# EXPRESSIONS OF CIRCLES ON A COMPLEX PROJECTIVE SPACE BY GEODESICS AND BY TRAJECTORIES ON GEODESIC SPHERES

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Abstract. We study whether circles can be seen as geodesics on geodesic spheres or not. If a circle on a complex projective space has complex torsion either $\pm 1$ or 0, it can be seen as a geodesic on some geodesic sphere. But if its complex torsion $\tau$ satisfies $0 < |\tau| < 1$, then it cannot be seen as geodesics on any geodesic spheres, and can be seen as trajectories for some Sasakian magnetic fields on geodesic spheres. We show that there are three kinds of such expressions up to congruency.

## 1. Introduction

The aim of this paper is to generalize the following elementary fact on circles on a Euclidean space to those on a complex projective space: If we take a circle of positive geodesic curvature in a Euclidean 3-space $\mathbb{R}^3$, then there is a standard 2-sphere of some radius where this circle can be seen as a geodesic, and such a sphere is uniquely determined up to the action of isometries of $\mathbb{R}^3$.

A circle on a Riemannian manifold is a helix of order two ([11]). It is a smooth curve parameterized by its arclength which satisfies the system of differential equations $\nabla_{\dot\gamma}\dot\gamma = k_\gamma Y_\gamma$ and $\nabla_{\dot\gamma} Y_\gamma = -k_\gamma \dot\gamma$ with a nonnegative constant $k_\gamma$ and a field $Y_\gamma$ of unit vectors along $\gamma$. This constant $k_\gamma$ is called the geodesic curvature of $\gamma$. Circles on a complex projective space, being different from those on a Euclidean space, they are not congruent to each other even if they have the same geodesic curvature. Circles of given positive geodesic curvature are classified by their complex torsions which measure angles of their velocity vectors to complex lines spanned by their acceleration vectors ([3]). A circle has complex torsion $\pm 1$ means that its velocity and acceleration vectors

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form a complex line in the tangent space at each point, and a circle has null structure torsion means that they form a real 2-plane. In this paper, first we show that a circle of positive geodesic curvature can be seen as a geodesic on some geodesic sphere if and only if its complex torsion is either $\pm 1$ or 0. There are many results how geodesics on submanifolds can be seen in their ambient spaces (see [12, 8, 9] and their references). But for our problem, the above shows that we are not enough to consider only geodesics.

To go through our study on expressions of circles on a complex projective space, we take a "nice" family of curves on geodesic spheres which includes geodesics. In this paper, from dynamical theoretical point of view, we consider trajectories for Sasakian magnetic fields on geodesic spheres. On each geodesic sphere in a complex projective space, we have an almost contact metric structure induced by the complex structure on the ambient space. A trajectory for such a magnetic field shows a motion of a charged particle under the influence of this contact structure. Since trajectories for the trivial magnetic field are geodesics, and since circles of complex torsion $\pm 1$ on a complex projective space are interpreted as trajectories for Kähler magnetic fields ([1]), we may say that this extension of a family of curves is reasonable.

In [6], Bao and the second author studied extrinsic shapes of trajectories on geodesic spheres in a complex projective space. But they did not consider congruency of expressions: If two extrinsic shapes coincide with each other, whether they are congruent with each other or not. In this paper, we show that every circle on a complex projective space can be seen as a non-geodesic trajectory for some Sasakian magnetic field on some geodesic sphere, and that up to congruency it has a unique expression when it has null complex torsion and has three kinds of expressions when its complex torsion $\tau$ satisfies $0 < |\tau| < 1$.

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## 2. Circles on a complex projective space

A smooth curve $\gamma$ parameterized by its arclength on a Riemannian manifold $\widetilde{M}$ is said to be a *circles* if it satisfies the equations

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = k_\gamma Y_\gamma, \\ \widetilde{\nabla}_{\dot\gamma} Y_\gamma = -k_\gamma \dot\gamma. \end{cases} \tag{2.1}$$

with a nonnegative constant $k_\gamma$ and a field $Y_\gamma$ of unit tangent vectors along $\gamma$. Here, $\widetilde{\nabla}$ denotes the Riemannian connection on $\widetilde{M}$. We call

$k_\gamma$ and $\{\dot\gamma, Y_\gamma\}$ the *geodesic curvature* and Frenet frame of $\gamma$, respectively. Since $\gamma$ is parameterized by its arclength, the equations (2.1) is equivalent to the differential equation

$$\widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma = -k_\gamma^{\,2}\,\dot\gamma. \tag{2.2}$$

For a circle $\gamma$ of positive geodesic curvature on a complex projective space $\mathbb{C}P^n$ which satisfies (2.1), by using the complex structure $J$ on $\mathbb{C}P^n$, we set $\tau_\gamma = \langle \dot\gamma, JY_\gamma\rangle$, and call it its *complex torsion*. Since $J$ is parallel, we find that it is constant along $\gamma$. When $\tau_\gamma = 0$, which is the case that the velocity and acceleration vectors form a real tangent plane at each point, we call this circle *totally real*. When $\tau_\gamma = \pm 1$, we can interpret this circle from dynamical theoretic point of view (see §4). Circles on $\mathbb{C}P^n$ are classified by their geodesic curvatures and absolute values of complex torsions under the congruency relation (see [7, 3]). We say two smooth curves $\gamma_1, \gamma_2$ on a Riemannian manifold $\widetilde M$ to be *congruent* to each other (in strong sense) if there is an isometry $\varphi$ of $\widetilde M$ satisfying $\varphi\circ\gamma_1(t) = \gamma_2(t)$ for all $t$. Since $\mathbb{C}P^n$ is a symmetric space of rank one and since every isometry $\varphi$ of $\mathbb{C}P^n$ is $\pm$-holomorphic, that is $d\varphi\circ J = \pm J\circ d\varphi$, we have the following.

**Lemma 1** (cf. [7]). *Two circles $\gamma_1, \gamma_2$ on $\mathbb{C}P^n(c)$ are congruent to each other if and only if they satisfy either $k_{\gamma_1} = k_{\gamma_2} = 0$ or $k_{\gamma_1} = k_{\gamma_2} > 0$ and $|\tau_{\gamma_1}| = |\tau_{\gamma_2}|$.*

Thus, the moduli space $\mathcal{C}(\mathbb{C}P^n)$, which is the set of all congruence classes of circles, is set theoretically identified with the band $[0,\infty)\times[0,1]/\sim$. Here, for $(k_1,\tau_1), (k_2,\tau_2) \in [0,\infty)\times[0,1]$, we define $(k_1,\tau_1)\sim(k_2,\tau_2)$ if and only if either $k_1 = k_2 = 0$ or $k_1 = k_2 > 0$ and $\tau_1 = \tau_2$.

We recall some basic properties of circles on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c$ ([3]).

1) Every circle is an orbit of one-parameter family of isometries of $\mathbb{C}P^n(c)$.
2) Every circle of geodesic curvature $k$ and of complex torsion $\pm 1$ is closed of length $2\pi/\sqrt{k^2+c}$. It lies on a totally geodesic $\mathbb{C}P^1$.
3) Every circle of geodesic curvature $k$ and of null complex torsion is closed of length $\pi/\sqrt{4k^2+c}$. It lies on a totally geodesic $\mathbb{R}P^2$
4) We have both closed and open circles of complex torsion $0 < |\tau| < 1$.

## 3. Expressions of circles by geodesics on geodesic spheres

For a curve $\sigma$ on a real hypersurface $M$ in $\mathbb{C}P^n(c)$, we regard it as a curve in $\mathbb{C}P^n(c)$ through an isometric immersion $\iota : M \to \mathbb{C}P^n(c)$.

