Minimal stretch maps between Euclidean triangles

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Abstract

Given an ordered pair of Euclidean triangles with marked vertices whose angles are all acute, we find a homeomorphism with the smallest Lipschitz constant among all homeomorphisms from the first one to the second that preseerve the marking and we give a formula for the Lipschitz constant of this map. We show that on the set of pairs of marked acute triangles with fixed area, the function which assigns the logarithm of the infimum of the Lipschitz constants of Lipschitz homeomorphisms between them induces a symmetric metric. We show that this metric is Finsler, we give a necessary and sufficient condition for a path in the resulting metric space to be geodesic and we determine the isometry group of this metric space.

This study is motivated by Thurston's asymmetric metric on the Teichmüller space of a hyperbolic surface, and the results in this paper constitute an analysis of a basic Euclidean analogue of Thurston's hyperbolic theory. Many interesting questions in the Euclidean setting deserve further attention.

Keywords: Thurston's asymmetric metric, Teichmüller theory, space of Euclidean triangles, geodesics, Finsler structure

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1 Introduction

In this paper, we study a metric on the moduli space of Euclidean triangles which is an analogue of Thurston's (asymmetric) metric on the Teichmüller space of a surface of finite type. Thurston introduced this metric in his 1985 preprint [22]. Gradually, the metric has been studied from various viewpoints. A first survey appeared in 2007 [18], in which properties of this metric were compared to analogous properties of the Teichmüller metric. A recent survey of work done on this metric is the paper [16], where, in particular, the limiting behavior of geodesics is studied. The paper [21], published in 2015, contains a set of open problems on this metric. The paper [17] contains several results on the comparison between the Thurston metric and the Teichmüller metric. Several new techniques have been introduced recently in the study of Thurston's metric, see in particular the papers [1, 10, 13]. This metric has been generalized to various settings, see [5, 9] for an analogous metric on spaces of geometrically finite hyperbolic manifolds and [12] for a generalization to the setting of higher Teichmüller theory. In the papers [6, 7], stretch maps, which constitute a fundamental tool for the study of Thurston's metric are constructed from the point of view of geometric analysis. The Finsler structure of this metric has been analysed leading to infinitesimal rigidity results, see [11] and [**?**]. Euclidean analogues of Thurston's metric have also been studied, see [2] for an analogue on the Teichmüller space of Euclidean tori and [14] for a recent sequel. The recent work [24] addresses the question of a Thurston-type metric on the moduli space of semi-translation surfaces. In the paper [20], a Thurston type metric is defined and studied on the space of singular flat metrics with a fixed quadrangulation.

There is a natural analogue of Thurston's metric on a basic model space, namely, the moduli space of Euclidean triangles. This elementary setting has not been investigated yet. Our aim in this article is to settle the case of the moduli space of acute Euclidean triangles (that is, triangles whose three angles are acute), which turns out to be a natural space to study.

We now present the main results of this paper.

Consider a triangle in the oriented Euclidean plane. Label its vertices by the set $\{v_1, v_2, v_3\}$ such that this labeling induces a counter-clockwise orientation on the boundary of the triangle. We call such a triangle *labeled*.

Consider two such labeled triangles T and T' and let $f : T \to T'$ be a label-preserving homeomorphism. The *Lipschitz constant of* f is defined as

$$L(f) = \sup_{x,y \in T, x \neq y} \frac{d_{euc}(f(x), f(y))}{d_{euc}(x, y)}$$

where d_{euc} is the metric in the Euclidean plane. Now let

$$L(T, T') = \log\big(\inf\{L(f) : f \text{ is a label-preserving homeomorphism between } T \text{ and } T'\}\big).$$

This formula induces a distance function on the space of Euclidean triangles which is an analogue of Thurston's Lipschitz metric defined in the hyperbolic setting (see [22, p. 4] where this distance function is also denoted by L).

We obtain the following:

1. In the case where T and T' are acute triangles, we give another formula for the distance $L(T, T')$, which we denote by $m(T, T')$, in terms of the lengths of the edges and altitudes of the triangles T and T'. The new formula is an analogue of Thurston's alternative form of his distance function in terms of lengths of simple closed geodesics (see [22, p. 4] where this distance function is denoted by K). The equality $L(T, T') = m(T, T')$ is an analogue of Thurston's equality between his two distance functions K and L (see [22, p. 40]).

2. For every $A > 0$, the metric induced by L on the space $\mathfrak{AT}_A$ of acute triangles having fixed area A is Finsler (this is an analogue of Thurston's result in [22, p. 20]).

3. We give a characterization of geodesics in $\mathfrak{AT}_A$: a path is geodesic if and only if the angle at each labeled vertex of a triangle in this family of triangles varies monotonically.

4. The isometry group of $\mathfrak{AT}_A$ is isomorphic to S_3, the symmetric group on three letters.

In the case of triangles that are not acute, there might not exist any best Lipschitz homeomorphism between them (see Remark 4) and the theory in this general setting needs more investigation. In the paper [15], further properties of the Thurston metric on the Teichmüller spaces of Euclidean triangles are obtained.

The paper is organized as follows. Section 2 is the technical heart of our work. For any two labeled triangles T and T', we define their Lipschitz distance, we introduce the distance $m(T, T')$ and we show that $m(T, T') = L(T, T')$ for any pair of acute triangles T and T'. In Section 3 we introduce several spaces of triangles and study some of their topological and metric properties. Among these spaces, the space of acute triangles and the space of non-obtuse triangles having area A will play central roles in this paper. We denote these spaces by $\mathfrak{AT}_A$ and $\overline{\mathfrak{AT}_A}$ respectively. In Section 4 we give a necessary and sufficient condition for a path in $\overline{\mathfrak{AT}_A}$ to be a geodesic. In Section 5 we prove that the metric L on $\mathfrak{AT}_A$ is Finsler. We determine the isometry group of $\overline{\mathfrak{AT}_A}$ in Section 6.

We now introduce some notation used throughout the paper. Let T be a labeled triangle, with vertices v_1, v_2, v_3. For $i \in \{1, 2, 3\}$, we denote the edge opposite to the vertex v_i by e_i, the angle at the vertex v_i by θ_i and the altitude from the vertex v_i by h_i. We let $||e_i||$ and $||h_i||$ be the lengths of e_i and h_i, respectively. We denote the intersection of the altitude h_i with the line containing the edge e_i by p_i. A triangle with vertices v_1, v_2, v_3 is denoted by $\Delta v_1 v_2 v_3$. The line segment between two points x and y in the Euclidean plane is denoted by $[x, y]$, and its length by $||[x, y]||$. Finally, Area(T) will denote the area of a triangle T.

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2 Acute triangles

Let T and T' be two acute triangles. We use the notation introduced above for the triangle T. Similarly, for T', we denote the vertices by v'_1, v'_2, v'_3, the edge opposite to the vertex v'_i by e'_i, the angle at the vertex e'_i by θ'_i, etc.

