# Connectivity of Julia sets of Hénon maps near the boundary of the Mandelbrot set

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**Abstract**

In this paper, we prove some new connectivity of the Julia sets $J$ of the complex Hénon maps $H(x,y)=(x^2+c+ay, ax)$ with sufficiently small $|a|$. We investigate the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set. We first give some conditions related to the connectivity of $J$ for sufficiently small $|a|$, which are useful for considering the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set. We consider a perturbation $\{H_{a,\lambda_t}\}_{a\in\mathbb{D}_{\delta_0},0\le t<\delta_0}$ of dissipative semi-parabolic Hénon maps $H_{a,\lambda_0}$ such that $\det DH_{a,\lambda_t}=-a^2$ and $H_{a,\lambda_t}$ has a fixed point $\mathbf{q}_{a,\lambda_t}$ for which $(DH_{a,\lambda_t})_{\mathbf{q}_{a,\lambda_t}}$ has an eigenvalue $\lambda_t$. Assume that $\lambda_t\to\lambda_0=\exp(2\pi i m/l)\in\partial\mathbb{D}$ as $t\to 0$ and $\lambda_t^l$ can be represented by $\exp(L_t+i\theta_t)$ with $L_t\neq 0$ for $0<t<\delta_0$. We prove that if $\theta_t=O(L_t)$, then the Julia sets $J_{a,\lambda_t}$ for $a\in\mathbb{D}_{\delta_0}, 0<t<\delta_0$ are connected by using the conditions above.

## 1 Introduction

In this paper, we deal with the connectivity of the Julia sets of complex Hénon maps.

In one-dimensional (1-D) complex dynamics, we consider a complex polynomial $f_c(x)=x^2+c, c\in\mathbb{C}$ and the *Julia set* $J_{f_c}$ of $f_c$. The Julia set $J_{f_c}$ of $f_c$ is defined by the boundary of the *filled Julia set* $K_{f_c}:=\{z\in\mathbb{C}:\{f_c^n(z)\}_{n\in\mathbb{N}}$ is bounded$\}$. Note that the notation $f_c^n$ is the $n$-fold composition of $f_c$. The Mandelbrot set $\mathcal{M}$ is defined by $\{c\in\mathbb{C}: J_{f_c}$ is connected$\}$. It is known that the Julia set $J_{f_c}$ of a polynomial $f_c$ is connected if and only if $K_{f_c}$ contains the critical point 0 of $f_c$ in $\mathbb{C}$ (see [13]). By using the fact, it is easy to find the boundary of the connectedness locus for parameters $c\in\mathbb{C}$. For example, the Julia set $J_{f_{1/4}}$ is connected, and the parameter $1/4$ belongs to the boundary of the connectedness locus. Indeed, $J_{f_{1/4+\varepsilon}}$ is connected if $\varepsilon<0$, and $J_{f_{1/4+\varepsilon}}$ is disconnected if $\varepsilon>0$. The parameter $c=1/4$ is called a *parabolic parameter* since $f_{1/4}$ has a parabolic fixed point $1/2$. Let us consider how the parameters $c$ for which $J_{f_c}$ is connected can approach parabolic parameters. Let us explain this by using perturbations of multipliers of parabolic fixed points. We say that a point $\alpha\in\mathbb{C}$ is a *parabolic fixed point* of $f_c$ if $f_c(\alpha)=\alpha$ and $f_c'(\alpha)$ is a root of unity. Here we consider the case where $f_{c_0}$ has a parabolic fixed point $\alpha_0$ with multipliers $\lambda_0:=\exp(2\pi i m/l)$, where $l\in\mathbb{Z}\setminus\{0\}$, $m\in\mathbb{Z}$ and $(m,l)=1$. Consider a one-parameter continuous family $\{\lambda_t\}_{t\in[0,\delta_0)}$, where $\delta_0>0$. Assume that $\lambda_t^l=\exp(L_t+i\theta_t)$ and $\mathbb{R}\ni\theta_t\to 0$ as $t\to 0$, where $L_t\in\mathbb{R}\setminus\{0\}$ and $\theta_t\in\mathbb{R}$

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for $0 < t < \delta_0$. Let $\{f_{c_t}\}_{t \in [0,\delta_0)}$ satisfy that $f_{c_t}$ has a fixed point $\alpha_t$ with multiplier $\lambda_t$. We say that $\lambda_t^l$ converges to 1 *radially* if $\theta_t = O(L_t)$. If $\theta_t = O(L_t)$, then $\{f_{c_t}\}_{t\in[0,\delta_0)}$ has nice properties (such as continuity of $J_{f_{c_t}}$, continuity of the Hausdorff dimension of $J_{f_{c_t}}$) (see [12]). In particular, we see that $J_{c_t}$ is connected for each $t \in (0, \delta_0)$, taking a smaller $\delta_0 > 0$ if necessary.

Let us explain radial convergence by observing the main cardioid $\mathcal{M}_0 := \{c = \lambda/2 - \lambda^2/4 : |\lambda| \leq 1\}$ of the Mandelbrot set $\mathcal{M}$ (Figure 1). For this purpose, we set $p_\lambda(x) := x^2 + \lambda/2 - \lambda^2/4$, which has a fixed point $\lambda/2$ with multiplier $\lambda$. The parameter $c = 1/4$ is a parabolic parameter. That is, the polynomial $p_1(x) = x^2 + 1/4$ has a parabolic fixed point $1/2$. Consider a family $\{p_{\lambda_t}\}$ with $\lambda_t \to 1$ as $t \to 0$. We set $c_t = \lambda_t/2 - \lambda_t^2/4$ and $\lambda_t = \exp(L_t + i\theta_t)$. If $\theta_t = 0$, then the parameters $c_t$ approach $1/4$ in $\operatorname{Int}\mathcal{M}_0 \cap \mathbb{R}$ (the first of Figure 1). If parameters $c_t$ in the sector in the second of Figure 1 approach $1/4$, then $\lambda_t$ satisfies $\theta_t = O(L_t)$. We remark that there is a family $\{\lambda_t\}$ such that the corresponding parameters $c_t \in \operatorname{Int}\mathcal{M}_0$ approach $1/4$ as $t \to 0$. For example, if parameters $c_t$ approach $1/4$ in the curve near $\partial\mathcal{M}_0$ in the third of Figure 1, then $\lambda_t$ satisfies $\theta_t^2 = o(L_t)$ and $\theta_t \neq O(L_t)$. On the other hand, if parameters $c_t$ approach $1/4$ in $\mathbb{R}_{>1/4}$, then $J_{f_{c_t}}$ is disconnected and $\theta_t \neq O(L_t)$ (see fourth of Figure 1).

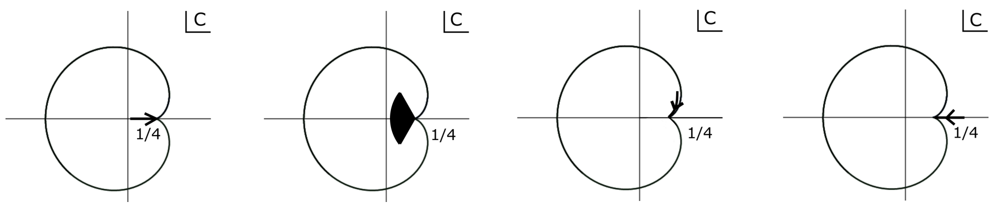


Figure 1: $\theta_t = 0$ (first), $\limsup_{t\to 0} |\theta_t/L_t| \leq 1/\sqrt{3}$ (second), $\theta_t^2 = o(L_t), \theta_t \neq O(L_t)$ (third) and disconnected (fourth)

In the case of two-dimensional (2-D) dynamics, for $(c, a) \in \mathbb{C}^2$, we consider the quadratic *Hénon map* of the form $H(x, y) = (x^2 + c + ay, ax)$. For a diffeomorphism $F(x, y) = (F_1(x, y), F_2(x, y))$ from an open set $U \subset \mathbb{C}^2$ to $\mathbb{C}^2$, we set

$$(DF)_{(x_0,y_0)}(\zeta, \eta) := \begin{pmatrix} (F_1)_x(x_0, y_0) & (F_1)_y(x_0, y_0) \\ (F_2)_x(x_0, y_0) & (F_2)_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$$

for $(x_0, y_0) \in U$ and $(\zeta, \eta) \in T_{(x_0,y_0)}U$. We have

$$(DH)_{(x,y)} = \begin{pmatrix} 2x & a \\ a & 0 \end{pmatrix}$$

for $(x, y) \in \mathbb{C}^2$. The map $H$ has constant Jacobian $-a^2$, i.e., $\det (DH)_{(x,y)} = -a^2$ for all $(x, y) \in \mathbb{C}^2$. Unlike 1-D dynamics, we can consider the inverse $H^{-1}$ of $H$ if $a \neq 0$. Let $K^\pm$ be the set of all points $(x, y) \in \mathbb{C}^2$ such that $\{H^{\pm n}(x, y)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}^2$. We consider the *Julia sets* $J^\pm := \partial K^\pm$ of $H$. Furthermore we denote $J$ by the intersection of $J^+$ and $J^-$. It is known that $J^\pm$ are connected (see [2]). The *Hénon connectedness locus* is the set of parameters $(c, a) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}$ for which the Julia set $J$ is connected. Let us consider the condition that $J$ is connected. Unlike one-dimensional dynamics, Hénon maps $H$ do not have critical points for $a \neq 0$. Instead, it was suggested to consider critical points of the Green functions along the unstable manifolds of the saddle points to compute the connectivity of the Julia sets (see [4]). The Julia set $J$ is connected if and only if the restriction of the Green function $G^+(x, y) := \lim_{n\to\infty}(1/2^n) \log^+ \|H^n(x, y)\|$ on the unstable manifold of some saddle point has no critical points in $K^+$ (see [4] and [10, Theorem 3.3]). However, it is not easy to find the boundary of the Hénon connectedness locus. Our result (Theorem 1.2) describes the local geometry near semi-parabolic parameters $(c, a)$ if $|a|$ is sufficiently small.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. To consider the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set, we consider a Hénon family for which each element of the family has a fixed point such that one of the eigenvalues of $DH$ at the fixed point is $\lambda$. Then, the set $\mathcal{P}_\lambda$ of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H(x, y) = (x^2 + c + ay, ax)$ has a fixed point $\mathbf{q}$ such that $\lambda$ is an eigenvalue of $(DH)_{\mathbf{q}}$ is the curve of equation

$$c = (1 - a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2. \tag{1}$$

We denote the right hand side of equation (1) by $c(a, \lambda)$. Moreover, we set $H_{a,\lambda}(x, y) = (x^2 + c(a, \lambda) + ax, ay)$ and $p_\lambda(x) = x^2 + c(0, \lambda)$. We denote the Julia sets of $H_{a,\lambda}$ by $J^{\pm}_{a,\lambda}$, $J_{a,\lambda}$ instead of $J^{\pm}$, $J$ respectively. Based on the above notations, we now present the first main result of this paper.

**Theorem 1.1.** *Assume that a Hénon family $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfies the vertical condition $(\mathrm{VC})_{\varepsilon,r}$ with respect to $\varepsilon > 0, r > 0$ (see Definition 3.1). Suppose that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$ for each $a \in \mathbb{D}_{\delta_0}$. Then the Julia sets $J_{a,\lambda}$ of the Hénon maps $H_{a,\lambda}$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$ are connected if and only if the Julia set $J_{p_\lambda}$ of the polynomial $p_\lambda$ is connected.*

We regard $\mathbb{D}_\varepsilon \times \mathbb{D}_r$ as a neighborhood of the critical point $z = 0$ of $p_\lambda$ in two dimensions. Note that most families $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$. Indeed, if $\lambda \neq 1$, then there is $\delta_0 > 0$ such that $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$ (see Lemma 3.2). The assumptions of Theorem 1.1 imply that the stable manifold of a saddle fixed point of $H_{a,\lambda}$ intersects transversely horizontal direction in $\mathbb{D}_r \times \mathbb{D}_r$. From this, we can construct a holomorphic motion of $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ over $a \in \mathbb{D}_{\delta_0}$ for each $y \in \mathbb{D}_r$ and can show that $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ is homeomorphic to the Julia set $J_{p_\lambda}$ of $p_\lambda$.

It is known that for a hyperbolic polynomial $x^2 + c$, there is a positive constant $\delta(c) > 0$ such that a small perturbation $\{H(x, y) = (x^2+c+ay, ax) : 0 < |a| < \delta(c)\}$ of $H(x, y) = (x^2+c, 0)$ is hyperbolic (see [6] and [9]). In particular, the Julia sets $J$ of the Hénon maps $H(x, y) = (x^2 + c + ay, ax)$ for $0 < |a| < \delta(c)$ are connected if and only if $J_{x^2+c}$ is connected. However, the proofs in [6] and [9] do not give any uniform estimate on the constant $\delta(c)$ from below for $c$ near the boundary of the Mandelbrot set. For example, it may be $\delta(c_n) \to 0$ as $\mathrm{Int}\,\mathcal{M}_0 \ni c_n \to \lambda/2 - \lambda^2/4 \in \partial\mathcal{M}_0$, where $|\lambda| = 1$. Therefore, we cannot apply methods of [6] and [9] to compute the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set. In our result, we only need to check that the $(\mathrm{VC})_{\varepsilon,r}$ and the condition $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$ hold for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. We can deduce the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set by using Theorem 1.1 (see the following Theorem 1.2 and Figure 2).

To present the second main result, we recall radial convergence. Let $\lambda_0 = \exp(2\pi i m/l)$, where $l \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{Z}$ and $(m, l) = 1$. Consider a one-parameter continuous family $\{\lambda_t\}_{t\in[0,\delta_0)}$, where $\delta_0 > 0$. Assume that $\lambda_t^l = \exp(L_t + i\theta_t)$ and $\mathbb{R} \ni \theta_t \to 0$ as $t \to 0$, where $L_t \in \mathbb{R} \setminus \{0\}$ and $\theta_t \in \mathbb{R}$ for $0 < t < \delta_0$. We say that $RD_{\lambda_t,\delta_0} := \{H_{a,\lambda_t} : a \in \mathbb{D}_{\delta_0} \text{ and } 0 < t < \delta_0\}$ is a *radial* perturbation if $\theta_t = O(L_t)$. For each $0 < t < \delta_0$, we will show that the section $\{H_{a,\lambda_t}\}_{a\in\mathbb{D}_{\delta_0}}$ of $RD_{\lambda_t,\delta_0}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$ and that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda_t} = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$. By applying Theorem 1.1 to the family $RD_{\lambda_t,\delta_0}$, we can show the second main result:

**Theorem 1.2.** *There is $\delta_0 > 0$ such that each $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ with $a \neq 0$ has connected Julia set $J_{a,\lambda_t}$.*

Note that $H_{a,\lambda_0}$ does not belong to $RD_{\lambda_t,\delta_0}$. A Hénon map $H_{a,\lambda_0}$ has connected Julia set for $a \in \mathbb{D}_{\delta_0}$ (see [17]). Radu and Tanase showed that there is $\delta_0 > 0$ such that $H_{a,\lambda_t}$ is hyperbolic for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$ if $\theta_t = 0$ for each $0 < t < \delta_0$ in [16]. In [16], by using hyperbolicity, the Julia sets are connected for the parameters if $\theta_t = 0$ for $0 < t < \delta_0$ (see the left of Figure 2). In our case, we consider a much wider range of eigenvalues than $\theta_t = 0$. In this case, we will show the Julia set $J_{a,\lambda}$ of $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ is connected without using hyperbolicity.

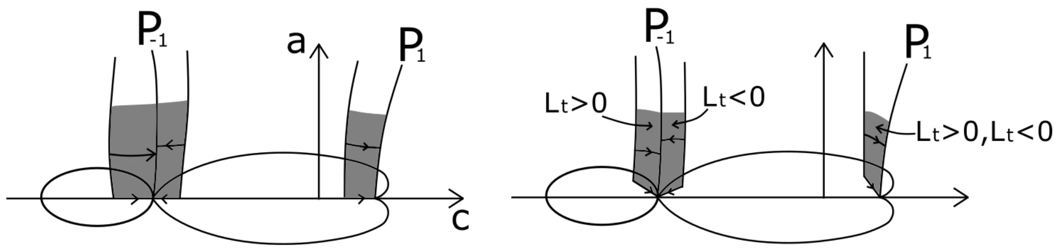


Figure 2: $\theta_t = 0$ (left) and radial perturbations (right). The set $\mathcal{P}_1$ (resp. $\mathcal{P}_{-1}$) is a semi-parabolic parameter given by equation (1) with $\lambda = 1$ (resp. $\lambda = -1$).