We say the curve $\iota \circ \sigma$ the *extrinsic shape* of $\sigma$. For a circle $\gamma$ on $\mathbb{C}P^n(c)$, if there are a real hypersurface $M$ of $\mathbb{C}P^n(c)$ and a smooth curve $\sigma$ on $M$ whose extrinsic shape coincides with $\gamma$, we say that $\gamma$ is expressed by $\sigma$, and say that $(M, \sigma)$ is an expression of $\gamma$. In order to make clear the difference of two expressions of a given curve, we give the notion of congruency of expressions. Let $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ be two expressions of a circle $\gamma$ on $\mathbb{C}P^n(c)$. Hence $\sigma_i$ is a smooth curve on a real hypersurface $M_i$ satisfying $\iota_i \circ \sigma_i(t) = \gamma(t)$ for all $t$, where $\iota_i : M_i \to \mathbb{C}P^n(c)$ is an isometric immersion for each $i = 1, 2$. We say these expressions to be *congruent* to each other if there is an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ with $\tilde{\varphi}(M_1) = M_2$ which either preserves $\gamma$ or reverse $\gamma$, that is, which satisfies either $\tilde{\varphi} \circ \gamma(t) = \gamma(t)$ for all $t$ or $\tilde{\varphi} \circ \gamma(t) = \gamma(-t)$ for all $t$.

In this section, corresponding to the elementary fact on circles on a Euclidean 3-space, we study expressions of circles on $\mathbb{C}P^n(c)$ by geodesics on geodesic spheres. We denote by $G(r)$ a geodesic sphere of radius $r$ $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$. Our results in this section are as follows.

**Theorem 1.** *Let $\gamma$ be a geodesic on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c$.*

(1) *We have infinitely many its expressions by geodesics on geodesic spheres which are not congruent to each other.*
(2) *For each $r$ with $\pi/(2\sqrt{c}\,) \leq r < \pi/\sqrt{c}$, it is uniquely expressed by a geodesic on a geodesic sphere of radius $r$ up to congruent relation.*
(3) *When $r < \pi/(2\sqrt{c}\,)$, it is not expressed by geodesics on geodesic spheres of radius $r$.*

In the above, the third assertion is clear. When the radius of a geodesic sphere is too small compared with the diameter of a geodesic we cannot express it as a curve on this geodesic sphere.

**Theorem 2.** *Let $\gamma$ be a circle on $\mathbb{C}P^n(c)$ of positive geodesic curvature $k_\gamma$ and of complex torsion $\tau_\gamma$.*

(1) *When $\tau_\gamma = \pm 1$, it has two kinds of expressions by geodesics on geodesic spheres up to congruency. The radii of these geodesic spheres are $\big(1/\sqrt{c}\big)\tan^{-1}\big(\sqrt{c}/k_\gamma\big)$ and $\big(1/\sqrt{c}\big)\big\{\pi - \tan^{-1}\big(\sqrt{c}/k_\gamma\big)\big\}$.*
(2) *When $\tau_\gamma = 0$, it has unique expression by a geodesic on a geodesic sphere up to congruency. The radius of this geodesic sphere is $\big(2/\sqrt{c}\,\big)\tan^{-1}\big(\sqrt{c}/(2k_\gamma)\big)$.*
(3) *When $0 < |\tau_\gamma| < 1$, it cannot be expressed by geodesics on geodesic spheres.*

Let $\mathcal{N}_M$ denote the inward unit normal of a geodesic sphere $M = G(r)$ of radius $r$ in $\mathbb{C}P^n(c)$. This geodesic sphere is endowed with an almost contact metric structure $(\xi, \eta, \phi, \langle\ ,\ \rangle)$ induced by the complex structure $J$ on $\mathbb{C}P^n(c)$. The characteristic vector field $\xi$ is defined by $\xi = -J\mathcal{N}_M$, the 1-form $\eta$ by $\eta(v) = \langle v, \xi\rangle$, the structure tensor field $\phi$ which is a $(1,1)$-tensor field by $\phi(v) = Jv - \eta(v)\mathcal{N}_M$, and $\langle\ ,\ \rangle$ is the induced metric. The shape operator $A_M$ of $M$ with respect to $\mathcal{N}_M$ satisfies $A_M\xi = \delta_M\xi$ and $A_M v = \lambda_M v$ with

$$\delta_M = \sqrt{c}\cot\bigl(\sqrt{c}\,r\bigr), \quad \lambda_M = \bigl(\sqrt{c}/2\bigr)\cot\bigl(\sqrt{c}\,r/2\bigr)$$

for each tangent vector $v \in TM$ orthogonal to $\xi$ (see [10], for example). In particular, the shape operator and the structure tensor field are simultaneously diagonalizable, that is, $A_M\phi = \phi A_M$.

For a geodesic $\sigma$ on a geodesic sphere $M$, we set $\rho_\sigma = \langle\dot\sigma, \xi\rangle$, and call it its *structure torsion*. By Weingarten formula which states $\widetilde{\nabla}_X\mathcal{N} = -A_M X$ for each vector field $X$ tangent to $M$, we have $\nabla_X\xi = \phi A_M X$. Therefore, we have

$$\frac{d}{dt}\rho_\sigma = \langle\dot\sigma, \phi A_M\dot\sigma\rangle = -\langle A_M\phi\dot\sigma, \dot\sigma\rangle$$

because $A_M$ is symmetric and $\phi$ is skew-symmetric. Hence we obtain

$$\frac{d}{dt}\rho_\sigma = \Bigl\langle\dot\sigma, \frac{1}{2}(\phi A_M - A_M\phi)\dot\sigma\Bigr\rangle = 0,$$

and find that the structure torsion is constant along $\sigma$. We can classify geodesics by their structure torsions.

**Lemma 2** ([4])**.** *Two geodesics on a geodesic sphere are congruent to each other if and only if the absolute values of their structure torsions coincide with each other.*

We here give a condition that the extrinsic shape of a geodesic on a geodesic sphere to be a circle on $\mathbb{C}P^n(c)$. For the sake of simplicity, we denote the extrinsic shape $\iota\circ\sigma$ of $\sigma$ also by $\sigma$.

**Lemma 3.** *The extrinsic shape of a geodesic $\sigma$ on a geodesic sphere $G(r)$ of radius $r$ in $\mathbb{C}P^n(c)$ is a circle if and only if one of the following conditions holds*:

i) $\rho_\sigma = \pm 1$,
ii) $\rho_\sigma = 0$,
iii) $|\rho_\sigma| = \cot\bigl(\sqrt{c}\,r/2\bigr)$ *when* $\pi/\bigl(2\sqrt{c}\bigr) < r < \pi/\sqrt{c}$.

*Corresponding to the above cases, the geodesic curvature $k_\sigma$ and complex torsion $\tau_\sigma$ of the extrinsic shape of $\sigma$ are as follows*:

i) $k_\sigma = |\delta_M|$ *and* $\tau_\sigma = \mp\mathrm{sgn}(\delta_M)\,1$*, where we ignore the complex torsion when* $r = \pi/\big(2\sqrt{c}\,\big)$ *and* $\mathrm{sgn}(\delta_M)$ *denotes the signature of* $\delta_M$

ii) $k_\sigma = \lambda_M$ *and* $\tau_\sigma = 0$;

iii) $k_\sigma = 0$.

