THE ORBITS OF PRIMITIVE REPRESENTATIONS OF A QUADRATIC FORM BY A QUADRATIC FORM

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Abstract. We study the primitive integral representations of a quadratic form in $n-1$ variables by a *reduced* quadratic form in n variables. Our aim is to describe the orbits of such representations under the action of the unit group of the reduced form. That description provides the mass of the representations, considered by Shimura. A formula for computation of the mass is also proved in the indefinite case.

Introduction

Let φ be a symmetric matrix in $GL_n(\mathbf{Q})$ and q a symmetric matrix in $GL_{n-1}(\mathbf{Q})$. We consider the set of *the primitive integral representations of q by φ* in the traditional sense:

$$\{q,\, \varphi\} = \{k \in \mathbf{Q}_n^{n-1} \mid k\varphi \cdot {}^t k = q,\ k\mathbf{Z}_1^n = \mathbf{Z}_1^{n-1}\}. \tag{0.1}$$

Here and throughout the paper we assume $n \geq 3$ and follow the notation and terminology in Shimura [8]; see also §1.1 in the text. Put

$$SO(\varphi) = \{\gamma \in SL_n(\mathbf{Q}) \mid \gamma\varphi \cdot {}^t\gamma = \varphi\}, \qquad \Gamma(\varphi) = SO(\varphi) \cap GL_n(\mathbf{Z}).$$

Then $\Gamma(\varphi)$ acts on $\{q,\, \varphi\}$ on the right. Hence, if $\{q,\, \varphi\} \neq \emptyset$, we can consider the orbits of $\{q,\, \varphi\}$ under the action of $\Gamma(\varphi)$. We denote by $\{q,\, \varphi\}/\Gamma(\varphi)$ the set of such orbits. It is noted that $\{q,\, \varphi\}/\Gamma(\varphi)$ is a finite set.

The purpose of this paper is to describe the $\Gamma(\varphi)$-orbits of $\{q,\, \varphi\}$ in terms of the $\Gamma(\varphi)$-orbits of the set $L[s,\, b\mathbf{Z}]$ (defined below) for a given reduced symmetric matrix φ. The term *reduced* was introduced by Shimura in [9, (6.2)] and should not be confused with the same term in the sense of Minkowski. We note a simple fact: If L_0 is a $\mathbf{Z}$-maximal lattice in $\mathbf{Q}_n^1$ with respect to a symmetric matrix φ_0, then the matrix φ representing φ_0 with respect to a basis of L_0 is reduced and the lattice $\mathbf{Z}_n^1$ is $\mathbf{Z}$-maximal with respect to φ; conversely, every reduced matrix can be obtained in this fashion.

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To state our result, we need some more notation. We put $\varphi(x, y) = x\varphi \cdot {}^t y$, $\varphi[x] = \varphi(x, x)$, and

$$L[s, b\mathbf{Z}] = \{v \in \mathbf{Q}_n^1 \mid \varphi[v] = s,\ \varphi(v, L) = b\mathbf{Z}\} \tag{0.2}$$

for a $\mathbf{Z}$-lattice L in $\mathbf{Q}_n^1$ and $s, b \in \mathbf{Q}^\times$. Since $\Gamma(\varphi)$ acts on $L[s, b\mathbf{Z}]$ on the right, the set $L[s, b\mathbf{Z}]/\Gamma(\varphi)$ is also defined. Suppose L is $\mathbf{Z}$-maximal with respect to φ; namely, L satisfies $\varphi[L] \subset \mathbf{Z}$ and it is maximal among such lattices. For a fixed element $v \in L[s, b\mathbf{Z}]$, by virtue of [10, Theorem 2.2] we can define an injection of $L[s, b\mathbf{Z}]/\Gamma(\varphi)$ into $H\backslash H_\mathbf{A}/(H_\mathbf{A} \cap C(L))$, which we denote by $u\Gamma(\varphi) \mapsto H\xi_u(H_\mathbf{A} \cap C(L))$ with $\xi_v = 1_n$. Here $W = (\mathbf{Q}v)^\perp$, $H = \{\gamma \in SO(\varphi) \mid v\gamma = v\}$, $C(L) = \{\alpha \in SO(\varphi)_\mathbf{A} \mid L\alpha = L\}$, and the subscript $\mathbf{A}$ means the adelization; $\xi_u \in H_\mathbf{A}$ is given by

$$\xi_u = \gamma_u^{-1}\alpha_u$$

with $\gamma_u \in SO(\varphi)$ such that $u\gamma_u = v$ and $\alpha_u \in C(L)$; see (1.6) in the text for details. For the lattice $L \cap W$ in W we also put $C(L \cap W) = \{\alpha \in H_\mathbf{A} \mid (L \cap W)\alpha = L \cap W\}$ and $\Gamma(L \cap W) = H \cap C(L \cap W)$. Let U be a complete set of representatives for $L[s, b\mathbf{Z}]/\Gamma(\varphi)$ containing v. Consider $A = \{u \in U \mid \xi_u \in HC(L \cap W)\}$ and for $u \in A$ take a complete set Z_u of representatives for $\Gamma(L \cap W)/(H \cap \eta_u^{-1}\xi_u C(L)\xi_u^{-1}\eta_u)$ with an element η_u of H such that $\xi_u \in \eta_u C(L \cap W)$. Then our main result of this paper is stated as follows.

Theorem. *Suppose $L = \mathbf{Z}_n^1$ and it is $\mathbf{Z}$-maximal with respect to φ. Let h be an element of $\{q, \varphi\}$. Fix arbitrary numbers $s, b \in \mathbf{Q}^\times$ satisfying $b^{-2}s = \det(\varphi)^{-1}\det(q)$. Take $v \in L[s, b\mathbf{Z}]$ so that $L \cap (\mathbf{Q}v)^\perp = \mathbf{Z}_{n-1}^1 h$. Set $W = (\mathbf{Q}v)^\perp$ and $H = \{\gamma \in SO(\varphi) \mid v\gamma = v\}$ for this v and take U, A, and Z_u as above. (Note that q is the matrix representing φ, restricted to $W = \mathbf{Q}_{n-1}^1 h$, with respect to the basis of $L \cap W$ determined by h.) Then $\{q, \varphi\}$ can be given by*

$$\{q, \varphi\} = \bigsqcup_{u \in A} \Gamma(q)h\eta_u^{-1}\gamma_u^{-1}\Gamma(\varphi) = \bigsqcup_{u \in A}\bigsqcup_{\zeta \in Z_u} h\zeta\eta_u^{-1}\gamma_u^{-1}\Gamma(\varphi).$$

Here $\Gamma(q) = SO(q) \cap GL_{n-1}(\mathbf{Z})$, which is isomorphic to $\Gamma(L \cap W)$ via $\gamma' \mapsto \gamma$ defined by the rule $\gamma' h = h\gamma$.

We can prove a similar result on the $\Gamma(L)$-orbits in a general setting, which will be stated and proved in Section 1. Theorem is derived by specializing that result. The maximality of L in the assumption means that φ is reduced. Hence our theorem describes $\{q, \varphi\}/\Gamma(\varphi)$ for *every reduced φ and every q* as long as $\{q, \varphi\} \neq \emptyset$. It is nontrivial to find an element v in the statement for a given $h \in \{q, \varphi\}$. The existence of such an element can be seen from our previous result [4, Theorem 2.1]; see §1.4 for details.

As an application, Theorem provides *the mass of the set $\{q, \varphi\}$* in the sense of Shimura in both definite and indefinite cases, which will be explained in Section

2. In fact, from the decomposition of $\{q, \varphi\}$ into $\Gamma(\varphi)$-orbits we have

$$\mathfrak{m}(\{q, \varphi\}) = \sum_{u \in A} \#(Z_u)\nu(\Gamma(\varphi))^{-1}, \tag{0.3}$$

where $\mathfrak{m}(\{q, \varphi\})$ denotes the mass of $\{q, \varphi\}$ and $\nu(\Gamma(\varphi))$ is a quantity defined with the measure of $\Gamma(\varphi)\backslash\mathcal{Z}$ with respect to an invariant measure on a symmetric space $\mathcal{Z}$ on which $SO(\varphi)$ acts; in particular, $\nu(\Gamma(\varphi)) = [\Gamma(\varphi) : 1]^{-1}$ and $\mathfrak{m}(\{q, \varphi\}) = \#\{q, \varphi\}$ if φ is definite. By using (0.3) in the definite case, we can get numerical examples of $\#\{q, \varphi\}$ in §4.1, where φ is a reduced matrix representing the sum of five squares.

In Section 3 we give a formula for $\nu(\Gamma(\varphi))$ when φ is indefinite, involving *the measure of the unit group of* 2φ in the sense of Siegel [14]. This formula plays a fundamental role in the proof of a relationship between the mass of the set of primitive solutions like $L[s, b\mathbf{Z}]$ and the measure of primitive representations of an integer by an indefinite quadratic form, considered by Siegel [13]; the relationship will be reported in [5].

In Section 4 we present an example of $\{q, \varphi\}/\Gamma(\varphi)$ for an indefinite form φ in 7 variables. In addition, we give not only $\mathfrak{m}(\{q, \varphi\})$ but also the mass of the set $L[s, 2^{-1}\mathbf{Z}]$ by means of the formula mentioned above.

1. Orbits of the solutions of a quadratic Diophantine equation

1.1. **Preliminaries.** We denote by $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ the ring of rational integers, the fields of rational numbers, and real numbers, respectively. If A is a commutative associative ring with identity element, then we write $A^\times$ for the group of all invertible elements of A and A_n^m the A-module of all $(m \times n)$-matrices with entries in A. The transpose of a matrix x of A_n^m is denoted by ${}^t x$; the determinant and trace of x of A_n^n are denoted by $\det(x)$ and $\mathrm{tr}(x)$. We denote the identity element of A_n^n by 1_n. We put $GL_n(A) = (A_n^n)^\times$ and $SL_n(A) = \{x \in GL_n(A) \mid \det(x) = 1\}$. We write $\mathrm{diag}[a_1, \cdots, a_s]$ for the matrix with square matrices $a_1, \cdots, a_s$ in the diagonal blocks and 0 in all other blocks. For a finite set X, we denote by $\#X$ the number of elements in X. If a union $\bigcup_{i \in I} X_i$ is disjoint, then we indicate it by writing $\bigsqcup_{i \in I} X_i$.

Let F be an algebraic number field of finite degree or its completion at a prime, and let $\mathfrak{g}$ be the maximal order of F. Let V be a vector space of dimension n over F and φ a nondegenerate symmetric F-bilinear form of $V \times V$ into F. The quadratic form on V is defined by $x \mapsto \varphi[x] = \varphi(x, x)$ for $x \in V$. We let $\mathrm{GL}(V)$ act on V on the right. We denote by $O^\varphi(V)$ the orthogonal group of φ and put $SO^\varphi(V) = \{\gamma \in O^\varphi(V) \mid \det(\gamma) = 1\}$. Let L be a $\mathfrak{g}$-lattice in V, that is, L is a finitely generated $\mathfrak{g}$-submodule of V that spans V over F, where F is a number field or a nonarchimedean local field. We set

$\widetilde{L} = \{v \in V \mid 2\varphi(v, L) \subset \mathfrak{g}\}$. We call L $\mathfrak{g}$-*maximal* with respect to φ if L satisfies $\varphi[L] \subset \mathfrak{g}$ and that if L' is such a lattice containing L, then $L' = L$.

Suppose F is a number field. We denote by $\mathbf{a}$ and $\mathbf{h}$ the sets of archimedean primes and nonarchimedean primes of F. For $v \in \mathbf{a} \cup \mathbf{h}$ we denote by F_v the completion of F at v and by $\mathfrak{g}_v$ the maximal order in F_v if $v \in \mathbf{h}$. We also denote by φ_v the F_v-bilinear extension of φ to the vector space $V_v = V \otimes_F F_v$ over F_v; we write L_v for the $\mathfrak{g}_v$-lattice in V_v generated by L over $\mathfrak{g}_v$ if $v \in \mathbf{h}$. We denote by $G_{\mathbf{A}}$ the adelization of $G = SO^{\varphi}(V)$ and by G_v for $v \in \mathbf{a} \cup \mathbf{h}$ the localization of G at v. We put

$$C(L) = \{\alpha \in G_{\mathbf{A}} \mid L\alpha = L\}, \quad \Gamma(L) = G \cap C(L). \tag{1.1}$$

Also put $C(L_v) = G_v \cap C(L)$ for $v \in \mathbf{h}$. We note that $\Gamma(L) = \Gamma(\widetilde{L})$.