Proposition 1. *For any two acute triangles T and T', we have*

$$\exp(L(T, T')) \geq \max\{\frac{||e'_1||}{||e_1||}, \frac{||e'_2||}{||e_2||}, \frac{||e'_3||}{||e_3||}, \frac{||h'_1||}{||h_1||}, \frac{||h'_2||}{||h_2||}, \frac{||h'_3||}{||h_3||}\}.$$

Proof. Since any label-preserving homeomorphism from T to T' sends e_1 to e_1', e_2 to e_2' and e_3 to e_3', it is clear that

$$\exp(L(T,T')) \geq \max\{\frac{||e_1'||}{||e_1||}, \frac{||e_2'||}{||e_2||}, \frac{||e_3'||}{||e_3||}\}.$$

Let $f : T \to T'$ be a label-preserving homeomorphism. Since T is acute, it follows that each p_i lies in the interior of the edge e_i. Therefore we have

$$d_{euc}(f(p_i), f(v_i)) = d_{euc}(f(p_i), v_i') \geq ||h_i'||.$$

Since

$$d_{euc}(p_i, v_i) = ||h_i||,$$

it follows that $L(f) \geq \frac{||h_i'||}{||h_i||}$. Therefore for any two acute triangles T and T', we have

$$\exp(L(T,T')) \geq \max\{\frac{||e_1'||}{||e_1||}, \frac{||e_2'||}{||e_2||}, \frac{||e_3'||}{||e_3||}, \frac{||h_1'||}{||h_1||}, \frac{||h_2'||}{||h_2||}, \frac{||h_3'||}{||h_3||}\},$$

which is the inequality we need. □

For two arbitrary labeled triangles T and T', we define

$$m(T,T') = \log(\max\{\frac{||e_1'||}{||e_1||}, \frac{||e_2'||}{||e_2||}, \frac{||e_3'||}{||e_3||}, \frac{||h_1'||}{||h_1||}, \frac{||h_2'||}{||h_2||}, \frac{||h_3'||}{||h_3||}\}). \quad (1)$$

Assume that we scale the triangle T by a factor λ and the triangle T' by a factor λ', where $\lambda, \lambda' > 0$. Let us denote the scaled triangles by λT and $\lambda' T'$. This means that the triangle λT has edge lengths $\lambda||e_1||$, $\lambda||e_2||$, $\lambda||e_3||$ and that the triangle $\lambda' T'$ has edge lengths $\lambda'||e_1'||$, $\lambda'||e_2'||$, $\lambda'||e_3'||$. The following formulae are clear:

$$\exp(L(\lambda T, \lambda' T')) = \frac{\lambda'}{\lambda} \exp(L(T,T')) \quad (2)$$

$$\exp(m(\lambda T, \lambda' T')) = \frac{\lambda'}{\lambda} \exp(m(T,T')). \quad (3)$$

Remark 1. We shall prove that for any two acute triangles T and T', $L(T,T') = m(T,T')$, see Theorem 2. We shall use the following fact in the proof: If $\lambda, \lambda' > 0$, then $L(T,T') = m(T,T')$ if and only if $L(\lambda T, \lambda' T') = m(\lambda T, \lambda' T)$. This follows from Identities (2) and (3).

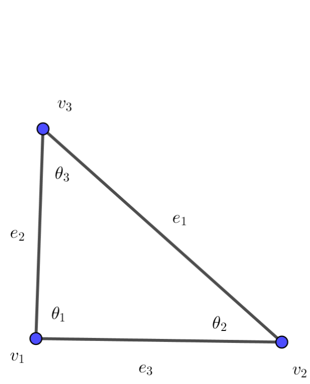

Figure 1: A best Lipschitz homeomorphism between the two triangles is the affine map (Subsection 2.1).

2.1 Right triangles

Right triangles appear naturally as sitting on the boundary of the moduli space of acute triangles, and they are also needed in our proofs of some statements concerning acute triangles.

Assume that T and T' are two right triangles where θ_1 and θ_1' are equal to $\frac{\pi}{2}$, see Figure 1. In this subsection we calculate $L(T,T')$.

Proposition 2. *Let T and T' be two right triangles such that $\theta_1 = \theta_1' = \frac{\pi}{2}$. Then*

$$\exp(L(T,T')) = \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}.$$

Proof. It is clear that

$$\exp(L(T,T')) \geq \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}. \tag{4}$$

Let us prove the reverse inequality.

We consider a Euclidean system of coordinates (x, y) in $\mathbb{R}^2$.

Up to performing some isometries to T and T', we may assume that the vertices v_1 and v_1' are at the origin, the edges e_3 and e_3' are on the x-axis, and the edges e_2 and e_2' are on the y-axis. Consider the following homeomorphism from T to T':

$$f : (x, y) \to (\frac{||e_3'||}{||e_3||}x, \frac{||e_2'||}{||e_2||}y).$$

It is clear that

$$L(f) \geq \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}.$$

Let $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ be two points in T.

Then

$$d_{euc}(f(q_1), f(q_2)) = d_{euc}((\frac{||e_3'||}{||e_3||}x_1, \frac{||e_2'||}{||e_2||}y_1), (\frac{||e_3'||}{||e_3||}x_2, \frac{||e_2'||}{||e_2||}y_2))$$

$$= \sqrt{(\frac{||e_3'||}{||e_3||})^2(x_2 - x_1)^2 + (\frac{||e_2'||}{||e_2||})^2(y_2 - y_1)^2}$$

$$\leq \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}d(q_1, q_2).$$

Therefore $L(f) \leq \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}$. Combined with (4), this gives $L(f) = \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}$. From this we conclude that $\exp(L(T, T')) = \max\{\frac{||e_3'||}{||e_3||}, \frac{||e_2'||}{||e_2||}\}$. □

Note that $\exp(L(T, T'))$ is given by an infimum and that this infimum is attained by the map we just constructed. Therefore the map we defined above is a *best Lipschitz homeomorphism*.

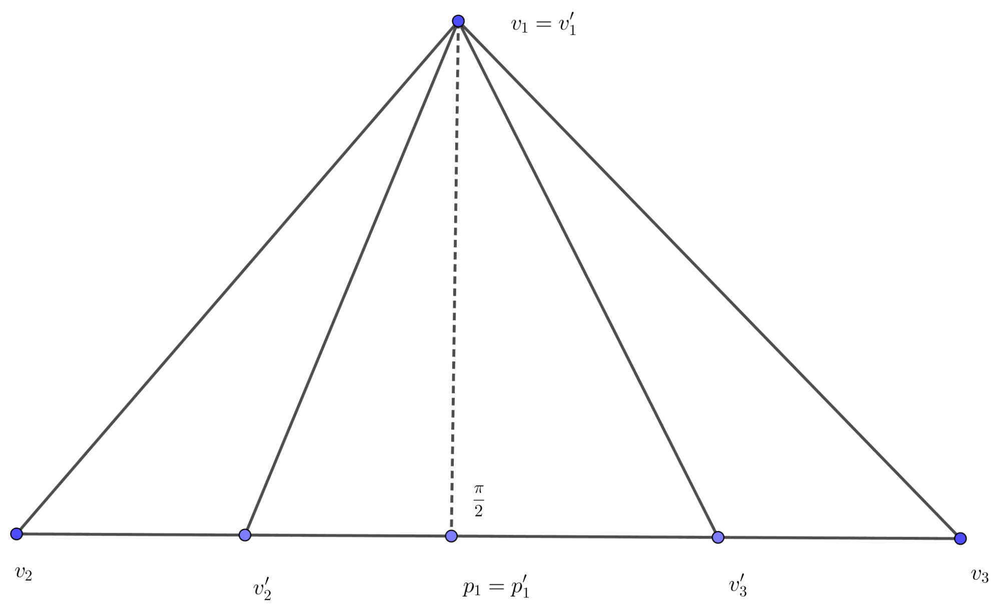

Figure 2: If the triangle T' is contained in the triangle T and T and if T' have the same altitudes, then $\exp(L(T,T')) = 1$(Proposition 3).