*Proof.* By Weingarten formula and by Gauss formula which states $\widetilde{\nabla}_X Y = \nabla_X Y + \langle A_M X, Y\rangle \mathcal{N}$ for arbitrary vector fields $X, Y$ tangent to $M$, we have

$$\widetilde{\nabla}_{\dot\sigma}\dot\sigma = \langle A_M\dot\sigma, \dot\sigma\rangle\mathcal{N} = \big\{\lambda_M + (\delta_M - \lambda_M)\rho_\sigma^2\big\}\mathcal{N},$$

$$\widetilde{\nabla}_{\dot\sigma}\mathcal{N} = -A_M\dot\sigma = -\lambda_M\big(\dot\sigma - \rho_\sigma\xi\big) - \delta_M\rho_\sigma\xi = -\lambda_M\dot\sigma + \rho_\sigma(\lambda_M - \delta_M)\xi.$$

Since we have $\delta_M - \lambda_M = -\big(\sqrt{c}/2\big)\tan\big(\sqrt{c}\,r/2\big)$, when $\pi/\big(2\sqrt{c}\,\big) \le r < \pi/\sqrt{c}$ and $\rho_\sigma^2 = \lambda_M/(\lambda_M - \delta_M) = \cot^2\big(\sqrt{c}\,r/2\big)$, we have $\widetilde{\nabla}_{\dot\sigma}\dot\sigma = 0$. Hence the extrinsic shape is a geodesic. In other case, since $\delta_M \ne \lambda_M$, we find that the extrinsic shape of $\sigma$ is a circle of positive geodesic curvature if and only if either $\rho_\sigma = \pm 1$, the case that $\dot\sigma$ is parallel to $\xi$, or $\rho_\sigma = 0$. When $\rho_\sigma = \pm 1$, the Frenet frame of the extrinsic shape is $\{\dot\sigma = \pm\xi, \mathrm{sgn}(\delta_M)\mathcal{N}_M\}$. Hence, we find that its geodesic curvature is $|\delta_M|$ and its complex torsion is $\mp\mathrm{sgn}(\delta_M)\,1$. When $\rho_\sigma = 0$, we clearly find that they are $\lambda_M$ and 0 because the Frenet frame is $\{\dot\sigma, \mathcal{N}_M\}$. □

The third assertions of Theorems 1 and 2 are direct consequences of Lemma 3. Since two expressions are not congruent to each other if their underlying geodesic spheres are not isometric to each other, we have the first assertion of Theorem 1 by Lemma 1 and the third condition in Lemma 3. Also, this condition guarantees the existence parts of the second assertion in Theorem 1.

If we vary the radius of geodesic sphere $M = G(r)$, then $\delta_M$ is monotone decreasing with respect to the radius and takes all values in the interval $(-\infty, \infty)$. We note that $\delta_M = 0$ if and only if $r = \pi/(2\sqrt{c}\,)$. Thus, by Lemma 3, we get the existence parts of the first assertion of Theorem 2. If fact, we take a geodesic sphere $M$ which satisfies $k = \delta_M$ $(> 0)$, In this case, the radius of $M$ is $\big(1/\sqrt{c}\,\big)\tan^{-1}\big(\sqrt{c}/k\big)$. Then, the circle is expressed by a geodesic $\sigma$ on $M$ with $\rho_\sigma = \mp 1$. Also, if we take a geodesic sphere $M$ which satisfies $k = -\delta_M$ $(> 0)$, which is the case that its radius is $\big(1/\sqrt{c}\,\big)\big\{\pi - \tan^{-1}\big(\sqrt{c}/k\big)\big\}$, the circle is expressed by a geodesic $\sigma$ on $M$ with $\rho_\sigma = \pm 1$. Similarly, if we vary the radius of geodesic sphere, then $\lambda_M$ is monotone decreasing with respect to the radius and takes all values in the interval $(0, \infty)$. Hence, we get the existence part of the second assertion of Theorem 2 by Lemma 3. For congruency of expressions, we have the following.

**Lemma 4.** *If we have two expressions of a circle on $\mathbb{C}P^n(c)$ by geodesics on geodesic spheres of the same radius, they are congruent to each other.*

*Proof.* Let $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ be two expressions of a circle $\gamma$ by geodesics on geodesic spheres of radius $r$. Since geodesic spheres are of the same radius, there is an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ satisfying $\tilde{\varphi}(M_1) = M_2$. Then $\tilde{\varphi}|_{M_1} \circ \sigma_1$ is a geodesic on $M_2$. Since $\tilde{\varphi}$ is $\pm$-holomorphic and since we have $d\tilde{\varphi}(\mathcal{N}_{M_1}) = \mathcal{N}_{M_2}$, we find

$$\rho_{\tilde{\varphi}\circ\sigma_1} = \langle d\tilde{\varphi} \circ \dot{\sigma}_1, -J\mathcal{N}_{M_2}\rangle = \pm\langle d\tilde{\varphi} \circ \dot{\sigma}_1, -d\tilde{\varphi}(J\mathcal{N}_{M_1})\rangle = \pm\rho_{\sigma_1}.$$

Considering the geodesic curvature and complex torsion of $\gamma$, by use of Lemma 3, we obtain $|\rho_{\tilde{\varphi}\circ\sigma_1}| = |\rho_{\sigma_2}|$. Hence we find that $\tilde{\varphi} \circ \sigma_1$ and $\sigma_2$ are congruent to each other by Lemma 2. We therefore have an isometry $\psi$ of $M_2$ with $\psi \circ (\tilde{\varphi} \circ \sigma_1)(t) = \sigma_2(t)$ for all $t$. It is known that there is an isometry $\tilde{\psi}$ of $\mathbb{C}P^n(c)$ satisfying $\tilde{\psi}\big|_{M_2} = \psi$. Considering the isometry $\tilde{\psi} \circ \tilde{\varphi}$ of $\mathbb{C}P^n(c)$, we find that it maps $M_1$ to $M_2$ and preserves $\gamma$. Thus, we find that $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ are congruent to each other. □

## 4. Expressions by trajectories on geodesic spheres

As we studied in the previous section, if a circle on $\mathbb{C}P^n(c)$ has complex torsion $\tau$ with $0 < |\tau| < 1$, it cannot be expressed by geodesics on geodesic spheres. Therefore, we need to extend the family of curves on geodesic spheres. Though circles are the simplest curves next to geodesics from the viewpoint of the Frenet-Serret formula, the family of circles is not suitable in our study by the following reason. If a circle on $\mathbb{R}^3$ is expressed as a small circle of a sphere, which is a circle on this sphere, then it is also expressed as a small circle of a sphere of larger radius. Therefore, we here consider curves from dynamical theoretic point of view.

Generally, a closed 2-form $\mathbb{B}$ on a Riemannian manifold $M$ is said to be a *magnetic field* because it can be regarded as a generalization of static magnetic fields in a Euclidean 3-space (see [13], for example). We define an endomorphism $\Omega_{\mathbb{B}}$ of the tangent bundle $TM$ of $M$ by $\mathbb{B}(v, w) = \langle v, \Omega_{\mathbb{B}}(w)\rangle$ for all $v, w \in T_pM$ at an arbitrary point $p \in M$, and consider it as the Lorentz force under the influence of $\mathbb{B}$. We say a smooth curve $\gamma$ parameterized by its arclength to be a *trajectory* for $\mathbb{B}$ if it satisfies the differential equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \Omega_{\mathbb{B}}\dot{\gamma}$ with the Riemannian connection $\nabla$ on $M$. When $\mathbb{B}$ is the null 2-form, its trajectories are geodesics. Hence, we may say that trajectories are natural generalizations of geodesics. Since trajectories are determined by their initial vectors, they induce a dynamical system on the unit tangent bundle of $M$. In this sense, we may say that trajectories are the simplest curve next to geodesics from the dynamical theoretic point of view.

On a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c$, as it is a Kähler manifold, we have a natural closed 2-form $\mathbb{B}_J$ called the Kähler form. It is given by $\mathbb{B}_J(v,w) = \langle v, Jw\rangle$. Its constant multiple $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ ($\kappa \in \mathbb{R}$) is said to be a *Kähler magnetic field* (see [1]). Its trajectory $\gamma$ is hence a smooth curve parameterized by its arclength which satisfies $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Since $J$ is parallel, it is a circle of geodesic curvature $|\kappa|$ and of complex torsion $-\mathrm{sgn}(\kappa)\,1$. Thus, the first assertions of Theorems 1 and 2 mean that every trajectory for an arbitrary Kähler magnetic field is expressed by a geodesic on some geodesic sphere.