Let X be a vector space of dimension $n-1(>1)$ over F. Given $k \in \mathrm{Hom}(X, V)$, we denote by $\varphi[k]$ the quadratic form on X defined by $x \mapsto \varphi(xk, xk)$ for $x \in X$. We also denote by $\varphi(k, \widetilde{L})$ the $\mathfrak{g}$-lattice in $\mathrm{Hom}(X, F)$ consisting of $\varphi(k, \ell)$ for all $\ell \in \widetilde{L}$, where $\varphi(k, \ell)$ is defined by $x \mapsto \varphi(xk, \ell)$ for $x \in X$. If $\varphi[k]$ is a nondegenerate quadratic form q on X, then k is injective.

Let W be the subspace $\{v \in V \mid \varphi(y, v) = 0\}$ of V for a fixed $y \in V$ such that $\varphi[y] \neq 0$ and ψ the restriction of φ to W. Suppose h is an element of $\mathrm{Hom}(X, V)$ such that $L \cap W = Mh$ with a $\mathfrak{g}$-lattice M in X. Then the special orthogonal group of the restriction of φ to Xh is $SO^{\psi}(W)$. Put $q = \varphi[h]$, which is nondegenerate on X because $Xh = W$ and ψ is nondegenerate on W. Then we understand $SO^{q}(X)$ by the set of all the elements η' determined by the rule $\eta' h = h\eta$ for $\eta \in SO^{\psi}(W)$.

1.2. **Statement in a general setting.** Let (V, φ), L, and X be as in §1.1 for an algebraic number field F. Hereafter until the end of this section, we put $G = SO^{\varphi}(V)$. For a nondegenerate quadratic form q on X and a $\mathfrak{g}$-lattice $\mathfrak{B}$ in $\mathrm{Hom}(X, F)$, we shall consider the set of solutions of a quadratic Diophantine equation $\varphi[x] = q$ in the following type:

$$\widetilde{L}[q, \mathfrak{B}] = \{k \in \mathrm{Hom}(X, V) \mid \varphi[k] = q,\ \varphi(k, \widetilde{L}) = \mathfrak{B}\}. \tag{1.2}$$

This set was introduced by Shimura in the theory of quadratic Diophantine equations [8, Section 13] as a generalization of the set defined by

$$L[s, \mathfrak{b}] = \{x \in V \mid \varphi[x] = s,\ \varphi(x, L) = \mathfrak{b}\} \tag{1.3}$$

for $0 \neq s \in F$ and a fractional ideal $\mathfrak{b}$ of F.

We consider the orbits of $L[s, \mathfrak{b}]$ or $\widetilde{L}[q, \mathfrak{B}]$ under the action of the stabilizer $\Gamma(L)$, that is, we have the sets of $\Gamma(L)$-orbits

$$L[s, \mathfrak{b}]/\Gamma(L), \qquad \widetilde{L}[q, \mathfrak{B}]/\Gamma(\widetilde{L}).$$

These are finite sets by [8, Theorem 13.3]. Hence if $\widetilde{L}[q,\, \mathfrak{B}]$ is nonempty, there are a finite number of $\Gamma(\widetilde{L})$-orbits of $\widetilde{L}[q,\, \mathfrak{B}]$ such that

$$\widetilde{L}[q,\, \mathfrak{B}] = \bigsqcup_{\zeta\in Z} h\zeta\Gamma(\widetilde{L})$$

with a finite subset Z of G for an arbitrarily fixed element $h \in \widetilde{L}[q,\, \mathfrak{B}]$. Now assume $L[s,\, \mathfrak{b}] \neq \emptyset$ for s and $\mathfrak{b}$ as in (1.3). We pick $x_0 \in L[s,\, \mathfrak{b}]$ and fix it. Put

$$W = (Fx_0)^{\perp} = \{x \in V \mid \varphi(x_0,\, x) = 0\}$$

and let ψ be the restriction of φ to W. We regard $SO^{\psi}(W)$ as the subgroup $H = \{\eta \in G \mid x_0\eta = x_0\}$ of G. Let h be an element of $\mathrm{Hom}(X,\, V)$ such that

$$L \cap W = Mh \tag{1.4}$$

with a $\mathfrak{g}$-lattice M in X. Put $q = \varphi[h]$ and $\mathfrak{B} = \varphi(h,\, \widetilde{L})$. Then our aim of this section is to describe the $\Gamma(\widetilde{L})$-orbits of $\widetilde{L}[q,\, \mathfrak{B}]$ in terms of the $\Gamma(L)$-orbits of $L[s,\, \mathfrak{b}]$ under the assumption that L is $\mathfrak{g}$-maximal with respect to φ. To state our result on $\widetilde{L}[q,\, \mathfrak{B}]$, Proposition 1.2 below, we need some more notation.

Take a set $\{L_i\}_{i\in I}$ of representatives for the G-classes in the G-genus of L for which $L_i[s,\, \mathfrak{b}] \neq \emptyset$. Note that I depends on s and $\mathfrak{b}$. Suppose L is $\mathfrak{g}$-maximal with respect to φ. Then by virtue of [10, Theorem 2.2] we have a bijection

(1.5)
$$\bigsqcup_{i\in I}\{L_i[s,\, \mathfrak{b}]/\Gamma(L_i)\} \ni x\Gamma(L_i) \mapsto H\xi[x](H_{\mathbf{A}} \cap C(L)) \in H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}} \cap C(L)),$$

where $\xi[x] \in H_{\mathbf{A}}$ is given as follows:

Set $L_i\alpha_i = L$ with $\alpha_i \in G_{\mathbf{A}}$ for $i \in I$. For $x \in L_i[s,\, \mathfrak{b}]$ there is $\gamma \in G$ such that $x\gamma = x_0$ by [8, Lemma 1.5(ii)] as $\varphi[x] = \varphi[x_0]$. For each $v \in \mathbf{h}$ observe that x_0 and $x(\alpha_i)_v$ belong to $L_v[s,\, \mathfrak{b}_v]$. Since L_v is $\mathfrak{g}_v$-maximal, by [10, Theorem 1.3] there is $\alpha_v \in C(L_v)$ such that $x_0 = x(\alpha_i)_v\alpha_v$. For $v \in \mathbf{a}$ we put $\alpha_v = (\alpha_i)_v^{-1}\gamma_v$. Let α be the element of $C(L)$ whose v-component is α_v for every prime v of F. Then assigning $\gamma^{-1}\alpha_i\alpha$ to x, we have the bijection given in (1.5) by [8, Theorem 11.6(i)]; see also [10, Theorem 2.2]. For our argument below, we denote such γ and α by $\gamma[x]$ and $\alpha[x]$, respectively; notice that $\gamma[x]$ and $\alpha[x]$ are not uniquely determined by $x \in L_i[s,\, \mathfrak{b}]$. We then put

$$\xi[x] = \gamma[x]^{-1}\alpha_i\alpha[x]. \tag{1.6}$$

For each $i \in I$ take a complete set U_i of representatives for $L_i[s,\, \mathfrak{b}]/\Gamma(L_i)$. Then by (1.5) we have

$$H_{\mathbf{A}} = \bigsqcup_{i\in I}\bigsqcup_{u\in U_i} H\xi[u](H_{\mathbf{A}} \cap C(L)).$$

Noticing that $H_{\mathbf{A}} \cap C(L) \subset C(L \cap W)$, we further take a subset U_i' of U_i such that

$$\bigcup_{u \in U_i} H\xi[u]C(L \cap W) = \bigsqcup_{u \in U_i'} H\xi[u]C(L \cap W). \tag{1.7}$$

Put $A_u^i = \{w \in U_i \mid H\xi[w]C(L \cap W) = H\xi[u]C(L \cap W)\}$ for a fixed $u \in U_i$.

We identify the special orthogonal group of the restriction of φ to Xh with H. Considering the orthogonal group $SO^q(X)$ in the manner of §1.1, set

$$\Delta_u = H \cap \xi[u]C(L \cap W)\xi[u]^{-1}, \qquad \Delta_u' = \{\delta' \in SO^q(X) \mid \delta \in \Delta_u\}$$

for $u \in U_i$. We note that $\Delta_u = H \cap C((L \cap W)\xi[u]^{-1}) = \Gamma((L \cap W)\xi[u]^{-1})$.

Lemma 1.1. *Let E be the subgroup of $H_{\mathbf{A}}$ defined by*

$$E = \{\varepsilon \in H_{\mathbf{A}} \mid \varepsilon'\mathfrak{B} = \mathfrak{B}\} \tag{1.8}$$

with the present h. Then $E = C(L \cap W)$. Moreover, $\Delta_u = H \cap \xi[u]E\xi[u]^{-1}$ and

$$[\Delta_u : H \cap \xi[u]C(L)\xi[u]^{-1}] \leq [C(L \cap W) : H_{\mathbf{A}} \cap C(L)] \tag{1.9}$$

for every $u \in U_i$ and $i \in I$.

Proof. Let $\varepsilon \in E$. Since $\varepsilon'\mathfrak{B} = \mathfrak{B}$, we have $\varphi(xh\varepsilon_v, \widetilde{L_v}) = \varphi(xh, \widetilde{L_v})$ for every $x \in M_v$ and $v \in \mathbf{h}$. Further since h satisfies (1.4), we have $\varphi(xh, \widetilde{L_v}) \subset \varphi(L_v, \widetilde{L_v}) \subset 2^{-1}\mathfrak{g}_v$. Thus $M_v h\varepsilon_v \subset L_v$. From this it follows that $(L \cap W)\varepsilon \subset L \cap W$, which proves $E \subset C(L \cap W)$. The opposite inclusion can be seen in a straightforward way. Hence we have the first assertion. By using this we see that $H \cap \xi[u]E\xi[u]^{-1} = \Delta_u \subset \xi[u]E\xi[u]^{-1}$. Then the map $x \mapsto \xi[u]^{-1}x\xi[u]$ gives an injection of $\Delta_u/(H \cap \xi[u]C(L)\xi[u]^{-1})$ into $E/(H_{\mathbf{A}} \cap C(L))$, which shows (1.9). □

For $w \in A_u^i$ take an element $\eta[w]$ of H such that $\xi[w] \in \eta[w]\xi[u]C(L \cap W)$ and a complete set Z_w^u of representatives for $\Delta_u/(H \cap \eta[w]^{-1}\xi[w]C(L)\xi[w]^{-1}\eta[w])$.

Proposition 1.2. *Assume that L is $\mathfrak{g}$-maximal with respect to φ. Let $x_0 \in L[s, \mathfrak{b}]$, take h as in (1.4), and put $q = \varphi[h]$ and $\mathfrak{B} = \varphi(h, \widetilde{L})$ with the notation above. Then for each $i \in I$ there exists a bijection*

$$L_i[s, \mathfrak{b}]/\Gamma(L_i) \longrightarrow \bigsqcup_{u \in U_i'} \left\{\Delta_u' \backslash \widetilde{L}_i[q, \xi[u]'\mathfrak{B}]/\Gamma(\widetilde{L}_i)\right\} \tag{1.10}$$

and $\#\left\{\Delta_u' \backslash \widetilde{L}_i[q, \xi[u]'\mathfrak{B}]/\Gamma(\widetilde{L}_i)\right\} = \#(A_u^i)$ *for $u \in U_i'$. Consequently we have*

$$\widetilde{L}_i[q, \xi[u]'\mathfrak{B}] = \bigsqcup_{w \in A_u^i} \bigsqcup_{\zeta \in Z_w^u} h\zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i) \tag{1.11}$$

for every $u \in U_i'$ and $i \in I$, where $\gamma[w]$ is an element of G such that $w\gamma[w] = x_0$.