2.2 A problem related to right triangles

We continue using the above notation.

Proposition 3. *Let T and T' be two acute triangles satisfying $\theta_2 \leq \theta'_2$ and $\theta_3 \leq \theta'_3$. Assume that $||h_1|| = ||h'_1||$. Then $\exp(L(T,T')) = 1$ and there is a label-preserving homeomorphism $f : T \to T'$ satisfying$L(f) = 1$.*

Proof. We can move the triangle T into T' so that their altitudes coincide, see Figure 2. Therefore we have two triangles $T = \Delta v_1 v_2 v_3$ and $T' = \Delta v'_1 v'_2 v'_3$ satisfying $v_1 = v'_1$ and $h_1 = h'_1$. We want to find $L(T,T')$. Since T and T' have a common altitude from $v_1 = v'_1$, we readily see that for any label-preserving homeomorphism $f : T \to T'$, we have $L(f) \geq 1$. Now we construct a label-preserving homeomorphism $f : T \to T'$ such that $L(f) = 1$; this will complete the proofs of the two statements in the proposition. Consider the homeomorphism f as in Section 2.1 which sends the triangle $\Delta v_1 v_2 p_1$ to the triangle $\Delta v_1 v'_2 p_1$. Likewise, consider the homeomorphism h as in Section 2.1 which sends the triangle $\Delta v_1 p_1 v_3$ to the triangle $\Delta v_1 p_1 v'_3$. Clearly $L(g) = L(h) = 1$ and since g and h agree on the altitude $h_1 = h'_1 = [v_1, p_1]$, they induce a homeomorphism $f : T \to T'$. Furthermore it is clear that $L(f) = 1$.

□

2.3 Acute triangles

We are now ready to prove the main result on the space of acute triangles. This is an analogue, in the case we are discussing, of a result of Thurston in [22] (see Corollary 8.5).

Theorem 2. *If T and T' are two acute triangles, then $L(T,T') = m(T,T')$. Furthermore, there exists a best Lipschitz homeomorphism between the two acute triangles.*

Proof. As already noted, we have $\exp(L(T,T')) \geq \exp(m(T,T'))$, see Proposition 1. We will show that $\exp(L(T,T')) \leq \exp(m(T,T'))$. There exist distinct $i, j \in \{1,2,3\}$ such that either

1. $\theta_i \leq \theta'_i$ and $\theta_j \leq \theta'_j$, or
2. $\theta_i \geq \theta'_i$ and $\theta_j \geq \theta'_j$.

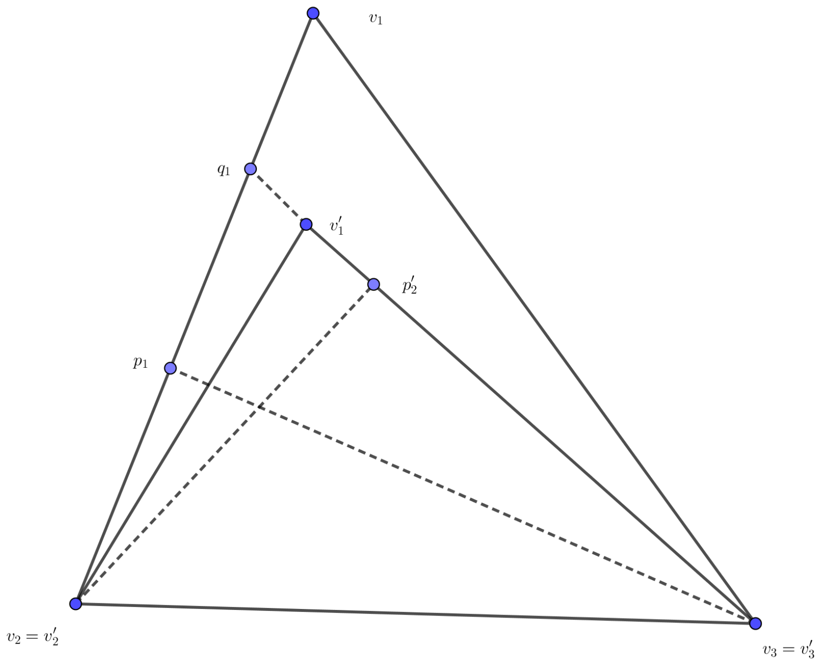

Figure 3: The case where $\theta_2 \geq \theta'_2$ and $\theta_3 \geq \theta'_3$ (Theorem 2).

Without loss of generality we suppose that $\{i, j\} = \{2, 3\}$.

Assume that the first case holds, that is, $\theta_2 \leq \theta'_2$ and $\theta_3 \leq \theta'_3$. We may scale the triangles T and T' so that they have altitudes of the same length from the vertex labeled by 1. That is, we may suppose that $||h_1|| = ||h'_1||$. In that case the triangle T' can be moved inside the triangle T so that their altitudes coincide, see Figure 2. Therefore there exists a homeomorphism $f : T \to T'$ such that $L(f) = 1$. Hence we have

$$\exp(L(T, T')) \leq L(f) = 1 \leq \exp(m(T, T')).$$

This completes the proof of the claim for the first case.

Assume now that the second case holds, that is, $\theta_2 \geq \theta'_2$ and $\theta_3 \geq \theta'_3$. After scaling the triangle T by $\frac{1}{||e_1||}$ and the triangle T' by $\frac{1}{||e'_1||}$, we may suppose that the lengths of e_1 and e'_1 are equal to 1. Also, we may assume that T and T' share an edge and two vertices, that is, $e_1 = e'_1$, $v_2 = v'_2$ and $v_3 = v'_3$. It follows that T contains T', see Figure 3. Consider the triangle $\Delta q_1 v_2 v_3$ where q_1 is the intersection of the line passing through the points v'_1

and v_3 with the line segment $[v_1, v_2]$. Consider the homeomorphism g sending $\Delta v_1 v_2 v_3$ to $\Delta q_1 v_2 v_3$ which is the identity on the triangle $\Delta p_1 v_2 v_3$ and which maps the triangle $\Delta v_1 p_1 v_3$ to the triangle $\Delta q_1 p_1 v_3$ as in the Section 2.1. Clearly $L(g) = 1$.

Now consider the homeomorphism h from $\Delta q_1 v_2 v_3$ to $\Delta v_1' v_2 v_3$ which is the identity on the triangle $\Delta p_2' v_2 v_3$ and which sends $\Delta q_1 v_2 p_2'$ to $\Delta v_1' v_2 p_2'$ as in Section 2.1. Clearly $L(h) = 1$.

Therefore $h \circ g$ is a homeomorphism between T and T' satisfying $L(h \circ g) = 1$. We have

$$\exp(L(T, T')) \leq L(h \circ g) = 1 \leq \exp(m(T, T')).$$

In particular, $L(T, T')$ is given by an infimum which is attained by the map constructed above. Thus, there exists a best Lipschitz homeomorphism between any two acute triangles. □

Remark 3. Assume that T and T' have the same area. Since

$$||e_i|| \cdot ||h_i|| = ||e_i'|| \cdot ||h_i'||,$$

it follows that

$$\frac{||e_i||}{||e_i'||} = \frac{||h_i'||}{||h_i||} \text{ for any } i \in \{1, 2, 3\}.$$

Therefore

$$m(T, T') = \log(\max\{\frac{||e_1'||}{||e_1||}, \frac{||e_2'||}{||e_2||}, \frac{||e_3'||}{||e_3||}, \frac{||e_1||}{||e_1'||}, \frac{||e_2||}{||e_2'||}, \frac{||e_3||}{||e_3'||}\})$$

$$= \max_i\{|\log(||e_i'||) - \log(||e_i||)|\}.$$

Observe that $m(T, T') = m(T', T)$.