On a geodesic sphere in $\mathbb{C}P^n(c)$, by using the structure tensor field $\phi$, we can define a 2-form $\mathbb{F}_\phi$ by $\mathbb{F}_\phi(v,w) = \langle v, \phi w\rangle$. Since the complex structure $J$ is parallel, we find that this 2-form is closed (see [5]). Its constant multiple $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ ($\kappa \in \mathbb{R}$) is said to be a *Sasakian* magnetic field or a contact magnetic field. A trajectory $\sigma$ for $\mathbb{F}_\kappa$ is hence a smooth curve parameterized by its arclength which satisfies the equation $\nabla_{\dot\sigma}\dot\sigma = \kappa\phi\dot\sigma$. As a family of representing curves, we adopt trajectories for Sasakian magnetic fields. Though the equations of trajectories for Kähler and Sasakian magnetic fields are quite resemble, they have different properties. For example, since $\phi$ is not parallel, trajectories for Sasakian magnetic fields are not circles, in general. Likewise geodesics, for a trajectory $\sigma$ for $\mathbb{F}_\kappa$ on a geodesic sphere, we set $\rho_\sigma = \langle\dot\sigma, \xi\rangle$, and call it its structure torsion. By the same computation as for structure torsions for geodesics, we find that it is constant along $\sigma$. Since we have $\|\nabla_{\dot\sigma}\dot\sigma\| = |\kappa|\sqrt{1-\rho_\sigma^2}$, we find that the norms of acceleration vectors depend on directions of trajectories. On contrary, for a trajectory $\gamma$ for a Kähler magnetic field $\mathbb{B}_\kappa$, we have $\|\nabla_{\dot\gamma}\dot\gamma\| = |\kappa|$ and find that it does not depend on $\gamma$. Thus, our study on expressions of circles by trajectories for Sasakian magnetic fields is not trivial.

First, we recall a condition that two trajectories on a geodesic sphere to be congruent to each other. Their structure torsions play an important role.

**Lemma 5** ([2])**.** *Let $\sigma_1$ and $\sigma_2$ be trajectories for $\mathbb{F}_{\kappa_1}$ and $\mathbb{F}_{\kappa_2}$, respectively, on a geodesic sphere $M$ in $\mathbb{C}P^n(c)$. They are congruent to each other if and only if they satisfy one of the following conditions*:

i) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$,
ii) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$, $|\kappa_1| = |\kappa_2|$ *and* $\kappa_1\rho_{\sigma_1} = \kappa_2\rho_{\sigma_2}$.

This lemma shows that the moduli space $\mathcal{T}(M)$ of trajectories for Sasakian magnetic fields on a geodesic sphere $M$ is set theoretically coincides with the set $[0,\infty)\times[-1,1]/\approx$. Here, for two points $(\kappa_1,\rho_1)$, $(\kappa_2,\rho_2)\in[0,\infty)\times[-1,1]$, we define $(\kappa_1,\rho_1)\approx(\kappa_2,\rho_2)$ if and only if either $(\kappa_1,\rho_1)=(\kappa_2,\rho_2)$ or $|\rho_1|=|\rho_2|=1$.

Now, we take a trajectory $\sigma$ for $\mathbb{F}_\kappa$ on a geodesic sphere $M$. By use of Gauss and Weingarten formulae, we have

$$\begin{aligned}\widetilde{\nabla}_{\dot\sigma}\widetilde{\nabla}_{\dot\sigma}\dot\sigma &= \widetilde{\nabla}_{\dot\sigma}\big\{\kappa J\dot\sigma + (\langle A_M\dot\sigma,\dot\sigma\rangle - \kappa\rho_\sigma)\mathcal{N}_M\big\}\\ &= -\Big\{\kappa^2(1-\rho_\sigma^2) + \big\{\lambda_M + (\delta_M-\lambda_M)\rho_\sigma^2\big\}^2\Big\}\dot\sigma\\ &\quad + \big\{\lambda_M - \kappa\rho_\sigma + (\delta_M-\lambda_M)\rho_\sigma^2\big\}\big\{\kappa + (\delta_M-\lambda_M)\rho_\sigma\big\}\big(\rho_\sigma\dot\sigma - \xi\big).\end{aligned}$$

Thus, we find that the extrinsic shape of $\sigma$ is a circle on $\mathbb{C}P^n$ if and only if the second term of the last expression vanishes. In this case, its geodesic curvature $k_\sigma$ and complex torsion $\tau_\sigma$ are given by

$$k_\sigma^2 = \kappa^2\big(1-\rho_\sigma^2\big) + \langle A_M\dot\sigma,\dot\sigma\rangle^2, \quad \tau_\sigma = -\big\{\kappa(1-\rho_\sigma^2) + \langle A_M\dot\sigma,\dot\sigma\rangle\rho_\sigma\big\}/k_\sigma,$$

where we ignore $\tau_\sigma$ when $k_\sigma=0$. Thus, we obtain the following.

**Lemma 6** ([6])**.** *Let $\sigma$ be a trajectory for $\mathbb{F}_\kappa$ on a geodesic sphere $M$ in $\mathbb{C}P^n(c)$. Its extrinsic shape is a circle on $\mathbb{C}P^n(c)$ if and only if one of the following condition holds*:

i) $\rho_\sigma = \pm 1$,
ii) $\lambda_M - \kappa\rho_\sigma + (\delta_M-\lambda_M)\rho_\sigma^2 = 0$,
iii) $\kappa + (\delta_M-\lambda_M)\rho_\sigma = 0$.

*Corresponding to these cases, the geodesic curvature $k_\sigma$ and the complex torsion $\tau_\sigma$ of the extrinsic shape of $\sigma$ are as follows*:

i) $k_\sigma = |\delta_M|$, $\tau_\sigma = \mp\mathrm{sgn}(\delta_M)1$,
ii) $k_\sigma = |\kappa|$, $\tau_\sigma = -\mathrm{sgn}(\kappa)\,1$,
iii) $k_\sigma = \sqrt{\kappa^2 - 2\lambda_M\kappa\rho_\sigma + \lambda_M^2}$, $\tau_\sigma = (2\kappa\rho_\sigma^2 - \kappa - \lambda_M\rho_\sigma)/k_\sigma$.

*Here, we ignore complex torsions in cases that the extrinsic shape is a geodesic.*

*Remark* 1. When $|\rho_\sigma| = 1$, we see that $\sigma$ is a geodesic on $M$ and does not depend on $\kappa$. The second and third conditions in Lemma 3 correspond to the third and the second conditions in this lemma.

The following is the main result on expressions of circles on $\mathbb{C}P^n$ by non-geodesic trajectories on geodesic spheres.

**Theorem 3.** *Let $\gamma$ be a circle of positive geodesic curvature $k_\gamma$ and of complex torsion $\tau_\gamma$ on $\mathbb{C}P^n(c)$.*

(1) *When $\tau_\gamma = \pm 1$, up to congruency, it has infinitely many kinds of expressions by non-geodesic trajectories for some Sasakian magnetic fields on geodesic spheres.*
(2) *When $\tau_\gamma = 0$, it is uniquely expressed by a non-geodesic trajectory on some geodesic sphere up to the congruence relation.*
(3) *When $0 < |\tau_\gamma| < 1$, we have three kinds of its expressions by non-geodesic trajectories on geodesic spheres up to the congruency relation.*

In order to show this theorem, we view Lemma 6 from a different angle. We denote by $\mathcal{E}(M)$ $\big(\subset \mathcal{T}(M)\big)$ the moduli space of trajectories on $M$ whose extrinsic shapes are circles. Lemma 6 tells us that we have a map $\Phi_M : \mathcal{E}(M) \to \mathcal{C}(\mathbb{C}P^n)$. To show the image of this map we study the second and the third cases in Lemma 6.

First we consider the case that trajectories satisfy the second condition in Lemma 6. Since $\lambda_M > 0$, we see $\rho_\sigma \neq 0$. As $\delta_M - \lambda_M < 0$, we find that the function $\kappa(\rho) = \lambda_M/\rho + (\delta_M - \lambda_M)\rho$ on the interval $(0, 1)$ is monotone decreasing and takes values in the interval $(\delta_M, \infty)$. Thus, geodesic curvatures of extrinsic shapes of trajectories on a given geodesic sphere which satisfy the second condition in Lemma 6 take all values in the intrerval $(\delta_M, \infty)$.