The rest of this paper is organized as follows. In Section 2, we present fundamental facts for Hénon maps. In Section 3, we introduce the *vertical condition* $(\mathrm{VC})_{\varepsilon,r}$ with respect to $\varepsilon, r$, and the condition $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$. By using these conditions, we construct a holomorphic motion of $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ over $a \in \mathbb{D}_{\delta_0}$ for each $y \in \mathbb{D}_r$. Using these, we show Theorem 1.1. In Section 4, we show Theorem 1.2 by using Theorem 1.1. In particular, we check the condition $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$ holds for $H_{a,\lambda_t}$ by using local coordinates near semi-parabolic fixed points.

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# 2 Preliminary

In this section, we recall some basic results on the dynamics of Hénon maps. See [2], [8], [14] and [16] for more details.

**Definition 2.1.** For $(c, a) \in \mathbb{C}^2$, let $H\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the map of the form

$$H(x, y) = (p(x) + ay, ax), \text{ where } p(x) = x^2 + c.$$

We call $H\colon \mathbb{C}^2 \to \mathbb{C}^2$ a *Hénon map*. If $a \neq 0$, the inverse is

$$H^{-1}(x, y) = \frac{1}{a}(y, x - p(y/a)).$$

**Remark 2.2.** *In [14], a holomorphic automorphism of $\mathbb{C}^2$ of the form*

$$F\colon (x, y) \mapsto (y, y^2 + c - \delta x), \quad \delta, c \in \mathbb{C}, \ \delta \neq 0$$

*is called a Hénon map. The form of the Hénon map $H$ in Definition 2.1 differs from the form $(x, y) \mapsto (y, y^2 + c - \delta x)$ given in [14]; however, $H$ is conjugate by a polynomial automorphism to $(x, y) \mapsto (y, y^2 + c + a^2 x)$.*

In [8], the dynamical space $\mathbb{C}^2$ is divided into the following three sets.

**Definition 2.3** ([8])**.** Let $r > 0$ be a large number. Consider the following three subsets of $\mathbb{C}^2$,

$$\mathbb{D}_r \times \mathbb{D}_r := \{(x, y) \in \mathbb{C}^2 : |x| < r, |y| < r\},$$

$$V^+ := \{(x, y) \in \mathbb{C}^2 : |x| \geq \max\{|y|, r\}\} \text{ and } V^- := \{(x, y) \in \mathbb{C}^2 : |y| \geq \max\{|x|, r\}\}.$$

Let $H$ be a Hénon map with $a \neq 0$. We define the *escaping sets* $U^\pm$ of $H$ by

$$U^+ := \bigcup_{k \geq 0} H^{-k}(V^+) \text{ and } U^- := \bigcup_{k \geq 0} H^k(V^-).$$

We consider the Julia sets and the filled Julia sets of Hénon maps.

**Definition 2.4.** For a Hénon map $H(x, y) = (p(x) + ay, ax)$ with $a \neq 0$, we define the *filled Julia sets* $K^\pm$ of $H$ as follows:

$$K^\pm := \{(x, y) \in \mathbb{C}^2 : \{H^{\pm n}(x, y)\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{C}^2\}.$$

We define the *Julia sets* $J^\pm$ and $J$ of $H$ as follows:

$$J^\pm := \partial K^\pm \text{ and } J := J^+ \cap J^-.$$

**Remark 2.5.** *For $a = 0$, we can also define $K^+$ and $J^+$. In this case, $K^+$ (resp. $J^+$) is the product set of the filled Julia set (resp. the Julia set) of $p$ and $\mathbb{C}$.*

Bedford and Smillie [2] showed that there is a positive constant $r > 0$ depending on $H$ such that

$$H(V^+) \subset V^+, H^{-1}(V^-) \subset V^-, U^+ = \mathbb{C}^2 \setminus K^+ \text{ and } U^- = \mathbb{C}^2 \setminus K^-. \tag{2}$$

It is easy to see that for a polynomial $x^2 + c_0$, there is $r > 0$ and $\delta$ such that $H(x, y) = (x^2 + c + ay, ax)$ satisfies the condition (2) with respect to $r$ for $(c, a) \in \mathbb{D}_\delta(c_0) \times \mathbb{D}_\delta \setminus \{0\}$.

In this paper, we consider the following three types of fixed points.

**Definition 2.6.** Suppose that a Hénon map $H$ has a fixed point $\mathbf{q}$. Let $\lambda$ and $\nu$ be the eigenvalues of $(DH)_\mathbf{q}$. We say that the fixed point $\mathbf{q}$ is

(i) *attracting* if $|\lambda| < 1$ and $|\nu| < 1$,
(ii) *semi-parabolic* if $|\nu| < 1$ and $\lambda = \exp(2\pi i p/l)$ for some $p/l \in \mathbb{Q}$,
(iii) a *saddle* if $|\nu| < 1$ and $|\lambda| > 1$.

We write that $A_n = O(B_n)$ if there are a positive constant $K > 0$ and a positive integer $N \in \mathbb{N}$ such that $|A_n| \leq K|B_n|$ for $n \geq N$. We set $\mathrm{Pr}_1 \colon \mathbb{C}^2 \to \mathbb{C}, \mathrm{Pr}_1(x, y) := x$ and $\mathrm{Pr}_2 \colon \mathbb{C}^2 \to \mathbb{C}, \mathrm{Pr}_2(x, y) := y$. We recall *stable manifolds* (see [14], [18]).

**Definition 2.7** ([14, p.311])**.** Let $H$ be a Hénon map and $r > 0$ satisfy the condition (2) with respect to $H$. For a saddle fixed point $\mathbf{q}$ of $H$, the *stable manifold* $W^s(\mathbf{q})$ of $\mathbf{q}$ is defined as

$$W^s(\mathbf{q}) := \{z \in \mathbb{C}^2 : \lim_{n \to \infty} \|H^n(z) - \mathbf{q}\| = 0\}$$

where $\|\cdot\|$ is the Euclidean metric of $\mathbb{C}^2$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. To consider the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set, we consider a Hénon family for which each element of the family has a fixed point such that one of the eigenvalues of $DH$ at the fixed point is $\lambda$. A Hénon map $H(x, y) = (x^2+c+ay, ax)$ has a fixed point $\mathbf{q}$ such that $\lambda \neq 0$ is an eigenvalue of $(DH)_{\mathbf{q}}$ if and only if

$$c = (1 - a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2. \tag{3}$$

Let $\mathcal{P}_\lambda$ be the set of parameters $(c, a) \in \mathbb{C}^2$ satisfying (3). We denote the right hand side of equation (3) by $c(a, \lambda)$. Moreover, we set $H_{a,\lambda}(x, y) = (x^2+c(a, \lambda)+ay, ax)$ and $p_\lambda(x) = x^2+c(0, \lambda)$. We denote the filled Julia sets and the Julia sets of $H_{a,\lambda}$ by $K^\pm_{a,\lambda}$ $J^\pm_{a,\lambda}$, $J_{a,\lambda}$ instead of $K^\pm$, $J^\pm$, $J$ respectively. We see that $H_{a,\lambda}$ has a fixed point

$$\mathbf{q}_{a,\lambda} := \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}, a\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)\right) \tag{4}$$

with eigenvalues $\lambda$ and $\nu := -a^2/\lambda$. We give the parametrization of $W^{ss}(\mathbf{q}_{a,\lambda})$.

**Lemma 2.8** ([7], [15, the proofs of Propositions 3.16, 3.17], [17, Proposition 5.2])**.** *Let $v = (-a/\lambda, 1)$ be an eigenvector for $\nu$. Assume that $\lambda$ satisfies $|-a^2/\lambda| < 1$ and $|\lambda| > |-a^2/\lambda|$. Then there exists the injective holomorphic map*

$$\Phi_{a,\lambda}\colon \mathbb{C} \to \mathbb{C}^2, \Phi_{a,\lambda}(z) = \lim_{k\to\infty} H^{-k}_{a,\lambda}(\mathbf{q}_{a,\lambda} + \nu^k zv) \tag{5}$$

*such that $\Phi_{a,\lambda}(\nu z) = H_{a,\lambda}(\Phi_{a,\lambda}(z))$ for $z \in \mathbb{C}$ and $a \neq 0$.*

*Let $\Phi_{0,\lambda}(z) := (\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda}, z)$. Then, $\Phi_{a,\lambda}$ is analytic with respect to $a$ and $\sup_{z\in K} \|\Phi_{a,\lambda}(z) - \Phi_{0,\lambda}(z)\| = O(a)$ for each compact subset $K$ of $\mathbb{C}$.*

**Remark 2.9.** *Fix $\lambda_0 \neq 0$. Consider a family $\{H_{a,\lambda}\}_{(a,\lambda)\in\mathbb{D}_{\delta_0}\times\mathbb{D}_{\delta_0}(\lambda_0)}$. Since the fixed point $\mathbf{q}_{a,\lambda}$ of $H_{a,\lambda}$ depends holomorphically on $a, \lambda$, we see that $\Phi_{a,\lambda}(z)$ is holomorphic with respect to $(a, \lambda, z) \in \mathbb{D}_{\delta_0} \setminus \{0\} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$ (see the proof of Theorem 6.43 in [14]). Since $\Phi_{a,\lambda}$ is holomorphic with respect to each variable separately when the other variables are fixed, $\Phi_{a,\lambda}(z)$ is holomorphic with $(a, \lambda, z) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$, taking a smaller $\delta_0 > 0$ if necessary.*

**Definition 2.10.** The curve $W^{ss}(\mathbf{q}) := \Phi(\mathbb{C})$ is called the *strong stable manifold* of a fixed point $\mathbf{q}$ for $a \neq 0$, where $\Phi$ is given by (5), and the definition of (5) is valid if at least one eigenvalue of the fixed point has absolute value less than 1 (see the proof of Theorem 6.43 in [14]).

For a fixed point $\mathbf{q}$ of $H$, the *local strong stable manifold* $W^{ss}_{\mathrm{loc}}(\mathbf{q})$ of $\mathbf{q}$ is defined by the component of $W^{ss}(\mathbf{q}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ that contains $\mathbf{q}$, with the topology induced by $\Phi\colon \mathbb{C} \to W^{ss}(\mathbf{q})$.

**Remark 2.11.** *When $a = 0$, we set $W^{ss}(\mathbf{q}) = \bigcup_{j\geq 0} p^{-j}(\{\mathrm{Pr}_1\, \mathbf{q}\}) \times \mathbb{C}$ and $W^{ss}_{\mathrm{loc}}(\mathbf{q}) = \{\mathrm{Pr}_1\, \mathbf{q}\} \times \mathbb{D}_r$.*

**Remark 2.12.** *If $\mathbf{q}$ is a saddle, then $W^s(\mathbf{q}) = W^{ss}(\mathbf{q})$. In this case, the* local stable manifold *$W^s_{\mathrm{loc}}(\mathbf{q})$ of $\mathbf{q}$ is defined by $W^{ss}_{\mathrm{loc}}(\mathbf{q})$.*

By using Lemma 2.8, we have the following.

**Lemma 2.13.** *If $\lambda_0 \in \mathbb{C} \setminus \{0, 1\}$, then there is a positive constant $\delta_0 > 0$ such that $H_{a,\lambda}$ has a saddle fixed point $\boldsymbol{s}_{a,\lambda}$ depending holomorphically on $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$. Let $\tilde{\Phi}_{0,\lambda}(z) := (\mathrm{Pr}_1\, \boldsymbol{s}_{0,\lambda}, z)$, and $\tilde{\Phi}_{a,\lambda}(z) := \lim_{k\to\infty} H^{-k}_{a,\lambda}(\boldsymbol{s}_{a,\lambda} + \tilde{\nu}^k z\tilde{v})$, where $\tilde{\nu}$ is the eigenvalue of $(DH_{a,\lambda})_{\boldsymbol{s}_{a,\lambda}}$ with $|\tilde{\nu}| < 1$ and $\tilde{v}$ is the eigenvector of $\tilde{\nu}$ of the form $(\cdot, 1)$ for $(a, \lambda) \in (\mathbb{D}_{\delta_0} \setminus \{0\}) \times \mathbb{D}_{\delta_0}(\lambda_0)$ (see (5)). Then, $\tilde{\Phi}_{a,\lambda}(z)$ is holomorphic with respect to $(a, \lambda, z) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$.*

*Proof.* If $|\lambda_0| > 1$, then $\mathbf{q}_{a,\lambda}$ is saddle for $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$, taking a smaller $\delta_0$ if necessary. By taking $\mathbf{s}_{a,\lambda}$ as $\mathbf{q}_{a,\lambda}$ and applying Lemma 2.8, we obtain the statement of Lemma 2.13.

We may assume that $\lambda_0 \neq 1$, $\lambda_0 \neq 0$ and $|\lambda_0| \leq 1$. Then $c(0, \lambda_0) \neq 1/4$ and $H_{0,\lambda_0}(x, y)$ has a saddle fixed point. By the implicit function theorem, there are a positive constant $\delta_0$ and a saddle fixed point $\mathbf{s}_{a,\lambda}$ of $H_{a,\lambda}$ depending holomorphically on $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$.

We show that $\tilde{\Phi}_{a,\lambda}(z)$ is holomorphic with respect to $(a, \lambda, z) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$ by using Lemma 2.8. It follows from $\mathbf{s}_{0,\lambda_0} \neq \mathbf{q}_{0,\lambda_0}$ that $\mathbf{s}_{a,\lambda} \neq \mathbf{q}_{a,\lambda}$ for $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$, taking $\delta_0$ small enough. By $\mathbf{s}_{a,\lambda} \neq \mathbf{q}_{a,\lambda}$ and the fixed point equation $(x^2 + c(a, \lambda) + ay, ax) = (x, y)$, we have

$$\mathbf{s}_{a,\lambda} = (1 - a^2 - \mathrm{Pr}_1\, \mathbf{q}_{a,\lambda}, a(1 - a^2 - \mathrm{Pr}_1\, \mathbf{q}_{a,\lambda})).$$

Consider the characteristic equation $\det((DH_{a,\lambda})_{\mathbf{s}_{a,\lambda}} - \tilde{\lambda} I) = 0$, which is equivalent to $-(2\mathrm{Pr}_1\, \mathbf{s}_{a,\lambda} - \tilde{\lambda})\tilde{\lambda} - a^2 = 0$. We set $f(a, \lambda, \tilde{\lambda}) := -(2\mathrm{Pr}_1\, \mathbf{s}_{a,\lambda} - \tilde{\lambda})\tilde{\lambda} - a^2$. We see that $f(0, \lambda_0, \tilde{\zeta}) = 0$, where $\tilde{\zeta} = 0, 2 - \lambda_0$ (see (4)). By $\partial_{\tilde{\lambda}} f(0, \lambda_0, 2 - \lambda_0) = 2 - \lambda_0 \neq 0$ and the Implicit Function Theorem, there exists a holomorphic map $\tilde{\lambda}(a, \lambda)$ with $|\tilde{\lambda}(a, \lambda)| > 1$ for $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$ such that $f(a, \lambda, \tilde{\lambda}(a, \lambda)) = 0$ and $\tilde{\lambda}(0, \lambda_0) = 2 - \lambda_0$, taking a smaller $\delta_0$ if necessary. We see that $c(a, \lambda) = c(a, \tilde{\lambda}(a, \lambda))$ for each $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$ since $H_{a,\lambda}$ has the fixed point $\mathbf{s}_{a,\lambda}$ with one eigenvalue $\tilde{\lambda}(a, \lambda)$ (see (3)). In particular, $H_{a,\lambda} = H_{a,\tilde{\lambda}(a,\lambda)}$ for each $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$. By the identity theorem, we have $\mathbf{s}_{a,\lambda} = \mathbf{q}_{a,\tilde{\lambda}(a,\lambda)}$ for $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$. Thus $\tilde{\Phi}_{a,\lambda} = \Phi_{a,\tilde{\lambda}(a,\lambda)}$ for $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$, where $\Phi_{a,\tilde{\lambda}(a,\lambda)}$ is the map in Lemma 2.8. By Remark 2.9 and the fact that $\tilde{\lambda}(a, \lambda)$ depends holomorphically on $(a, \lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$, it follows that $\tilde{\Phi}_{a,\lambda}(z)$ is holomorphic with respect to $(a, \lambda, z) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$. □

# 3 Vertical Condition

For the rest of the section, we assume that the Jacobians of Hénon maps $H_{a,\lambda}$ are less than 1 in the absolute value, and $\lambda \neq 0$. In this section, we show the first main result (Theorems 3.10 and 3.11). We construct a holomorphic motion of $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ over $a \in \mathbb{D}_{\delta_0}$ for each $y \in \mathbb{D}_r$ to obtain the first main result. In order to construct it, we consider the *vertical condition* and the condition $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$. We consider the vertical cone field $\{C^v_{(x,y)}\}_{(x,y)\in\mathbb{D}_r\times\mathbb{D}_r}$ given by

$$C^v_{(x,y)} := \{(\zeta, \eta) \in T_{(x,y)}\mathbb{C}^2 : |\eta| > |\zeta|\} \tag{6}$$

for $(x, y) \in \mathbb{D}_r \times \mathbb{D}_r$. We first introduce the *vertical condition* as follows.