Remark 4. (The case of obtuse triangles.) Consider Figure 4. Let T and T' be the triangles $\Delta v_1 v_2 v_3$ and $\Delta v_1' v_2' v_3'$, respectively. Note that $v_2 = v_2'$ and $v_3 = v_3'$. We will show that $L(T, T') > m(T, T')$. Indeed it is easy to see that

$$\exp(m(T, T')) = \frac{||h_1'||}{||h_1||} = \frac{3}{1} = 3.$$

Consider the point p_1 which is the intersection of the altitude from the vertex v_1 and the edge e_1. If f is any label-preserving homeomorphism from T to T', then $f(p_1)$ is in the interior of the edge e_1. It follows that

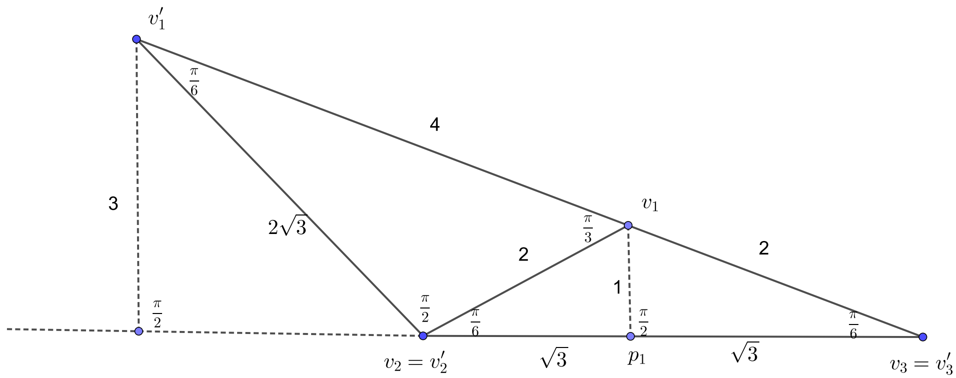

Figure 4: Two triangles T and T' such that $L(T, T') > m(T, T')$ (Remark 4).

$$d_{euc}(f(p_1), f(v_1)) = d_{euc}(f(p_1), v_1') > d_{euc}(v_2, v_1) = 2\sqrt{3}.$$

Hence

$$L(f) > \frac{d_{euc}(f(p_1), f(v_1))}{d_{euc}(p_1, v_1)} = 2\sqrt{3}. \tag{5}$$

Thus $\exp(L(T, T')) \geq 2\sqrt{3} > 3 = \exp(m(T, T'))$. Note that one can scale the triangles T and T' to get λT and $\lambda' T'$ so that they have the same area. In this case we have $L(\lambda T, \lambda' T') > m(\lambda T, \lambda' T')$.

Also, for each $\epsilon > 0$ one can find a homeomorphism $f_\epsilon : T \to T'$ such that

$$2\sqrt{3} < L(f_\epsilon) < 2\sqrt{3} + \epsilon.$$

It follows that $L(T, T') = \log(2\sqrt{3})$ and there is no best Lipschitz homeomorphism between T and T'.

2.4 Some Facts about Two Non-obtuse Triangles Sharing an Edge

Let T be a triangle with vertices v_1, v_2, v_3 and T' a triangle with vertices v_1', v_2', v_3' such that $v_1 = v_1'$ and $v_2 = v_2'$. Suppose that T and T' are non-

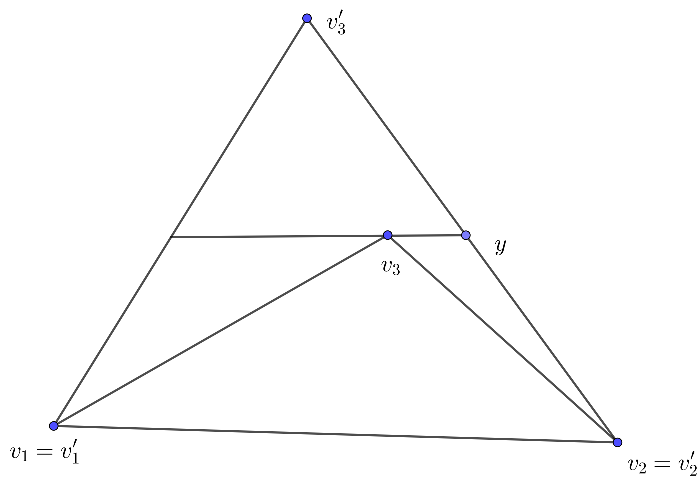

Figure 5: The case where T and T' are non-obtuse and where T is contained in T' (Subsection 2.4).

obtuse and that T is contained in T', as in Figure 5. We claim that

$$\exp(m(T,T')) = \frac{||h_3'||}{||h_3||}.$$

Consider the line which passes through v_3 and which is parallel to the line passing through v_1 and v_2. Let y be the point of intersection of this line with the edge e_1'. Then we have

$$\frac{||h_3'||}{||h_3||} = \frac{||e_1'||}{||[v_2,y]||} \geq \frac{||e_1'||}{||e_1||}.$$

Similarly,

$$\frac{||h_3'||}{||h_3||} \geq \frac{||e_2'||}{||e_2||}.$$

We also have

$$\frac{||h_3'||}{||h_3||} \geq 1 = \frac{||e_3'||}{||e_3||}.$$

Now we show that

$$\frac{||h'_3||}{||h_3||} \geq \frac{||h'_1||}{||h_1||}.$$

We have

$$\frac{||h'_3||}{||h_3||} = \frac{\text{Area}(T')}{\text{Area}(T)} = \frac{||h'_1||}{||h_1||}\frac{||e'_1||}{||e_1||}.$$

Since $\frac{||e'_1||}{||e_1||} \geq 1$, we get $\frac{||h'_3||}{||h_3||} \geq \frac{||h'_1||}{||h_1||}$. Similarly, $\frac{||h'_3||}{||h_3||} \geq \frac{||h'_2||}{||h_2||}$. Hence,

$$\exp(m(T, T')) = \frac{||h'_3||}{||h_3||},$$

which proves the claim.

Now we want to scale T so that the new triangle and T' have the same area. Clearly we need to scale T by the factor $\lambda = \frac{\sqrt{||h'_3||}}{\sqrt{||h_3||}}$. Let $T'' = \lambda T$. Then Equality 3 implies that

$$\exp(m(T'', T')) = \lambda = \frac{\sqrt{||h'_3||}}{\sqrt{||h_3||}} = \frac{||e''_3||}{||e'_3||},$$

where e''_3 is the edge of T'' which is opposite to the vertex with label 3, v''_3. Let us summarize the above discussion as a lemma.

