\{(1, v, s^k v), \ 1 \leq k \leq d\}$, when $v \neq 0$, and at $(1, 0, 0)$ when $v = 0$. So, $N(K_{l_x})$ is the mapping torus of the homeomorphism defined on the d-pinched disc $\mathcal{D}_1$ by the first return map along the circles l_v.

To compute $H_1(L_X, \mathbb{Z})$, we use the Mayer-Vietoris sequence associated to the decomposition of L_X as the union $N(K_{l_x}) \cup \nu(D \times S)$. The homology classes $\bar{m}$ and $\bar{l}_1$ of the curves m and l_1 form a basis of $H_1(b(N(K_{l_x})), \mathbb{Z})$. But, $\bar{m}$ is a generator of $H_1(\nu(D \times S), \mathbb{Z})$ and is equal to 0 in $H_1(N(K_{l_x}), \mathbb{Z})$. As the class $\bar{l}_0$ of K_{l_x} is a generator of $H_1(N(K_{l_x}), \mathbb{Z})$ and as $\bar{l}_1 = d.\bar{l}_0$, the Mayer-Vietoris sequence has the following shape:

$$... \xrightarrow{\delta_2} H_1(b(N(K_{l_x})), \mathbb{Z}) \xrightarrow{\Delta_1} \mathbb{Z}.\bar{m} \oplus \mathbb{Z}.\bar{l}_0 \xrightarrow{i_1} H_1(L_X, \mathbb{Z}) \to 0$$

where $\Delta_1(\bar{l}_1) = (0, d.\bar{l}_0)$ and $\Delta_1(\bar{m}) = (\bar{m}, 0)$. So, $H_1(L_X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/d.\mathbb{Z}$.

End of proof

Notation : Let $(\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/ \sim)$ be a singular pinched solid torus of sheets $\mathcal{C}_{d_j}, 1 \leq j \leq n$, where $k = d_1 + \ldots + d_j + \ldots + d_n > 1$, and core $\tilde{l}_0 = \Gamma(l_{0_j})$, as defined in Section 1 (see 2. in Definition III). As the orientation preserving homeomorphism class of $(\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/ \sim)$ only depends on the integers n and $d_j, 1 \leq j \leq n$, we take the following notation:

$$T(n; d_j, 1 \leq j \leq n) = (\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/ \sim)$$

Lemma 2.5 *Let $T(n; d_j, 1 \leq j \leq n)$ be a singular pinched solid torus of sheets $\mathcal{C}_{d_j}, 1 \leq j \leq n$, and core $\tilde{l}_0$. As $k = d_1 + \ldots d_j \ldots + d_n > 1$, the core $\tilde{l}_0$ is the topological singular locus of $T(n; d_j, 1 \leq j \leq n)$. Moreover, each sheet $\mathcal{C}_{d_j}$ of $T(n; d_j, 1 \leq j \leq n)$ is the closure of a connected component of $(T(n; d_j, 1 \leq j \leq n) \setminus \tilde{l}_0)$. On the boundary of each sheet $\mathcal{C}_{d_j}$ we have a well defined meridian curve m_j.*

Proof
By definition $(T(n; d_j, 1 \leq j \leq n) \setminus \tilde{l}_0)$ is a three dimensional manifold. A neighbourhood of $x \in \tilde{l}_0$ is homeomorphic to trivial product of a k-pinched disc by an interval. As $k > 1$, $T(n; d_j, 1 \leq j \leq n)$ is not a three dimensional manifold around $x \in \tilde{l}_0$. The definition of the identification morphism

$$\Gamma : (\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j}) \to ((\coprod_{1 \leq j \leq n} \mathcal{C}_{d_j})/ \sim),$$

implies that each sheet of $T(n; d_j, 1 \leq j \leq n)$ is the closure of a connected component of $(T(n; d_j, 1 \leq j \leq n) \setminus \tilde{l}_0)$. By construction the boundary of $T(n; d_j, 1 \leq j \leq n)$ is the disjoint union of the n tori, each of them being the boundary τ_j of a sheet $\mathcal{C}_{d_j}$. By the previous corollary 2.2, there exists a well defined meridian curve m_j on τ_j.