**Definition 3.1.** Let $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ be a Hénon family. Fix $\varepsilon > 0$, and fix $r > 0$ such that $H_{a,\lambda}$ satisfies the condition (2) with respect to $r$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$. Moreover, we assume that $\delta_0 < \min\{1/2, \varepsilon\}$. We say that $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfies the *vertical condition* $(\mathrm{VC})_{\varepsilon,r}$ with respect to $\varepsilon, r$ if the following three conditions hold:

- (i) $H^{-1}_{a,\lambda}((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_r \times \mathbb{D}_{r/2})) \subset V^-$ and $|\mathrm{Pr}_2\, H^{-1}_{a,\lambda}(x, y)| > 2|y|$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$ and $(x, y) \in V^- \cup ((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_r \times \mathbb{D}_{r/2}))$.
- (ii) $(DH^{-1}_{a,\lambda})_{(x,y)}(C^v_{(x,y)}) \subset C^v_{H^{-1}_{a,\lambda}(x,y)}$ and $|\mathrm{Pr}_2\, (DH^{-1}_{a,\lambda})_{(x,y)}(\zeta, \eta)| > 2|\eta|$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$, $(x, y) \in H_{a,\lambda}((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_\varepsilon \times \mathbb{D}_r)) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ and $(\zeta, \eta) \in C^v_{(x,y)}$.
- (iii) There are a saddle fixed point $\mathbf{s}_{a,\lambda}$ of $H_{a,\lambda}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$ and a holomorphic map $f_a\colon \mathbb{D}_r \to \mathbb{C}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$ such that $W^s_{loc}(\mathbf{s}_{a,\lambda}) = \{(f_a(y), y) : y \in \mathbb{D}_r\}$ and $T_{(x,y)}W^s_{loc}(\mathbf{s}_{a,\lambda}) \subset C^v_{(x,y)}$ for $(x, y) \in W^s_{loc}(\mathbf{s}_{a,\lambda})$.

We see that most families $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfy the condition $(\mathrm{VC})_{\varepsilon,r}$ by the following lemma.

**Lemma 3.2.** *Fix $\lambda \neq 1$. Then there is a positive constant $\delta_0 > 0$ such that $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$.*

*Proof.* Fix $\varepsilon > 0$. We can take $r > 0$ such that each $H_{a,\lambda}$ satisfies the condition (2) with respect to $r$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$ by taking a smaller $\delta_0$ if necessary. Assume that $\delta_0 < \min\{1/2, \varepsilon\}$.

We show that the condition ( i ) in Definition 3.1 holds. For $(x, y) \in V^- \cup ((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_r \times \mathbb{D}_{r/2}))$, we set $(x_1, y_1) := H^{-1}_{a,\lambda}(x, y) = (y/a, (x - y^2/a^2 - c(a, \lambda))/a)$. We show $|y_1| > 2|y|$. Since $|x| \leq \max\{|y|, r\}$, we have

$$|ay_1| = |x - y^2/a^2 - c(a, \lambda)| \geq |y/a|^2 - |c(a, \lambda)| - \max\{|y|, r\}.$$

To obtain $|y_1| > 2|y|$, it suffices to show that $|y/a|^2 - 2|a||y| - |c(a, \lambda)| - \max\{|y|, r\} > 0$ which is equivalent to $|y|^2 - 2|a|^3|y| - |a|^2|c(a, \lambda)| - |a|^2 \max\{|y|, r\} > 0$. Note that $|y| \geq r/2$ by $(x, y) \in V^- \cup ((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_r \times \mathbb{D}_{r/2}))$. If $a = 0$, we have $|y|^2 \geq r^2/4 > 0$. Thus, we can take $\delta_0 > 0$ such that $|y_1| > 2|y|$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$. Similarly, we have $(x_1, y_1) \in V^-$ if $(x, y) \in (\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_r \times \mathbb{D}_{r/2})$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$, taking a smaller $\delta_0$ if necessary. Thus, $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}\setminus\{0\}}$ satisfies the condition ( i ) in Definition 3.1.

We next show that the condition (ii) in Definition 3.1 holds. Fix $(x, y) \in H_{a,\lambda}((\mathbb{D}_r \times \mathbb{D}_r) \setminus (\mathbb{D}_\varepsilon \times \mathbb{D}_r)) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ and $(\zeta, \eta) \in C^v_{(x,y)}$. We set $(x_1, y_1) = H^{-1}_{a,\lambda}(x, y)$ and $(\zeta_1, \eta_1) = (DH^{-1}_{a,\lambda})_{(x,y)}(\zeta, \eta)$. By $(x_1, y_1) = H^{-1}_{a,\lambda}(x, y) = (y/a, (x - y^2/a^2 - c(a, \lambda))/a)$, we have $\zeta_1 = \eta/a$ and $\eta_1 = (\zeta - 2y\eta/a^2)/a$. By $x_1 = y/a$, we have $\eta_1 = (\zeta - 2x_1\eta/a)/a$. Thus we have

$$|\eta_1| > \frac{1}{|a|}(\frac{|2x_1\eta|}{|a|} - |\zeta|) \geq \frac{1}{|a|}(\frac{2\varepsilon}{|a|} - 1)|\eta| > \frac{1}{\delta_0}(\frac{2\varepsilon}{\delta_0} - 1)|\eta| > 2|\eta|,$$

by $\delta_0 < \min\{1/2, \varepsilon\}$. In particular, we have $|\eta_1| > 1/|a|(2\varepsilon - |a|)|\zeta_1| > (2\varepsilon/|\delta_0| - 1)|\zeta_1| > |\zeta_1|$ by $\eta = a\zeta_1$ and $\delta_0 < \min\{1/2, \varepsilon\}$. Thus the condition (ii) holds for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$.

Finally, we show that the condition (iii) in Definition 3.1 holds. There exists a saddle fixed point $\mathbf{s}_{a,\lambda}$ of $H_{a,\lambda}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$ by Lemma 2.13. By Lemmas 2.8 and 2.13, $W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) = \{(f_a(y), g_a(y)) : y \in g_a^{-1}(\mathbb{D}_r)\}$ for some holomorphic maps $f_a, g_a$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$. Here, we remark that by Rouché's theorem and $g_0(y) = y$, there exists $g_a^{-1}$ in $\mathbb{D}_r$ for $a \in \mathbb{D}_{\delta_0}$, taking a smaller $\delta_0$ if necessary. Thus we have $W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) = \{(f_a(g_a^{-1}(y)), y) : y \in \mathbb{D}_r\}$. In the case of $a = 0$, we have $W^s_{\mathrm{loc}}(\mathbf{s}_{0,\lambda}) = \{\mathrm{Pr}_1\, \mathbf{s}_{0,\lambda}\} \times \mathbb{D}_r$ and $T_{(x,y)}W^s_{\mathrm{loc}}(\mathbf{s}_{0,\lambda}) = \{(\zeta, \eta) : \zeta = 0\} \subset C^v_{(x,y)}$. Since $W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda})$ depends on $a$ holomorphically, by taking $\delta_0$ sufficiently small, we have $|(f_a(g_a^{-1}(y)))'| < 1$ for $y \in \mathbb{D}_r$. □

**Remark 3.3.** *The proof of Lemma 3.2 is still valid for perturbations $\{H_{a,\lambda}\}_{(a,\lambda)\in\mathbb{D}_{\delta_0}\times\mathbb{D}_{\delta_0}(\lambda_0)}$, where $\lambda_0 \neq 1$. That is, there is $\delta_0 > 0$ such that the family $\{H_{a,\lambda}\}_{(a,\lambda)\in\mathbb{D}_{\delta_0}\times\mathbb{D}_{\delta_0}(\lambda_0)}$ satisfies* ( i ), (ii), *and* (iii) *in* $(\mathrm{VC})_{\varepsilon,r}$.

**Remark 3.4.** *By the proof of Lemma 3.2,* ( i ) *and* (ii) *in Definition 3.1 hold for $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}\setminus\{0\}}$ without the condition $\lambda \neq 1$ in Lemma 3.2. Moreover, for each $\varepsilon > 0$, there is $\delta_0 > 0$ such that* (ii) *in Definition 3.1 holds for $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$.*

Consider a Hénon family $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ satisfying the $(\mathrm{VC})_{\varepsilon,r}$, and

$$(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset \tag{7}$$

for $a \in \mathbb{D}_{\delta_0}$ (see Remark 2.5 for the case $a = 0$). Let $\mathbf{s}_{a,\lambda}$ be a saddle fixed point of $H_{a,\lambda}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$. Let $\mathbf{v}_{a,\lambda} := (\bigcup_{j \in \mathbb{Z}_{\geq 0}} H_{a,\lambda}^{-j}(W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}))) \cap (\mathbb{D}_r \times \mathbb{D}_r)$. We say that $v$ is a *vertical component* of $\mathbf{v}_{a,\lambda}$ if $v$ is a connected component of $H_{a,\lambda}^{-m}(W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}))) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ for some $m \in \mathbb{Z}_{\geq 0}$. Under the assumption (7), we have

$$\mathbf{v}_{a,\lambda} \subset (\mathbb{D}_r \setminus \mathbb{D}_\varepsilon) \times \mathbb{D}_r \tag{8}$$

for $a \in \mathbb{D}_{\delta_0}$ by $W^s(\mathbf{s}_{a,\lambda}) \subset J^+_{a,\lambda}$ and (2). To construct a holomorphic motion of $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ over $a \in \mathbb{D}_{\delta_0}$ for each $y \in \mathbb{D}_r$, we prove the following two lemmas.

**Lemma 3.5.** *Suppose that $\{H_{a,\lambda}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies the* $(\mathrm{VC})_{\varepsilon,r}$ *and the condition (7) holds for $a \in \mathbb{D}_{\delta_0}$. Let $v_a$ be a vertical component of $\mathbf{v}_{a,\lambda}$ represented by $\{(f_a(y), y) : y \in \mathbb{D}_r\}$ for some holomorphic map $f_a$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$. Then for each $a \in \mathbb{D}_{\delta_0}$, $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ is the union of two distinct vertical components $v_{a,1}$ and $v_{a,2}$. Moreover, $v_{a,j}$ can be represented by $\{(f_{a,j}(y), y) : y \in \mathbb{D}_r\}$ for some holomorphic map $f_{a,j} \colon \mathbb{D}_r \to \mathbb{C}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$ for $j = 1, 2$.*

*Proof.* Let $v_a$ be a vertical component of $\mathbf{v}_{a,\lambda}$ represented by $\{(f_a(y), y) : y \in \mathbb{D}_r\}$ for some holomorphic map $f_a$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$.

We first show that the set $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{C} \times \{w\})$ consists of exactly two points for $w \in \mathbb{D}_r$ and $a \in \mathbb{D}_{\delta_0}$. For $a = 0$, $f_0$ is constant and $H_{0,\lambda}(x, y) = (x^2 + c(0, \lambda), 0)$. Clearly, $H_{0,\lambda}^{-1}(v_0) \cap (\mathbb{C} \times \{w\})$ consists of exactly two points for $w \in \mathbb{D}_r$ since the critical value $c(0, \lambda)$ of $x^2 + c(0, \lambda)$ does not belong to $f_0(\mathbb{D}_r)$ (see (7)). For $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$, recall that $H_{a,\lambda}^{-1}(x, y) = (y/a, (x - y^2/a^2 - c(a, \lambda))/a)$. Consider the equation

$$f_a(y) - y^2/a^2 - c(a, \lambda) = aw \tag{9}$$

for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$. We set $g_a(y) := f_a(y) - y^2/a^2 - c(a, \lambda) - aw$ and $h_a(y) := -f_a(y) + c(a, \lambda) + aw$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$. By $v_a \subset \mathbf{v}_{a,\lambda} \subset \mathbb{D}_r \times \mathbb{D}_r$, we have $|f_a(y)| \leq r$ for $y \in \mathbb{D}_r$. If $|a| < 1$, then for $y \in \partial \mathbb{D}_{\sqrt{(4r+4|c(a,\lambda)|)|a|^2}}$, we have

$$\begin{aligned} |g_a(y)| &\geq |y^2/a^2| - |c(a, \lambda)| - |aw| - |f_a(y)| \geq (3 - |a|)r + 3|c(a, \lambda)| > r + |c(a, \lambda)| + |a|r \\ &\geq |f_a(y)| + |c(a, \lambda)| + |aw| \geq |h_a(y)|. \end{aligned}$$

By Rouché's theorem, $g_a$ and $g_a + h_a$ have the same number of zeros inside $\mathbb{D}_{\sqrt{(4r+4|c(a,\lambda)|)|a|^2}}$. Since $g_a(y) + h_a(y) = -y^2/a^2$, the map $g_a$ has two zeros in $\mathbb{D}_{\sqrt{(4r+4|c(a,\lambda)|)|a|^2}}$. Moreover, the map $g_a$ has two distinct zeros by using the condition (ii) and (iii) in Definition 3.1. Indeed, let $g_a(y_0) = 0$. Then we have $(y_0/a, w) \in H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{C} \times \{w\})$. By (8), we have $|y_0/a| > \varepsilon$. Moreover, we have $|f_a'(y_0)| < 1$ by the condition (ii), (iii) in Definition 3.1, and (6). Hence we have

$$|g_a'(y_0)| \geq |2y_0/a^2| - |f_a'(y_0)| > 2\varepsilon/|a| - 1 > 2\varepsilon/\delta_0 - 1 > 0, \tag{10}$$

by $\delta_0 < \min\{\varepsilon, 1/2\}$ (see Definition 3.1). Thus, $g_a$ has two distinct zeros in $\mathbb{D}_{\sqrt{(4r+4|c(a,\lambda)|)|a|^2}}$. On the other hand, by $\delta_0 < \min\{\varepsilon, 1/2\}$, if $y \in \mathbb{D}_r$ satisfies $|y| \geq \sqrt{(4r + 4|c(a, \lambda)|)|a|^2}$, then

$$|\mathrm{Pr}_2\, H_{a,\lambda}^{-1}(f_a(y), y)| = \frac{|f_a(y) - y^2/a^2 - c(a, \lambda)|}{|a|} \geq \frac{|y^2/a^2| - |f_a(y)| - |c(a, \lambda)|}{|a|} \geq \frac{3r + 3|c(a, \lambda)|}{|a|} > r.$$

Hence, there are exactly two distinct solutions of (9) with respect to $y \in \mathbb{D}_r$, which belong to $\mathbb{D}_{\sqrt{(4r+4|c(a,\lambda)|)|a|^2}}$. This implies that the set $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{C} \times \{w\})$ consists of exactly two points for

$w \in \mathbb{D}_r$. Note that $H_{a,\lambda}^{-1}(v_a) \cap ((\mathbb{C} \setminus \mathbb{D}_r) \times \{w\}) = \emptyset$ by (2), otherwise $W^s(\mathbf{s}_{a,\lambda}) \cap V^+ \neq \emptyset$. Thus, $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ is the union of two vertical components of $\mathbf{v}_{a,\lambda}$.

Recall that $f_0$ is constant. Assume $f_0 \equiv A$ for some $A \in \mathbb{C}$. Let $z^2 + c(0,\lambda) - A = (z - A_1)(z - A_2)$. Then we have $H_{0,\lambda}^{-1}(v_0) \cap (\mathbb{D}_r \times \mathbb{D}_r) = (\{A_1\} \cup \{A_2\}) \times \mathbb{D}_r$. There are positive constants $\delta_1, \varepsilon_1 > 0$ such that $\delta_1 < \delta_0$, $\mathbb{D}_{\varepsilon_1}(A_1) \cap \mathbb{D}_{\varepsilon_1}(A_2) = \emptyset$, $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_{\varepsilon_1}(A_j) \times \mathbb{D}_r) \neq \emptyset$ for $j = 1, 2$ and $a \in \mathbb{D}_{\delta_1}$, and $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r) \subset (\mathbb{D}_{\varepsilon_1}(A_1) \cup \mathbb{D}_{\varepsilon_1}(A_2)) \times \mathbb{D}_r$ for $a \in \mathbb{D}_{\delta_1}$. For $a \in \mathbb{D}_{\delta_1}$ and $j \in \{1, 2\}$, we let $v_{a,j}$ be the component of $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ which is contained in $\mathbb{D}_{\varepsilon_1}(A_j) \times \mathbb{D}_r$.