Lemma 1. *Let T and T' be labeled non-obtuse triangles having the same area. There exists $i \in \{1, 2, 3\}$ such that*

1. *$\theta_j \leq \theta'_j$ and $\theta_k \leq \theta'_k$, or*
2. *$\theta_j \geq \theta'_j$ and $\theta_k \geq \theta'_k$,*

where $j, k \in \{1, 2, 3\}$, $j, k \neq i$. Then, we have

$$\exp(m(T, T')) = \max\{\frac{||e_i||}{||e'_i||}, \frac{||e'_i||}{||e_i||}\},$$

or, equivalently,

$$m(T, T') = |\log(||e'_i||) - \log(||e_i||)|.$$

More precisely,

1. If $\theta_j \le \theta'_j$ and $\theta_k \le \theta'_k$ then

$$\exp(m(T,T')) = \frac{||e_i||}{||e'_i||}, \tag{6}$$

2. and if $\theta_j \ge \theta'_j$ and $\theta_k \ge \theta'_k$ then

$$\exp(m(T,T')) = \frac{||e'_i||}{||e_i||}. \tag{7}$$

Lemma 2. *Let T and T' be two labeled triangles of the same area with angles $(\theta_1, \theta_2, \theta_3)$ and $(\theta'_1, \theta'_2, \theta'_3)$. Let $\{i,j,k\} = \{1,2,3\}$.*

1. If $\theta_i > \theta'_i$, $\theta_j < \theta'_j$ and $\theta_k \le \theta'_k$, then

$$\exp(m(T,T')) = \frac{||e_i||}{||e'_i||} > \max\{\frac{||e_k||}{||e'_k||}, \frac{||e'_k||}{||e_k||}\}.$$

2. If $\theta_i > \theta'_i$, $\theta_j < \theta'_j$ and $\theta_k > \theta'_k$, then

$$\exp(m(T,T')) = \frac{||e'_j||}{||e_j||} > \max\{\frac{||e_k||}{||e'_k||}, \frac{||e'_k||}{||e_k||}\}.$$

Proof. Without loss of generality, we will assume that $i=1, j=2, k=3$.

1. By Lemma 1, have $\exp(m(T,T')) = \frac{||e_1||}{||e'_1||}$, and clearly $\exp(m(T,T')) > 1$. This means that $\frac{||e_1||}{||e'_1||} \ge \max\{\frac{||e_3||}{||e'_3||}, \frac{||e'_3||}{||e_3||}\}$. Assume that $\frac{||e_3||}{||e'_3||} = \frac{||e_1||}{||e'_1||}$ or $\frac{||e'_3||}{||e_3||} = \frac{||e_1||}{||e'_1||}$. If $\frac{||e'_3||}{||e_3||} = \frac{||e_1||}{||e'_1||}$, then

$$||e'_1|| \cdot ||e'_3|| \sin\theta'_2 > ||e_1|| \cdot ||e_3|| \sin\theta_2,$$

since $\theta_2 < \theta'_2 < \frac{\pi}{2}$. This is a contradiction since T and T' have the same area. If $\frac{||e_3||}{||e'_3||} = \frac{||e_1||}{||e'_1||}$, since

$$||e'_2|| \cdot ||e'_3|| \sin\theta'_1 = ||e_2|| \cdot ||e_3|| \sin\theta_1,$$

we have

$$\frac{||e'_2||}{||e_2||} = \frac{||e_3||}{||e'_3||}\frac{\sin\theta_1}{\sin\theta'_1} > \frac{||e_3||}{||e'_3||} = \frac{||e_1||}{||e'_1||} = \exp(m(T,T')),$$

which is impossible. This completes the proof of the claim.

2. We know that

$$1 < \exp(m(T,T')) = \frac{||e_2'||}{||e_2||} \geq \max\{\frac{||e_3||}{||e_3'||}, \frac{||e_3'||}{||e_3||}\}.$$

If $\frac{||e_2'||}{||e_2||} = \frac{||e_3||}{||e_3'||}$, then

$$||e_2|| \cdot ||e_3|| \sin\theta_1 > ||e_2'|| \cdot ||e_3'|| \sin\theta_1',$$

which is a contradiction. If $\frac{||e_2'||}{||e_2||} = \frac{||e_3'||}{||e_3||}$, since

$$||e_1'|| \cdot ||e_3'|| \sin\theta_2' = ||e_1|| \cdot ||e_3|| \sin\theta_2,$$

we have

$$\frac{||e_1||}{||e_1'||} = \frac{||e_3'||}{||e_3||}\frac{\sin\theta_2'}{\sin\theta_2} > \frac{||e_3'||}{||e_3||} = \frac{||e_2'||}{||e_2||} = \exp(m(T,T')),$$

which is impossible. This completes the proof of the claim.

□

Remark 5. Let T and T' be two labeled triangles of equal area such that $\theta_i > \theta_i'$ and $\theta_j < \theta_j'$. Scale T to get a new triangle T'' satisfying $||e_k''|| = ||e_k'||$, see Figure 6. Then it follows from the discussion before Lemma 1 that

$$\max\{\frac{||e_k||}{||e_k'||}, \frac{||e_k'||}{||e_k||}\} = \max\{\sqrt{\frac{||h_k''||}{||h_k'||}}, \sqrt{\frac{||h_k'||}{||h_k''||}}\}.$$

Thus Lemma 2 implies that

$$\exp(m(T,T')) > \max\{\sqrt{\frac{||h_k''||}{||h_k'||}}, \sqrt{\frac{||h_k'||}{||h_k''||}}\}.$$

3 The space of triangles

In this section, we consider the set of isometry classes of triangles. First, we define the notion of metric. The definition we give is different from the usual definition of a metric since we drop the symmetry axiom.

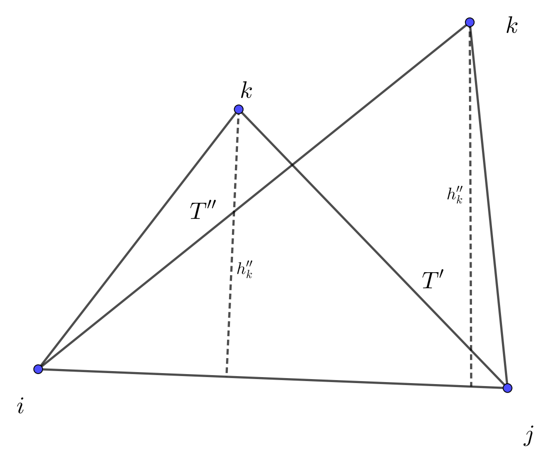

Figure 6: The case of two triangles sharing an edge and such that one is not contained in the other (see Remark 5).

Definition 6. A metric on a set X is a function $\eta : X \times X \to \mathbb{R}$ such that

- $\eta(x,x) = 0$ for all $x \in X$,
- $\eta(x,y) > 0$ if $x \neq y$,
- $\eta(x,y) + \eta(y,z) \geq \eta(x,z)$ for all $x, y, z \in X$.

The pair (X,d) or the set X is called a metric space. If $\eta(x,y) = \eta(y,x)$ for all $x, y \in X$, then the metric is said to be symmetric. Otherwise, the metric is said to be asymmetric.

For a fixed $A > 0$, let $\mathfrak{T}_A$ be the set of equivalence classes of labeled triangles with area A, where two triangles are equivalent if there is an isometry between them which respects the labeling.

We show that L defines a metric on $\mathfrak{T}_A$. If T is a labeled triangle with area A, then we denote its equivalence class in $\mathfrak{T}_A$ by $[T]$. It is clear that L gives a well-defined function on $\mathfrak{T}_A \times \mathfrak{T}_A$. We denote this function by L as well.

The proof of the following lemma is similar to that of Proposition 2.1 of [22].