Next we consider the case that trajectories satisfy the third condition in Lemma 6. If a trajectory $\sigma$ satisfies the third condition, since $\kappa = -(\delta_M - \lambda_M)\rho_\sigma = c\rho_\sigma/(4\lambda_M)$, we obtain that geodesic curvature $k_\sigma$ and the complex torsion $\tau_\sigma$ of the extrinsic shape of $\sigma$ are expressed as follows:

$$k_\sigma = \sqrt{\lambda_M^2 - \frac{c\rho_\sigma^2}{2} + \frac{c^2\rho_\sigma^2}{16\lambda_M^2}}, \qquad \tau_\sigma = \frac{\rho_\sigma(2c\rho_\sigma^2 - c - 4\lambda_M^2)}{4k_\sigma\lambda_M}. \tag{4.1}$$

We study how $k_\sigma$ varies with respect to $\rho_\sigma$. Since we have $|\rho_\sigma| < 1$ and $\delta_M = 0$ if and only if $\lambda_M = \sqrt{c}/2$, we have

- $\lambda_M \le k_\sigma < -\delta_M$, when $\lambda_M < \sqrt{2c}/4$,
- $k_\sigma \equiv \sqrt{2c}/4$, when $\lambda_M = \sqrt{2c}/4$,
- $-\delta_M < k_\sigma \le \lambda_M$, when $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$,
- $\delta_M < k_\sigma \le \lambda_M$, when $\lambda_M \ge \sqrt{c}/2$.

We next study $\tau_\sigma$. When $\lambda_M = \sqrt{2c}/4$, we have $\tau_\sigma = \rho_\sigma(4\rho_\sigma^2 - 3)$. Hence, if we vary $\rho_\sigma$ in the interval $(-1, 1)$, then $\tau_\sigma$ takes all values in the interval $[-1, 1]$, and takes three times for each value in the interval $(-1, 1)$ and takes once for $\pm 1$. To study other cases, we need to recall the study in [6]. Since it is deeply related to our study, we state more clearly. We set a function $g_M$ on the intervals corresponding to these cases by

$$g_M(k) = \frac{(k^2-\lambda_M^2)(32\lambda_M^2k^2+4c\lambda_M^2-c^2)^2}{c(c-8\lambda_M^2)^3k^2}$$

By using two equalities in (4.1), we have ${\tau_\sigma}^2 = g_M(k_\sigma)$. We set

$$\alpha_M = \sqrt{2c(c-4\lambda_M^2)}/(8\lambda_M), \quad \text{when } \lambda_M \le \sqrt{c}/2,$$
$$\beta_M = \sqrt{8\lambda_M^2 - 2c}/4, \qquad \text{when } \lambda_M \ge \sqrt{c}/2.$$

Computing the differential of $g_M$, we find that this function satisfies the following properties.

(1) When $\lambda_M < \sqrt{2c}/4$,

- it is monotone increasing in the union of intervals $[\lambda_M, \sqrt{2c}/4] \cup [\alpha_M, -\delta_M)$;
- it is monotone decreasing in the interval $[\sqrt{2c}/4, \alpha_M]$;
- we have $g_M(\lambda_M) = g_M(\alpha_M) = 0$ and $g_M(\sqrt{2c}/4) = \lim_{k\uparrow -\delta_M} g_M(k) = 1$.

(2) When $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$,

- it is monotone decreasing in the union of intervals $(-\delta_M, \alpha_M] \cup [\sqrt{2c}/4, \lambda_M]$;
- it is monotone increasing in the interval $[\alpha_M, \sqrt{2c}/4]$;
- we have $g_M(\lambda_M) = g_M(\alpha_M) = 0$ and $g_M(\sqrt{2c}/4) = \lim_{k\downarrow -\delta_M} g_M(k) = 1$.

(3) When $\lambda_M = \sqrt{c}/2$,

- it is monotone increasing in the interval $(0, \sqrt{2c}/4]$;
- it is monotone decreasing in the interval $[\sqrt{2c}/4, \lambda_M]$;
- we have $g(\sqrt{2c}/4) = 1,\ \lim_{k\downarrow 0} g_M(k) = g_M(\lambda_M) = 0$.

(4) When $\sqrt{c}/2 < \lambda_M < \sqrt{2c}/2$,

- it is monotone decreasing in the union of intervals $(\delta_M, \beta_M] \cup [\sqrt{2c}/4, \lambda_M]$;
- it is monotone increasing in the interval $[\beta_M, \sqrt{2c}/4]$;
- we have $\lim_{k\downarrow\delta_M} g_M(k) = g(\sqrt{2c}/4) = 1,\ g_M(\lambda_M) = 0$ and $0 < g_M(\beta_M) < 1$;
- the value $g_M(\beta_M)$ is monotone increasing with respect to $\lambda_M$.

(5) When $\lambda_M \ge \sqrt{2c}/2$,

- it is monotone decreasing in the interval $(\delta_M, \lambda_M]$;
- we have $\lim_{k\downarrow\delta_M} g_M(k) = 1,\ g_M(\lambda_M) = 0$.

Therefore, the image of $\Phi_M$ is like the following figures. We note that

- when $\lambda_M = \sqrt{2c}/4$, we have $-\delta_M = \alpha_M = \sqrt{2c}/4$,

- when $\lambda = \sqrt{c}/2$, we have $\delta_M = \alpha_M = \beta_M = 0$,
- when $\lambda_M = \sqrt{2c}/2$, we have $\delta_M = \beta_M = \sqrt{2c}/4$ and $g_M(\beta_M) = 1$,

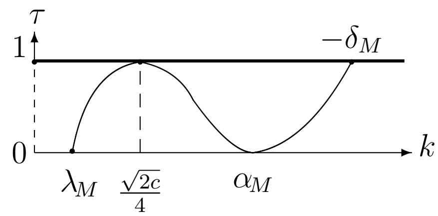


Fig. 1. $\lambda_M < \frac{\sqrt{2c}}{4}$

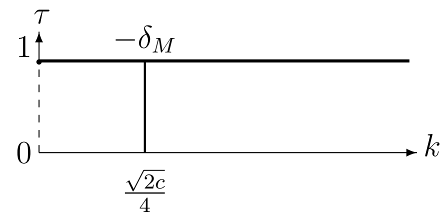


Fig. 2. $\lambda_M = \frac{\sqrt{2c}}{4}$

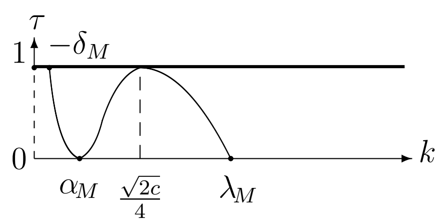


Fig. 3. $\frac{\sqrt{2c}}{4} < \lambda_M < \frac{\sqrt{c}}{2}$

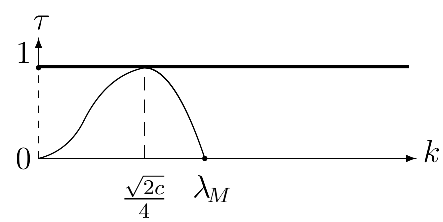


Fig. 4. $\lambda_M = \frac{\sqrt{c}}{2}$

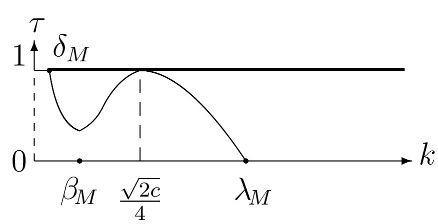


Fig. 5. $\frac{\sqrt{c}}{2} < \lambda_M < \frac{\sqrt{2c}}{2}$

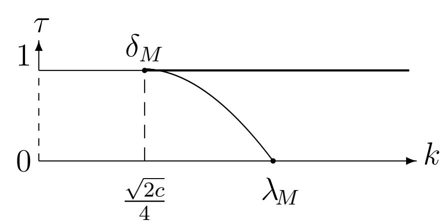


Fig. 6. $\lambda_M = \frac{\sqrt{2c}}{2}$

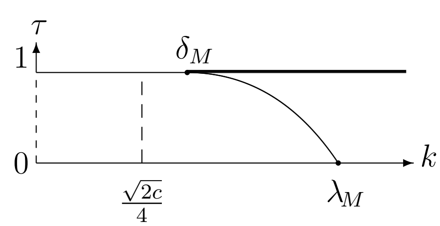


Fig. 7. $\lambda_M > \frac{\sqrt{2c}}{2}$

In order to show our result, we need to study congruency of two expressions by trajectories.