We show that $v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_1}$ for each fixed $w \in \mathbb{D}_r$ by using the implicit function theorem. We set $F(a, y) := f_a(y) - y^2/a^2 - c(a, \lambda) - aw$ for $a \in \mathbb{D}_{\delta_1} \setminus \{0\}$. Since $T_{(f_a(y),y)} v \subset C^v_{(f_a(y),y)}$ for $y \in \mathbb{D}_r$, we have $|\partial_y f_a(y)| < 1$ for $y \in \mathbb{D}_r$. Fix arbitrary points $a \in \mathbb{D}_{\delta_1} \setminus \{0\}$ and $\tilde{z}_j = \tilde{z}_j(a)$ with $F(a, \tilde{z}_j(a)) = 0$ and $\tilde{z}_j(a)/a \in \mathbb{D}_{\varepsilon_1}(A_j)$. Then, we have $(\tilde{z}_j/a, (f_a(\tilde{z}_j) - \tilde{z}_j^2/a^2 - c(a,\lambda))/a) = (\tilde{z}_j/a, w) \in v_{a,j} \cap (\mathbb{C} \times \{w\})$. By (8), we have $|\tilde{z}_j/a| > \varepsilon$. Since $F(a, \tilde{z}_j) = g_a(\tilde{z}_j)$, we have $|\partial_y F(a, \tilde{z}_j)| > 0$ (see (10)). By the implicit function theorem, $\{\tilde{z}_j/a, w\} = v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_1} \setminus \{0\}$. Moreover, $\tilde{z}_j(a)^2/a^2 = f_a(\tilde{z}_j(a)) - c(a, \lambda) - aw$ since $F(a, \tilde{z}_j(a)) = 0$. Note that $\tilde{z}_j(a) \in \mathbb{D}_{r/2}$ by $r > |w| = |\mathrm{Pr}_2\, H_{a,\lambda}^{-1}(f_a(\tilde{z}_j), \tilde{z}_j)|$ and ( i ) in Definition 3.1. Since $f_a(z)$ is holomorphic with respect to $a \in \mathbb{D}_{\delta_0}$ and $z \in \mathbb{D}_r$, $f_a(z) \to f_0(z)$ uniformly on $\overline{\mathbb{D}_{r/2}}$ as $a \to 0$. This implies that $\tilde{z}_j(a)/a \to A_j$ as $a \to 0$ since $f_a(z) \to f_0(z) \equiv A$ as $a \to 0$. Thus $\{\tilde{z}_j/a\} = v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_1}$.

For each $a \in \mathbb{D}_{\delta_0} \setminus \mathbb{D}_{\delta_1}$ and each $j \in \{1, 2\}$, $v_{a,j} \cap (\mathbb{C} \times \{w\})$ can be analytically continued along a path connecting $a$ and a point in $\mathbb{D}_{\delta_1}$. By the monodromy theorem, for each $a \in \mathbb{D}_{\delta_0}$ and each $j \in \{1, 2\}$, there is a component $v_{a,j}$ of $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ such that $v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_0}$ for each fixed $w \in \mathbb{D}_r$. We show that $v_{a,1} \neq v_{a,2}$ for each $a \in \mathbb{D}_{\delta_0} \setminus \mathbb{D}_{\delta_1}$. Assume that $v_{a,1} = v_{a,2}$ for some $a \in \mathbb{D}_{\delta_0} \setminus \mathbb{D}_{\delta_1}$. There exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $\{a \in \mathbb{D}_{\delta_0} : v_{a,1} = v_{a,2}\}$ such that $a_n \to a_0$ as $n \to \infty$ and

$$|a_0| = \inf\{|a| \in \mathbb{D}_{\delta_0} : v_{a,1} = v_{a,2}\}. \tag{11}$$

Clearly, we have $|a_0| \geq \delta_1$ by the argument above. Moreover, we have $v_{a_0,1} = v_{a_0,2}$. Otherwise, $v_{a,1} \neq v_{a,2}$ for all $a$ in a small neighborhood of $a_0$, which implies that $a_n \notin \{a \in \mathbb{D}_{\delta_0} : v_{a,1} = v_{a,2}\}$ for sufficiently large $n$. Consider the vertical component $v_{a_0}$ of $\mathbf{v}_{a_0,\lambda}$. Let $H_{a_0,\lambda}^{-1}(v_{a_0}) \cap (\mathbb{D}_r \times \mathbb{D}_r) = \tilde{v}_{a_0,1} \cup \tilde{v}_{a_0,2}$ for some vertical components $\tilde{v}_{a_0,1}, \tilde{v}_{a_0,2}$ of $\mathbf{v}_{a,\lambda}$. There are open neighborhoods $U_1, U_2$ of $\tilde{v}_{a_0,1}, \tilde{v}_{a_0,2}$ respectively such that $U_1 \cap U_2 = \emptyset$. We may assume that $v_{a_0,1} (= v_{a_0,2}) = \tilde{v}_{a_0,1}$. We take a positive constant $\delta_2 > 0$ such that $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r) \cap U_j \neq \emptyset$ for $j = 1, 2$ and $a \in \mathbb{D}_{\delta_2}(a_0)$, and $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r) \subset U_1 \cup U_2$ for $a \in \mathbb{D}_{\delta_2}(a_0)$. Recall that $F(a, y) = f_a(y) - y^2/a^2 - c(a, \lambda) - aw$ for $w \in \mathbb{D}_r$. We see that $|\partial_y F(a_0, \tilde{z}_1)| > 0$, where $\tilde{z}_1$ satisfies that $F(a_0, \tilde{z}_1) = 0$ and $(\tilde{z}_1/a_0, w) \in U_1$ (see (10)). By the implicit function theorem, $v_{a,1} \cap (\mathbb{C} \times \{w\}) \subset U_1$ moves holomorphically over $\mathbb{D}_{\delta_2}(a_0)$ for each fixed $w \in \mathbb{D}_r$. We can take $\tilde{a} \in \mathbb{D}_{\delta_2}(a_0)$ with $|\tilde{a}| < |a_0|$ such that $v_{\tilde{a},1} = v_{\tilde{a},2}$. This contradicts (11). Thus, $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r) = v_{a,1} \cup v_{a,2}$ and $v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_0}$ for each fixed $w \in \mathbb{D}_r$ and $j \in \{1, 2\}$.

Finally, we show that there are holomorphic maps $f_{a,1}, f_{a,2}$ such that $v_{a,j} = \{(f_{a,j}(y), y) : y \in \mathbb{D}_r\}$ for $j = 1, 2$. Since $\mathrm{Pr}_2 \colon v_{a,j} \to \mathbb{D}_r$ is a bijective holomorphic map for $a \in \mathbb{D}_{\delta_0}$ and $j \in \{1, 2\}$, there are holomorphic maps $f_{a,1}, f_{a,2}$ such that $v_{a,j} = \{(f_{a,j}(y), y) : y \in \mathbb{D}_r\}$ for $a \in \mathbb{D}_{\delta_0}$ and $j = 1, 2$. □

Let $W_{a,\lambda}$ be the vertical component of $\mathbf{v}_{a,\lambda}$ such that $H_{a,\lambda}(W_{a,\lambda}) \subset W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda})$ and $W_{a,\lambda} \cap W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$.

**Lemma 3.6.** *Suppose that $\{H_{a,\lambda}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$ and the condition (7) holds for $a \in \mathbb{D}_{\delta_0}$. Then the two sets $H_{a,\lambda}^{-n}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ and $H_{a,\lambda}^{-m}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ do not intersect for $n \neq m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{D}_{\delta_0}$.*

*Proof.* Fix $a \in \mathbb{D}_{\delta_0}$. Assume that there are components $v_n$ and $v_m$ of $H_{a,\lambda}^{-n}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ and $H_{a,\lambda}^{-m}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ respectively such that $v_n \cap v_m \neq \emptyset$. We may assume that $n > m$. We see that $H_{a,\lambda}^{m}(v_n)$ is a subset of some component of $H_{a,\lambda}^{-n+m}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$. On the other hand, $H_{a,\lambda}^{m}(v_m) \subset W_{a,\lambda}$. Thus, an intersection point of $v_n$ and $v_m$ is mapped under $H_{a,\lambda}^{m}$ into $W_{a,\lambda} \cap H_{a,\lambda}^{-n+m}(W_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$. Hence, we have $H_{a,\lambda}^{n-m}(W_{a,\lambda}) \cap W_{a,\lambda} \neq \emptyset$. This contradicts $H_{a,\lambda}(W_{a,\lambda}) \subset W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda})$, $H_{a,\lambda}(W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda})) \subset W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda})$ and $W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) \cap W_{a,\lambda} = \emptyset$. Thus we have proved Lemma 3.6. □

Let us consider the section $S_{a,\lambda,y} := W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{C} \times \{y\})$ for $y \in \mathbb{D}_r$. Recall that $H_{0,\lambda}(x,y) = (p_\lambda(x), 0)$. We denote the Julia set of $p_\lambda$ by $J_{p_\lambda}$. We now construct holomorphic motions.

**Lemma 3.7.** *Suppose that* $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ *satisfies the* $(\mathrm{VC})_{\varepsilon,r}$ *and the condition (7) holds for each* $a \in \mathbb{D}_{\delta_0}$. *Then there exists a holomorphic motion* $h_{\lambda,y}\colon \mathbb{D}_{\delta_0} \times S_{0,\lambda,y} \to \mathbb{C} \times \{y\}$ *such that* $h_{\lambda,y}(a,(x,y)) \in S_{a,\lambda,y}$ *for* $y \in \mathbb{D}_r$, $a \in \mathbb{D}_{\delta_0}$ *and* $(x,y) \in S_{0,\lambda,y}$. *In particular,* $h_{\lambda,y}$ *can be extended to a holomorphic motion* $\tilde{h}_{\lambda,y}\colon \mathbb{D}_{\delta_0} \times (\mathbb{C} \times \{y\}) \to \mathbb{C} \times \{y\}$ *and* $\tilde{h}_{\lambda,y}(\{a\} \times (J_{p_\lambda} \times \{y\})) = \overline{S_{a,\lambda,y}}$.

*Proof.* By Lemma 3.5, $H_{a,\lambda}^{-j}(W_{a,\lambda}) \cap (\mathbb{C} \times \{y\})$ has exactly $2^j$ elements for each $j \in \mathbb{Z}_{\geq 0}$. By Lemma 3.6, we have

$$S_{a,\lambda,y} = (W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) \cap (\mathbb{C} \times \{y\})) \sqcup \bigsqcup_{j\in\mathbb{Z}_{\geq 0}} (H_{a,\lambda}^{-j}(W_{a,\lambda}) \cap (\mathbb{C} \times \{y\})).$$

By (iii) in Definition 3.1, we have $W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) = \{(f_a(y), y) : y \in \mathbb{D}_r\}$ for some holomorphic map $f_a\colon \mathbb{D}_r \to \mathbb{C}$ depending holomorphically on $a \in \mathbb{D}_{\delta_0}$.

Let $v_a$ be a vertical component of $\mathbf{v}_{a,\lambda}$ such that $v_a \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_0}$ for each $w \in \mathbb{D}_r$. By Lemma 3.5, there are two vertical components $v_{a,1}$ and $v_{a,2}$ of $\mathbf{v}_{a,\lambda}$ such that $H_{a,\lambda}^{-1}(v_a) \cap (\mathbb{D}_r \times \mathbb{D}_r) = v_{a,1} \cup v_{a,2}$, and $v_{a,j} \cap (\mathbb{C} \times \{w\})$ moves holomorphically over $\mathbb{D}_{\delta_0}$ for each fixed $w \in \mathbb{D}_r$ and $j \in \{1,2\}$. Thus, we can construct $h_{\lambda,y}\colon \mathbb{D}_{\delta_0} \times S_{0,\lambda,y} \to \mathbb{C} \times \{y\}$ such that $h_{\lambda,y}(a,(x,y)) \in S_{a,\lambda,y}$ for $y \in \mathbb{D}_r$, $a \in \mathbb{D}_{\delta_0}$ and $(x,y) \in S_{0,\lambda,y}$. By Lemma 3.6, for each fixed $a \in \mathbb{D}_{\delta_0}$ and $y \in \mathbb{D}_r$, $h_{\lambda,y}(a,(x,y))$ is injective with respect to $x$ with $(x,y) \in S_{0,\lambda,y}$. The map $h_{\lambda,y}$ can be extended to a holomorphic motion $\tilde{h}_{\lambda,y}\colon \mathbb{D}_{\delta_0} \times \mathbb{C} \times \{y\} \to \mathbb{C} \times \{y\}$ and $\tilde{h}_{\lambda,y}(\{a\} \times (J_{p_\lambda} \times \{y\})) = \overline{S_{a,\lambda,y}}$ (see [11] and [19]). □

**Corollary 3.8.** *Suppose that* $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ *satisfies the* $(\mathrm{VC})_{\varepsilon,r}$ *and the condition (7) holds for* $a \in \mathbb{D}_{\delta_0}$. *Then* $J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \bigcup_{y\in\mathbb{D}_r} \overline{S_{a,\lambda,y}}$. *In particular,* $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ *is path connected for* $y \in \mathbb{D}_r$ *if* $J_{p_\lambda}$ *is connected.*

*Proof.* We first show that

$$\overline{W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \overline{W^s(\mathbf{s}_{a,\lambda})} \cap (\mathbb{D}_r \times \mathbb{D}_r) \tag{12}$$

as follows. Let $(z_1, w_1) \in \overline{W^s(\mathbf{s}_{a,\lambda})} \cap (\mathbb{D}_r \times \mathbb{D}_r)$. Since $(z_1, w_1) \in \mathbb{D}_r \times \mathbb{D}_r$, for each $n \in \mathbb{N}$, there exists $(z_{1,n}, w_{1,n}) \in W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ such that $(z_{1,n}, w_{1,n}) \to (z_1, w_1)$ as $n \to \infty$. Thus $(z_1, w_1) \in \overline{W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)}$. Hence $(z_1, w_1) \in \overline{W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r)} \cap (\mathbb{D}_r \times \mathbb{D}_r)$. The opposite inclusion is obvious. Thus we have shown (12).

We next show that

$$\overline{\bigcup_{y\in\mathbb{D}_r} (S_{a,\lambda,y})} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \bigcup_{y\in\mathbb{D}_r} \overline{S_{a,\lambda,y}} \tag{13}$$

as follows. Let $(z_2, w_2) \in \overline{\bigcup_{y\in\mathbb{D}_r} (S_{a,\lambda,y})} \cap (\mathbb{D}_r \times \mathbb{D}_r)$. By $(z_2, w_2) \in \mathbb{D}_r \times \mathbb{D}_r$, we can take $(z_{2,n}, w_{2,n}) \in (\bigcup_{y\in\mathbb{D}_r} S_{a,\lambda,y}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ such that $(z_{2,n}, w_{2,n}) \to (z_2, w_2)$ as $n \to \infty$. Let $v_n$ be the vertical component $\mathbf{v}_{a,\lambda}$ which contains $(z_{2,n}, w_{2,n})$. Let $x_n \in \mathbb{D}_r$ such that $(x_n, w_2) \in v_n$. To show $(z_2, w_2) \in \bigcup_{y \in \mathbb{D}_r} \overline{S_{a,\lambda,y}}$, we show that $(x_n, w_2) \to (z_2, w_2)$ as $n \to \infty$. Since $\|(x_n, w_2) - (z_2, w_2)\| \leq \|(x_n, w_2) - (z_{2,n}, w_{2,n})\| + \|(z_{2,n}, w_{2,n}) - (z_2, w_2)\|$, it suffices to show that $\|(x_n, w_2) - (z_{2,n}, w_{2,n})\| \to 0$ as $n \to \infty$. Since $\mathrm{Pr}_2 \colon v_n \to \mathbb{D}_r$ is a homeomorphism, we can take a curve $\gamma \colon [0,1] \to v_n$, represented by $\gamma(s) = (f(s), g(s))$, between $(x_n, w_2)$ and $(z_{2,n}, w_{2,n})$ such that $\mathrm{length}_E(\mathrm{Pr}_2 \gamma) = |w_{2,n} - w_2|$, where $\mathrm{length}_E(\mathrm{Pr}_2 \gamma)$ is the length of the curve $\mathrm{Pr}_2 \gamma$ with respect to the Euclidean metric. Note that $\mathrm{Pr}_2 \gamma$ is the segment between $w_{2,n}$ and $w_2$. By $(f'(s), g'(s)) \in C^v_{\gamma(s)}$ (see (ii) in Definition 3.1), we have

$$\|(x_n, w_2) - (z_{2,n}, w_{2,n})\| \leq \mathrm{length}_E \gamma \leq \sqrt{2} \int_0^1 \max\{|f'(s)|, |g'(s)|\} ds = \sqrt{2} \int_0^1 |g'(s)| ds.$$

Since $\int_0^1 |g'(s)| ds = |w_{2,n} - w_2| \to 0$ as $n \to \infty$, we have that $\|(x_n, w_2) - (z_{2,n}, w_{2,n})\| \to 0$ as $n \to \infty$. Hence we have $(z_2, w_2) \in \bigcup_{y \in \mathbb{D}_r} \overline{S_{a,\lambda,y}}$. Therefore $\overline{\bigcup_{y \in \mathbb{D}_r} (S_{a,\lambda,y})} \cap (\mathbb{D}_r \times \mathbb{D}_r) \subset \bigcup_{y \in \mathbb{D}_r} \overline{S_{a,\lambda,y}}$. The opposite inclusion is obvious. Thus we have shown (13).