End of proof

3 The topology of the normalization

Let $(X,0)$ be a reduced surface germ, let $(\Sigma,0)$ be its singular locus and let $\nu : (X',p') \to (X,p)$ be its normalization. As in Definition II (Section 1), if σ is an irreducible component of Σ, let $\{\sigma'_j, 1 \leq j \leq n(\sigma)\}$ be the set of the the $n(\sigma)$ irreducible components of $\nu^{-1}(\sigma)$, and let d_j be the degree of ν restricted to σ'_j. Moreover, let $k(\sigma) =: d_1 + \ldots + d_j + \ldots + d_{n(\sigma)}$ be the total degree of ν above σ.
Let Σ_+ be the union of the irreducible components σ of Σ such that $k(\sigma) > 1$. In L_X, let K_{Σ_+} be the link of Σ_+. We choose a compact regular neighbourhood $N(K_{\Sigma_+})$ of K_{Σ_+}. Let $E(K_{\Sigma_+})$ be the closure of $L_X \setminus N(K_{\Sigma_+})$. By definition $E(K_{\Sigma_+})$ is the (compact) exterior of K_{Σ_+}.

Lemma 3.1 *1. The restriction of ν to $\nu^{-1}(E(K_{\Sigma_+}))$ is an homeomorphism and $(L_X \setminus K_{\Sigma_+})$ is a topological manifold.*
2. The link K_{Σ_+} is the topological singular locus of L_X.
3. The homeomorphism class of L_X determines the homeomorphism class of $N(K_{\Sigma_+})$ and $E(K_{\Sigma_+})$.
4. The number of connected components of $E(K_{\Sigma_+})$ is equal to the number of irreducible components of $(X,0)$.

Proof
If $(X,0)$ has an isolated singular point at the origin, the normalization is bijective and as the links L_X and $L_{X'}$ are compact, ν restricted to $L_{X'}$ is a homeomorphism.

If Σ is one dimensional, let K_Σ be the link of Σ. Let σ be an irreducible component of Σ and let $N(K_\sigma)$ be the connected component of $N(K_\Sigma)$ which contains the link K_σ.

When $k(\sigma) = 1$, ν restricted to $\nu^{-1}(N(K_\sigma))$ is a bijection. So, the restriction of ν to $\nu^{-1}(E(K_{\Sigma_+}))$ is an homeomorphism. Moreover ν restricted to $\nu^{-1}(X \setminus \Sigma)$ is an analytic isomorphism. So, $(L_X \setminus K_{\Sigma_+})$ is a topological manifold. This ends the proof of Statement 1.

If $k(\sigma) > 1$, ν restricted to $\nu^{-1}(K_\sigma)$ is not injective. Let p be a point of K_σ. The number of the irreducible components σ'_j of $\nu^{-1}(\sigma)$ is denoted $n(\sigma)$. So, $\nu^{-1}(\sigma) = \cup_{1\leq j\leq n(\sigma)}\sigma'_j$. Let d_j be the degree of ν restricted to σ'_j. The intersection $\nu^{-1}(p) \cap \sigma'_j$ has d_j points $\{p_{i(j)}, 1 \leq i \leq d_j\}$. As (X',p') is normal, $(X' \setminus p')$ is smooth and $(\sigma'_j \setminus p')$ is a smooth curve germ at any point $z_j \in (\sigma'_j \setminus p')$. In $(X' \setminus p')$, we can choose at the points $p_{i(j)}$, a smooth germ of curve $(\gamma_{i(j)}, p_{i(j)})$ which cuts σ'_j transversally at $p_{i(j)}$ and such that $D'_{i(j)} = \nu^{-1}(N(K_\sigma)) \cap \gamma_{i(j)}$ is a disc centered at $p_{i(j)}$. Let $D_{i(j)}$ be $\nu(D'_{i(j)})$. By construction p is the common center of the topological discs $D_{i(j)}$. So, $(\cup_{1\leq j\leq n(\sigma)}(\cup_{1\leq i\leq d_j} D_{i(j)}))$ is a $k(\sigma)$-pinched disc centered at p. As $k(\sigma) > 1$, L_X is not a topological manifold at p. This ends the proof of Statement 2. Statements 1 and 2 imply that (K_{Σ_+}) is the set of the topologically singular points of L_X. It implies 3.