**Lemma 7.** *Let* $(M_1, \sigma_1)$ *and* $(M_2, \sigma_2)$ *be two expressions of a circle* $\gamma$ *on* $\mathbb{C}P^n(c)$ *by trajectories for Sasakian magnetic fields* $\mathbb{F}_{\kappa_1}$ *and* $\mathbb{F}_{\kappa_2}$ *on geodesic spheres in* $\mathbb{C}P^n(c)$. *They are congruent to each other if and only if these underlying geodesic spheres have the same radius and trajectories satisfy one of the following conditions*:

(1) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$,
(2) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$ *and* $\kappa_1 \rho_{\sigma_1} = \kappa_2 \rho_{\sigma_2}$.

*Proof.* Through isometric embeddings, we regard geodesic spheres $M_1, M_2$ as subsets of $\mathbb{C}P^n(c)$.

Suppose that these two expressions are congruent to each other. Then there is an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ with $\tilde{\varphi}(M_1) = M_2$ which satisfies either $\tilde{\varphi} \circ \sigma_1(t) = \sigma_2(t)$ for all $t$ or $\tilde{\varphi} \circ \sigma_1(t) = \sigma_2(-t)$ for all $t$. In particular, base geodesic spheres have the same radius. We hence have $(d\tilde{\varphi} \circ \dot{\sigma}_1)(t) = \dot{\sigma}_2(t)$ or $(d\tilde{\varphi} \circ \dot{\sigma}_1)(t) = -\dot{\sigma}_2(-t)$. We set

$$\epsilon = \begin{cases} 1, & \text{when } \tilde{\varphi} \circ \sigma_1(t) = \sigma_2(t) \text{ holds}, \\ -1, & \text{when } \tilde{\varphi} \circ \sigma_1(t) = \sigma_2(-t) \text{ holds}. \end{cases}$$

Since we have $d\tilde{\varphi}(\mathcal{N}_{M_1}) = \mathcal{N}_{M_2}$, and since $\tilde{\varphi}$ is $\pm$-holomorphic, we find

$$\rho_{\sigma_2} = \epsilon \langle d\tilde{\varphi}(\dot{\sigma}_1), -J d\tilde{\varphi}(\mathcal{N}_{M_1}) \rangle = \pm \epsilon \langle d\tilde{\varphi}(\dot{\sigma}_1), d\tilde{\varphi}(\xi_{M_1}) \rangle = \pm \epsilon \rho_{\sigma_1}.$$

In particular, we have $|\rho_{\sigma_1}| = |\rho_{\sigma_2}|$. Also we have

$$\begin{aligned} \kappa_2 \phi \dot{\sigma}_2 &= \nabla_{\dot{\sigma}_2} \dot{\sigma}_2 = \nabla_{d\tilde{\varphi} \circ \dot{\sigma}_1} (d\tilde{\varphi} \circ \dot{\sigma}_1) = d\tilde{\varphi}\big(\nabla_{\dot{\sigma}_1} \dot{\sigma}_1\big) = d\tilde{\varphi}(\kappa_1 \phi \dot{\sigma}_1) \\ &= \kappa_1 d\tilde{\varphi}(J\dot{\sigma}_1 - \rho_{\sigma_1} \mathcal{N}_{M_1}) = \pm \kappa_1 J (d\tilde{\varphi} \circ \dot{\sigma}_1) - \kappa_1 \rho_{\sigma_1} \mathcal{N}_{M_2} \\ &= \pm \epsilon \kappa_1 J \dot{\sigma}_2 \mp \epsilon \kappa_1 \rho_{\sigma_2} \mathcal{N}_{M_2} = \pm \epsilon \kappa_1 \phi \dot{\sigma}_2. \end{aligned}$$

When $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$, this tells nothing. When $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1$, we find $|\kappa_1| = |\kappa_2|$ and $\kappa_2 \rho_{\sigma_2} = \kappa_1 \rho_{\sigma_1}$. Thus, if two expressions $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ of $\gamma$ are congruent to each other, then the conditions hold.

On the other hand, we suppose that $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ satisfy the conditions in the assertion. Since $M_1, M_2$ are geodesic spheres of the same radius $r$ in $\mathbb{C}P^n(c)$, we have an isometry $\tilde{\varphi}$ of $\mathbb{C}P^n(c)$ with $\tilde{\varphi}(M_1) = M_2$. By the same computation as above, we find $\rho_{\tilde{\varphi} \circ \sigma_1} = \pm \rho_{\sigma_1}$ and $\nabla_{d\tilde{\varphi} \circ \dot{\sigma}_1} (d\tilde{\varphi} \circ \dot{\sigma}_1) = \pm \kappa_1 \phi (d\tilde{\varphi} \circ \dot{\sigma}_1)$ because $\tilde{\varphi}$ is $\pm$-holomorphic. Thus, by our conditions we find that $\tilde{\varphi} \circ \sigma_1$ and $\sigma_2$ are trajectories on $M_2$ which satisfy the conditions in Lemma 5. We therefore have an isometry $\psi$ of $M_2$ satisfying $\psi \circ (\tilde{\varphi} \circ \sigma_1)(t) = \sigma_2(t)$. It is well known that isometries on a geodesic sphere are equivariant. This means that for the isometry $\psi$ of $M_2$ there is an isometry $\tilde{\psi}$ of $\mathbb{C}P^n(c)$ with $\tilde{\psi}\big|_{M_2} = \psi$. We hence find that the isometry $\tilde{\psi} \circ \tilde{\varphi}$ satisfies $(\tilde{\psi} \circ \tilde{\varphi})(M_1) = M_2$ and

$(\tilde{\psi} \circ \tilde{\varphi}) \circ \sigma_1(t) = \sigma_2(t)$ for all $t$. Hence, we see $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ are congruent to each other as expressions of $\gamma$. □

This lemma guarantees that we only need to study images of maps into $\mathcal{C}(\mathbb{C}P^n)$. We now show Theorem 3.

*Proof of Theorem 3.* (1) We study the second case in Lemma 6. As we see in the study of the image of $\Phi_M$, structure torsions of trajectories satisfying this condition are not null and geodesic curvatures of their extrinsic shapes take values in the interval $(\delta_M, \infty)$. If we vary radii of geodesic spheres, their principal curvatures $\delta_M$ vary in the interval $(-\infty, \infty)$. We hence get the first assertion.