By $J^+_{a,\lambda} = \overline{W^s(\mathbf{s}_{a,\lambda})}$ (see [3]), $W^s(\mathbf{s}_{a,\lambda}) \cap (\mathbb{D}_r \times \mathbb{D}_r) = \bigcup_{y \in \mathbb{D}_r} S_{a,\lambda,y}$, (12) and (13), we have

$$J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \overline{W^s(\mathbf{s}_{a,\lambda})} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \overline{\bigcup_{y \in \mathbb{D}_r} (S_{a,\lambda,y})} \cap (\mathbb{D}_r \times \mathbb{D}_r) = \bigcup_{y \in \mathbb{D}_r} \overline{S_{a,\lambda,y}}.$$

By the above, we have $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\}) = \overline{S_{a,\lambda,y}}$ for $y \in \mathbb{D}_r$. Note that the condition $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda} = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$ implies that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap (J_{p_\lambda} \times \mathbb{C}) = \emptyset$ since $J^+_{0,\lambda} = J_{p_\lambda} \times \mathbb{C}$. In particular, the critical point 0 of $p_\lambda$ belongs to $\mathbb{C} \setminus J_{p_\lambda}$. Since $J_{p_\lambda}$ is connected, $p_\lambda$ has an attracting or parabolic periodic point. Hence $J_{p_\lambda}$ is the image of unit circle under a continuous map (see [5]). In particular, $J_{p_\lambda}$ is path connected. Since $\overline{S_{a,\lambda,y}}$ is homeomorphic to $J_{p_\lambda}$ (see Lemma 3.7), the section $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y\})$ is path connected for each $y \in \mathbb{D}_r$. □

The following lemma is useful for checking whether $J_{a,\lambda}$ is disconnected.

**Lemma 3.9.** *Suppose that $\{H_{a,\lambda}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies the condition* $(\mathrm{VC})_{\varepsilon,r}$ *and the condition (7) holds for $a \in \mathbb{D}_{\delta_0}$. Then each vertical component $v$ of $\mathbf{v}_{a,\lambda}$ contains a point of $J_{a,\lambda}$ for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$.*

*Proof.* Let $v_0 := v$ be a vertical component of $\mathbf{v}_{a,\lambda}$. Inductively, let $v_n$ be a component of $H^{-1}_{a,\lambda}(v_{n-1}) \cap (\mathbb{D}_r \times \mathbb{D}_r)$ for each $n \in \mathbb{N}$. Then we have the nested compact sequence $\{H^n_{a,\lambda}(\overline{v_n})\}_{n \geq 0}$ with $H^n_{a,\lambda}(\overline{v_n}) \subset H^{n-1}_{a,\lambda}(\overline{v_{n-1}})$ for $n \in \mathbb{N}$. For any $a \in \mathbb{D}_1$, we have $J^-_{a,\lambda} = K^-_{a,\lambda}$ (see Lemma 5.5 in [2]). A point of $\bigcap_{n \in \mathbb{N}} H^{n-1}_{a,\lambda}(\overline{v_{n-1}})$ belongs to $J_{a,\lambda}$ since its backward orbit is bounded and $\overline{v}$ is a subset of $J^+_{a,\lambda}$. □

We now prove the first main result of this paper, divided into Theorem 3.10 and Theorem 3.11. Theorem 3.10 relates to the connected case, and Theorem 3.11 to the disconnected case.

**Theorem 3.10.** *Let $\{H_{a,\lambda}\}_{a \in \mathbb{D}_{\delta_0}}$ be a Hénon family satisfying the* $(\mathrm{VC})_{\varepsilon,r}$. *Assume that the condition (7) holds for $a \in \mathbb{D}_{\delta_0}$. If the Julia set $J_{p_\lambda}$ of the polynomial $p_\lambda$ is connected, then the Julia set $J_{a,\lambda}$ of the Hénon map $H_{a,\lambda}$ is connected for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$.*

*Proof.* Assume that $J_{p_\lambda}$ is connected and $\lambda \neq 1$. We first show that $J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r)$ is connected. We take any distinct points $(x_1, y_1)$ and $(x_2, y_2)$ in $J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r)$. We construct a path between these points as follows. Let $v$ be a vertical component of $\mathbf{v}_{a,\lambda}$, and $(z_j, y_j)$ be the intersection of $v \cap (\mathbb{C} \times \{y_j\})$ for $j = 1, 2$. Since $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y_j\})$ is homeomorphic to $J_{p_\lambda}$ for $j = 1, 2$, and $J_{p_\lambda}$ is a path connected, there exists a path between $(z_j, y_j)$ and $(x_j, y_j)$ in $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y_j\})$ for each $j = 1, 2$. We can take a path between $(z_1, y_1)$ and $(z_2, y_2)$ in $v$ since $v$ is path connected. Thus, $J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r)$ is path connected, which implies that it is connected.

We now show that $J_{a,\lambda}$ is connected. We see that $J_{a,\lambda} = \bigcap_{k\geq 0} \overline{(H^k_{a,\lambda}(J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r)))}$. Moreover, we have $\overline{H^{k+1}_{a,\lambda}(J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r))} \subset \overline{H^k_{a,\lambda}(J^+_{a,\lambda} \cap (\mathbb{D}_r \times \mathbb{D}_r))}$. Hence $J_{a,\lambda}$ is a nested intersection of connected compact subsets. Thus $J_{a,\lambda}$ is connected. □

We next show the following theorem.

**Theorem 3.11.** *Let $\{H_{a,\lambda}\}_{a\in\mathbb{D}_{\delta_0}}$ be a Hénon family satisfying the condition $(\mathrm{VC})_{\varepsilon,r}$. Assume that the condition (7) holds for $a \in \mathbb{D}_{\delta_0}$. If the Julia set $J_{p_\lambda}$ of the polynomial $p_\lambda$ is disconnected, then the Julia set $J_{a,\lambda}$ of the Hénon map $H_{a,\lambda}$ is disconnected for $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$.*

*Proof.* Assume that $J_{p_\lambda}$ is disconnected. That is, it is a Cantor set. We show that $J_{a,\lambda}$ is disconnected. We can take a Jordan curve $\gamma_j \subset \mathbb{C}$ for each $j = 1, 2$ with $\gamma_1 \cap \gamma_2 = \emptyset$ such that $J_{p_\lambda} \subset U_1 \cup U_2$ and $J_{p_\lambda} \cap U_j \neq \emptyset$ for $j = 1, 2$, where $U_j$ is the bounded domain with the boundary $\gamma_j$ for $j = 1, 2$. Let $\tilde{h}_{\lambda,y} \colon \mathbb{D}_{\delta_0} \times (\mathbb{C} \times \{y\}) \to \mathbb{C} \times \{y\}$ be the holomorphic motion given in Lemma 3.7.

Fix $a \in \mathbb{D}_{\delta_0} \setminus \{0\}$. Let $\gamma_{a,y,j} := \mathrm{Pr}_1\, \tilde{h}_{\lambda,y}(\{a\} \times (\gamma_j \times \{y\})) \subset \mathbb{C}$ and $U_{a,y,j} = \mathrm{Pr}_1\, \tilde{h}_{\lambda,y}(\{a\} \times (U_j \times \{y\})) \subset \mathbb{C}$ for $j = 1, 2$ and $y \in \mathbb{D}_r$. Since $(\gamma_{a,y,j} \times \{y\}) \cap J^+_{a,\lambda} = \emptyset$ for $j = 1, 2$ and $y \in \mathbb{D}_r$, there exists $\varepsilon_1(a, y)$ with $0 < \varepsilon_1(a, y) < r - |y|$ such that

$$(\bigcup_{y\in\mathbb{D}_r} (\gamma_{a,y,j} \times \overline{\mathbb{D}_{\varepsilon_1(a,y)}(y)})) \cap J^+_{a,\lambda} = \emptyset \tag{14}$$

for each $j = 1, 2$ and each $y \in \mathbb{D}_r$. We now show that

$$J^+_{a,\lambda} \cap (\mathbb{C} \times \overline{\mathbb{D}_{\varepsilon_1(a,y)}(y)}) \subset (\bigcup_{j=1}^{2} U_{a,y,j}) \times \overline{\mathbb{D}_{\varepsilon_1(a,y)}(y)} \tag{15}$$

for each $y \in \mathbb{D}_r$. Fix $y_0 \in \mathbb{D}_r$. Let $(z, w) \in (\mathbb{C} \setminus \bigcup_{j=1}^2 U_{a,y_0,j}) \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)}$. In order to show (15), it suffices to show that $(z, w) \notin J^+_{a,\lambda} \cap (\mathbb{C} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)})$. If $(z, w) \in \bigcup_{j=1}^2 \gamma_{a,y_0,j} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)}$, then $(z, w) \notin J^+_{a,\lambda}$ by (14). Thus we may assume that $(z, w) \in (\mathbb{C} \setminus \bigcup_{j=1}^2 \overline{U_{a,y_0,j}}) \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)}$. By assuming that $(z, w) \in J^+_{a,\lambda} \cap (\mathbb{C} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)})$, we derive a contradiction as follows. Note that $w \in \mathbb{D}_r$ by $\varepsilon_1(a, y_0) < r - |y_0|$. By $\overline{W^s(\mathbf{s}_{a,\lambda})} = J^+_{a,\lambda}$, there exists a vertical component $v_0$ of $\mathbf{v}_{a,\lambda}$ and a point $\hat{z} \in \mathbb{C}$ such that $\{(\hat{z}, w)\} = v_0 \cap (\mathbb{C} \times \{w\}) \subset (\mathbb{C} \setminus \bigcup_{j=1}^2 \overline{U_{a,y_0,j}}) \times \{w\}$. Let $x_0 \in \mathbb{C}$ be the point such that $\{(x_0, y_0)\} = v_0 \cap (\mathbb{C} \times \{y_0\})$. By $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y_0\}) = \overline{S_{a,\lambda,y_0}}$ (see Corollary 3.8), we have $J^+_{a,\lambda} \cap (\mathbb{C} \times \{y_0\}) \subset (\bigcup_{j=1}^2 U_{a,y_0,j}) \times \{y_0\}$. Hence, we have $\{(x_0, y_0)\} = v_0 \cap (\mathbb{C} \times \{y_0\}) \subset (\bigcup_{j=1}^2 U_{a,y_0,j}) \times \{y_0\}$. Since $\mathrm{Pr}_2 \colon v_0 \to \mathbb{D}_r$ is a homeomorphism, we can take a path $\gamma_0 \subset v_0 \cap (\mathbb{C} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)})$ between $(x_0, y_0)$ and $(\hat{z}, w)$. Then we have $(\mathrm{Pr}_1\, \gamma_0) \cap (\bigcup_{j=1}^2 \gamma_{a,y_0,j}) \neq \emptyset$ which implies that $v_0 \cap ((\bigcup_{j=1}^2 \gamma_{a,y_0,j}) \times \overline{\mathbb{D}_{\varepsilon_1(a,y_0)}(y_0)}) \neq \emptyset$. By $v_0 \subset J^+_{a,\lambda}$ and (14), we have a contradiction. Thus we have (15) for each $y \in \mathbb{D}_r$.

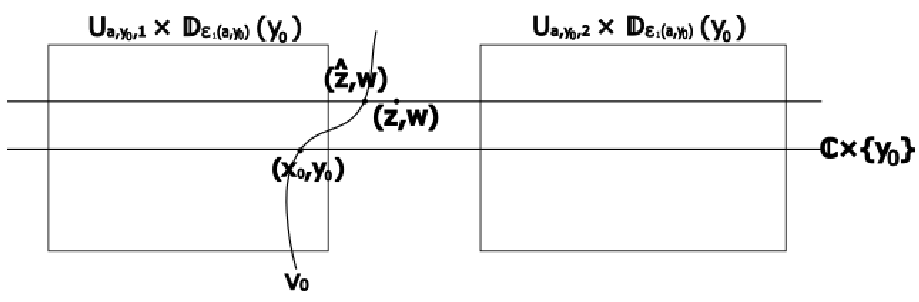


Figure 3: The case $(x_0, y_0) \in U_{a,y_0,1} \times \{y_0\}$

Let $\mathbf{U}_1 = \bigcup_{y\in\mathbb{D}_r}(U_{a,y,1} \times \mathbb{D}_{\varepsilon_1(a,y)/2}(y))$. Since $\overline{\mathbb{D}_{r/2}}$ is compact, we can take a positive integer $N \in \mathbb{N}$ and points $y_1, y_2, \cdots, y_N \in \mathbb{D}_r$ such that $\mathbf{U}_{1,N} := \bigcup_{j=1}^{N}(U_{a,y_j,1} \times \mathbb{D}_{\varepsilon_1(a,y_j)/2}(y_j))$ satisfies $\overline{\mathbb{D}_{r/2}} \subset \mathrm{Pr}_2\, \mathbf{U}_{1,N}$. Let $\mathbf{v}_{a,\lambda,j}$ be the union of all vertical components of $\mathbf{v}_{a,\lambda}$ which intersect $U_{a,0,j} \times \{0\}$ for $j = 1, 2$. We now show that

$$\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) \subset \mathbf{U}_{1,N}. \tag{16}$$

Take a vertical component $v_1 \subset \mathbf{v}_{a,\lambda,1}$ of $\mathbf{v}_{a,\lambda}$ and a point $w_1 \in \mathbb{D}_{r/2}$. To show (16), it suffices to show that $v_1 \cap (\mathbb{D}_r \times \{w_1\}) \subset \mathbf{U}_{1,N}$. Let $\tilde{h}_{a,\lambda,y} \colon \mathbb{C} \times \{y\} \to \mathbb{C} \times \{y\}, \tilde{h}_{a,\lambda,y}(x, y) := \tilde{h}_{\lambda,y}(a, (x, y))$ for $y \in \mathbb{D}_r$ and $x \in \mathbb{C}$. By the construction of $\tilde{h}_{\lambda,y}$ (see the proof of Lemma 3.7), $\mathrm{Pr}_1 \bigcup_{y\in\mathbb{D}_r} \tilde{h}^{-1}_{a,\lambda,y}(v_1 \cap (\mathbb{C} \times \{y\}))$ consists of a single point of $J_{p_\lambda}$, say $b_1$. We have $b_1 \in U_1$ by $v_1 \cap (U_{a,0,1} \times \{0\}) \neq \emptyset$. Since $w_1 \in \mathbb{D}_{r/2}$, we can take a positive integer $k_1$ with $1 \leq k_1 \leq N$ such that $w_1 \in \mathbb{D}_{\varepsilon_1(a,y_{k_1})/2}(y_{k_1})$. Let $x_1 \in \mathbb{C}$ be the point such that $\{(x_1, y_{k_1})\} = v_1 \cap (\mathbb{C} \times \{y_{k_1}\})$. By $\mathrm{Pr}_1\, \tilde{h}^{-1}_{a,\lambda,y_{k_1}}(U_{a,y_{k_1},1} \times \{y_{k_1}\}) = U_1 \ni b_1$, we have $(x_1, y_{k_1}) \in U_{a,y_{k_1},1} \times \mathbb{D}_{\varepsilon_1(a,y_{k_1})/2}(y_{k_1})$. This implies that $v_1 \cap (U_{a,y_{k_1},1} \times \{y_{k_1}\}) \neq \emptyset$. By (14) and (15), we have $\{(z_1, w_1)\} := v_1 \cap (\mathbb{D}_r \times \{w_1\}) \subset U_{a,y_{k_1},1} \times \{w_1\}$. Otherwise, for any path in $v_1 \cap (\mathbb{C} \times \mathbb{D}_{\varepsilon_1(a,y_{k_1})/2}(y_{k_1}))$ between $(z_1, w_1)$ and $(x_1, y_{k_1})$, the path necessarily intersects $\gamma_{a,y_{k_1},1} \times \mathbb{D}_{\varepsilon_1(a,y_{k_1})/2}(y_{k_1})$. This leads to a contradiction (see the proof of (15)). Thus we have $v_1 \cap (\mathbb{D}_r \times \{w_1\}) \subset U_{a,y_{k_1},1} \times \{w_1\}$, which implies that $v_1 \cap (\mathbb{D}_r \times \{w_1\}) \subset U_{a,y_{k_1},1} \times \mathbb{D}_{\varepsilon_1(a,y_{k_1})/2}(y_{k_1}) \subset \mathbf{U}_{1,N}$. Thus we have (16).