Lemma 3. *Let T and T' be two labeled triangles of equal area. If $L(T,T') \leq 0$, then $L(T,T') = 0$ and T and T' are isometric.*

Proof. For any $\lambda \geq 0$ the set of λ-Lipschitz homeomorphisms from T to T' which respect the labeling is equicontinuous. Suppose that $L(T,T') \leq 0$. Then we can pick a map $\phi : T \to T'$ which has minimum global Lipschitz constant $e^{L(T,T')} \leq 1$.

If $L(T,T') < 0$, we can assume that ϕ is a homeomorphism and that its Lipschitz constant (which might be larger than $e^{L(T,T')}$) is < 1. But then we can cover the surface T by a countable family of discs of different radii such that the interiors of these discs are pairwise disjoint and their complement has measure zero, and such that ϕ maps one of these discs to a disc of strictly smaller radius. This is impossible since T and T' have the same area. Thus, $L(T,T') = 0$.

Now we consider a covering of the surface T by a countable family of discs of different radii such that the interiors of these discs are pairwise disjoint and their complement has measure zero. Since the Lipschitz constant of the map is 1, as in the previous paragraph, we conclude that each such disc is mapped by f surjectively onto a disc of the same radius. Repeating the same argument with a covering of T by discs whose radii tend uniformly to zero, we see that ϕ is an isometry. □

Note that Lemma 3 implies that for any $[T], [T'] \in \mathfrak{T}_A$ we have $L([T], [T']) \geq 0$ and $L([T], [T']) = 0$ if and only if $[T] = [T']$.

Lemma 4. *If T, T', T'' are three labeled triangles of the same area, then*

$$L(T, T') + L(T', T'') \geq L(T, T'').$$

Proof. Let $f : T \to T'$ and $g : T' \to T''$ be label-preserving homeomorphisms. The assertion follows from the fact that

$$L(g \circ f) \leq L(g)L(f).$$

□

The following theorem follows immediately from Lemma 3 and Lemma 4.

Theorem 7. *The function L is a metric on $\mathfrak{T}_A$.*

Clearly the function m defined by Equation (1) induces a function on $\mathfrak{T}_A \times \mathfrak{T}_A$. We denote this function by m as well. Our next objective is to prove that m is a symmetric metric on $\mathfrak{T}_A$.

Theorem 8. *m is a symmetric metric on $\mathfrak{T}_A$.*

Proof. Let $[T], [T'] \in \mathfrak{T}_A$. Then T has edges of length $||e_1||, ||e_2||, ||e_3||$ and T' has edges of length $||e'_1||, ||e'_2||, ||e'_3||$. Since T and T' have the same area, by Remark 3, we have

$$m(T, T') = \log(\max\{\frac{||e'_1||}{||e_1||}, \frac{||e'_2||}{||e_2||}, \frac{||e'_3||}{||e_3||}, \frac{||e_1||}{||e'_1||}, \frac{||e_2||}{||e'_2||}, \frac{||e_3||}{||e'_3||}\}).$$

It is clear from this formula that m separates points and m is symmetric. Let T'' be another triangle with area A having edges of lengths $||e''_1||, ||e''_2||, ||e''_3||$. It follows from the following inequality that m satisfies the triangle inequality:

$$\max\{\frac{||e'_1||}{||e_1||}, \frac{||e'_2||}{||e_2||}, \frac{||e'_3||}{||e_3||}, \frac{||e_1||}{||e'_1||}, \frac{||e_2||}{||e'_2||}, \frac{||e_3||}{||e'_3||}\}$$

$$\times \max\{\frac{||e''_1||}{||e'_1||}, \frac{||e''_2||}{||e'_2||}, \frac{||e''_3||}{||e'_3||}, \frac{||e'_1||}{||e''_1||}, \frac{||a'_2||}{||a''_2||}, \frac{||e'_3||}{||e''_3||}\}$$

$$\geq \max\{\frac{||e''_1||}{||e_1||}, \frac{||e''_2||}{||e_2||}, \frac{||e''_3||}{||e_3||}, \frac{||e_1||}{||e''_1||}, \frac{||e_2||}{||e''_2||}, \frac{||e_3||}{||e''_3||}\}.$$

□

The fact that the metric m is symmetric implies that it induces on the underlying set a well-defined topology, and that one may talk about Cauchy sequences and completeness in the usual sense, and we shall do this in the sequel. (Note that in the case of a non-symmetric metric, these notions are more complicated to define, see [3, Chapter 1].)

3.1 The space of acute triangles

If there is no risk of confusion, we will denote the equivalence class of a labeled triangle T in $\mathfrak{T}_A$ by T as well.

We denote the set of equivalence classes of acute triangles by $\mathfrak{AT}$. For each fixed $A > 0$, let $\mathfrak{AT}_A$ be the set of equivalence classes of acute triangles having area A. By Theorem 2 the restrictions of L and m on $\mathfrak{AT}_A$ give the same metric.

3.2 Two models for $\mathfrak{AT}_A$

In this section, we introduce two models of $\mathfrak{AT}_A$. Since each labeled triangle is determined by the length of its edges, there is an injection $\mathfrak{AT}_A \to (\mathbb{R}_+^*)^3$ sending the class of a labeled triangle T to $(||e_1||, ||e_2||, ||e_3||)$, where e_1, e_2, e_3 are the edges of T. The image of this map is a 2-dimensional submanifold of $(\mathbb{R}_+^*)^3$, namely, the submanifold defined as the following subset of $(\mathbb{R}_+^*)^3$:

$$\{(a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3 : a_1^2 + a_2^2 - a_3^2 > 0, a_2^2 + a_3^2 - a_1^2 > 0,$$
$$a_3^2 + a_1^2 - a_2^2 > 0, \mathrm{Area}(a_1, a_2, a_3) = A\},$$

where $\mathrm{Area}(a_1, a_2, a_3)$ denotes the area of a triangle with edge lengths a_1, a_2 and a_3. The inequalities

$$a_1^2 + a_2^2 - a_3^2 > 0,\ a_2^2 + a_3^2 - a_1^2 > 0,\ a_3^2 + a_1^2 - a_2^2 > 0$$

follow from the law of cosines, since the angles of the triangles are acute.

Furthermore m induces a metric on this manifold, which we will denote by m as well. Clearly, if $a = (a_1, a_2, a_3)$ and $a' = (a'_1, a'_2, a'_3)$ are points in this manifold, then

$$\begin{aligned} m(a, a') &= \log(\max\{\frac{a_1}{a'_1}, \frac{a'_1}{a_1}, \frac{a_2}{a'_2}, \frac{a'_2}{a_2}, \frac{a_3}{a'_3}, \frac{a'_3}{a_3}\}) \\ &= \max\{|\log a'_1 - \log a_1|, |\log a'_2 - \log a_2|, |\log a'_3 - \log a_3|\}. \end{aligned} \tag{8}$$

If there is no risk of confusion, we will denote this submanifold by $\mathfrak{A}\mathfrak{T}_A$ as well. We shall call this model the *edge model*.

Now we introduce the second model. We first note that each labeled triangle having area A is determined by its angles, that is, the map $\mathfrak{A}\mathfrak{T}_A \to \mathbb{R}_+^{*3}$ sending a triangle T to $(\theta_1, \theta_2, \theta_3)$ is injective, where $\theta_1, \theta_2, \theta_3$ are the angles at the vertices of T. Its image is the interior of a Euclidean equilateral triangle in a hyperplane in $\mathbb{R}^3$. If there is no risk of confusion, we shall also denote this image by $\mathfrak{A}\mathfrak{T}_A$. We shall denote by $(\theta_1, \theta_2, \theta_3)$ the triangle having angles $\theta_1, \theta_2, \theta_3$. We shall call this model the *angle model*.