1.3. **Proof of Proposition 1.2.** We first show that

$$\bigcup_{u\in U_i}\{H\backslash(H\xi[u]E\cap G\xi[u]C(L))/D\}=\{H\xi[u]D\mid u\in U_i\}, \tag{1.12}$$

where E is as in (1.8) and $D=H_{\mathbf{A}}\cap C(L)$. Since $\xi[u]\in H\xi[u]E\cap G\xi[u]C(L)$, we have $H\xi[u]D\in H\backslash(H\xi[u]E\cap G\xi[u]C(L))/D$ for every $u\in U_i$. Conversely, let $\sigma\in H\xi[u]E\cap G\xi[u]C(L)$ with $u\in U_i$. Since $H\xi[u]E\cap G\xi[u]C(L)\subset H_{\mathbf{A}}\cap G\alpha_iC(L)$, by applying [10, Theorem 2.2], σ can be regarded as a representative of the image of some element $x\Gamma(L_i)$ under the bijection

$$(V\cap x_0C(L)\alpha_i^{-1})/\Gamma(L_i)\longrightarrow H\backslash(H_{\mathbf{A}}\cap G\alpha_iC(L))/D$$

defined in [10, Theorem 2.2(ii)] with x_0, $C(L)$, and α_i in place of h, D, and y there. Here $x_0C(L)\alpha_i^{-1}$ is meaningful as a subset of $V\otimes_F F_{\mathbf{A}}$. Thus we can put $\sigma=\gamma^{-1}\alpha_i\delta^{-1}$ with $\gamma\in G$ and $\delta\in C(L)$ such that $x\gamma=x_0$ and $x=x_0\delta\alpha_i^{-1}$. By virtue of [10, (2.5)], we have $V\cap x_0C(L)\alpha_i^{-1}=L_i[s,\,\mathfrak{b}]$. Hence x belongs to $L_i[s,\,\mathfrak{b}]$. Take $\xi[x]=\gamma[x]^{-1}\alpha_i\alpha[x]$ in the manner explained in (1.6). Then we see that $\sigma=\xi[x]\alpha[x]^{-1}\delta^{-1}$ and $x_0\alpha[x]^{-1}\delta^{-1}=x\alpha_i\delta^{-1}=x_0$. Hence we have $\sigma\in H\xi[x]D$. Since we can find $u\in U_i$ such that $H\xi[x]D=H\xi[u]D$ via (1.5), the desired (1.12) follows.

Let us take a subset U_i' in (1.7). Clearly U_i can be given by $U_i=\bigsqcup_{u\in U_i'}A_u^i$, where $A_u^i=\{w\in U_i\mid H\xi[w]E=H\xi[u]E\}$ for $u\in U_i$. We observe that $(H\xi[u]E\cap G\xi[u]C(L))\cap(H\xi[w]E\cap G\xi[w]C(L))\neq\emptyset$ if and only if $w\in A_u^i$ for $u,\ w\in U_i$, in which case these two sets coincide. Hence the left-hand side of (1.12) coincides with the disjoint union

$$\bigsqcup_{u\in U_i'}\{H\backslash(H\xi[u]E\cap G\alpha_iC(L))/D\}\,. \tag{1.13}$$

Then (1.12) leads to the following fact:

$$H\xi[u]E\cap G\alpha_iC(L)=\bigsqcup_{w\in A_u^i}H\xi[w]D \tag{1.14}$$

for every $u\in U_i'$.

Now we put $\mathcal{V}=\mathrm{Hom}(X,\,V)$ and consider the set

$$\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1}$$

for $u\in U_i'$. Here $h\xi[u]EC(\widetilde{L})\alpha_i^{-1}$ is meaningful as a subset of $\mathcal{V}\otimes_F F_{\mathbf{A}}$ and $C(\widetilde{L})=C(L)$. Suppose that $\left\{\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1}\right\}\cap\left\{\mathcal{V}\cap h\xi[w]EC(\widetilde{L})\alpha_i^{-1}\right\}$ $\neq\emptyset$ for $u,\,w\in U_i'$. Put $h\xi[u]=h\xi[w]\varepsilon\delta\varepsilon_1$ with $\varepsilon,\,\varepsilon_1\in E$ and $\delta\in C(L)$. By [10, (2.8)] we have $\xi[u]=\xi[w]\varepsilon\delta\varepsilon_1$. Observe that $\delta\in D\subset E$. Then

$H\xi[u]C(L\cap W)=H\xi[w]\varepsilon\delta\varepsilon_1 C(L\cap W)=H\xi[w]C(L\cap W)$ as $E=C(L\cap W)$. Thus $u=w$ by our choice of U_i'. Hence we have a disjoint union

$$\bigsqcup_{u\in U_i'}\left\{\Delta_u'\backslash(\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1})/\Gamma(\widetilde{L}\alpha_i^{-1})\right\},$$

where $\Delta_u'\backslash(\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1})/\Gamma(\widetilde{L}\alpha_i^{-1})$ is meaningful under the isomorphism of $SO^q(X)$ onto H explained in §1.1. By [10, Theorem 2.3] there exists a bijection

$$H\backslash(H\xi[u]E\cap G\alpha_i C(L))/D\to\Delta_u'\backslash(\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1})/\Gamma(\widetilde{L}\alpha_i^{-1}) \tag{1.15}$$

via $\sigma\mapsto h\eta^{-1}\gamma$ with $\gamma\in G$ and $\eta\in H$ such that $\sigma\in\eta\xi[u]E\cap\gamma\alpha_i C(L)$. Furthermore, by applying [10, (2.12)] with $C(\widetilde{L})$ in place of D there, we have

$$\mathcal{V}\cap h\xi[u]EC(\widetilde{L})\alpha_i^{-1}=(\widetilde{L}\alpha_i^{-1})[q,\,\xi[u]'\mathfrak{B}].$$

Combining this with (1.15) and composing it with the bijection of $L_i[s,\,\mathfrak{b}]/\Gamma(L_i)$ onto the set of (1.13), defined by the same map as (1.5), we obtain (1.10). This bijection can be given by

$$w\Gamma(L_i)\mapsto H\xi[w]D\mapsto\Delta_u'h\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)$$

for $w\in A_u^i$ with $u\in U_i'$, since $\xi[w]=\gamma[w]^{-1}\alpha_i\alpha[w]\in\eta[w]\xi[u]E\cap\gamma[w]^{-1}\alpha_i C(L)$ with $\eta[w]\in H$. Notice that $h\eta[w]^{-1}\gamma[w]^{-1}\in\widetilde{L}_i[q,\,\xi[u]'\mathfrak{B}]$ for any $w\in A_u^i$. Then (1.10) concludes that $\Delta_u'\backslash\widetilde{L}_i[q,\,\xi[u]'\mathfrak{B}]/\Gamma(\widetilde{L}_i)$ consists of $\Delta_u'h\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)$ for all $w\in A_u^i$. Thus we have $\widetilde{L}_i[q,\,\xi[u]'\mathfrak{B}]=\bigsqcup_{w\in A_u^i}h\Delta_u\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)$ for every $u\in U_i'$.

To prove (1.11), observe that $\zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)=\zeta'\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)$ if and only if $\zeta^{-1}\zeta'\in H\cap\eta[w]^{-1}\xi[w]C(L)\xi[w]^{-1}\eta[w]$ for $\zeta,\,\zeta'\in\Delta_u$. From this we have $\Delta_u\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)=\bigsqcup_{\zeta\in Z_w^u}\zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)$ with Z_w^u given before the statement. Suppose $h\zeta'\in h\zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)\gamma[w]\eta[w]$ with $\zeta,\,\zeta'\in Z_w^u$. Then $\zeta'\in\zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)\gamma[w]\eta[w]$ by [10, (2.8)]. Since $\zeta^{-1}\zeta'\in H\cap\eta[w]^{-1}\xi[w]C(L)\xi[w]^{-1}\eta[w]$, it follows that $\zeta=\zeta'$, which proves (1.11). This completes the proof of Proposition 1.2.

Proposition 1.2 is not valid for arbitrary $h\in\widetilde{L}[q,\,\mathfrak{B}]$; it needs to take h as in (1.4). Such an h exists if $L[s,\,\mathfrak{b}]\neq\emptyset$. Conversely, we can prove the following.

Lemma 1.3. *Assume that F has class number 1. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Suppose $\widetilde{L}[q,\,\mathfrak{B}]\neq\emptyset$ with q and $\mathfrak{B}$ as in (1.2). Let s and b be arbitrarily fixed numbers of $F^\times$ such that $b^{-2}s=\det(\varphi)^{-1}\det(q)$. For every $k\in\widetilde{L}[q,\,\mathfrak{B}]$ there exists $x_0\in L[s,\,b\mathfrak{g}]$ such that $L\cap(Fx_0)^\perp=Mk$ with a $\mathfrak{g}$-lattice M in X.*

Proof. With respect to a $\mathfrak{g}$-basis of L we may identify V with F_n^1, L with $\mathfrak{g}_n^1$, and φ with a symmetric element of $GL_n(F)$. Also, fixing a $\mathfrak{g}$-basis of $\mathfrak{B}$ and taking the dual basis of X, we may identify $\mathfrak{B}$ with $2^{-1}\mathfrak{g}_1^{n-1}$ and X with F_{n-1}^1. Then $\widetilde{L} = \mathfrak{g}_n^1(2\varphi)^{-1}$ and $\widetilde{L}[q, \mathfrak{B}]$ coincides with the set $\{q, \varphi\}$ in (0.1), where ${}^tq = q \in GL_{n-1}(F)$. Given $k \in \{q, \varphi\}$, through the mapping λ of [4, (1.5)] with $m = n-1$, k corresponds to an element x of $\{r, \varphi^{-1}\}$ with $r = \det(\varphi)^{-1}\det(q)$, where $\{r, \varphi^{-1}\} = \{x \in F_n^1 \mid x\varphi^{-1} \cdot {}^tx = r, x\mathfrak{g}_1^n = \mathfrak{g}\}$. Set $r = sb^{-2}$ with $s, b \in F^\times$. Then $bx\varphi^{-1}$ belongs to $L[s, b\mathfrak{g}]$. Also if we put $x_0 = bx\varphi^{-1}$, k can be viewed as an inverse image of $x_0^*(= x)$ under λ. Hence by [4, Theorem 2.1] we have $L \cap (Fx_0)^\perp = \mathfrak{g}_m^1 k$. This proves the desired fact. $\square$

1.4. **Proof of Theorem.** Let $V = \mathbf{Q}_n^1$ and let φ be a symmetric matrix in $GL_n(\mathbf{Q})$. We put $G = SO(\varphi) = \{\gamma \in SL_n(\mathbf{Q}) \mid \gamma\varphi \cdot {}^t\gamma = \varphi\}$. For the $\mathbf{Z}$-lattice $L = \mathbf{Z}_n^1$ in V and for $s, b \in \mathbf{Q}^\times$ let $C(L)$ and $\Gamma(L)$ be as in (1.1) and $L[s, b\mathbf{Z}]$ as in (0.2). Taking X to be $\mathbf{Q}_m^1$ in the notation of §1.1, we may put $\mathrm{Hom}(X, \mathbf{Q}) = \mathbf{Q}_1^m$, $\mathrm{Hom}(X, V) = \mathbf{Q}_n^m$, and $\varphi(k, \ell) = k\varphi \cdot {}^t\ell$ for $k \in \mathbf{Q}_n^m$ and $\ell \in \mathbf{Q}_n^1$, where $m = n-1 > 1$.

We assume that L is $\mathbf{Z}$-maximal with respect to φ; in other words, we treat a *reduced* symmetric matrix φ in the sense of [9, (6.2)]. Let us consider the set $\{q, \varphi\}$ in (0.1) and prove Theorem in the introduction by applying Proposition 1.2 to $\widetilde{L}[q, 2^{-1}\mathbf{Z}_1^m]$ for a given $q = {}^tq \in GL_m(\mathbf{Q})$.