By using (16), we show that

$$\overline{\mathbf{v}_{a,\lambda,1}} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) \subset \mathbf{U}_{1,N}. \tag{17}$$

We have $\overline{\mathbf{v}_{a,\lambda,1}} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) = \bigcup_{y\in\mathbb{D}_{r/2}} \overline{\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \{y\})}$ by the same argument as in the proof of (13). Thus, to show that (17), it suffices to show that $\overline{\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \{y\})} \subset \mathbf{U}_{1,N}$ for $y \in \mathbb{D}_{r/2}$. Fix $\hat{w}_1 \in \mathbb{D}_{r/2}$. By using the argument in the proof of (16), we can take $\hat{k}_1 \in \mathbb{N}$ with $1 \leq \hat{k}_1 \leq N$ such that $\hat{w}_1 \in \mathbb{D}_{\varepsilon_1(a,y_{\hat{k}_1})/2}(y_{\hat{k}_1})$ and $\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \{\hat{w}_1\}) \subset U_{a,y_{\hat{k}_1},1} \times \{\hat{w}_1\}$. By (14), $\overline{\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \{\hat{w}_1\})} \subset U_{a,y_{\hat{k}_1},1} \times \{\hat{w}_1\}$. In particular, we have $\overline{\mathbf{v}_{a,\lambda,1} \cap (\mathbb{D}_r \times \{\hat{w}_1\})} \subset U_{a,y_{\hat{k}_1},1} \times \mathbb{D}_{\varepsilon_1(a,y_{\hat{k}_1})/2}(y_{\hat{k}_1}) \subset \mathbf{U}_{1,N}$. Thus we have (17).

We next show that

$$\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) \cap \overline{\mathbf{U}_{1,N}} = \emptyset. \tag{18}$$

Note that $\overline{\mathbf{U}_{1,N}} = \bigcup_{j=1}^{N}(\overline{U_{a,y_j,1}} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_j)/2}(y_j)})$. Assume that there is $(z_2, w_2) \in \mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) \cap \overline{\mathbf{U}_{1,N}}$. Let $v_2 \subset \mathbf{v}_{a,\lambda,2}$ be the vertical component of $\mathbf{v}_{a,\lambda}$ which contains $(z_2, w_2)$. By (14), we may assume that $(z_2, w_2) \in U_{a,y_{k_2},1} \times \overline{\mathbb{D}_{\varepsilon_1(a,y_{k_2})/2}(y_{k_2})}$ for some $k_2$ with $1 \leq k_2 \leq N$. Let $x_2 \in \mathbb{C}$ be the point such that $\{(x_2, y_{k_2})\} = v_2 \cap (\mathbb{C} \times \{y_{k_2}\})$. By (14) and $z_2 \in U_{a,y_{k_2},1}$, we have $\{(x_2, y_{k_2})\} = v_2 \cap (\mathbb{C} \times \{y_{k_2}\}) \subset U_{a,y_{k_2},1} \times \{y_{k_2}\}$. Otherwise, for any path in $v_2 \cap (\mathbb{C} \times \mathbb{D}_{\varepsilon_1(a,y_{k_2})/2}(y_{k_2}))$ between $(x_2, y_{k_2})$ and $(z_2, w_2)$, the path necessarily intersects $\gamma_{a,y_{k_2},1} \times \mathbb{D}_{\varepsilon_1(a,y_{k_2})/2}(y_{k_2})$. This leads to a contradiction (see the proof of (15)). Thus we have $v_2 \cap (\mathbb{C} \times \{y_{k_2}\}) \subset U_{a,y_{k_2},1} \times \{y_{k_2}\}$, which implies that $\mathrm{Pr}_1\, \tilde{h}^{-1}_{a,\lambda,y_{k_2}}(v_2 \cap (\mathbb{C} \times \{y_{k_2}\})) \subset U_1$. On the other hand, $v_2 \cap (U_{a,0,2} \times \{0\}) \neq \emptyset$ since $v_2 \subset \mathbf{v}_{a,\lambda,2}$ is a vertical component of $\mathbf{v}_{a,\lambda}$. Thus $\mathrm{Pr}_1\, \tilde{h}^{-1}_{a,\lambda,0}(v_2 \cap (\mathbb{C} \times \{0\})) \subset U_2$. Since $U_1 \cap U_2 = \emptyset$ and $\mathrm{Pr}_1 \bigcup_{y\in\mathbb{D}_r} \tilde{h}^{-1}_{a,\lambda,y}(v_2 \cap (\mathbb{C} \times \{y\}))$ consists of a single point, we have a contradiction. This contradiction implies that (18) holds.

By using (18), we show that

$$\overline{\mathbf{v}_{a,\lambda,2}} \cap (\mathbb{D}_r \times \mathbb{D}_{r/2}) \cap \overline{\mathbf{U}_{1,N}} = \emptyset. \tag{19}$$

By the same argument as in the proof of (17), it suffices to show that $\overline{\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \{y\})} \cap \overline{\mathbf{U}_{1,N}} = \emptyset$ for $y \in \mathbb{D}_{r/2}$. Fix $\hat{w}_2 \in \mathbb{D}_{r/2}$. By (18), we have $\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \{\hat{w}_2\}) \cap \overline{\mathbf{U}_{1,N}} = \emptyset$. Note that

$\overline{\mathbf{U}_{1,N}} \cap (\mathbb{D}_r \times \{\hat{w}_2\})$ can be represented by a finite union of the sets $\overline{U_{a,y_j,1}} \times \{\hat{w}_2\}$. Thus, by (14), we have $\overline{\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \{\hat{w}_2\})} \cap \overline{\mathbf{U}_{1,N}} \cap (\mathbb{D}_r \times \{\hat{w}_2\}) = \emptyset$. Hence we have (19) by $\overline{\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \{\hat{w}_2\})} \cap \overline{\mathbf{U}_{1,N}} \cap (\mathbb{D}_r \times \{\hat{w}_2\}) = \overline{\mathbf{v}_{a,\lambda,2} \cap (\mathbb{D}_r \times \{\hat{w}_2\})} \cap \overline{\mathbf{U}_{1,N}}$.

Finally, we show that $J_{a,\lambda}$ is disconnected. We set $\mathbf{U}_2 := (\mathbb{C} \times \mathbb{D}_{r/2}) \setminus \overline{\mathbf{U}_{1,N}}$. Then $J_{a,\lambda} \subset \mathbf{U}_{1,N} \cup \mathbf{U}_2$. Indeed, we have $J_{a,\lambda} \subset \mathbb{D}_r \times \mathbb{D}_{r/2}$ by (2) and ( i ) in Definition 3.1. Thus $J_{a,\lambda} \subset (\overline{\mathbf{v}_{a,\lambda,1}} \cup \overline{\mathbf{v}_{a,\lambda,2}}) \cap (\mathbb{D}_r \times \mathbb{D}_{r/2})$. Therefore, by (17) and (19), $J_{a,\lambda} \subset \mathbf{U}_{1,N} \cup \mathbf{U}_2$. Clearly, we have $J_{a,\lambda} \cap \mathbf{U}_{1,N} \neq \emptyset$ by (16) and Lemma 3.9. Similarly, we have $J_{a,\lambda} \cap \mathbf{U}_2 \neq \emptyset$ by (18) and Lemma 3.9. Thus $J_{a,\lambda}$ is disconnected. □

# 4 Application for radial perturbations of semi-parabolic Hénon maps

In this section, we apply Theorem 3.10 to perturbations of semi-parabolic Hénon maps. To consider the connectivity of $J$ for the parameters near the boundary of the Mandelbrot set, we consider perturbations of semi-parabolic Hénon maps by using a perturbation of one eigenvalue of semi-parabolic fixed points. Let $\lambda_0 = \exp(2\pi i m/l)$, where $l \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{Z}$ and $(m, l) = 1$. Let $\{\lambda_t\}_{t \in [0,\delta_0)}$ be a one-parameter continuous family of complex numbers, where $\delta_0 > 0$. Assume that $\lambda_t^l = \exp(L_t + i\theta_t)$ and $\mathbb{R} \ni \theta_t \to 0$ as $t \to 0$, where $L_t \in \mathbb{R} \setminus \{0\}$ and $\theta_t \in \mathbb{R}$ for $0 < t < \delta_0$.

**Definition 4.1** (Radial perturbations)**.** We say that a family $RD_{\lambda_t,\delta_0} := \{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}, 0<t<\delta_0}$ is a *radial perturbation* of the semi-parabolic Hénon family $\{H_{a,\lambda_0}\}_{a \in \mathbb{D}_{\delta_0}}$ if $\theta_t = O(L_t)$.

In order to apply Theorem 3.10 to $RD_{\lambda_t,\delta_0}$, we first check that the section $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ of $RD_{\lambda_t,\delta_0}$ satisfies the condition $(\mathrm{VC})_{\varepsilon,r}$ for each $t$ with $0 < t < \delta_0$.

**Lemma 4.2.** *There is $\delta_0 > 0$ such that the section $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ of $RD_{\lambda_t,\delta_0}$ satisfies the condition $(\mathrm{VC})_{\varepsilon,r}$ for each $t$ with $0 < t < \delta_0$.*

*Proof.* If $\lambda_0 \neq 1$, then the section $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ of $RD_{\lambda_t,\delta_0}$ satisfies the $(\mathrm{VC})_{\varepsilon,r}$ for $0 < t < \delta_0$ by Lemma 3.2, taking a smaller $\delta_0 > 0$ if necessary (see Remark 3.3).

Assume that $\lambda_0 = 1$. We may assume that each $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ with $a \neq 0$ satisfies the condition (2) with respect to $r$ by taking a smaller $\delta_0$ and a larger $r$ if necessary. Fix $\varepsilon > 0$. We can show that $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies ( i ), (ii) in Definition 3.1 for $0 < t < \delta_0$ in the same way as in the proof of Lemma 3.2, taking a smaller $\delta_0 > 0$ if necessary (see Remark 3.4). Thus it suffices to show that $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies (iii) in Definition 3.1 for $0 < t < \delta_0$, taking a smaller $\delta_0 > 0$ if necessary.

To show that (iii) in Definition 3.1 holds, we first show that $H_{a,\lambda_t}$ has a saddle fixed point for $0 < t < \delta_0$. In order to show this, we show that $H_{a,\lambda_t}$ has two distinct fixed points. Consider the equation $(x^2 + c(a, \lambda_t) + ay, ax) = (x, y)$. By $y = ax$, we have

$$x^2 + (a^2 - 1)x + c(a, \lambda_t) = 0. \tag{20}$$

Assume that $(x - \alpha)(x - \beta) = x^2 + (a^2 - 1)x + c(a, \lambda_t)$. If $\alpha = \beta$, then $\alpha = 1/2 - a^2/2$. In this case, $H_{a,\lambda_t}$ has only one fixed point $(1/2 - a^2/2, a/2 - a^3/2)$. Since $H_{a,\lambda_t}$ has a fixed point $\mathbf{q}_{a,\lambda_t} = (\lambda_t/2 - a^2/(2\lambda_t), a(\lambda_t/2 - a^2/(2\lambda_t)))$ (see (4)), we have $\lambda_t = 1$ or $\lambda_t = -a^2$. Since $\lambda_0 = 1$, we have $\lambda_t = 1$ for $a \in \mathbb{D}_{\delta_0}$, by taking $\delta_0 > 0$ so that $0 < \delta_0 < 1/2$ and $|\lambda_t| > 1/2$ for $0 < t < \delta_0$. This contradicts $\lambda_t = \exp(L_t + i\theta_t)$ with $L_t \neq 0$ for $0 < t < \delta_0$. This contradiction shows that $H_{a,\lambda_t}$ has two distinct fixed points. The other fixed point of $H_{a,\lambda}$ is $\mathbf{u}_{a,\lambda} := (1 - a^2 - \mathrm{Pr}_1\, \mathbf{q}_{a,\lambda}, a(1 - a^2 - \mathrm{Pr}_1\, \mathbf{q}_{a,\lambda}))$ by (20). These fixed points $\mathbf{q}_{a,\lambda}, \mathbf{u}_{a,\lambda}$ depend holomorphically for $(a, \lambda)$ in a small neighborhood of $(0, 1)$. At least one of the fixed points $\mathbf{q}_{a,\lambda_t}$, $\mathbf{u}_{a,\lambda_t}$ is a saddle by $\mathbf{q}_{a,\lambda_t} \neq \mathbf{u}_{a,\lambda_t}$ (see [14, Theorem 7.1.16, p.234]). Hence, there is a fixed point $\mathbf{s}_{a,\lambda}$ of $H_{a,\lambda}$ depending holomorphically for $(a,\lambda)$ in a small neighborhood of $(0,\lambda_0)$, and $\mathbf{s}_{a,\lambda_t}$ is a saddle fixed point of $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$, by taking a smaller $\delta_0$ if necessary.

Finally, we show that (iii) in Definition 3.1 holds. Let $\tilde{\Phi}_{0,\lambda}(z) := (\mathrm{Pr}_1\, \mathbf{s}_{0,\lambda}, z)$ and $\tilde{\Phi}_{a,\lambda}$ be the parametrization of $W^{ss}(\mathbf{s}_{a,\lambda})$ for $(a,\lambda) \in \mathbb{D}_{\delta_0} \setminus \{0\} \times \mathbb{D}_{\delta_0}(\lambda_0)$ given by (5). In the same way as in the proof of Lemma 2.13, $\tilde{\Phi}_{a,\lambda}(z)$ is holomorphic with $(a,\lambda,z) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0) \times \mathbb{C}$. We have $W^{ss}_{\mathrm{loc}}(\mathbf{s}_{a,\lambda}) = \{(\tilde{\Phi}_{a,\lambda,1}(\tilde{\Phi}^{-1}_{a,\lambda,2}(y)), y) : y \in \mathbb{D}_r\}$ for $(a,\lambda) \in \mathbb{D}_{\delta_0} \times \mathbb{D}_{\delta_0}(\lambda_0)$, taking a smaller $\delta_0$ if necessary, where $\tilde{\Phi}_{a,\lambda} = (\tilde{\Phi}_{a,\lambda,1}, \tilde{\Phi}_{a,\lambda,2})$ (see the proof of Lemma 3.2). We have $\mathbf{s}_{a,\lambda_0} = \mathbf{q}_{a,\lambda_0}$ since $\mathbf{u}_{a,\lambda_t}, \mathbf{q}_{a,\lambda_t} \to \mathbf{q}_{a,\lambda_0}$ as $t \to 0$. Since $W^{ss}_{\mathrm{loc}}(\mathbf{q}_{0,\lambda_0}) = \{\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_0}\} \times \mathbb{D}_r$, we see that $T_{(x,y)} W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda_t}) \subset C^v_{(x,y)}$ for $(x,y) \in W^s_{\mathrm{loc}}(\mathbf{s}_{a,\lambda_t})$, $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$, taking a smaller $\delta_0 > 0$ if necessary. Thus, $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ satisfies (iii) in Definition 3.1 for $0 < t < \delta_0$. □

We next prepare local coordinates near semi-parabolic fixed points to check that the condition (7) holds for $RD_{\lambda_t,\delta_0}$.