(2) and (3). We study the third case in Lemma 6. If a trajectory $\sigma$ of null structure torsion satisfies this condition, then it is a geodesic. We divide the image $\mathcal{ET}(M)$ $\big(\subset \mathcal{C}(\mathbb{C}P^n)\big)$ of the moduli space of non-geodesic trajectories on $M$ whose extrinsic shapes are circles through $\Phi_M$. We set

$$\mathcal{ET}_0(M) = \big\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \bigm| \tau^2 = g_M(k),\ -\delta_M < k \le \alpha_M\big\},$$

when $\sqrt{2c}/4 < \lambda_M < \sqrt{c}/2$. Similarly, we set

$$\mathcal{ET}_1(M) = \begin{cases} \left\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \,\middle|\, \begin{array}{l} \tau^2 = g_M(k), \\ \lambda_M < k \le \sqrt{2c}/4 \end{array}\right\}, & \text{when } \lambda_M < \dfrac{\sqrt{2c}}{4} \\[2ex] \left\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \,\middle|\, \begin{array}{l} \tau^2 = g_M(k), \\ \alpha_M < k \le \sqrt{2c}/4 \end{array}\right\}, & \text{when } \dfrac{\sqrt{2c}}{4} < \lambda_M < \dfrac{\sqrt{c}}{2}, \\[2ex] \left\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \,\middle|\, \begin{array}{l} \tau^2 = g_M(k), \\ \delta_M < k \le \sqrt{2c}/4 \end{array}\right\}, & \text{when } \dfrac{\sqrt{c}}{2} \le \lambda_M < \dfrac{\sqrt{2c}}{2}, \end{cases}$$

$$\mathcal{ET}_2(M) = \begin{cases} \left\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \,\middle|\, \begin{array}{l} \tau^2 = g_M(k), \\ \sqrt{2c}/4 < k \le \alpha_M \end{array}\right\}, & \text{when } \lambda_M < \dfrac{\sqrt{2c}}{4}, \\[2ex] \left\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \,\middle|\, \begin{array}{l} \tau^2 = g_M(k), \\ \sqrt{2c}/4 < k < \lambda_M \end{array}\right\}, & \text{when } \dfrac{\sqrt{2c}}{4} < \lambda_M < \dfrac{\sqrt{2c}}{2}, \end{cases}$$

and set

$$\mathcal{ET}_3(M) = \big\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \bigm| \tau^2 = g_M(k),\ \alpha_M < k < -\delta_M\big\},$$

when $\lambda_M < \sqrt{2c}/4$. Then we have

$$\mathcal{ET}(M) = \begin{cases} \mathcal{ET}_1(M) \cup \mathcal{ET}_2(M) \cup \mathcal{ET}_3(M), & \text{when } \lambda_M < \sqrt{2c}/4, \\ \mathcal{ET}_0(M) \cup \mathcal{ET}_1(M) \cup \mathcal{ET}_2(M), & \text{when } \sqrt{2c}/4 < \lambda_M < \sqrt{c}/2, \\ \mathcal{ET}_1(M) \cup \mathcal{ET}_2(M), & \text{when } \sqrt{c}/2 \leq \lambda_M < \sqrt{2c}/2. \end{cases}$$

In order to study how the set $\mathcal{ET}(M)$ depends on the radius of $M$, we consider the function $g_M$ as a function on $\lambda_M$ (see [6]). That is, we define a function $f_k$ on a suitable domain by

$$f_k(x) = \frac{(k^2 - x)\{4(8k^2 + c)x - c^2\}^2}{ck^2(c - 8x)^3}.$$

Its differential is given as

$$\frac{df_k}{dx} = \frac{(8k^2 - c)^2(4x + c)\{4(8k^2 + c)x - c^2\}}{ck^2(8x - c)^4}.$$

Hence, we find that $\dfrac{df_k}{dx}(\lambda_M^2) > 0$ if and only if $\lambda_M > \epsilon_k := c/\big(2\sqrt{8k^2 + c}\,\big)$.

We take the following five families of subsets in $\mathcal{C}(\mathbb{C}P^n)$:

$$\mathcal{F}_0 = \big\{\mathcal{ET}_0(M) \bigm| \sqrt{2c}/4 < \lambda_M < \sqrt{c}/2\big\},$$
$$\mathcal{F}_1 = \big\{\mathcal{ET}_1(M) \setminus \{[\sqrt{2c}/4, 1]\} \bigm| \lambda_M < \sqrt{2c}/4\big\},$$
$$\mathcal{F}_2 = \big\{\mathcal{ET}_1(M) \bigm| \sqrt{2c}/4 < \lambda_M < \sqrt{2c}/2\big\},$$
$$\mathcal{F}_3 = \big\{\mathcal{ET}_3(M) \bigm| \lambda_M < \sqrt{2c}/4\big\},$$
$$\mathcal{F}_4 = \big\{\mathcal{ET}_2(M) \bigm| \sqrt{2c}/4 < \lambda_M < \sqrt{2c}/2\big\} \cup \big\{\mathcal{ET}(M) \bigm| \lambda_M \geq \sqrt{2c}/2\big\}.$$

Since we have

$$\lim_{\lambda_M \downarrow \sqrt{2c}/4} (-\delta_M) = \lim_{\lambda_M \downarrow \sqrt{2c}/4} \alpha_M = \frac{\sqrt{2c}}{4}, \quad \lim_{\lambda_M \uparrow \sqrt{c}/2} (-\delta_M) = \lim_{\lambda_M \uparrow \sqrt{c}/2} \alpha_M = 0,$$

the sets in $\mathcal{F}_0$ cover the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}H^n) \bigm| 0 < k < \sqrt{2c}/4,\ \tau < 1\big\}$ (see Fig. 8). For each $M$ in this family, when $-\delta_M < k < \alpha_M$, we have $\lambda_M < \epsilon_k$. This means that if we increase $\lambda_M$, then $\mathcal{ET}_0(M)$ moves down in $\mathcal{C}(\mathbb{C}P^n) \equiv [0,\infty) \times [0,1]/\sim$. Thus, we find that $\mathcal{F}_0$ forms a foliation of the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \bigm| 0 < k < \sqrt{2c}/4,\ \tau < 1\big\}$. Similarly, since we have $\lim_{\lambda_M \downarrow 0} g_M(k) = 1$, the sets in $\mathcal{F}_1$ cover the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}H^n) \bigm| 0 < k < \sqrt{2c}/4,\ 0 < \tau < 1\big\}$ (see Fig. 9). For each $M$ in this family, when $\lambda_M < k < \sqrt{2c}/4$, as we have $\epsilon_k > \sqrt{2c}/4$, if we decrease $\lambda_M$, then $\mathcal{ET}_0(M)$ moves up in $\mathcal{C}(\mathbb{C}P^n)$. Thus, we find that $\mathcal{F}_1$ forms a foliation of the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \bigm| 0 < k < \sqrt{2c}/4,\ 0 < \tau < 1\big\}$.

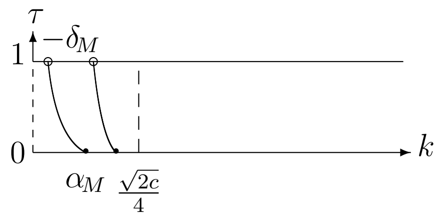


FIG. 8. foliation $\mathcal{F}_0$

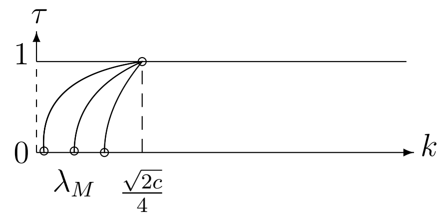


FIG. 9. foliation $\mathcal{F}_1$

Next, we study the family $\mathcal{F}_3$. Since we have

$$\lim_{\lambda_M \uparrow \sqrt{2c}/4} (-\delta_M) = \lim_{\lambda_M \uparrow \sqrt{2c}/4} \alpha_M = \sqrt{2c}/4, \quad \lim_{\lambda_M \downarrow 0} (-\delta_M) = \lim_{\lambda_M \downarrow 0} \alpha_M = \infty$$

we find that the sets in $\mathcal{F}_3$ cover the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}H^n) \;\big|\; k > \sqrt{2c}/4,\ \tau < 1\big\}$ (see Fig. 10). For each $M$ in $\mathcal{F}_3$, when $\alpha_M < k < -\delta_M$, as we have $\lambda_M < \epsilon_{-\delta_M} < \epsilon_k$, if we increase $\lambda_M$, then the set $\mathcal{ET}_3(M)$ moves down in $\mathcal{C}(\mathbb{C}P^n)$. Thus, we find that $\mathcal{F}_3$ forms a foliation of the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}H^n) \;\big|\; k > \sqrt{2c}/4,\ \tau < 1\big\}$. Similarly, since we have

$$\lim_{\lambda_M \downarrow \sqrt{2c}/2} \delta_M = \frac{\sqrt{2c}}{4}, \qquad \lim_{\lambda_M \to \infty} \delta_M = \infty,$$

we find that the sets in $\mathcal{F}_4$ cover the set $\big\{[k,\tau] \in \mathcal{C}(\mathbb{C}P^n) \;\big|\; k > \sqrt{2c}/4,\ 0 < \tau < 1\big\}$ (see Fig. 11). For each $M$ in $\mathcal{F}_4$, when $\sqrt{2c}/4 < \lambda_M < \sqrt{2c}/2$ and $\sqrt{2c}/4 < k < \lambda_M$ or when $\lambda_M \geq \sqrt{2c}/2$ and $\delta_M < k < \lambda_M$, as we have $\epsilon_k < \epsilon_{\sqrt{2c}/4} = \sqrt{2c}/4$, if we increase $\lambda_M$, then the sets $\mathcal{ET}_3(M)$ and $\mathcal{ET}(M)$ in our case move up in $\mathcal{C}(\mathbb{C}P^n)$.