We first observe that $\widetilde{L} = \mathbf{Z}_n^1(2\varphi)^{-1}$. Then $\varphi(k, \widetilde{L}) = 2^{-1}\mathbf{Z}_1^m$ if and only if k is primitive in the sense that $k\mathbf{Z}_1^n = \mathbf{Z}_1^m$ for $k \in \mathbf{Q}_n^m$. Hence we have

$$\widetilde{L}[q, 2^{-1}\mathbf{Z}_1^m] = \{q, \varphi\}. \tag{1.16}$$

Assuming $\{q, \varphi\} \neq \emptyset$, pick h from $\{q, \varphi\}$. By Lemma 1.3 there exists $v \in L[s, b\mathbf{Z}]$ such that

$$L \cap (\mathbf{Q}v)^\perp = \mathbf{Z}_m^1 h \tag{1.17}$$

with fixed $s, b \in \mathbf{Q}^\times$ satisfying $b^{-2}s = \det(\varphi)^{-1}\det(q)$. We set $W = (\mathbf{Q}v)^\perp$, $\psi = \varphi|_W$, and $H = SO(\psi) = \{\gamma \in G \mid v\gamma = v\}$. Then Proposition 1.2 is applicable to h, where we take L_1 to be L and take U as a complete set U_1 of representatives for $L[s, b\mathbf{Z}]/\Gamma(L)$ containing v; we may assume $v \in U'$ and $\xi[v] = 1_n$ with the notation of Proposition 1.2. We write simply A and Z_u for A_v^1 and Z_u^v; also put $\xi_v = \xi[v]$, $\gamma_u = \gamma[u]$, and $\eta_u = \eta[u]$. Clearly $\Gamma(L) = G \cap GL_n(\mathbf{Z}) = \Gamma(\varphi)$. As for $\Gamma(L \cap W)$, the map $\gamma \mapsto \gamma'$ determined by the rule $\gamma' h = h\gamma$ for $\gamma \in \Gamma(L \cap W)$ gives an isomorphism of $\Gamma(L \cap W)$ onto $\Gamma(q) = SO(q) \cap GL_m(\mathbf{Z})$. Thus from (1.11) our theorem follows.

2. The mass of $\widetilde{L}[q,\, \mathfrak{B}]$

2.1. **Preliminaries for the mass.** In this section we assume that F is *totally real* and keep the notation of Section 1. We shall recall some notation to define the mass. If the readers are familiar with this subject, we recommend them to proceed directly into §2.2 in which $\mathfrak{m}(\widetilde{L}[q,\, \mathfrak{B}])$ is discussed.

We first represent φ by a symmetric matrix in $GL_n(F)$ with a fixed basis of V over F. Then $SO^{\varphi}(V)$ is given by $G = \{\gamma \in SL_n(F) \mid \gamma\varphi \cdot {}^t\gamma = \varphi\}$. For each $v \in \mathbf{a}$ we denote by φ_v the image of φ under the embedding of F into $\mathbf{R}$ over $\mathbf{Q}$ at v; we put $G_v = SO(\varphi_v) = \{\alpha \in SL_n(\mathbf{R}) \mid \alpha\varphi_v \cdot {}^t\alpha = \varphi_v\}$. By a Witt decomposition, φ_v can be represented by

$$\begin{bmatrix} 0 & 0 & -1_{r_v} \\ 0 & \theta_v & 0 \\ -1_{r_v} & 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $r_v \geq 0$ and θ_v is an element of $GL_{t_v}(\mathbf{R})$ which is positive or negative definite. (If $t_v = 0$, we ignore θ_v.) We take $\kappa \in F^{\times}$ so that $\kappa_v\varphi_v$ has signature $(r_v + t_v,\, r_v)$ for every $v \in \mathbf{a}$. Further, we fix $\sigma_v \in GL_n(\mathbf{R})$ such that

$$\kappa_v\sigma_v\varphi_v \cdot {}^t\sigma_v \text{ is of the form (2.1) with } 0 < \theta_v = {}^t\theta_v \in GL_{t_v}(\mathbf{R}). \tag{2.2}$$

Put $\varphi'_v = \kappa_v\sigma_v\varphi_v \cdot {}^t\sigma_v$. We then define a set $\mathcal{Z}^{\varphi}_v$ by

(2.3)

$$\mathcal{Z}^{\varphi}_v = \mathcal{Z}(r_v,\, \theta_v) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^{r_v+t_v}_{r_v} \;\middle|\; x \in \mathbf{R}^{r_v}_{r_v},\, y \in \mathbf{R}^{t_v}_{r_v},\, {}^tx + x > {}^ty\theta_v^{-1}y \right\}.$$

When $r_v = 0$, we understand that $\mathcal{Z}^{\varphi}_v$ consists of a single point written as $\mathbf{1}_v$. Notice that $\mathcal{Z}^{\varphi}_v$ depends on the choice of (2.2).

For $\alpha \in SO(\varphi'_v)$ and $z \in \mathcal{Z}^{\varphi}_v$ we can define $\alpha(z) \in \mathcal{Z}^{\varphi}_v$ in the manner explained in [8, §16.3]. Then by [8, Proposition 16.6(i), (iii), and (v)], $SO(\varphi'_v)$ acts transitively on $\mathcal{Z}^{\varphi}_v$, $C'_v = \{\gamma \in SO(\varphi'_v) \mid \gamma(\mathbf{1}_v) = \mathbf{1}_v\}$ is a maximal compact subgroup of $SO(\varphi'_v)$, and $\{\alpha \in SO(\varphi'_v) \mid \alpha(z) = z$ for every $z \in \mathcal{Z}^{\varphi}_v\} = SO(\varphi'_v) \cap \{\pm 1_n\}$ if $r_v > 0$, where $\mathbf{1}_v = \begin{bmatrix} 1_{r_v} \\ 0 \end{bmatrix}$. If $r_v = 0$, we let $SO(\varphi'_v)$ act trivially on $\mathcal{Z}^{\varphi}_v = \{\mathbf{1}_v\}$.

We set

$$\mathcal{Z} = \prod_{v \in \mathbf{a}} \mathcal{Z}(r_v,\, \theta_v), \quad G_{\mathbf{a}} = \prod_{v \in \mathbf{a}} G_v, \quad C_{\mathbf{a}} = \prod_{v \in \mathbf{a}} C_v, \quad C_v = \sigma_v^{-1}C'_v\sigma_v.$$

Since $SO(\kappa_v\varphi_v) = SO(\varphi_v)$ and $\sigma_v SO(\varphi_v)\sigma_v^{-1} = SO(\varphi'_v)$, we can define the action of $G_{\mathbf{a}}$ on $\mathcal{Z}$ by $\alpha(z) = ((\sigma_v\alpha_v\sigma_v^{-1})(z_v))_{v\in\mathbf{a}}$ for $\alpha \in G_{\mathbf{a}}$ and $z \in \mathcal{Z}$. Hence G acts on $\mathcal{Z}$ via the projection of G into $G_{\mathbf{a}}$. Put $\mathbf{1} = (\mathbf{1}_v)_{v\in\mathbf{a}}$.

Suppose $G_{\mathbf{a}}$ is not compact for a moment. Let $v \in \mathbf{a}$ such that $r_v > 0$. Since $\mathcal{Z}_v^{\varphi}$ is connected, this can be viewed as a Riemannian manifold with the G_v-invariant metric defined by

$$ds_v^2 = \mathrm{tr}\left({}^t dz_v \cdot \xi'(z_v)^{-1} \cdot dz_v \cdot \xi(z_v)^{-1}\right), \tag{2.4}$$

where $z_v = [(z_v)_{ij}] = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{Z}(r_v, \theta_v)$, $dz_v = [d(z_v)_{ij}]$, $\xi(z_v) = x + {}^t x - {}^t y\theta_v^{-1}y$, and $\xi'(z_v) = \begin{bmatrix} x + {}^t x & {}^t y \\ y & \theta_v \end{bmatrix}$. Furthermore, $\mathcal{Z}_v^{\varphi}$ is a symmetric space in the following sense:

Lemma 2.1. *$\mathcal{Z}_v^{\varphi}$ is a simply connected Riemannian globally symmetric space of the noncompact type with the metric ds_v^2 and it is analytically diffeomorphic to the symmetric space $(G_v)_0/C_v$ under the map $\alpha_v(\mathbf{1}_v) \mapsto \alpha_v C_v$ for $\alpha_v \in G_v$ if $r_v > 0$. Here $(G_v)_0$ is the identity component of G_v.*

We omit the proof of Lemma 2.1 because this fact seems to be well known. As for the proof, [2, II, Proposition 4.3(a), V, §2, Example 1, and VI, Theorem 1.1(iii)], and [8, (16.10a and b)] may be referred.

Now, $\mathcal{Z}$ is a symmetric space on which $G_{\mathbf{a}}$ acts transitively, where the $G_{\mathbf{a}}$-invariant metric is defined by $ds^2 = \sum_{v\in\mathbf{b}} ds_v^2$ with $\mathbf{b} = \{v \in \mathbf{a} \mid r_v > 0\}$. We define a $G_{\mathbf{a}}$-invariant measure on $\mathcal{Z}$ by

$$\mathbf{d}z = \prod_{v\in\mathbf{b}} \left\{ \det(2^{-1}\xi(z_v))^{-n/2} \prod_{i=1}^{r_v+t_v} \prod_{j=1}^{r_v} d(z_v)_{ij} \right\}, \tag{2.5}$$

where z_v is as in (2.4). Let $\Gamma(L)$ be as in (1.1). We have the image Γ of $\Gamma(L)$ under the projection of G into $G_{\mathbf{a}}$. Then Γ is a discrete subgroup of $G_{\mathbf{a}}$ in the relative topology. Hence Γ acts properly discontinuously on $\mathcal{Z}$ and $\Gamma\backslash\mathcal{Z}$ is a locally compact Hausdorff space with the quotient topology. We denote by $\mathrm{vol}(\Gamma\backslash\mathcal{Z})$ the measure of $\Gamma\backslash\mathcal{Z}$ with respect to $\mathbf{d}z$ and assume that $\mathrm{vol}(\Gamma\backslash\mathcal{Z})$ is finite. We then put

$$\nu(\Gamma) = [\Gamma \cap T : 1]^{-1}\mathrm{vol}(\Gamma\backslash\mathcal{Z}), \tag{2.6}$$

where $T = \{\gamma \in G \mid \gamma = \mathrm{id.\ on\ } \mathcal{Z}\}$.

To discuss the mass in §2.2 in a unified way, we employ the symbol $\nu(\Gamma)$ also when $G_{\mathbf{a}}$ is compact. Namely, we take the measure of $G_{\mathbf{a}}$ to be 1 and set $\mathrm{vol}(\Gamma\backslash\mathcal{Z}) = [\Gamma : \Gamma \cap T]^{-1}$. Hence $\nu(\Gamma) = [\Gamma : 1]^{-1}$ if $G_{\mathbf{a}}$ is compact.

Let $\{L_i\}_{i\in I_0}$ be a complete set of representatives for the G-classes in the G-genus of L. Then *the mass of G relative to $C(L)$* is defined by

$$\mathfrak{m}(G,\, C(L)) = \sum_{i\in I_0} \nu(\Gamma(L_i)). \tag{2.7}$$

This is independent of the choice of $\{L_i\}_{i\in I_0}$. Also when $G_{\mathbf{a}}$ is not compact, $\mathfrak{m}(G,\, C(L))$ depends on the choice of (2.2).

Let X be a vector space of dimension m over F for $n > m > 1$ and put $\mathcal{V} = \mathrm{Hom}(X,\, V)$. Let us take a subset S of $\mathcal{V}$ such that

$$S = \bigsqcup_{\beta\in B} k\beta\Gamma.$$

Here k is a fixed element of $\mathcal{V}$ of $\det(\varphi[k]) \neq 0$, B is a finite subset of G, and $\Gamma = G \cap D$ with an open subgroup D of $G_{\mathbf{A}}$ such that D contains $G_{\mathbf{a}}$ and $G_{\mathbf{h}} \cap D$ is compact. We regard $SO((Xk)^\perp)$ as $\{\gamma \in G \mid k\gamma = k\}$ and observe that $k\beta\gamma = k\beta$ with some $\gamma \in \Gamma$ if and only if $\gamma \in \beta^{-1}SO((Xk)^\perp)\beta \cap \Gamma$. Following [10, (3.4)], *the mass of the set S* is then defined by

$$\mathfrak{m}(S) = \sum_{\beta\in B} \nu(SO((Xk)^\perp) \cap \beta\Gamma\beta^{-1})/\nu(\Gamma). \tag{2.8}$$

Here $\nu(SO((Xk)^\perp) \cap \beta\Gamma\beta^{-1})$ is defined in a similar manner to (2.6). It can be shown that $\mathfrak{m}(S)$ is independent of the choice of B and Γ; the latter means that $\mathfrak{m}(S)$ defined with respect to a decomposition of S into Γ'-orbits is equal to (2.8) for $\Gamma' = G \cap D'$ with an open subgroup D' of $G_{\mathbf{A}}$ such that D' contains $G_{\mathbf{a}}$ and $G_{\mathbf{h}} \cap D'$ is compact; for the proof, see after [11, Theorem 10]. We note that $\mathfrak{m}(S)$ depends on the choice of matrices equivalent to φ and the restriction $\varphi|_{(Xk)^\perp}$ over $\mathbf{R}$ as in (2.2). Also by [10, (3.5)], $\mathfrak{m}(S) = \#(S)$ if $G_{\mathbf{a}}$ is compact.