**Lemma 4.3** ([16, Theorem 3.5 and its proof])**.** *Let $r > 3$ be a fixed constant, $\lambda_0 := \exp(2\pi i m/l)$ and $\lambda_t := (1+t)\lambda_0$ where $t \in \mathbb{R}$. Then, there exist $\delta > 0, \delta' > 0$ such that for $|a| < \delta$ and $|t| < \delta'$ there exists a coordinate transformation $\phi_{a,t} \colon B = \mathbb{D}_{\rho'}(\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_t}) \times \mathbb{D}_r \to \mathbb{D}_\rho \times \mathbb{D}_{r+O(|a|)}$ such that $\phi_{a,t}(\mathbf{q}_{a,\lambda_t}) = (0,0)$, $W^{ss}_{\mathrm{loc}}(\mathbf{q}_{a,\lambda_t}) \subset \mathbb{D}_{\rho'}(\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_t}) \times \mathbb{D}_r$, $\phi_{a,t}(W^{ss}_{\mathrm{loc}}(\mathbf{q}_{a,\lambda_t})) \subset \{0\} \times \mathbb{C}$, the image of any horizontal curve $\mathbb{D}_{\rho'}(\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_t}) \times \{y_1\}$ under $\phi_{a,t}$ is a subset of $\mathbb{C} \times \{y_2\}$ for some $y_2 \in \mathbb{D}_{r+O(|a|)}$, and $\tilde{H}_{a,\lambda_t} = \phi_{a,t} \circ H_{a,\lambda_t} \circ \phi^{-1}_{a,t}$, $\tilde{H}_{a,\lambda_t}(x,y) = (X_1, Y_1)$ has the form*

$$(X_1, Y_1) = (\lambda_t(x + x^{l+1} + C_{a,t} x^{2l+1} + b_{a,t,2l+2}(y) x^{2l+2} + \cdots), \nu_{a,t} y + x h_{a,t}(x,y)), \tag{21}$$

*where $C_{a,t}$ is a constant depending only on $a$ and $t$, $x h_{a,t}(x,y) = O(a)$ and $\nu_{a,t}$ is the other eigenvalue of $(DH_{a,\lambda_t})_{\mathbf{q}_{a,\lambda_t}}$. Moreover, the transformation $\phi_{a,t}$ is analytic for $a$ and $t$,*

$$\lim_{a \to 0} \phi_{a,t}(x,y) = (\phi_t(x), y)$$

*uniformly for $t$. The map $\phi_t \colon \mathbb{D}_{\rho'}(\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_t}) \to \mathbb{D}_\rho$ is the transformation of the polynomial $p_{\lambda_t}$ and*

$$\phi_t \circ p_{\lambda_t} \circ \phi_t^{-1}(x) = \lambda_t(x + x^{l+1} + C_{0,t} x^{2l+1} + O(x^{2l+2})).$$

For $r > 3$, $\lambda_0 = \exp(2\pi i m/l)$, if $(c,a) \in \mathcal{P}_{\lambda_0}$ with sufficiently small $|a|$, then the sets $U^\pm$ given in Definition 2.3 satisfy the equations (2). The condition $\lambda_t = (1+t)\lambda_0$ in Lemma 4.3 corresponds to $\theta_t = 0$ for $0 < t < \delta_0$ in Definition 4.1. To see that $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ has the form (21) (Lemma 4.4), we sketch the proof of Lemma 4.3.

**Sketch of the proof of Lemma 4.3.** The proof of Lemma 4.3 breaks into four steps.

**Step 1.** Let $\Phi_{a,\lambda_t} = (\Phi_{a,\lambda_t,1}, \Phi_{a,\lambda_t,2})$ be given in Lemma 2.8. Then $\Phi_{a,\lambda_t}(y) = \Phi_{0,\lambda_t}(y) + O(a)$ by Lemma 2.8. For sufficiently small $|a|$ and $|t|$, we may assume that there exists $\Phi^{-1}_{a,\lambda_t,2}$ in $\mathbb{D}_r$ by Rouché's theorem and $\Phi_{a,\lambda_t,2}(z) = z$. For $(x,y) \in \mathbb{C} \times \mathbb{D}_r$, consider the transformation

$$(X,Y) = (x - \Phi_{a,\lambda_t,1}(\Phi^{-1}_{a,\lambda_t,2}(y)), \Phi^{-1}_{a,\lambda_t,2}(y)) \text{ with inverse } (x,y) = (X + \Phi_{a,\lambda_t,1}(Y), \Phi_{a,\lambda_t,2}(Y)), \tag{22}$$

which maps $W^{ss}_{\mathrm{loc}}(\mathbf{q}_{a,\lambda_t})$ into $\{0\} \times \mathbb{C}$. By using the transformation (22), $H_{a,\lambda_t}$ and $H_{0,\lambda_t}$ have the forms

$$(b_{a,t,1}(Y)X + b_{a,t,2}(Y)X^2 + \cdots, \nu_{a,t} Y + X h_{a,t}(X,Y)) \text{ and } (\lambda_t X + X^2, 0)$$

respectively. Note that $b_{a,t,1}(0) = \lambda_t$ and $b_{a,t,1}(Y) = \lambda_t + O(a)$ since $\Phi_{a,t}(y) = \Phi_{0,t}(y) + O(a)$.

**Step 2.** Suppose that $H_{a,\lambda_t}(x,y)$ has the form

$$(b_{a,t,1}(y)x + b_{a,t,2}(y)x^2 + \cdots, \nu_{a,t} y + x h_{a,t}(x,y)). \tag{23}$$

Let us reduce the function $b_{a,t,1}(y)$ to $b_{a,t,1}(0) = \lambda_t$ (see Proposition 3.2 in [17]). Since $\dfrac{b_{a,t,1}(\nu_{a,t}^n y)}{\lambda_t} = 1 + O(\nu_{a,t}^n y)$, the product

$$u_{a,t}(y) = \prod_{n \geq 0} \left( \frac{b_{a,t,1}(\nu_{a,t}^n y)}{\lambda_t} \right)$$

converges for $y \in \mathbb{D}_r$. By using $(X, Y) = (u_{a,t}(y)x, y)$ with inverse $(x, y) = (X/u_{a,t}(Y), Y)$, (23) has the form

$$\begin{aligned} & u_{a,t}(\nu_{a,t}Y + Xh_{a,t}(X/u_{a,t}(Y), Y)/u_{a,t}(Y)) \times (b_{a,t,1}(Y)X/u_{a,t}(Y) + b_{a,t,2}(Y)(X/u_{a,t}(Y))^2 + \cdots) \\ = \ & \frac{u_{a,t}(\nu_{a,t}Y)b_{a,t,1}(Y)}{u_{a,t}(Y)} X + O(X^2) = \lambda_t X + O(X^2) \end{aligned}$$

in the first coordinate.

**Step 3.** We may assume that $H_{a,\lambda_t}$ has the form

$$(\lambda_t x + b_{a,t,2}(y)x^2 + b_{a,t,3}(y)x^3 + \cdots, \nu_{a,t}y + xh_{a,t}(x, y)).$$

We next reduce the function $b_{a,t,k}(y)$ to constants by induction on $2 \leq k \leq 2l + 1$. Consider

$$(\lambda_t x + b_{a,t,2}x^2 + b_{a,t,3}x^3 + \cdots + b_{a,t,k-1}x^{k-1} + b_{a,t,k}(y)x^k + \cdots, \nu_{a,t}y + xh_{a,t}(x, y)), \tag{24}$$

where $b_{a,t,j}$ is constant for $j = 1, 2, \ldots, k-1$. We set

$$v_{a,t}(y) = \sum_{n=0}^{\infty} (b_{a,t,k}(\nu_{a,t}^n y) - b_{a,t,k}(0))\lambda_t^{n(k-1)-1}.$$

This series converges since $|\nu_{a,t}\lambda_t^{k-1}| < |\nu_{a,t}\lambda_t^{2l}| < 1$ for sufficiently small $t$ and $|a|$. By using local coordinate $(X, Y) = (x + v_{a,t}(y)x^k, y)$ with inverse $(x, y) = (X - v_{a,t}(Y)X^k + \cdots, Y)$, (24) has the form

$$\begin{aligned} & \lambda_t X + \cdots + b_{a,t,k-1}X^{k-1} + (b_{a,t,k}(Y) + \lambda_t^k v_{a,t}(\nu_{a,t}Y) - \lambda_t v_{a,t}(Y))X^k + O(X^{k+1}) \\ = \ & \lambda_t X + \cdots + b_{a,t,k-1}X^{k-1} + b_{a,t,k}(0)X^k + O(X^{k+1}) \end{aligned}$$

in the first coordinate.

**Step 4.** We may assume that $H_{a,\lambda_t}$ has the form

$$\lambda_t(x + b_{a,t,2}x^2 + \cdots + b_{a,t,2l+1}x^{2l+1} + b_{a,t,2l+2}(y)x^{2l+2} + \cdots)$$

in the first coordinate. We reduce $b_{a,t,k}$ to 0 for each $k$ with $2 \leq k \leq 2l + 1$ and $k - 1 \notin l\mathbb{N}$ by induction. We first assume that $H_{a,\lambda_t}$ has the form

$$\lambda_t(x + b_{a,t,k}x^k + \cdots + b_{a,t,l+1}x^{l+1} + \cdots) \tag{25}$$

in the first coordinate. By the local coordinate

$$(X, Y) = (x - \lambda_t b_{a,t,k}x^k/(\lambda_t^k - \lambda_t), y) \text{ with inverse } (x, y) = (X + \lambda_t b_{a,t,k}X^k/(\lambda_t^k - \lambda_t) + \cdots, Y), \tag{26}$$

(25) has the form

$$\lambda_t X + (\lambda_t b_{a,t,k} - (\lambda_t^k - \lambda_t)\lambda_t b_{a,t,k}/(\lambda_t^k - \lambda_t))X^k + \cdots = \lambda_t X + O(X^{k+1}).$$

By induction and a linear transformation, we may assume that the first coordinate of $H_{a,\lambda_t}$ has the form

$$\lambda_t(x + x^{l+1} + b_{a,t,l+2}x^{l+2} + \cdots + b_{a,t,2l+1}x^{2l+1} + b_{a,t,2l+2}(y)x^{2l+2} + \cdots).$$

We next assume that $H_{a,\lambda_t}$ has the form

$$\lambda_t(x + x^{l+1} + b_{a,t,k}x^k + \cdots + b_{a,t,2l+1}x^{2l+1} + \cdots) \tag{27}$$

in the first coordinate. By induction, we reduce $b_{a,t,k}$ to 0 for each $k$ with $l+2 \le k \le 2l$. By (26), (27) has the form

$$\lambda_t(X + X^{l+1}) + (\lambda_t b_{a,t,k} - (\lambda_t^k - \lambda_t)\lambda_t b_{a,t,k}/(\lambda_t^k - \lambda_t))X^k + \cdots = \lambda_t(X + X^{l+1}) + O(X^{k+1})$$

in the first coordinate. Therefore, we may assume that $H_{a,\lambda_t}$ has the form

$$\lambda_t(x + x^{l+1} + C_{a,t}x^{2l+1} + b_{a,t,2l+2}(y)x^{2l+2} + O(x^{2l+3}))$$

in the first coordinate. Hence we have Theorem 4.3. Note that, by repeating Step 3 and Step 4, $H_{a,\lambda_t}$ has the form

$$(\lambda_t(x + x^{l+1} + C_{a,t}x^{2l+1} + b_{a,t,3l+1}(y)x^{3l+1} + \cdots), \nu_{a,t}y + xh_{a,t}(x,y)). \tag{28}$$

We obtain the following lemma by the same computation as in the proof of Theorem 3.5 in [16].

**Lemma 4.4.** *There is $\delta_0 > 0$ such that by a coordinate transformation $\phi_{a,t}$, $\phi_{a,t} \circ H_{a,\lambda_t} \circ \phi_{a,t}^{-1}$ has the form (21) for each $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$.*

We now prove the second main result of this paper.

**Theorem 4.5.** *There is $\delta_0 > 0$ such that each $H_{a,\lambda_t} \in RD_{\lambda_t,\delta_0}$ with $a \neq 0$ has connected Julia set $J_{a,\lambda_t}$.*

*Proof.* It suffices to show the statement of Theorem 4.5 for $H_{a,\lambda_t^{-1}}$ instead of $H_{a,\lambda_t}$. Moreover, by Theorem 3.10 and Lemma 4.2, it suffices to show that there are $\varepsilon > 0$ and $\delta_0 > 0$ such that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda_t^{-1}} = \emptyset$ for $H_{a,\lambda_t^{-1}} \in RD_{\lambda_t^{-1},\delta_0}$, and $J_{p_{\lambda_t^{-1}}}$ is connected for $0 < t < \delta_0$.

We first consider the case $\lambda_0 = 1$. By a transformation $\phi_{a,t}$ (see Lemma 4.4), $\phi_{a,t} \circ H_{a,\lambda_t^{-1}} \circ \phi_{a,t}^{-1}$ is of the form

$$(x_1, y_1) = (\lambda_t^{-1}(x + x^2 + C_{a,t}x^3 + O_y(x^4)), \nu_{a,t}y + xh_{a,t}(x,y)) \tag{29}$$

in $\mathbb{D}_\rho \times \mathbb{D}_{r+O(|a|)}$, where the notation $O_y(x^\alpha)$ represents a holomorphic map of $(x,y)$ which is bounded by $K|x|^\alpha$ for some $K$. By the transformation $\psi_t(x,y) = (-1/(\lambda_t x), y)$, the map $\psi_t \circ \phi_{a,t} \circ H_{a,\lambda_t^{-1}} \circ \phi_{a,t}^{-1} \circ \psi_t^{-1}(X,Y) = (X_1, Y_1)$ is of the form

$$(\lambda_t X + 1 + g_{a,t}(X,Y), \nu_{a,t}Y + f_{a,t}(X,Y)) = \left(\lambda_t X + 1 + D_{a,t}/X + O_Y\left(1/|X|^2\right), \nu_{a,t}Y + O_Y\left(1/|X|\right)\right). \tag{30}$$

We take a constant $M > 0$ such that

$$|g_{a,t}(X,Y)| \le \frac{M}{|X|} \text{ and } |f_{a,t}(X,Y)| \le \frac{M}{|X|} \tag{31}$$

for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. Recall that $\phi_{a,t}$ is a transformation from $\mathbb{D}_{\rho'}(\mathrm{Pr}_1\, \mathbf{q}_{0,\lambda_t}) \times \mathbb{D}_r$ to $\mathbb{D}_\rho \times \mathbb{D}_{r+O(|a|)}$. Since $\mathrm{Pr}_2\, \phi_{a,t}(x,y) = y + O(a)$ (see the sketch of the proof of Lemma 4.3), we may assume that the forms (30) are defined in $\{|X| > 1/\rho\} \times \mathbb{D}_r$ and the inequalities (31) hold for $(X,Y) \in$

$\{|X| > 1/\rho\} \times \mathbb{D}_r$. Note that $\psi_t \circ \phi_{a,t}(\mathbf{q}_{a,\lambda_t^{-1}}) = (\infty, 0)$. We set $(X_0, Y_0) := \psi_0 \circ \phi_{0,0} \circ H^N_{0,\lambda_0^{-1}}(0,0)$ for some large $N \in \mathbb{N}$. We have $\operatorname{Re} X_0 > 1/\rho$, by taking a larger $N$ if necessary, since the forward orbit of critical point 0 under $p_{\lambda_0^{-1}}$ converges to its parabolic fixed point (see [1], p.120). Let $\gamma > 0$ be a number such that $\mathbb{D}_\gamma(X_0) \times \mathbb{D}_r \subset \{X \in \mathbb{C} : \operatorname{Re} X > 1/\rho\} \times \mathbb{D}_r$. We set $E := \mathbb{D}_\gamma(X_0) \times \mathbb{D}_r$. We consider the affine transformations

$$Q_t(z,w) = \left( \frac{z - b_t}{X_0 - b_t}, w \right) \quad \text{and} \quad Q_t^{-1}(z,w) = ((X_0 - b_t)z + b_t, w), \text{ where } b_t := \frac{1}{1-\lambda_t}.$$

We set

$$F_{a,t}(z,w) = Q_t \circ \psi_t \circ \phi_{a,t} \circ H_{a,\lambda_t^{-1}} \circ \phi_{a,t}^{-1} \circ \psi_t^{-1} \circ Q_t^{-1}(z,w).$$

Then, we have

$$\operatorname{Pr}_1 F_{a,t}(z,w) = \operatorname{Pr}_1 Q_t \circ \psi_t \circ \phi_{a,t} \circ H_{a,\lambda_t^{-1}} \circ \phi_{a,t}^{-1} \circ \psi_t^{-1} \circ Q_t^{-1}(z,w) = \lambda_t z + \frac{g_{a,t}((X_0 - b_t)z + b_t, w)}{X_0 - b_t}.$$

We set

$$G_{a,t}(z,w) := \frac{g_{a,t}((X_0 - b_t)z + b_t, w)}{X_0 - b_t}.$$

Further we set $E'(= E'_t) := Q_t(E)$ and $U'(= U'_t) := Q_t(U)$, where $U = \{|z| < 1/\rho\} \times \mathbb{D}_r$. Then we have

$$E' = \left\{ z \in \mathbb{C} : |z - 1| < \frac{\gamma}{|b_t - X_0|} \right\} \times \mathbb{D}_r \quad \text{and} \quad U' = \left\{ z \in \mathbb{C} : \left| z - \frac{b_t}{b_t - X_0} \right| < \frac{1}{\rho |b_t - X_0|} \right\} \times \mathbb{D}_r.$$

Clearly, if $\rho$ and $\delta_0$ are sufficiently small, then for each $a \in \mathbb{D}_{\delta_0}$ and each $t \in (0, \delta_0)$, we have

$$\operatorname{Pr}_2 F_{a,t}((\mathbb{C} \times \mathbb{D}_r) \setminus U') \subset \mathbb{D}_r. \tag{32}$$

Indeed, $|\operatorname{Pr}_2 \psi_t \circ \phi_{a,t} \circ H_{a,\lambda_t^{-1}} \circ \phi_{a,t}^{-1} \circ \psi_t^{-1}(X,Y)| \le |\nu_{a,t} Y| + M/|X| < |\nu_{a,t}| r + M\rho$ for $(X,Y) \in \{|X| > 1/\rho\} \times \mathbb{D}_r$ (see (30) and (31)). Recall that $\nu_{a,t} = -a^2/\lambda_t^{-1}$ and $|\lambda_0| = 1$. By taking $\rho$ and $\delta_0$ with sufficiently small, we have $|\nu_{a,t}| r + M\rho < r$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. Since $\operatorname{Pr}_2 Q_t(z,w) = w$, we have (32).