The number r of irreducible components of $(X,0)$ is equal to the number of connected components of $L_{X'}$. But, $L_{X'}$ and $\nu^{-1}(E(K_{\Sigma_+}))$ have the same number of connected components since $(L_{X'} \setminus \nu^{-1}(E(K_{\Sigma_+})))$ is a regular neighbourhood of the differential link $\nu^{-1}(K_{\Sigma_+})$. Statement 1 implies that r is also the number of connected components of $E(K_{\Sigma_+})$. This proves 4.

End of proof

Lemma 3.2 *Let σ be an irreducible component of Σ_+, let K_σ be the link of σ in L_X. We choose, in L_X, a compact regular neighbourhood $N(K_\sigma)$ of K_σ. The link K_σ is a deformation retract of $N(K_\sigma)$. If l_σ is the homotopy class of K_σ in $\pi_1(N(K_\sigma))$, then $\pi_1(N(K_\sigma) = \mathbb{Z}.l_\sigma$. We have:*

1. The tubular neighbourhood $\nu^{-1}(N(K_\sigma))$ of $\nu^{-1}(K_\sigma)$ is the disjoint union of $n(\sigma)$ solid tori $T'_j, 1 \leq j \leq n(\sigma)$, and the boundary of $N(K_\sigma)$ is the disjoint union of $n(\sigma)$ tori.

2. Let T'_j be one of the $n(\sigma)$ connected components of $\nu^{-1}(N(K_\sigma))$. Then, $N(K_\sigma)$ is homeomorphic to a singular pinched torus $T(n(\sigma); d_j, 1 \leq j \leq n(\sigma))$ wich has $n(\sigma)$ sheets equal to $\nu(T'_j) = C_j$. In particular, $\nu(T'_j) = C_j$ is a d_j-curling.

3. On each connected component τ_j of the boundary of $N(K_\sigma)$, the homeomorphism class of $N(K_\sigma)$ determines a unique (up to isotopy) meridian curves m_j. If l_j is a parallel curve on τ_j the homotopy class of l_j in $\pi_1(N(K_\sigma))$ is equal to $d_j.K^1_\sigma$ where K^1_σ is the homotopy class of K_σ.

Proof
The link $L_{X'}$ of the normalization $\nu : (X', p') \to (X, p)$ of $(X, 0)$ is a three dimensional Waldhausen graph manifold. Let σ is an irreducible component of Σ_+. Let σ'_j be one of the $n(\sigma)$ irreducible components of $\nu^{-1}(\sigma)$, and let d_j be the degree of ν restricted to σ'_j.

In $L_{X'}$, $\nu^{-1}(K_\sigma)$ is a differentiable one dimensional link with $n(\sigma)$ connected components $K_{\sigma'_j}, 1 \leq j \leq n(\sigma)$. But, $\nu^{-1}(N(K_\sigma))$ is a regular compact neighbourhood of the link $(\coprod_{1\leq j\leq n(\sigma)} K_{\sigma'_j})$. So, $\nu^{-1}(N(K_\sigma))$ is the disjoint union of $n(\sigma)$ solid tori $(\coprod_{1\leq j\leq n(\sigma)} T'_j)$. Moreover, let τ'_j be the boundary of T'_j. As ν restricted to the boundary of $\nu^{-1}(N(K_\sigma))$ is a homeomorphism, the boundary of $N(K_\sigma)$ is the disjoint union of the $n(\sigma)$ tori $\tau_j = \nu(\tau'_j)$. Statement 1 is proved.