2.2. **Results on the mass.** We return to the setting of §1.2. With E, A_u^i, and Z_w^u there, we observe by noticing $\xi[w] \in \eta[w]\xi[u]E$ for $w \in A_u^i$ that $\#(Z_w^u) = [\Delta_w : H \cap \xi[w]C(L)\xi[w]^{-1}]$, where $\Delta_w = H \cap \xi[w]E\xi[w]^{-1}$. Also, because of $\dim(W^\perp) = 1$,

$$\nu(SO(W^\perp) \cap \zeta\eta[w]^{-1}\gamma[w]^{-1}\Gamma(\widetilde{L}_i)\gamma[w]\eta[w]\zeta^{-1}) = 1$$

for every $\zeta \in Z_w^u$ and $w \in A_u^i$. Thus by (1.11) with h of (1.4) the mass of $\widetilde{L}_i[q,\, \xi[u]'\mathfrak{B}]$ can be given as follows:

$$\mathfrak{m}(\widetilde{L}_i[q,\, \xi[u]'\mathfrak{B}]) = \sum_{w\in A_u^i} [\Delta_w : H \cap \xi[w]C(L)\xi[w]^{-1}]\nu(\Gamma(\widetilde{L}_i))^{-1} \tag{2.9}$$

for $u \in U_i'$ and $i \in I$. It is noted that the elements $h\zeta\eta[w]^{-1}\gamma[w]^{-1}\gamma$ with $w \in A_u^i$, $\zeta \in Z_w^u$, and $\gamma \in \Gamma(\widetilde{L}_i)$ constitute the set $\widetilde{L}_i[q,\, \xi[u]'\mathfrak{B}]$ without repetition. This is because if $h\zeta\eta[w]^{-1}\gamma[w]^{-1}\gamma = h\zeta\eta[w]^{-1}\gamma[w]^{-1}$ with $\gamma \in$

$\Gamma(\widetilde{L}_i)$, then $\gamma = 1$ by [10, (2.8)] as $\dim(W^{\perp}) = 1$. Hence $\#\{\widetilde{L}_i[q, \xi[u]'\mathfrak{B}]\} = \sum_{w\in A_u^i} \#(Z_w^u)[\Gamma(\widetilde{L}_i) : 1]$ if $G_{\mathbf{a}}$ is compact, which is consistent with (2.9).

We consider the set $L_i[s, \mathfrak{b}] = \{x \in V \mid \varphi[x] = s, \varphi(x, L_i) = \mathfrak{b}\}$. Since $L_i[s, \mathfrak{b}]/\Gamma(L_i)$ is a finite set, the mass $\mathfrak{m}(L_i[s, \mathfrak{b}])$ can be defined by (2.8) with $\mathbf{d}z$ and with the $H_{\mathbf{a}}$-invariant measure on a symmetric space associated with $\psi = \varphi|_W$ in the sense of §2.1, where $H = SO^{\psi}(W)$ and $W = (Fx_0)^{\perp}$ with a fixed $x_0 \in L[s, \mathfrak{b}]$.

Lemma 2.2. *For each $u \in U_i'$, $\{\xi[w] \mid w \in A_u^i\}$ is a complete set of representatives for $H\backslash(H\xi[u]E \cap G\alpha_i C(L))/(H_{\mathbf{A}} \cap C(L))$. Moreover,*

$$\nu(\Gamma(L_i))\mathfrak{m}(L_i[s, \mathfrak{b}]) = \sum_{u\in U_i'} \nu(\Delta_u) \sum_{w\in A_u^i} [\Delta_w : H \cap \xi[w]C(L)\xi[w]^{-1}].$$

Proof. The first assertion follows from (1.14) with $D = H_{\mathbf{A}} \cap C(L)$. Put $\mathcal{E}_u = \{\xi[w] \mid w \in A_u^i\}$. We observe that $H\backslash(H_{\mathbf{A}} \cap G\alpha_i C(L))/D$ coincides with $\bigsqcup_{u\in U_i'}\{H\backslash(H\xi[u]E \cap G\alpha_i C(L))/D\}$. Hence $\bigsqcup_{u\in U_i'} \mathcal{E}_u$ gives a complete set of representatives for $H\backslash(H_{\mathbf{A}} \cap G\alpha_i C(L))/D$. Then the following equality is a special case of the equality in the line 6 of page 347 of [10] with V, $C(L)$, x_0, and α_i in place of $\mathcal{V}$, D, h, and y there:

$$\nu(\Gamma(L_i))\mathfrak{m}(L_i[s, \mathfrak{b}]) = \sum_{u\in U_i'} \sum_{w\in A_u^i} \nu(H \cap \xi[w]C(L)\xi[w]^{-1}).$$

Applying [11, Theorem 10(iii) and (iv)], we have further

$$\begin{aligned} \nu(H \cap \xi_w C(L)\xi_w^{-1}) &= [\Delta_w : H \cap \xi_w C(L)\xi_w^{-1}]\nu(\Delta_w) \\ &= [\Delta_w : H \cap \xi_w C(L)\xi_w^{-1}]\nu(\Delta_u). \end{aligned}$$

Combining this with the above equality, we have the second assertion. □

Proposition 2.3. *Let the notation be as in §1.2. Suppose that $\{\xi[u] \mid u \in U_i', i \in I\}$ is a complete set of representatives for $H\backslash H_{\mathbf{A}}/C(L \cap W)$. Then the following assertions hold for every $u \in U_i'$ and $i \in I$:*

$$\sum_{w\in A_u^i} [\Delta_w : H \cap \xi[w]C(L)\xi[w]^{-1}] = [C(L \cap W) : H_{\mathbf{A}} \cap C(L)],$$

$$\mathfrak{m}(\widetilde{L}_i[q, \xi[u]'\mathfrak{B}]) = [C(L \cap W) : H_{\mathbf{A}} \cap C(L)]\nu(\Gamma(\widetilde{L}_i))^{-1}.$$

Proof. By applying [10, Theorem 3.2] to $L[s, \mathfrak{b}]$ combined with [10, (3.3)], we obtain

$$\sum_{i\in I} \nu(\Gamma(L_i))\mathfrak{m}(L_i[s, \mathfrak{b}]) = [E : D]\mathfrak{m}(H, E), \tag{2.10}$$

where $E = C(L \cap W)$ and $D = H_{\mathbf{A}} \cap C(L)$. The mass $\mathfrak{m}(H, E)$ can be written as $\sum_{i\in I} \sum_{u\in U_i'} \nu(\Delta_u)$ by our assumption. On the other hand, by Lemma 2.2 the left-hand side of (2.10) equals $\sum_{i\in I}\sum_{u\in U_i'}\nu(\Delta_u)\sum_{w\in A_u^i}[\Delta_w : H\cap\xi_w C(L)\xi_w^{-1}]$. Therefore, in view of (1.9), $\sum_{w\in A_u^i}[\Delta_w : H\cap\xi_w C(L)\xi_w^{-1}]$ must be equal to $[E : D]$ for every $u\in U_i'$ and every $i\in I$. This shows the first assertion. The second assertion follows from this combined with (2.9). $\square$

Let us apply Proposition 2.3 to $\mathfrak{m}(\{q,\varphi\})$ of (0.3) in Introduction. To do this, we assume that

$$H\varepsilon(H_{\mathbf{A}}\cap C(L)) = H\varepsilon C(L\cap W)\quad\text{for every } \varepsilon\in H_{\mathbf{A}}. \tag{2.11}$$

Then Proposition 2.3 is applicable to (1.16) in the setting of §1.4 because $U_i' = U_i$ for every $i\in I$. Noticing that $A(=A_v^1)=\{v\}$ in our theorem in this case, we have the following.

Corollary 2.4. *In the setting of Theorem suppose* (2.11). *Then $A=\{v\}$ and $\#(Z_v) = [C(L\cap W) : H_{\mathbf{A}}\cap C(L)]$. Consequently the mass of the primitive representations of q by φ is given by*

$$\begin{cases}\#\{q,\varphi\} = [C(L\cap W) : H_{\mathbf{A}}\cap C(L)][\Gamma(L):1] & \textit{if } \varphi \textit{ is definite},\\ \mathfrak{m}(\{q,\varphi\}) = [C(L\cap W) : H_{\mathbf{A}}\cap C(L)]\nu(\Gamma(L))^{-1} & \textit{if } \varphi \textit{ is indefinite},\end{cases}$$

where $\nu(\Gamma(L))$ is defined by (2.6) *and $\Gamma(L)=\Gamma(\varphi)$ in the introduction.*

3. Formula for computation of the mass

This section is divided into three subsections; §3.1 is based on [8, §16.8] by Shimura; §3.2 is based on [14, Chapters 3 and 4] by Siegel. In §3.3 we shall combine them.

3.1. **Invariant measure on a ball $\mathcal{B}$.** Let φ be an *indefinite* symmetric matrix in $GL_n(\mathbf{Q})$. Let $O(\varphi)$ be the orthogonal group of φ and put $O(\varphi)_\infty = \{\gamma\in GL_n(\mathbf{R}) \mid \gamma\varphi\cdot{}^t\gamma=\varphi\}$. Set $\mathcal{S}_+^m = \{{}^tP = P\in\mathbf{R}_m^m \mid P>0\}$ for $0<m\in\mathbf{Z}$.

We consider a Witt decomposition of φ over $\mathbf{R}$ and set

$$\varphi_\infty = \kappa\sigma\varphi\cdot{}^t\sigma = \begin{bmatrix}0 & 0 & -1_r\\ 0 & \theta & 0\\ -1_r & 0 & 0\end{bmatrix},\qquad 0<\theta={}^t\theta\in GL_t(\mathbf{R}), \tag{3.1}$$

as in (2.2) with $\kappa\in\mathbf{Q}^\times$ and $\sigma\in GL_n(\mathbf{R})$; we may assume $\kappa\in\{\pm1\}$ with a suitable change of σ. Then φ_∞ has signature $(r+t,\, r)$ with $r>0$. Put $p=r+t$. Since $O(\varphi)_\infty$ acts transitively on the space $\mathcal{Z}^\varphi$ defined by (2.3) with φ_∞, we have a diffeomorphism $gC_\infty\mapsto g(\mathbf{1})$ of $O(\varphi)_\infty/C_\infty$ onto $\mathcal{Z}^\varphi$, where $C_\infty = \{\gamma\in O(\varphi)_\infty \mid \gamma(\mathbf{1})=\mathbf{1}\}$ and $\mathbf{1}=\begin{bmatrix}1_r\\ 0\end{bmatrix}$.

Let $S_0 = \begin{bmatrix} 1_p & 0 \\ 0 & -1_r \end{bmatrix}$ and consider $O(S_0) = \{\alpha \in GL_n(\mathbf{R}) \mid \alpha S_0 \cdot {}^t\alpha = S_0\}$. We can let $O(S_0)$ act on the ball $\mathcal{B}$ defined by

$$\mathcal{B} = \{x \in \mathbf{R}_r^p \mid 1_r - {}^t xx > 0\}.$$

To be precise, we put

$$\mathcal{Y}(S_0) = \{Y \in GL_n(\mathbf{R}) \mid {}^t Y S_0 Y = \mathrm{diag}[A,\, -B] \text{ with } A \in \mathcal{S}_+^p,\ B \in \mathcal{S}_+^r\},$$

$$B(x) = \begin{bmatrix} 1_p & x \\ {}^t x & 1_r \end{bmatrix}$$

for $x \in \mathcal{B}$. Then the mapping $(x,\, \kappa,\, \mu) \mapsto B(x)\mathrm{diag}[\kappa,\, \mu]$ gives a bijection of $\mathcal{B} \times GL_p(\mathbf{R}) \times GL_r(\mathbf{R})$ onto $\mathcal{Y}(S_0)$. This can be shown in a similar way to the proof of [8, Lemma 16.2]. For $\alpha \in O(S_0)$ and $x \in \mathcal{B}$ we have $\alpha B(x) \in \mathcal{Y}(S_0)$ and hence we can define $\alpha(x) \in \mathcal{B}$ by the relation $\alpha B(x) = B(\alpha(x))\mathrm{diag}[\kappa,\, \mu]$ with $(\kappa,\, \mu) \in GL_p(\mathbf{R}) \times GL_r(\mathbf{R})$. Note that $\beta(\alpha(x)) = (\beta\alpha)(x)$ for $\alpha,\, \beta \in O(S_0)$. We denote by $\mathbf{0}$ the zero matrix of $p \times r$, which belongs to $\mathcal{B}$.