By $X_0 = \operatorname{Pr}_1 \psi_0 \circ \phi_{0,0} \circ H^N_{0,\lambda_0^{-1}}(0,0)$, $E = \mathbb{D}_\gamma(X_0) \times \mathbb{D}_r$ and (32), we have $Q_t \circ \psi_t \circ \phi_{a,t} \circ H^N_{a,\lambda_t^{-1}}(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \subset E'$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$, by taking $\delta_0 > 0$ and $\varepsilon > 0$ sufficiently small if necessary. Thus, to show that there are $\varepsilon > 0$ and $\delta_0 > 0$ such that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J^+_{a,\lambda_t^{-1}} = \emptyset$ for $H_{a,\lambda_t^{-1}} \in RD_{\lambda_t^{-1},\delta_0}$, it suffices to show that there is $\delta_0 > 0$ such that

$$\bigcup_{k \in \mathbb{N}} F^k_{a,t}(E') \cap U' (= \bigcup_{k \in \mathbb{N}} F^k_{a,t}(E'_t) \cap U'_t) = \emptyset \tag{33}$$

for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$, taking smaller $\gamma > 0$, $\rho > 0$ and a larger $N \in \mathbb{N}$ if necessary. Indeed, if (33) holds for $F_{a,t}$, then $\bigcup_{n \ge 0} H^n_{a,\lambda_t^{-1}}(\mathbb{D}_\varepsilon \times \mathbb{D}_r)$ is a bounded set of $\mathbb{C}^2$, which implies that $\mathbb{D}_\varepsilon \times \mathbb{D}_r \subset \operatorname{Int} K_{a,\lambda_t^{-1}}$. To obtain (33), we show the following Claims 1,2,3.

**Claim 1.** For sufficiently small $\delta_0$, the following hold. If $L_t > 0$ for $0 < t < \delta_0$, then $U' \subset \mathbb{D} \times \mathbb{D}_r$ for $0 < t < \delta_0$. If $L_t < 0$ for $0 < t < \delta_0$, then $U' \cap (\mathbb{D} \times \mathbb{D}_r) = \emptyset$ for $0 < t < \delta_0$.

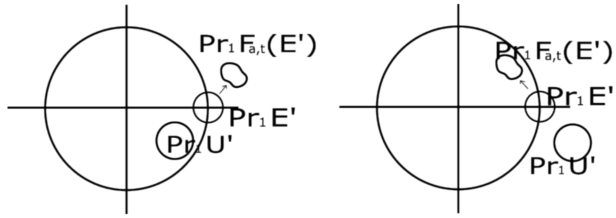


Figure 4: $L_t > 0$ and $\theta_t > 0$ (left), $L_t < 0$ and $\theta_t > 0$ (right)

We prove Claim 1. Assume that $L_t > 0$ for each $0 < t < \delta_0$. We show that $\dfrac{|b_t|}{|b_t - X_0|} + \dfrac{1}{\rho|b_t - X_0|} < 1$ for sufficiently small $t$ to obtain the conclusion. The inequality is equivalent to

$$|1 - X_0(1-\lambda_t)| - |1-\lambda_t|/\rho - 1 > 0. \tag{34}$$

Let $x_1 := \mathrm{Re}(\lambda_t - 1)$ and $y_1 := \mathrm{Im}(\lambda_t - 1)$. Then, $x_1 = L_t + O(L_t^2)$ and $y_1 = \theta_t + O(\theta_t L_t)$. First, we have

$$\begin{aligned} |1 + X_0(\lambda_t - 1)| &= \sqrt{(1 + \mathrm{Re}(X_0)x_1 - \mathrm{Im}(X_0)y_1)^2 + (\mathrm{Re}(X_0)y_1 + \mathrm{Im}(X_0)x_1)^2} \\ &= \sqrt{1 + 2\mathrm{Re}(X_0)x_1 - 2\mathrm{Im}(X_0)y_1 + O((x_1+y_1)^2)} \\ &= \sqrt{1 + 2\mathrm{Re}(X_0)x_1 - 2\mathrm{Im}(X_0)y_1 + O(L_t^2)} \\ &= 1 + \mathrm{Re}(X_0)x_1 - \mathrm{Im}(X_0)y_1 + O(L_t^2). \end{aligned}$$

We now show the inequality (34). By the above computation, we have

$$|1 - X_0(1-\lambda_t)| - |1-\lambda_t|/\rho - 1 = \mathrm{Re}(X_0)x_1 - \mathrm{Im}(X_0)y_1 + O(L_t^2) - |1-\lambda_t|/\rho.$$

Recall that $X_0 = \mathrm{Pr}_1\, \psi_0 \circ \phi_{0,0} \circ H^N_{0,\lambda_0^{-1}}(0,0)$. Since $\mathrm{Re}\, X_0 > 0$, $0 < x_1 \asymp L_t$, $y_1 = O(\theta_t)$ and $|1-\lambda_t| = O(L_t)$, we have the assertion, taking a large $N \in \mathbb{N}$ so that $|\mathrm{Im}\, X_0|/\mathrm{Re}\, X_0$ and $1/(\rho \mathrm{Re}\, X_0)$ are sufficiently small (see [1], p.120), where $x_1 \asymp L_t$ means that there is $K > 0$ such that $L_t/K < x_1 < KL_t$ for sufficiently small $t$.

Assume that $L_t < 0$ for $0 < t < \delta_0$. We claim that $\dfrac{|b_t|}{|b_t - X_0|} - \dfrac{1}{\rho|b_t - X_0|} > 1$ for sufficiently small $t$. The inequality is equivalent to $1 - |1-\lambda_t|/\rho - |1 - X_0(1-\lambda_t)| > 0$. We have

$$1 - |1-\lambda_t|/\rho - |1 - X_0(1-\lambda_t)| = -|1-\lambda_t|/\rho - \mathrm{Re}(X_0)x_1 + \mathrm{Im}(X_0)y_1 + O(L_t^2).$$

By $\mathrm{Re}\, X_0 > 0$ and $x_1 < 0$, we have the assertion, taking a larger $N \in \mathbb{N}$ so that $|\mathrm{Im}\, X_0|/\mathrm{Re}\, X_0$ and $1/(\rho\mathrm{Re}\, X_0)$ are sufficiently small if necessary (see [1], p.120). Thus, we have proved Claim 1.

**Claim 2.** For sufficiently small $\delta_0$, the following hold. If $L_t > 0$ for $0 < t < \delta_0$, then $|\mathrm{Pr}_1\, F_{a,t}(z,w)| > e^{L_t/2}|z| > |z|$ for $a \in \mathbb{D}_{\delta_0}, 0 < t < \delta_0$ and $(z,w) \in (\{|z| > 1/2\} \times \mathbb{D}_r) \setminus U'$. If $L_t < 0$ for $0 < t < \delta_0$, then $|\mathrm{Pr}_1\, F_{a,t}(z,w)| < e^{L_t/2}|z| < |z|$ for $a \in \mathbb{D}_{\delta_0}, 0 < t < \delta_0$ and $(z,w) \in (\{|z| > 1/2\} \times \mathbb{D}_r) \setminus U'$.

We prove Claim 2. Assume that $(z,w) \in (\{|z| > 1/2\} \times \mathbb{D}_r) \setminus U'$. By using the inequality (31), we have

$$\frac{|G_{a,t}(z,w)|}{||\lambda_t z| - e^{L_t/2}|z||} < \frac{2|G_{a,t}(z,w)|}{|e^{L_t} - e^{L_t/2}|} \le \frac{2M\rho}{|e^{L_t} - e^{L_t/2}||X_0 - b_t|} = \frac{2M\rho|1-\lambda_t|}{e^{L_t/2}|e^{L_t/2} - 1||X_0(1-\lambda_t) - 1|}.$$

Since $|1-\lambda_t| = O(L_t)$, there exists a positive constant $\delta_0$ such that the ratio is less than $1/2$ if $0 < t < \delta_0$, taking $\rho > 0$ sufficiently small if necessary. We note that the constant $\delta_0$ does not depend on $z$. By the inequality

$$|\lambda_t z| - e^{L_t/2}|z| - |G_{a,t}(z)| \le |\mathrm{Pr}_1 F_{a,t}(z,w)| - e^{L_t/2}|z| \le |\lambda_t z| - e^{L_t/2}|z| + |G_{a,t}(z,w)|, \tag{35}$$

the statement of Claim 2 holds.

**Claim 3.** For sufficiently small $\delta_0$, the following hold. If $L_t > 0$ for $0 < t < \delta_0$, then $F_{a,t}(E') \cap (\mathbb{D} \times \mathbb{D}_r) = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. If $L_t < 0$ for $0 < t < \delta_0$, then $F_{a,t}(E') \subset \mathbb{D} \times \mathbb{D}_r$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$.

We prove Claim 3. Let $(z,w) \in E'$. Then, we have $z = 1 + z_0$, where $|z_0| < \gamma/|b_t - X_0|$. Since

$$|\mathrm{Pr}_1 F_{a,t}(z,w)| = |\lambda_t(1+z_0) + G_{a,t}(1+z_0,w)| = |\lambda_t + \lambda_t z_0 + G_{a,t}(1+z_0,w)|,$$

we have

$$e^{L_t} - |\lambda_t z_0 + G_{a,t}(1+z_0,w)| \le |\mathrm{Pr}_1 F_{a,t}(z,w)| \le e^{L_t} + |\lambda_t z_0 + G_{a,t}(1+z_0,w)|.$$

Since $|z_0| < \gamma/|b_t - X_0| = O(L_t)$ and $|G_n(1+z_0)| < M\rho/|X_0 - b_t| = O(L_t)$, we have the assertion, taking smaller $\gamma, \rho$ if necessary. Hence, we have proved Claim 3.

We now show that there is $\delta_0 > 0$ such that $F^k_{a,t}(E') \cap U' = \emptyset$ for $a \in \mathbb{D}_{\delta_0}, 0 < t < \delta_0$ and each $k \in \mathbb{N}$ by using Claims 1, 2, and 3. First, assume that $L_t > 0$ for $0 < t < \delta_0$. By Claim 1, $U' \subset \mathbb{D} \times \mathbb{D}_r$. By Claim 3, $F_{a,t}(E') \cap (\mathbb{D} \times \mathbb{D}_r) = \emptyset$, and so $F_{a,t}(E') \cap U' = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. By $U' \subset \mathbb{D} \times \mathbb{D}_r$ and Claim 2, we have $|\mathrm{Pr}_1 F_{a,t}(z,w)| > |z| > 1$ for $(z,w) \in (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r$, and so $F_{a,t}((\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r) \subset (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r$. By using $F_{a,t}(E') \subset (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r$, for $k \ge 2$, we have

$$F^k_{a,t}(E') = F^{k-1}_{a,t}(F_{a,t}(E')) \subset F^{k-1}_{a,t}((\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r) \subset (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r.$$

Hence, we have $F^k_{a,t}(E') \cap U' = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$, $0 < t < \delta_0$ each $k \in \mathbb{N}$ by $U' \subset \mathbb{D} \times \mathbb{D}_r$.

Assume that $L_t < 0$ for $0 < t < \delta_0$. Similarly, we have $F_{a,t}(E') \cap U' = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. We note that $U' \subset (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r$ by Claim 1. Therefore, $F_{a,t}$ is defined in $\mathbb{D} \times \mathbb{D}_r$. It is easy to see that

$$F_{a,t}(\overline{\mathbb{D}_{1/2}} \times \mathbb{D}_r) \subset \mathbb{D}_{2/3} \times \mathbb{D}_r \tag{36}$$

for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$, taking a smaller $\delta_0$ if necessary. Indeed, if $(z,w) \in \overline{\mathbb{D}_{1/2}} \times \mathbb{D}_r$, then $|\mathrm{Pr}_1 F_{a,t}(z,w)| \le |\lambda_t z| + |G_{a,t}(z,w)| \le |\lambda_t|/2 + M\rho/(|X_0 - b_t|) < 2/3$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$, taking a smaller $\delta_0$ if necessary. Thus we have (36). By Claim 2, $U' \subset (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_r$ and (36), we have $|\mathrm{Pr}_1 F_{a,t}(z,w)| < |z| < 1$ or $|\mathrm{Pr}_1 F_{a,t}(z,w)| < 2/3$ for $(z,w) \in \mathbb{D} \times \mathbb{D}_r$, and so $F_{a,t}(\mathbb{D} \times \mathbb{D}_r) \subset \mathbb{D} \times \mathbb{D}_r$. For $k \ge 2$, we have

$$F^k_{a,t}(E') = F^{k-1}_{a,t}(F_{a,t}(E')) \subset F^{k-1}_{a,t}(\mathbb{D} \times \mathbb{D}_r) \subset \mathbb{D} \times \mathbb{D}_r.$$

Hence, we have $F^k_{a,t}(E') \cap U' = \emptyset$ for $a \in \mathbb{D}_{\delta_0}, 0 < t < \delta_0$ and each $k \in \mathbb{N}$ by $U' \cap (\mathbb{D} \times \mathbb{D}_r) = \emptyset$. Hence we obtain (33). This implies that there are $\delta_0 > 0$ and $\varepsilon > 0$ such that $(\mathbb{D}_\varepsilon \times \mathbb{D}_r) \cap J_{a,\lambda_t^{-1}} = \emptyset$ for $a \in \mathbb{D}_{\delta_0}$ and $0 < t < \delta_0$. For the constant $\varepsilon > 0$, $\{H_{a,\lambda_t}\}_{a \in \mathbb{D}_{\delta_0}}$ satisfies the condition $(\mathrm{VC})_{\varepsilon,r}$ for $0 < t < \delta_0$ by Lemma 4.2, taking a smaller $\delta_0$ if necessary (see Remark 3.4).

In order to apply Theorem 3.10 to $\{H_{a,\lambda_t^{-1}}\}_{a \in \mathbb{D}_{\delta_0}}$ for $0 < t < \delta_0$, we show that $J_{p_{\lambda_t^{-1}}} \subset \mathbb{C}$ is connected for $0 < t < \delta_0$. It suffices to show that $p_{\lambda_t^{-1}}$ has an attracting fixed point. Since $p_{\lambda_t^{-1}}$ has a fixed point with multiplier $\lambda_t^{-1} = \exp(-L_t - i\theta_t)$, if $L_t > 0$ for $0 < t < \delta_0$, then $p_{\lambda_t^{-1}}$ has an attracting fixed point. If $L_t < 0$ for $0 < t < \delta_0$, then $F_{0,t}(\overline{\mathbb{D} \times \mathbb{D}_r}) \subset \mathbb{D} \times \mathbb{D}_r$ by Claim 2 and (36). Thus $p_{\lambda_t^{-1}}$ has an attracting fixed point.

In the case of $l \geq 2$, by the transformation $X = -1/(lx^l), Y = y$ and a linear transformation, the form of (28) with $\lambda_t$ replaced by $\lambda_t^{-1}$ is conjugate to

$$(X_1, Y_1) = \lambda_t^l X + 1 + D_{a,t}/X + O_Y(1/X^2), \nu_n Y + O_Y(1/|X|^{1/l})).$$

Similar to the case of $l = 1$, we can show the statement in Theorem 4.5 in the case of $l \geq 2$. □

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