We consider $C_j = \nu(T'_j)$. Let $\nu_j : T'_j \to C_j$ be ν restricted to T'_j. Moreover, $K_{\sigma'_j}$ is the core of the solid torus T'_j. As ν_j restricted to $T'_j \setminus K_{\sigma'_j}$ is an orientation preserving homeomorphism and as ν_j restricted to $K_{\sigma'_j}$ has degree d_j, ν_j is a d_j-curling morphism as defined Section 1 (1. of Definition III). But, for all $(j, l), 1 \leq j \leq n, 1 \leq l \leq n$, we have $\nu_j(K_{\sigma'_j}) = \nu_l(K_{\sigma'_l})$. This equality shows that ν restricted to $\nu^{-1}(N(K_\sigma))$ is an identification morphism as defined Section 1 (2. of Definition III). So, $N(K_\sigma)$ is homeomorphic to a singular pinched torus $T(n(\sigma); d_j, 1 \leq j \leq n(\sigma))$ wich has $n(\sigma)$ sheets homeomorphic to the d_j-curlings $\nu(T'_j) = C_j, 1 \leq j \leq n(\sigma)$. Statement 2 is proved.

By Statement 2, for all $j, 1 \leq j \leq n(\sigma)$, $\nu(T'_j) = C_j$ is a d_j-curling of core K_σ. By 2.2, the boundary τ_j of C_j is given with a well defined oriented meridian curve m_j and a chosen oriented parallel curve l_j. Let us denote by K^1_σ, m^1_j and l^1_j the homotopy classes of K_σ, m_j and l_j in $\pi_1(C_j)$. Moreover, as K_σ is a deformation retract of C_j, we have $\pi_1(C_j) = \mathbb{Z}.K^1_\sigma$.
By Lemma 2.1, the kernel of the homomorphism $i_1 : \pi_1(\tau_j) \to \pi_1(C_j)$, induced by the inclusion, is infinite cyclic generated by m^1_j and, in $\pi_1(C_j)$, $l^1_j = d_j.K^1_\sigma$. By Statement 2, $N(K_\sigma)$ is a singular pinched torus of sheets $C_j, 1 \leq j \leq n(\sigma)$, and core K_σ. Then, we can retract $N(K_\sigma)$ by deformation onto its core K_σ. By Lemma 2.5, C_j is the closure in $N(K_\sigma)$ of a connected component of $(N(K_\sigma) \setminus K_\sigma)$ and τ_j is a connected component of the boundary of $N(K_\sigma)$. So, the well defined meridian curve m_j and the chosen parallel curve l_j are meridian and parallel curves on each connected component τ_j of the boundary of $N(K_\sigma)$.

End of proof

4 The proof of the main theorem

Theorem 4.1 *Let $(X,0)$ be a reduced surface germ. Let $\nu : (X',p') \to (X,p)$ be the normalization morphism of $(X,0)$. The homeomorphism class of L_X determines the homeomorphism class of the pair of links $(L_{X'}, K_{\Sigma'_+})$.*

Proof
If L_X is a topological manifold, Statements 1 and 2 of Lemma 3.1 state that K_{Σ_+} is empty. So, $E(K_{\Sigma_+}) = L_X$ and ν is a homeomorphism. When L_X is a topological manifold the theorem is trivial.

If L_X is not a topological manifold, let $\mathcal{K}$ be the set of its singular points. By Statement 2 of Lemma 3.1, we know that $\mathcal{K} = K_{\Sigma_+}$ is a disjoint union of circles. Let $N(\mathcal{K})$ be a regular compact neighbourhood of $\mathcal{K}$. By definition, the exterior $E(\mathcal{K})$ of $\mathcal{K}$ is the closure of $(L_X \setminus N(\mathcal{K}))$.