Lemma 3.1. *The above action is transitive. Also put $C_0 = \{\alpha \in O(S_0) \mid \alpha(\mathbf{0}) = \mathbf{0}\}$. Then $C_0 = O(S_0) \cap O(1_n)$, $\cong O(1_p) \times O(1_r)$, is a maximal compact subgroup of $O(S_0)$, where $O(1_m) = \{\gamma \in \mathbf{R}_m^m \mid \gamma \cdot {}^t\gamma = 1_m\}$.*

Proof. All assertions, except the last one, can be seen from the definition. The last assertion is a well known fact. □

Take $\tau \in GL_t(\mathbf{R})$ so that $\tau \cdot {}^t\tau = 2\theta$ and set

$$\delta = \sigma^{-1} \begin{bmatrix} 1_r & 0 & 1_r \\ 0 & \tau & 0 \\ -1_r & 0 & 1_r \end{bmatrix}. \tag{3.2}$$

Then we see that $\delta S_0 \cdot {}^t\delta = \kappa 2\varphi$, $\delta O(S_0)\delta^{-1} = O(\varphi)_\infty$, and $\delta C_0 \delta^{-1} = C_\infty$. On the other hand, by [8, (16.15b)] we have a diffeomorphism

$$\mathfrak{t} : \mathcal{B} \ni \begin{bmatrix} u \\ v \end{bmatrix} \longmapsto \begin{bmatrix} (1+u)(1-u)^{-1} \\ \tau v(1-u)^{-1} \end{bmatrix} \in \mathcal{Z}^\varphi.$$

Clearly, $\mathfrak{t}(\mathbf{0}) = \mathbf{1}$. The following can be verified in a straightforward way.

Lemma 3.2. *For $\alpha \in O(S_0)$ we have $\mathfrak{t}(\alpha(\mathbf{0})) = \delta\alpha\delta^{-1}(\mathbf{1})$.*

From this lemma we obtain $\mathfrak{t}(\alpha(x)) = \delta\alpha\delta^{-1}(\mathfrak{t}(x))$ for $\alpha \in O(S_0)$ and $x \in \mathcal{B}$. Hence the action of $O(S_0)$ on $\mathcal{B}$ corresponds to the action of $O(\varphi)_\infty$ on $\mathcal{Z}^\varphi$.

Now we define an $O(S_0)$-invariant measure on $\mathcal{B}$ by

$$\mathbf{d}x = \det(1_r - {}^t xx)^{-n/2} \prod_{i=1}^{p} \prod_{j=1}^{r} dx_{ij} \tag{3.3}$$

for $x = [x_{ij}] \in \mathcal{B}$. Then it can be found in [8, §16.8] that $\mathfrak{t}$ sends the measure $\mathbf{d}z$ on $\mathscr{Z}^{\varphi}$ back to $2^{rn/2} \det(\theta)^{r/2}$ times the measure $\mathbf{d}x$ on $\mathcal{B}$, which we denote by $\mathfrak{t}^*\mathbf{d}z$; namely,

$$\mathfrak{t}^*\mathbf{d}z = 2^{rn/2} \det(\theta)^{r/2}\mathbf{d}x. \tag{3.4}$$

3.2. **Measure of an unit group acting on a space $\mathfrak{H}$.** For a symmetric element S of $GL_n(\mathbf{Q})$ with the same signature as S_0 we put

$$\Omega(S) = \{\gamma \in \mathbf{R}_n^n \mid {}^t\gamma S\gamma = S\}, \qquad \Delta = \Omega(S) \cap GL_n(\mathbf{Z}).$$

Notice that $\Omega(S_0) = O(S_0)$. Set

$$\mathfrak{H} = \{P \in \mathcal{S}_+^n \mid PS^{-1}P = S\}, \qquad \mathfrak{H}_0 = \{P \in \mathcal{S}_+^n \mid PS_0^{-1}P = S_0\}.$$

We let $\Omega(S)$ act on $\mathfrak{H}$ by $P \mapsto P[\gamma]$ for $\gamma \in \Omega(S)$ and $P \in \mathfrak{H}$, where $P[\gamma] = {}^t\gamma P\gamma$. (In this section we do not use the symbol $X[y]$ for $yX \cdot {}^ty$.) It can be seen that this action is transitive and Δ is discrete in $\Omega(S)$. Then $\mathfrak{H}$ is a symmetric space on which Δ acts discontinuously.

There is a diffeomorphism Φ of $\mathcal{B}$ onto $\mathfrak{H}_0$ defined by

$$\Phi : x \longmapsto \begin{bmatrix} (1_p - x \cdot {}^tx)^{-1}(1_p + x \cdot {}^tx) & -2(1_p - x \cdot {}^tx)^{-1}x \\ -2 \cdot {}^tx(1_p - x \cdot {}^tx)^{-1} & (1_r + {}^txx)(1_r - {}^txx)^{-1} \end{bmatrix}.$$

Also, the mapping $x \mapsto \delta\Phi(x) \cdot {}^t\delta$ gives a diffeomorphism of $\mathcal{B}$ onto $\mathfrak{H}$, where δ is a fixed element of $GL_n(\mathbf{R})$ such that $\delta S_0 \cdot {}^t\delta = S$. In particular, the dimension of $\mathfrak{H}$ is pr.

For $\alpha \in \Omega(S_0)$ and $x \in \mathcal{B}$ we define $\alpha[x] \in \mathcal{B}$ by $\Phi(\alpha[x]) = \Phi[x][{}^t\alpha]$. Then $\delta\Phi(\alpha[x]) \cdot {}^t\delta = \delta\Phi(x) \cdot {}^t\delta[{}^t(\delta\alpha\delta^{-1})]$ and ${}^t\delta^{-1}\Omega(S_0) \cdot {}^t\delta = \Omega(S)$. Hence the action $x \mapsto \alpha[x]$ of $\Omega(S_0)$ on $\mathcal{B}$ corresponds to the action $P \mapsto P[\gamma]$ of $\Omega(S)$ on $\mathfrak{H}$.

An $\Omega(S_0)$-invariant metric on $\mathfrak{H}_0$ is defined by $8^{-1}\mathrm{tr}(P^{-1}dPP^{-1}dP)$, where $dP = [dp_{ij}]$ for $P = [p_{ij}] \in \mathfrak{H}_0$. Its pullback under Φ is given by

$$\mathrm{tr}\left((1_p - x \cdot {}^tx)^{-1}dx(1_r - {}^txx)^{-1} \cdot {}^tdx\right)$$

with $dx = [dx_{ij}]$ for $x = [x_{ij}] \in \mathcal{B}$. The corresponding $\Omega(S_0)$-invariant measure on $\mathcal{B}$ is then given by $\mathbf{d}x$ of (3.3). We define the $\Omega(S)$-invariant measure $\mathbf{d}P$ on $\mathfrak{H}$ by $\mathbf{d}x$ through the mapping $x \mapsto \delta\Phi(x) \cdot {}^t\delta$. It can be seen from [14, Chapter 3, Section 4 and Chapter 4, Theorem 5] that there exists a fundamental domain for Δ in $\mathfrak{H}$ and it has finite measure with respect to $\mathbf{d}P$ if $n > 2$. We denote this measure by $\mathrm{vol}(\mathfrak{H}/\Delta)$.

Let us consider the mapping $X \mapsto S[X]$ for $X \in GL_n(\mathbf{R})$ and denote the image $S[X]$ by W. By the inverse function theorem the n^2 variables of X can be given by differentiable functions of $n(n+1)/2$ independent variables in W and of $n(n-1)/2$ new variables $y = (y_1, \cdots, y_{n(n-1)/2})$. Let $J(W, y)$ be the Jacobian of this transformation of variables. Observe that $y_1, \cdots, y_{n(n-1)/2}$ give local coordinates on $\Omega(S, W)$ for a fixed W, where we put $\Omega(S, W) = \{X \in \mathbf{R}_n^n \mid S[X] = W\}$. Then $dv = \det(WS^{-1})^{1/2}|J(W, y)|dy_1 \cdots dy_{n(n-1)/2}$ defines a volume element on $\Omega(S, W)$, which is independent of the choice of y and W. The group $\Omega(S)$ acts on $\Omega(S, W)$ via $X \mapsto \gamma X$ for $\gamma \in \Omega(S)$ and dv is invariant under this action. Take a fundamental domain for Δ in $\Omega(S, W)$ and let $\mu(S)$ be the volume of that set computed with dv. Following Siegel [14, Chapter 4, (101)], we call $\mu(S)$ *the measure of the unit group* Δ.

It is shown by [14, Chapter 4, Theorem 7] that

$$2\mu(S) = \rho_p\rho_r|\det(S)|^{-(n+1)/2}\mathrm{vol}(\mathfrak{H}/\Delta), \tag{3.5}$$

where $\rho_m = \prod_{k=1}^m \pi^{k/2}/\Gamma(k/2)$. Also, $\mu(S)$ equals the quantity $\rho(S)$ treated in [12]. This fact can be verified by comparing the equality in the line 6 from the bottom of [13, Page 609] with [12, Hilfssatz 10]. Hence if S is integral and if the genus of S consists of a single class with respect to O, $\mu(S)$ can be stated by Siegel's product formula [12, (3)] as follows:

$$\mu(S) = 2\left\{\prod_p 2^{-1}e_p(S)\right\}^{-1}. \tag{3.6}$$

Here the product is taken over all primes p and $e_p(S)$ is the representation density of S at p defined by $e_p(S) = \lim_{m\to\infty} p^{-mn(n-1)/2}\#\{a \in (\mathbf{Z}_p)_n^n/p^m(\mathbf{Z}_p)_n^n \mid {}^taSa - S \in p^m(\mathbf{Z}_p)_n^n\}$.

3.3. **Relationship between $\nu(\Gamma(\varphi))$ and $\mu(2\varphi)$.** We shall apply the argument in §3.2 to the case $S = \kappa 2\varphi$ with δ of (3.2).

The mapping $\alpha C_0 \mapsto C_0 \cdot {}^t\alpha$ for $\alpha \in O(S_0)$ induces the diffeomorphism $\alpha(\mathbf{0}) \mapsto \alpha[\mathbf{0}]$ of $\mathcal{B}$ onto itself. Thus the composition

$$\Psi : z \longmapsto \mathfrak{t}^{-1}(z) = \alpha(\mathbf{0}) \longmapsto \alpha[\mathbf{0}] \longmapsto \delta\Phi(\alpha[\mathbf{0}]) \cdot {}^t\delta$$

for $z \in \mathcal{Z}^\varphi$ with $\alpha \in O(S_0)$ defines a diffeomorphism of $\mathcal{Z}^\varphi$ onto $\mathfrak{H}$. Then we observe that $\Psi(\gamma(z)) = \Psi(z)[{}^t\gamma]$ for $\gamma \in O(\varphi)_\infty$, where ${}^t\gamma \in \Omega(\varphi)$. Therefore *the action of $O(\varphi)_\infty$ on $\mathcal{Z}^\varphi$ corresponds to the action of $\Omega(\varphi)$ on $\mathfrak{H}$*. Also, we obtain $(\Psi^{-1})^*\mathbf{d}z = 2^{rn/2}\det(\theta)^{r/2}\mathbf{d}P$ by (3.4).