3.3 Topology of $\mathfrak{T}_A$

We consider the edge model for $\mathfrak{T}_A$. It is equipped with two topologies: the one which comes from the Euclidean metric (that is, the one induced from the ambient space) and the one which comes from m. We denote these topologies by $\mathcal{T}_{euc}$ and $\mathcal{T}_m$, respectively.

Proposition 4. *$\mathcal{T}_{euc}$ and $\mathcal{T}_m$ coincide.*

Proof. It suffices to show that the identity map of $\mathfrak{T}_A$ is a homeomorphism. This can be done by proving that for a sequence $T_n = (a_n, b_n, c_n)$, we have $T_n \to (a, b, c)$ with respect to m if and only if $T_n \to (a, b, c)$ with respect to d_{euc}. If $(a_n, b_n, c_n) \to (a, b, c)$ with respect to d_{euc}, then $\log a_n \to \log a$, $\log b_n \to \log b$ and $\log c_n \to \log c$ as $n \to \infty$. Thus $(a_n, b_n, c_n) \to (a, b, c)$ with respect to m as $n \to \infty$. If $(a_n, b_n, c_n) \to (a, b, c)$ with respect to m, then $|\log a_n - \log a| \to 0$, $|\log b_n - \log b| \to 0$ and $|\log c_n - \log c| \to 0$ as $n \to \infty$. Then $a_n \to a$, $b_n \to b$ and $c_n \to c$ as $n \to \infty$, but this means that $(a_n, b_n, c_n) \to (a, b, c)$ with respect to d_{euc} as $n \to \infty$. □

3.4 The Space of non-obtuse triangles

We will prove that $\mathfrak{T}_A$ is complete and introduce the space of non-obtuse triangles as the closure of $\mathfrak{AT}_A$ in $\mathfrak{T}_A$.

Proposition 5. $(\mathfrak{T}_A, m)$ *is complete.*

Proof. We use the edge model of $\mathfrak{T}_A$. Let $T_n = (a_n, b_n, c_n)$ be a Cauchy sequence of triangles with edge lengths a_n, b_n, c_n. It follows from Equation 8 that $\log a_n, \log b_n, \log c_n$ are Cauchy sequences. Then we have

$$\log a_n \to \log a, \log b_n \to \log b, \log c_n \to \log c,$$

where $\log a, \log b, \log c \in \mathbb{R}$. Thus, $(a_n, b_n, c_n) \to (a, b, c)$ (with respect to m), where $a, b, c > 0$. Note that (a, b, c) is a triangle with area A since area is a continuous function of the length of edges of triangles, and (a, b, c) satisfies strict triangle inequalities since the corresponding "triangle" is non-degenerate. Indeed any degenerate triangle has zero area. □

Furthermore, the set of non-obtuse triangles is a closed subset of $\mathfrak{T}_A$. This follows from the fact that the angle functions are continuous. The set of acute triangles $\mathfrak{AT}_A$ is an open subset of $\mathfrak{T}_A$ and it is easy to see that its closure in $\mathfrak{T}_A$ is the set of non-obtuse triangles of area A, which we denote by $\overline{\mathfrak{AT}_A}$.

3.5 The spaces $\mathfrak{T}_A$ and $\mathfrak{T}_{A'}$ are isometric

For any two triangles T and T', we have

$$m(\lambda T, \lambda T') = m(T, T'),$$

It follows that $\mathfrak{T}_A$ and $\mathfrak{T}_{\lambda^2 A}$ are isometric by the map sending the class of a triangle T to the class of the triangle λT. Similarly, for any $A, A' > 0$, $\overline{\mathfrak{AT}_A}$ and $\overline{\mathfrak{AT}_{A'}}$ are isometric.

4 Geodesics in $\overline{\mathfrak{AT}_A}$

Let (X, d) be a metric space where d is not necessarily symmetric. We use the following version of the notion of geodesic (this is the usual notion of geodesic in a non-symmetric metric space, see e.g. Def. 3.2 in [19]).

Definition 9. A geodesic in X is a map $h : I \to X$ where I is an interval of $\mathbb{R}$ such that for any triple $x_1, x_2, x_3 \in I$ satisfying $x_1 \leq x_2 \leq x_3$, we have

$$d(h(x_1), h(x_3)) = d(h(x_1), h(x_2)) + d(h(x_2), h(x_3)).$$

Note that if d is symmetric, a map obtained from h by reversing the direction of a geodesic is also a geodesic. But this is not necessarily true for asymmetric metrics. Note also that if d is asymmetric, one needs, in the above definition, to be careful about the order of the arguments.

For $A > 0$, let $T = (a_1, a_2, a_3)$ and $T' = (a'_1, a'_2, a'_3)$ be two different elements in $\overline{\mathfrak{AT}_A}$, where we use the edge model. In this section, we prove that T and T' can be joined by a geodesic. In other words, we show that the metric space $(\overline{\mathfrak{AT}_A}, m)$ is geodesic.

As before, there is unique $i \in 1, 2, 3$ such that one of the following holds:

1. $\theta_j \leq \theta'_j$ and $\theta_k \leq \theta'_k$,
2. $\theta_j \geq \theta'_j$ and $\theta_k \geq \theta'_k$,

where $j, k \in \{1, 2, 3\}$ and $j, k \neq i$. Without loss of generality, we assume that $i = 3$ and

$$\theta_1 \geq \theta'_1 \text{ and } \theta_2 \geq \theta'_2,$$

and we shall find a geodesic from T to T'. Since geodesics are reversible in a symmetric metric space, this argument should cover the case

$$\theta_1 \leq \theta'_1 \text{ and } \theta_2 \leq \theta'_2.$$

Consider the scaled triangles $\bar{T} = \frac{1}{a_3}T = (\frac{a_1}{a_3}, \frac{a_2}{a_3}, 1)$ and $\bar{T}' = \frac{1}{a'_3}(\frac{a'_1}{a'_3}, \frac{a'_2}{a'_3}, 1)$. The fact that they have an edge of equal length and the conditions on the angles imply that $\bar{T}'$ can be drawn inside $\bar{T}$, see Figure 7. Let the set of vertices of $\bar{T}'$ be $\{\bar{v}'_1, \bar{v}'_2, \bar{v}'_3\}$ and the set of vertices of $\bar{T}$ be $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$. Consider a family of triangles $\{\bar{T}_t\}$, $t \in [0, 1]$, with angles $(\theta_1(t), \theta_2(t), \theta_3(t))$ having the following properties:

1. For each t, $\bar{T}_t$ is a triangle with two vertices being $\bar{v}_1$ and $\bar{v}_2$.
2. $\bar{T}_0 = \bar{T}$ and $\bar{T}_1 = \bar{T}'$.
3. Each $\theta_i(t)$ is either a decreasing or increasing function.

4. Each $\theta_i(t)$ is a continuous function.

5. $(\theta_1(t), \theta_2(t), \theta_3(t))$ is not constant on any subinterval of $[0, 1]$.

See Figure 7 for an example of such a family.