Let K be a connected component of $\mathcal{K}$ and let $N(K)$ be the connected component of $N(\mathcal{K})$ which contains K. There exists an irreducible component σ of Σ_+ such that $K = K_\sigma$. By Lemma 3.2, $N(K)$ is homeomorphic to a singular pinched torus and K is its core. The number n of sheets of $N(K)$ depends only on the topology of $N(K)$. So, n is equal to $n(\sigma)$. As, a sheet C_j is the closure, in $N(K)$, of a connected component of $(N(K) \setminus K)$, the topology of the δ_j-curling C_j only depends on the topology of $N(K)$. By 2.2 we have $\delta_j = d_j$. So, $N(K)$ is homeomorphic to the singular pinched torus $T(n(\sigma); d_j, 1 \leq j \leq n(\sigma))$ wich has $n(\sigma)$ sheets. By Lemma 2.5, on each connected component τ_j of the boundary of $N(K)$, we have a well defined meridian curve m_j. It is the key point of this proof.
As the boundary of $E(\mathcal{K})$ is equal to the boundary of $N(\mathcal{K})$, the well defined family of meridian curves, on the boundary of $N(\mathcal{K})$, gives an oriented essential simple closed curve on each connected component of the boundary of $E(\mathcal{K})$. This allows us to perform Dehn fillings. As justified below, to take $E(\mathcal{K})$ and to perform, on each connected component τ_j of its boundary, a Dehn filling associated to m_j, produce a closed manifold homeomorphic to $L_{X'}$. Let us be more precise.

The Dehn filling construction:
Let T be a solid torus given with a meridian disc D and let m_T be the boundary of D. By definition m_T is a meridian curve on the boundary of T. Let $U(D)$ be a compact regular neighbourhood of D in T and let B be the closure of $T \setminus U(D)$. By construction B is a 3-dimensional ball. In the boundary $b(T)$ of T, the closure of the complement of the annulus $U(m_T) = U(D) \cap b(T)$ is also an annulus $E(m_T) \subset b(B)$.
On the other hand, we suppose that a torus τ is a boundary component of an oriented compact three-dimensional manifold M. Let γ be an oriented essential simple closed curve on τ. So, τ is the union of two annuli, $U(\gamma)$, a compact regular neighbourhood of γ, and the closure $E(\gamma)$ of $\tau \setminus U(\gamma)$. There is a unique way to glue T to M by an orientation reversing homeomorphism between the boundary of T and τ which send m_T to γ. Indeed, the gluing of $U(m_T)$ onto $U(\gamma)$ determines a union M' between $U(D)$ and M. This gluing extends to the gluing of $E(m_T)$ onto $E(\gamma)$ which determines a union between B and M'.
So, the result of such a gluing is unique up to orientation preserving homeomorphism and it is called the **Dehn filling** of M associated to γ.
One can find a presentation of the Dehn filling construction in S. Boyer [4].

The topology of the link L_X determines the exterior $E(\mathcal{K})$ of the singular locus $\mathcal{K}$ of L_X and also the well defined meridian curves of $N(\mathcal{K})$ on each connected component of the boundary of $E(\mathcal{K})$. Let $\sigma_i, 1 \leq i \leq r$ be the r irreducible components of Σ_+. So, the boundary of $E(\mathcal{K})$ has $n = \sum_{1 \leq i \leq r} n(\sigma_i)$ connected components. Let $T_j = S \times D$ be a solid torus and let m_{T_j} be a meridian curve of T_j. Let τ_j be one connected component of the boundary of $E(\mathcal{K})$ given with its already chosen curve m_j which is a meridian curve of $N(\mathcal{K})$. By 2.2, m_j is an essential simple closed curve on τ_j. We glue T_j to $E(\mathcal{K})$ with the help of an orientation reversing homeomorphism

$$f_{m_j} : b(T_j) \to \tau_j$$

defined on the boundary $b(T_j)$ of T_j such that $f_{m_j}(m_{T_j}) = m_j$.

We perform such a Dehn filling, associated to the given curve m_j, on each of the n connected components of the boundary of $E(\mathcal{K})$. So, we obtain a closed 3-dimensional Waldhausen graph manifold L. Let $\nu : (X', p') \to (X, p)$ be the normalization morphism of $(X, 0)$. Let Σ'_+ be $\nu^{-1}(\Sigma_+)$.