Put $\Gamma^\cdot(\varphi) = O(\varphi) \cap GL_n(\mathbf{Z})$. We then consider $\Gamma^\cdot(\varphi)\backslash\mathcal{Z}^\varphi$ in a similar manner to §2.2 with $L = \mathbf{Z}_n^1$. Since $\#(T \cap \Gamma^\cdot(\varphi)) = 2$, by using [11, Theorem 10(iii)] we see that $\nu(\Gamma(\varphi)) = 2^{-1}[\Gamma^\cdot(\varphi) : \Gamma(\varphi)]\mathrm{vol}(\Gamma^\cdot(\varphi)\backslash\mathcal{Z}^\varphi)$.

Proposition 3.3. *Let φ be an indefinite symmetric matrix in $GL_n(\mathbf{Q})$ with $n > 2$. Take a pair (κ, σ) as in* (3.1) *so that $0 < {}^t\theta = \theta \in GL_t(\mathbf{R})$ and $\kappa \in \{\pm 1\}$ (n=2r+t) and define $\nu(\Gamma(\varphi))$ by* (2.6)*. Then $\nu(\Gamma(\varphi))$ can be given by*

$$\nu(\Gamma(\varphi)) = 2^{rn/2}[\Gamma^{\cdot}(\varphi) : \Gamma(\varphi)] \det(\theta)^{r/2}\rho_{r+t}^{-1}\rho_r^{-1}|\det(2\varphi)|^{(n+1)/2}\mu(2\varphi).$$

Proof. Since $\kappa \in \mathbf{Z}^\times$, the present $\mathfrak{H}$ and Δ are given by $\{P \in \mathcal{S}_+^n \mid P(2\varphi)^{-1}P = 2\varphi\}$ and $\Omega(2\varphi) \cap GL_n(\mathbf{Z})$, respectively. Let F be a fundamental domain for $\mathfrak{H}/\Delta$. Then $\Psi^{-1}(F)$ is a fundamental domain for $\Gamma^{\cdot}(\varphi)\backslash\mathcal{Z}^\varphi$. By virtue of (3.5) we see that

$$\begin{aligned}\int_{\Psi^{-1}(F)} \mathbf{d}z &= \int_F (\Psi^{-1})^*\mathbf{d}z = 2^{rn/2}\det(\theta)^{r/2}\int_F \mathbf{d}P \\ &= 2^{rn/2}\det(\theta)^{r/2}\rho_{r+t}^{-1}\rho_r^{-1}|\det(\kappa 2\varphi)|^{(n+1)/2}\cdot 2\mu(\kappa 2\varphi).\end{aligned}$$

From this we obtain the desired formula. □

Corollary 3.4. *In the setting of Proposition 3.3 suppose φ is semi-integral. If the genus of φ consists of a single class with respect to SO (that is, $\#\{SO(\varphi)\backslash SO(\varphi)_{\mathbf{A}}/C(L)\} = 1$ with $L = \mathbf{Z}_n^1$), then $\nu(\Gamma(\varphi))$ can be given by*

$$\nu(\Gamma(\varphi)) = 2^{2+rn/2}\det(\theta)^{r/2}\rho_{r+t}^{-1}\rho_r^{-1}|\det(2\varphi)|^{(n+1)/2}\left\{\prod_p 2^{-1}e_p(2\varphi)\right\}^{-1},$$

where p runs over all prime numbers.

Proof. In view of [7, Lemma 5.6(1)] under the assumption on the genus of φ we see that $[\Gamma^{\cdot}(\varphi) : \Gamma(\varphi)] = 2$. Since 2φ is integral, the assertion follows from Proposition 3.3 combined with Siegel's formula (3.6). □

Let us apply this corollary to a $\mathbf{Z}$-*maximal* lattice L in a (nondegenerate) quadratic space (V, φ) of dimension $n > 2$ over $\mathbf{Q}$, where φ is indefinite. Let φ_0 be the matrix representing φ with respect to a $\mathbf{Z}$-basis of L. Then φ_0 is semi-integral and $\mathbf{Z}_n^1$ is $\mathbf{Z}$-maximal with respect to φ_0. Since φ is indefinite, we have $\#\{SO^\varphi(V)\backslash SO^\varphi_{\mathbf{A}}(V)/C(L)\} = 1$ by [8, Theorem 9.26] and [10, Remark 2.4(5)]. Hence we can apply Corollary 3.4 to φ_0.

Since L is $\mathbf{Z}$-maximal, the local density $e_p(2\varphi_0)$ can be computed for each prime p. In fact, by [7, Theorem 8.6(2)] the computation of $e_p(2\varphi_0)$ can be reduced to a group index, which is given by [7, Proposition 3.9] if φ_0 satisfies the condition that $\det(\varphi_0) \in \mathbf{Z}_p^\times\mathbf{Q}_p^{\times 2}$ if n is odd. If φ_0 does not satisfy this condition, [6, Lemma 2.5] gives the index in question. As for $\det(2\varphi_0)$, we have $[\widetilde{L} : L] = |\det(2\varphi_0)|$, which can be computed by using [9, Theorem 6.2].

4. Examples

4.1. **A positive definite form in five variables.** We shall consider the quadratic form defined by

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $L = \mathbf{Z}_5^1$. It is known that φ is equivarent to 1_5 over $\mathbf{Q}$, L is $\mathbf{Z}$-maximal with respect to φ, and the genus of L consists of a single $SO(\varphi)$-class. In [4, Introduction] we took up this quadratic form and showed that

$$\sum_{i=1}^{3} \frac{\#\{q_i,\, \varphi\}}{[\Gamma(q_i) : 1]} = \#L[29,\, \mathbf{Z}] = 720. \tag{4.1}$$

Here $v_1 = (-2,\, -2,\, -2,\, 4,\, 5)$, $v_2 = (-4,\, -4,\, -2,\, 8,\, 3)$, $v_3 = (-2,\, -2,\, -2,\, 8,\, 1)$; $H_i = \{\gamma \in SO(\varphi) \mid v_i\gamma = v_i\}$, $W_i = (\mathbf{Q}v_i)^{\perp}$, $L \cap W_i = \mathbf{Z}_4^1 k_i$ with

$$k_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -1 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad k_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix},$$

and $q_i = \varphi[k_i]$ for $i = 1,\, 2,\, 3$ are given by

$$q_1 = \begin{bmatrix} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 5/2 \\ 5/2 & 5/2 & 5/2 & 26 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 13 & 7/2 \\ 1/2 & 1/2 & 7/2 & 2 \end{bmatrix},$$

$$q_3 = \begin{bmatrix} 5 & 4 & 4 & 21/2 \\ 4 & 5 & 4 & 21/2 \\ 4 & 4 & 5 & 21/2 \\ 21/2 & 21/2 & 21/2 & 26 \end{bmatrix}.$$

These v_1, v_2, v_3 form a complete set of representatives for $L[29,\, \mathbf{Z}]/\Gamma(L)$. Also $\{q_i\}_{i=1}^3$ is a complete set of representatives for the SO-classes in the genus of q_1.

Let us determine the numerator $\#\{q_i,\, \varphi\}$ in (4.1) for each i. To do this, we shall apply Theorem to $k_i \in \{q_i,\, \varphi\}$ with $v_i \in L[29,\, \mathbf{Z}]$. We first observe that k_i indeed belongs to $\{q_i,\, \varphi\}$ because it is primitive by [4, Corollary 2.3]. It can be seen from the proof of [8, Theorem 12.14(iv)] that $[C(L\cap W_i) : (H_i)_{\mathbf{A}} \cap C(L)] = 2$ and (2.11) is satisfied with W_i and H_i in place of W and H there. From these together with Proposition 2.3 it follows that $[\Gamma(L \cap W_i) : H_i \cap C(L)] = [C(L \cap W_i) : (H_i)_{\mathbf{A}} \cap C(L)] = 2$. Then our theorem tells that

$$\{q_i,\, \varphi\} = \bigsqcup_{\zeta} k_i \zeta \Gamma(\varphi),$$

where ζ runs over a complete set of representatives for $\Gamma(L \cap W_i)/(H_i \cap C(L))$. Thus in view of [10, (2.8)], we obtain

$$\#\{q_i,\, \varphi\} = \#\{\Gamma(L \cap W_i)/(H_i \cap C(L))\} \cdot \#\Gamma(\varphi) = 2304 \tag{4.2}$$

for every i, since $[\Gamma(\varphi) : 1] = 1152$ as noted in [4, §4.2]. As also mentioned there, it can be computed that $\#\Gamma(q_1) = 48$, $\#\Gamma(q_2) = 8$, and $\#\Gamma(q_3) = 6$. By using these together with (4.2) we can check equality (4.1).

We can also find that $\#\{q,\, \varphi\} = 2304$ for every odd prime number d with a fixed element $v \in L[d,\, \mathbf{Z}]$ in a similar and simpler way by applying Corollary 2.4 and employing [8, Lemma 12.13 and Theorem 12.14], where $q = \varphi[k]$ and $L \cap (\mathbf{Q}v)^{\perp} = \mathbf{Z}_4^1 k$.

4.2. **An indefinite form in seven variables.** Let us consider the quadratic form defined by

$$\varphi = \begin{bmatrix} 1_4 & 0 \\ 0 & \beta^{\circ} \end{bmatrix}.$$

Here β is the norm form of a quaternion algebra B over $\mathbf{Q}$ ramified exactly at 2 and an odd prime ℓ, and β° is the restriction of β to the 3-dimensional subspace $B^{\circ} = \{x \in B \mid x^{\iota} = -x\}$ with the main involution ι of B; we identify β° with the matrix representing it with respect to a $\mathbf{Q}$-basis of B°. It is noted that φ is isotropic, since every indefinite quadratic form in n variables over $\mathbf{Q}$ is isotropic if $n > 4$.

Let L be a $\mathbf{Z}$-maximal lattice in $\mathbf{Q}_7^1$ with respect to φ. We identify φ with the matrix representing φ with respect to a $\mathbf{Z}$-basis of L and L with $\mathbf{Z}_7^1$. Put $G = SO(\varphi)$.

Proposition 4.1. *Let s be a squarefree positive integer and suppose that s is prime to ℓ and $s \equiv 1 \pmod 4$. Pick $v \in L$ such that $\varphi[v] = s$. Set $W = (\mathbf{Q}v)^{\perp}$, $\psi = \varphi|_W$, and $H = \{\gamma \in G \mid v\gamma = v\}$. Take $h \in \mathbf{Z}_7^6$ so that $L \cap W = \mathbf{Z}_6^1 h$ and put $q = \varphi[h]$. Let Z be a complete set of representatives for $\Gamma(L \cap W)/(H \cap C(L))$.*

Then $v \in L[s,\, 2^{-1}\mathbf{Z}]$, $[\Gamma(L\cap W) : H \cap C(L)] = 2^{\lambda+1}$, *and*

$$\{q,\, \varphi\} = \Gamma(q)h\Gamma(\varphi) = \bigsqcup_{\zeta\in Z} h\zeta\Gamma(\varphi), \tag{4.3}$$

where λ *is the number of prime factors of* s.

The existence of v in the statement can be seen from Lagrange's theorem that every positive integer is a sum of four squares. We also note that φ and q satisfy $\det(\varphi) = 2^{-6}\ell^2$ and $\det(q) = 2^{-4}\ell^2 s$, which will be found in the proof.