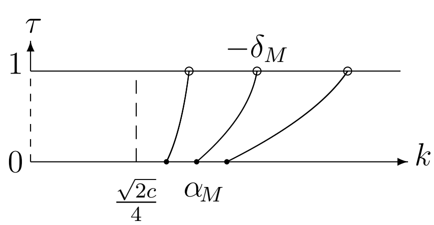


FIG. 10. foliation $\mathcal{F}_3$

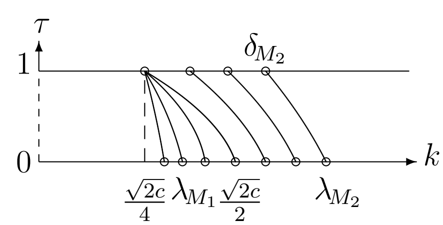


FIG. 11. foliation $\mathcal{F}_4$

Thirdly, we study the family $\mathcal{F}_2$. Since we have

$$\lim_{\lambda_M \downarrow \sqrt{2c}/4} \alpha_M = \sqrt{2c}/4, \qquad \lim_{\lambda_M \to \sqrt{c}/2} \delta_M = \lim_{\lambda_M \to \sqrt{c}/2} \alpha_M = 0,$$

$$\lim_{\lambda_M \to \sqrt{2c}/2} \delta_M = \sqrt{2c}/4, \qquad \lim_{\lambda_M \uparrow \sqrt{2c}/2} g_M(\beta_M) = 1,$$

we find that the sets in $\mathcal{F}_2$ covers the set $\{[k,\tau]\in\mathcal{C}(\mathbb{C}P^n) \mid 0<k<\sqrt{2c}/4,\ 0<\tau<1\}$ (see Figs. 12, 13). In Figs. 12 and 13, we set

$$\mathcal{F}_2^- = \{\mathcal{ET}_1(M) \mid \sqrt{2c}/4<\lambda_M<\sqrt{c}/2\},$$
$$\mathcal{F}_2^+ = \{\mathcal{ET}_1(M) \mid \sqrt{c}/2\le\lambda_M<\sqrt{2c}/2\}.$$

The dotted curves in Fig. 12 and Fig. 13 show $\mathcal{ET}(M)$ and $\mathcal{ET}_2(M)$ when $\lambda_M=\sqrt{c}/2$, respectively. When a geodesic sphere $M$ satisfies $\sqrt{2c}/4<\lambda_M<\sqrt{c}/2$, for $\alpha_M<k<\sqrt{2c}/4$, as we have $\lambda_M=\epsilon_k$ if and only if $k=\alpha_M$, if we increase $\lambda_M$, then the set $\mathcal{ET}_1(M)$ moves up in $\mathcal{C}(\mathbb{C}P^n)$. When a geodesic sphere $M$ satisfies $\sqrt{c}/2\le\lambda_M<\sqrt{2c}/2$, for $\delta_M<k<\sqrt{2c}/4$, as we have $\epsilon_k<\epsilon_0=\sqrt{c}/2<\lambda_M$, if we increase $\lambda_M$, then the set $\mathcal{ET}_1(M)$ moves up in $\mathcal{C}(\mathbb{C}P^n)$. Thus, we find that $\mathcal{F}_2$ forms a foliation of the set $\{[k,\tau]\in\mathcal{C}(\mathbb{C}P^n) \mid 0<k<\sqrt{2c}/4,\ 0<\tau<1\}$.

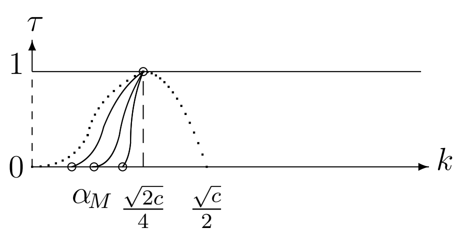


FIG. 12. $\mathcal{F}_2^-$

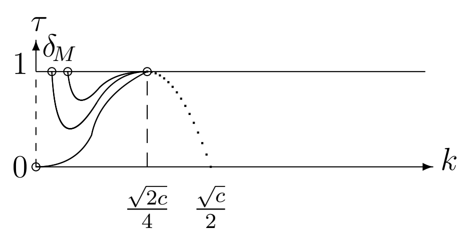


FIG. 13. $\mathcal{F}_2^+$

When $\lambda_M=\sqrt{2c}/4$, through $\Phi_M\big|_{\mathcal{ET}(M)}$, the set $\mathcal{ET}(M)$ is mapped onto the set $\{[k,\tau]\in\mathcal{C}(\mathbb{C}P^n) \mid k=\sqrt{2c}/4\}$. Considering the behavior of the function $h(\rho)=\rho(4\rho^2-3)$ on the interval $(-1,1)$ which satisfies $h(-\rho)=-h(\rho)$, we find that the inverse image $\big(\Phi_M\big|_{\mathcal{ET}(M)}\big)^{-1}([\sqrt{2c}/4,\tau])$ consists of a single point when $\tau=0,1$, and consists of three points in the others. We explain more on the case $\tau=0$. The equality $h(\rho)=0$ tells $\rho=0,\pm\sqrt{3}/2$. By the third condition in Lemma 6, we find that $\kappa=0$ when $\rho=0$, and $\kappa=\pm\sqrt{6c}/4$ when $\rho=\pm\sqrt{3}/2$. Since we study expressions by non-geodesic trajectories, the case $\rho=0$ is not related to our case, and trajectories for $\mathbb{F}_{\sqrt{6c}/4}$ of structure torsion $\sqrt{3}/2$ and those for $\mathbb{F}_{-\sqrt{6c}/4}$ of structure torsion $-\sqrt{3}/2$ are congruent to each other by Lemma 5. Thus, the inverse image consists of a single point when $\tau=0$.

By taking account of all cases, we get the second and the third assertions. $\square$

By the proof of Theorem 3, a circle $\sigma$ of null structure torsion on $\mathbb{C}P^n(c)$ is expressed by a non-geodesic trajectory on a geodesic sphere $M$ satisfying $k_\sigma = \alpha_M$.

**Proposition 1.** *Each circle $\sigma$ of positive geodesic curvature $k_\sigma$ and of null complex torsion on $\mathbb{C}P^n(c)$ is expressed by a non-geodesic trajectory on a geodesic sphere of radius $r$ with $r = \left(2/\sqrt{c}\right)\tan^{-1}\sqrt{(8k_\sigma^2+c)/c}$.*

Paying attention to radii of geodesic spheres, our proof of the first assertion of Theorem 3 shows the following.

**Proposition 2.** (1) *When $r > \pi/\left(2\sqrt{c}\right)$, every circle of complex torsion $\pm 1$ on $\mathbb{C}P^n(c)$ is uniquely expressed by a trajectory on a geodesic sphere of radius $r$ up to congruency.*

(2) *When $r \leq \pi/\left(2\sqrt{c}\right)$, if a circle $\sigma$ of complex torsion $\pm 1$ on $\mathbb{C}P^n(c)$ has geodesic curvature $k_\sigma$ with $k_\sigma > \sqrt{c}\cot\sqrt{c}\,r$, then it is is uniquely expressed by a trajectory on a geodesic sphere of radius $r$ up to congruency.*

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