As $\mathcal{K} = K_{\Sigma_+}$, ν restricted to $\nu^{-1}(E(\mathcal{K})) = \nu^{-1}(E(K_{\Sigma_+})) = E(K_{\Sigma'_+})$ is a homeomorphism.

But, $\nu^{-1}(N(K_{\Sigma_+}))$ is a tubular neighbourhood of the differential link $\nu^{-1}(K_{\Sigma_+})$ which has $n = \sum_{1 \leq i \leq r} n(\sigma_i)$ connected components. So, $\nu^{-1}(N(K_{\Sigma_+}))$ is a disjoint union of n solid tori. As in the proof of Lemma 3.2, let T'_j be one of these solid tori. Then, $C_j = \nu(T'_j)$ is a sheet of $N(K_\sigma)$ where σ is an irreducible component of Σ_+. But C_j is a d_j-curling and ν restricted to T'_j is a quotient morphism associated to this d_j-curling. Let m_j be the chosen meridian on C_j. Lemma 2.1 implies that $\nu^{-1}(m_j)$ is a meridian curve of T'_j. By the unicity of the Dehn filling construction, there exists an orientation preserving homeomorphism $f : L_{X'} \to L$.

As τ_j is the boundary of the d_j-curling C_j, the chosen gluing morphism $f_{m_j} : b(T_j) \to \tau_j$ extends to a d_j-curling morphism $\tilde{f}_{m_j} : T_j \to C_j$. Only using the topology of L_X, we can construct a continuous map $\nu_{top} : L \to L_X$ such that ν_{top} restricted to $E(\mathcal{K})$ is equal to the identity and ν_{top} restricted to each solid torus T_j is the d_j-curling morphism $\tilde{f}_{m_j}$. By construction $\nu_{top}^{-1}(\mathcal{K})$ is the disjoint union of the cores of the solid tori T_j. So, the topology of the pair $(L, \nu_{top}^{-1}(\mathcal{K}))$ only depends on the topology of L_X. By construction we have $f(K_{\Sigma'_+}) = \nu_{top}^{-1}(\mathcal{K})$. We have proved that f is a homeomorphism of pairs between $(L_{X'}, K_{\Sigma'_+})$ and $(L, \nu_{top}^{-1}(\mathcal{K}))$. As the topology of L_X determines the topology of the pair $(L, \nu_{top}^{-1}(\mathcal{K}))$, the topology of L_X determines the homeomorphism class of the pair of links $(L_{X'}, K_{\Sigma'_+})$.

End of proof

5 Surface germs with simply connected links

This section is devoted to the proof of the following proposition.

Proposition 5.1 *Let $(X, 0)$ be an irreducible surface germ. If the link L_X of $(X, 0)$ is simply connected, then the normalization $\nu : (X', p') \to (X, 0)$ is a homeomorphism and (X', p') is smooth at p'. In particular, L_X is a topological three manifold and the normalization is the good minimal resolution of $(X, 0)$.*

Proof
By Lemma 3.1 (or Proposition 3.12 in [8]), if L_X is a topological manifold the normalization $\nu : (X', p') \to (X, p)$ is a homeomorphism. Then, the link $L_{X'}$ is also simply connected and by Mumford's theorem [10] (X', p') is smooth at p'.
Now, we suppose that L_X is not a topological manifold. Then, the following two statements I and II prove that L_X is not simply connected. As before, Σ_+ is the union of the irreducible components σ of the singular locus of $(X, 0)$, which have a total degree, $k(\sigma) = d_1 + \ldots + d_j + \ldots + d_{n(\sigma)}$, stricly greater than one. By Lemma 3.1, if L_X is not a topological manifold, Σ_+ has at least one irreducible component σ.

Statement I If there exists an irreducible component σ of Σ_+ with $n(\sigma) > 1$, the rank of $H_1(L_X, \mathbb{Z})$ is greater than or equal to $(n(\sigma) - 1)$, in particular $H_1(L_X, \mathbb{Z})$ has infinite order.