Proof. We start with the invariants of φ in the sense of Shimura [9], which consist of the four data denoted by $\{\dim(\mathbf{Q}_7^1),\, \mathbf{Q}(\sqrt{-\det(\varphi)}),\, Q(\varphi),\, s_\infty(\varphi)\}$, where $Q(\varphi)$ is the characteristic algebra of φ and $s_\infty(\varphi)$ is the index of φ. We have $A^+(\varphi) = M_4(Q(\varphi))$ by definition, where $A^+(\varphi)$ is the even Clifford algebra of φ. We may regard $A^+(\varphi)$ as $A^+(\mathrm{diag}[1_4,\, \beta^\circ])$. Then applying [9, Lemma 2.8(ii)] to β°, we see that

$$A^+(\mathrm{diag}[1_4,\, \beta^\circ]) \cong A^+(\beta^\circ) \otimes_{\mathbf{Q}} A(1_4) = B \otimes_{\mathbf{Q}} M_2(B_{2,\infty}) \cong M_4(B_{\ell,\infty}),$$

where $B_{2,\infty}$ (resp. $B_{\ell,\infty}$) is a definite quaternion algebra over $\mathbf{Q}$ ramified exactly at 2 (resp. ℓ) and it is known that $A^+(\beta^\circ) = B$ and $A(1_4) = M_2(B_{2,\infty})$. This shows $Q(\varphi) = B_{\ell,\infty}$. Since the signature of φ is (5, 2), the index $s_\infty(\varphi)$ is 3 by definition. Also $\det(\varphi) = 2^{-7}[\widetilde{L} : L] = 2^{-6}\ell^2$ with the discriminant ideal $[\widetilde{L} : L]$ of φ below. To sum up, the invariants of φ are

$$\{7,\, \mathbf{Q}(\sqrt{-1}),\, B_{\ell,\infty},\, 3\}. \tag{4.4}$$

The invariants of ψ are then given by

$$\{6,\, \mathbf{Q}(\sqrt{-s}),\, Q,\, 2\}, \tag{4.5}$$

where Q is an indefinite quaternion algebra over $\mathbf{Q}$ of discriminant $2\ell P$ and P is the product of all prime factors p of s satisfying $p \equiv 3 \pmod 4$; (4.5) will be verified below. We assert that $L \cap W$ is $\mathbf{Z}$-maximal with respect to ψ. Indeed, the discriminant ideal of φ (resp. ψ) is given by $2\ell^2\mathbf{Z}$ (resp. $4s\ell^2\mathbf{Z}$) by [9, Theorem 6.2]. These combined with [3, (4.1) and (4.4)] show that $\varphi(v,\, L)$ must be $2^{-1}\mathbf{Z}$. The maximality of $L\cap W$ follows from this and [3, (4.2)]. At the same time, we have $v \in L[s,\, 2^{-1}\mathbf{Z}]$. It is known that the core dimension of φ at p is 3 if and only if p is ramified at $Q(\varphi)$, that is, $p = \ell$. In view of this together with (4.4) and (4.5), by [6, Theorem 3.8] we see that

$$[C(L\cap W) : H_{\mathbf{A}} \cap C(L)] = \prod_{p|2s} [C(L_p \cap W_p) : H_p \cap C(L_p)] = 2^{\lambda+1}. \tag{4.6}$$

Now, both φ and ψ are indefinite. Hence [8, Theorems 9.26 and 12.1(ii)] are applicable. These (and the proof of Theorem 12.1) show that

$$\#\{G\backslash G_{\mathbf{A}}/C(L)\} = 1,$$

$$\#\left\{L[s,\, 2^{-1}\mathbf{Z}]/\Gamma(L)\right\} = \#\left\{H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}} \cap C(L))\right\} = 1. \tag{4.7}$$

From this it follows that $\#\{H\backslash H_{\mathbf{A}}/C(L\cap W)\} = 1$. Thus (2.11) is satisfied in the present case. Hence Proposition 2.3 is applicable to $v \in L[s,\, 2^{-1}\mathbf{Z}]$ and H; namely, we have

$$A_v^1 = \{v\}, \qquad [\Gamma(L\cap W) : H \cap C(L)] = [C(L\cap W) : H_{\mathbf{A}} \cap C(L)],$$

where A_v^1 is as in Proposition 1.2 with L in place of L_1 and we may take $\xi[v]$ to be 1_7. This combined with (4.6) proves the second assertion. As for the last assertion, observe that $h \in \{q,\, \varphi\}$ by [4, Corollary 2.3]. Then Theorem in the introduction with $\gamma_v = \eta_v (= \xi_v) = 1_7$ gives (4.3).

To verify (4.5), let p be a prime number. If $p \nmid 2\ell s$, by [3, Theorem 1.1(2)] and by noticing the discriminant of $\mathbf{Q}(\sqrt{-s})$, p is unramified in the characteristic algebra $Q(\psi)$. Since ℓ is ramified in $Q(\varphi)$ by (4.4), the same theorem tells that ℓ is ramified in $Q(\psi)$ in both cases $(-4s/\ell) = \pm 1$, where $(-4s/p)$ is the quadratic residue symbol. Also since $s > 0$ and $s_\infty(\varphi) = 3$, we have $Q(\psi) \otimes_{\mathbf{Q}} \mathbf{R} = M_2(\mathbf{R})$. For each $p \mid s$, we recall that -1 is a norm of $\mathbf{Q}_p(\sqrt{-s})/\mathbf{Q}_p$ if and only if $(-1,\, -s)_p = 1$, where $(-1,\, -s)_p$ is the Hilbert symbol over $\mathbf{Q}_p$. The well known formula shows that

$$(-1,\, -s)_p = \left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Combining this with [3, Theorem 1.1(2)], we see that p is ramified (resp. unramified) in $Q(\psi)$ for every $p \mid P$ (resp. $p \nmid P$). Since $\#P$ is even as $s \equiv 1 \pmod 4$, by the theory of central simple algebras 2 must be ramified in $Q(\psi)$. Summing up these, we have $Q(\psi) = Q$ in (4.5), which completes the proof. $\square$

Lemma 4.2. *In the setting of Proposition 4.1 we have*

$$\mathfrak{m}(\{q,\, \varphi\}) = \frac{2^{\lambda+1}}{\nu(\Gamma(\varphi))} = \frac{2^\lambda \cdot 3^5 \cdot 5^2 \cdot 7(\ell+1)}{2\pi^5(\ell^6 - 1)} \tag{4.8}$$

for every squarefree positive integer s such that $\ell \nmid s$ and $s \equiv 1 \pmod 4$, where $\mathfrak{m}(\{q,\, \varphi\})$ is defined with $\mathcal{Z}^\varphi = \mathcal{Z}(2,\, 2 \cdot 1_3)$. Moreover, the mass of the set $L[s,\, 2^{-1}\mathbf{Z}]$ is given by

(4.9)

$$\mathfrak{m}(L[s,\, 2^{-1}\mathbf{Z}]) = \frac{2^{\lambda+1}\nu(\Gamma(L\cap W))}{\nu(\Gamma(\varphi))} = \frac{2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot s^{5/2} L(3,\, \chi)(\ell^2 - \chi(\ell))}{\pi(2\pi)^3(\ell^3 + \chi(\ell))},$$

where $L(s, \chi)$ is the L-function of the primitive Dirichlet character χ corresponding to $\mathbf{Q}(\sqrt{-s})$ and $\mathfrak{m}(L[s, 2^{-1}\mathbf{Z}])$ is defined with $\mathcal{Z}^\varphi$ above and $\mathcal{Z}^\psi = \mathcal{Z}(2, 2\cdot 1_2)$.

Proof. Recall that $q = \varphi[h]$, $L \cap W = \mathbf{Z}_6^1 h$, and $W = (\mathbf{Q}v)^\perp$ for $v \in L[s, 2^{-1}\mathbf{Z}]$, where L is the $\mathbf{Z}$-maximal lattice $\mathbf{Z}_7^1$ with respect to φ. The first equality of (4.8) follows from Proposition 4.1 and Corollary 2.4. We can define $\mathcal{Z}^\varphi$ by $\mathcal{Z}(2, 2\cdot 1_3)$ with a matrix as in (2.2). Then Corollary 3.4 is applicable to φ with $r = 2$, $\theta = 2\cdot 1_3$, and $\kappa = 1$:

$$\nu(\Gamma(\varphi)) = 2^{12}\rho_5^{-1}\rho_2^{-1}(2\ell^2)^4\left\{\prod_p 2^{-1}e_p(2\varphi)\right\}^{-1}. \tag{4.10}$$

We have $\rho_5 = 2^3\pi^6/3$, $\rho_2 = \pi$, and $e_p(2\varphi) = p^{7\mathrm{ord}_p(2)-21+\mathrm{ord}_p(\det(2\varphi))}[C^\varphi : D_1^\varphi]$ by [7, Theorem 8.6(2)]. Here ord_p is the order function of the completion $\mathbf{Q}_p$ of $\mathbf{Q}$ at p and $[C^\varphi : D_1^\varphi]$ is a group index depending on p; this is given by [7, Proposition 3.9] as follows:

$$[C^\varphi : D_1^\varphi] = 2p^{21}\begin{cases}\prod_{i=1}^3(1-p^{-2i}) & \text{if } p \nmid \ell,\\ 2(1+p)\prod_{i=1}^2(1-p^{-2i}) & \text{if } p \mid \ell.\end{cases}$$

Combining these with (4.10), we have

$$\nu(\Gamma(\varphi)) = \frac{2^4\cdot 3(\ell^6-1)}{\pi^7(\ell+1)}\zeta(2)\zeta(4)\zeta(6) = \frac{2^2\pi^5(\ell^6-1)}{3^5\cdot 5^2\cdot 7(\ell+1)}. \tag{4.11}$$

This proves the second equality of (4.8).

Let us next compute $\mathfrak{m}(L[s, 2^{-1}\mathbf{Z}])$. We have $L[s, 2^{-1}\mathbf{Z}] = v\Gamma(L)$ by (4.7). The first equality of (4.9) follows from this and Lemma 2.2. Observe that q is the matrix representing ψ with respect to the basis h of $L \cap W$, and so $\mathbf{Z}_6^1$ is $\mathbf{Z}$-maximal with respect to q. Hence we shall compute $\nu(\Gamma(L\cap W)) = \nu(\Gamma(q)) = 2^{-1}\mathrm{vol}(\Gamma(q)\backslash\mathcal{Z}^q)$, where $\mathcal{Z}^q = \mathcal{Z}(2, 2\cdot 1_2)$. By Corollary 3.4 we have $\nu(\Gamma(q)) = 2^{10}\rho_4^{-1}\rho_2^{-1}(4s\ell^2)^{7/2}\{\prod_p 2^{-1}e_p(2q)\}^{-1}$. For each prime p, $2^{-1}e_p(2q)$ is given by

$$p^{6\mathrm{ord}_p(2)+\mathrm{ord}_p(\det(2q))}\begin{cases}(1-\chi(p)p^{-3})\prod_{i=1}^2(1-p^{-2i}) & \text{if } p\nmid 2s\ell,\\ 2\prod_{i=1}^2(1-p^{-2i}) & \text{if } p \mid 2s,\\ 2(1+p)(1-\chi(p)p^{-2})^{-1}\prod_{i=1}^2(1-p^{-2i}) & \text{if } p\mid \ell.\end{cases}$$

We have thus

$$\nu(\Gamma(q)) = \frac{2^7\cdot s^{5/2}(\ell^3-\chi(\ell))(\ell^2-\chi(\ell))}{2^{\lambda+1}\pi^5(\ell+1)}\zeta(2)\zeta(4)L(3, \chi).$$

The second equality of (4.9) follows from this and (4.11). $\square$

We shall add a remark on the mass $\mathfrak{m}(G, C(L))$ of (2.7) in the present case. Since $\#\{G\backslash G_{\mathbf{A}}/C(L)\} = 1$, the computation of $\nu(\Gamma(\varphi))$ can be reduced to that of $\mathfrak{m}(G, C(L))$. If we define such a mass by fixing $\varphi'_\infty = \sigma'\varphi \cdot {}^t\sigma' = \begin{bmatrix} 0 & 0 & -1_2 \\ 0 & 1_3 & 0 \\ -1_2 & 0 & 0 \end{bmatrix}$ with $\sigma' \in GL_7(\mathbf{R})$ instead of φ_∞ in the proof above, then $\mathfrak{m}(G, C(L))$ can be computed by the formula due to Hanke [1, Theorem 5.1]. That formula is applicable to any isotropic quadratic form φ_0 in n variables over $\mathbf{Q}$ for which L is maximal under the condition that $\det(\varphi_0) \in \mathbf{Z}_p^\times \mathbf{Q}_p^{\times 2}$ for every prime p if n is odd (≥ 3), where $\mathbf{Q}_p^{\times 2} = \{a^2 \mid a \in \mathbf{Q}_p^\times\}$. Hence we can check our result on the $\nu(\Gamma(\varphi))$ computed with φ'_∞ by comparing with $\mathfrak{m}(G, C(L))$ computed by Hanke's formula.

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Corrections to [4].
Page 420, line 11 from the bottom: ϕ should be read as φ.
Page 426, line 11: $L[s]$ should be read as $\#L[s]$.

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