Proof of Statement I
Let $\Sigma_+ = (\sigma \cup_{1\leq i\leq r} \sigma_i)$, be the decomposition of Σ_+ as the union of its irreducible components. As $(X, 0)$ is irreducible, $E(K_{\Sigma_+})$ is connected by Lemma 3.1.
Then, $E(K_\sigma) = E(K_{\Sigma_+}) \cup_{1\leq i\leq r} N(K_{\sigma_i})$ and L_X are also connected. But $N(K_\sigma)$ which is a singular pinched torus (Lemma 3.2) is connected with $n(\sigma) > 1$ boundary components. We consider the Mayer-Vietoris exact sequence associated to the decomposition of L_X as the union $E(K_\sigma) \cup N(K_\sigma)$.

$$... \to H_1(L_X, \mathbb{Z}) \xrightarrow{\delta_1} H_0(E(K_\sigma) \cap N(K_\sigma), \mathbb{Z}) \xrightarrow{\Delta_0} H_0(E(K_\sigma), \mathbb{Z}) \oplus H_0(N(K_\sigma), \mathbb{Z}) \xrightarrow{i_0} H_0(L_X, \mathbb{Z}) \to 0$$

But $E(K_\sigma) \cap N(K_\sigma)$ is the disjoint union of $n(\sigma)$ disjoint tori. So, the rank of $H_0(E(K_\sigma) \cap N(K_\sigma), \mathbb{Z})$ is equal to $n(\sigma)$. Since σ is irreducible, $H_0(N(K_\sigma), \mathbb{Z})$ has rank one. Since $(X, 0)$ is irreducible, $H_0(E(K_\sigma), \mathbb{Z})$ and $H_0(L_X, \mathbb{Z})$ have rank one. So, the rank of $Ker(\Delta_0) = \delta_1(H_1(L_X, \mathbb{Z}))$ is equal to $(n(\sigma) - 1)$. This ends the proof of Statement I.

Statement II If there exists an irreducible component σ of Σ_+ with $n(\sigma) = 1$ and $k(\sigma) = d > 1$ the order of $H_1(L_X, \mathbb{Z})$ is at least d.

Proof of Statement II
By Lemma 2.3, if $n(\sigma) = 1$ and $d > 1$, $N(K_\sigma)$ is a d-curling and the boundary of $N(K_\sigma)$ is a torus τ. By Lemma 2.1, τ is given with a meridian curve m and a parallel curve l. Let $\bar{m}$, $\bar{l}$ and l_σ be the classes of m, l and K_σ, in $H_1(N(K_\sigma), \mathbb{Z})$. Moreover, we have $H_1(N(K_\sigma), \mathbb{Z}) = \mathbb{Z}.l_\sigma$ and $\bar{l} = d.l_\sigma$. We consider the Mayer-Vietoris exact sequence associated to the decomposition of L_X as the union $E(K_\sigma) \cup N(K_\sigma)$.

$$... \to H_2(L_X, \mathbb{Z}) \xrightarrow{\delta_2} H_1(E(K_\sigma) \cap N(K_\sigma), \mathbb{Z}) \xrightarrow{\Delta_1} H_1(E(K_\sigma), \mathbb{Z}) \oplus H_1(N(K_\sigma), \mathbb{Z})) \xrightarrow{i_1} H_1(L_X, \mathbb{Z}) \to ...$$

As $E(K_\sigma) \cap N(K_\sigma) = \tau$, the image of Δ_1 is generated by $\Delta_1(\bar{m}) = (x, 0)$ and $\Delta_1(\bar{l}) = (y, d.l_\sigma)$ where x and y are in $H_1(E(K_\sigma), \mathbb{Z})$. So, the image of Δ_1 is included in $H_1(E(K_\sigma), \mathbb{Z}) \oplus \mathbb{Z}\ d.l_\sigma$. It implies that the order of the cokernel of Δ_1 is at least d. This ends the proof of Statement II.
The two above statements imply Proposition 5.1.

End of proof

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