If we scale each $\bar{T}_t$ by a factor $\lambda(t)$ such that the area of $\lambda(t)\bar{T}_t$ is A, then we get a family of triangles $T_t = (a_1(t), a_2(t), a_3(t))$ such that $T_0 = (a_1, a_2, a_3)$ and $T_1 = (a'_1, a'_2, a'_3,)$. By Lemma 1, this family satisfies the following property:

$$\exp(m(T_{t_1}, T_{t_2})) = \frac{a_3(t_2)}{a_3(t_1)} \tag{9}$$

for any $t_1 \leq t_2$, $t_1, t_2 \in [0, 1]$. It follows that if $t_1 \leq t_2 \leq t_3$ then

$$\exp(m(T_{t_1}, T_{t_3})) = \frac{a_3(t_3)}{a_3(t_2)} \frac{a_3(t_2)}{a_3(t_1)}.$$

Thus,

$$m(T_{t_1}, T_{t_3}) = m(T_{t_1}, T_{t_2}) + m(T_{t_2}, T_{t_3}).$$

This means that the 1-parameter family of triangles T_t is a geodesic. Therefore we proved the following.

Theorem 10. *The space $(\overline{\mathfrak{AT}_A}, m)$ is geodesic, that is, any two points of $\overline{\mathfrak{AT}_A}$ can be joined by a geodesic.*

In fact, we also proved that any two distinct point in $\overline{\mathfrak{AT}_A}$ can be joined by a special geodesic which is a straight line segment in the angle model of $\overline{\mathfrak{AT}_A}$. To be more precise, let $T = (\theta_1, \theta_2, \theta_3)$ and $T' = (\theta'_1, \theta'_2, \theta'_3)$. Let

$$\theta_i(t) = (1-t)\theta_i + t\theta'_i, t \in [0, 1], i \in \{1, 2, 3\}.$$

Then clearly $T_t = (\theta_1(t), \theta_2(t), \theta_3(t))$ is a geodesic joining T and T'.

Remark 11. If T and T' are two elements in $\overline{\mathfrak{AT}_A}$ that have different angles at each vertex, then there are uncountably many geodesics joining them up to parametrization. But if T and T' have an equal angle at some vertex, then up to parametrization there is a unique geodesic joining them. This is implied by the fact that the angle sum is equal to π.

Now we determine when a path in $\overline{\mathfrak{AT}_A}$ is a geodesic.

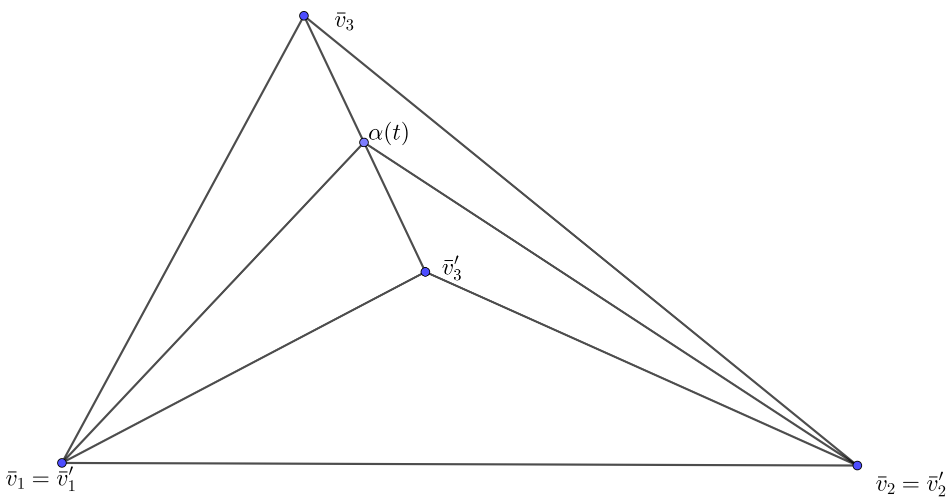

Figure 7: $\alpha(t)$ is a parametrization of the line segment between $\bar{v}_3$ and $\bar{v}'_3$. The family of triangles $\bar{T}_t = \Delta\bar{v}_1\bar{v}_2\alpha(t)$ is a geodesic in $\overline{\mathfrak{AT}_A}$. See §4.

Theorem 12. *Let $T_t = (\theta_1(t), \theta_2(t), \theta_3(t))$, $t \in [a, b]$ be a continuous family of triangles. Then T_t is a geodesic if and only if each $\theta_i(t)$ is monotone, that is, either $\theta_i(t)$ is non-decreasing or non-increasing.*

Proof. The above argument shows that if each $\theta_i(t)$ is monotone, then T_t is a geodesic. To prove the converse, we assume that there is a family T_t which is a geodesic but not all $\theta_i(t)$ are monotone. We also assume that $\theta_1(a) \leq \theta_1(b)$, $\theta_2(a) \leq \theta_2(b)$ and $\theta_3(a) > \theta_3(b)$. It follows that one of the $\theta_i(t)$ is not monotone. Since it is not possible that both $\theta_1(t)$ and $\theta_2(t)$ are non-decreasing, but $\theta_3(t)$ is not monotone, we should only consider the cases where $\theta_1(t)$ or $\theta_2(t)$ are not non-decreasing. By symmetry, we only need to assume that θ_1 is not non-decreasing. Since $\theta_1(a) \leq \theta_1(b)$, there exists $t_1, t_2, t_3 \in [a, b]$ such that $t_1 < t_2 < t_3$ and $\theta_1(t_1) < \theta_1(t_2) > \theta_1(t_3)$.

Consider the triangles T_{t_1}, T_{t_2} ans T_{t_3}. Scale these triangles so that the edges opposite to the vertex with label 3 have the same length, see Figure 8. Let $\bar{T}_{t_i}$ be the scaled triangles. Let $h_3(t_i)$ be the altitude of $\bar{T}_{t_i}$ from the

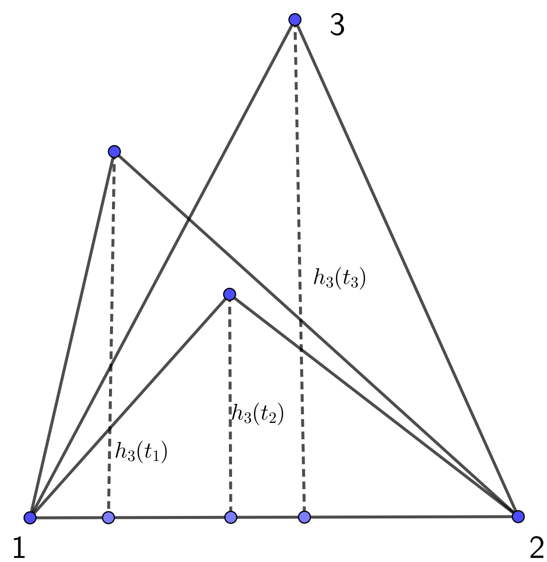

Figure 8: A continuous family of triangles forming a geodesic, see Theorem 12

vertex labeled by 3. We have

$$m(T_{t_1}, T_{t_3}) = \log(\frac{\sqrt{||h_3(t_3)||}}{\sqrt{||h_3(t_1)||}}),$$
$$m(T_{t_1}, T_{t_2}) \geq \log(\frac{\sqrt{||h_3(t_2)||}}{\sqrt{||h_3(t_1)||}}),$$
$$m(T_{t_2}, T_{t_3}) > \log(\frac{\sqrt{||h_3(t_3)||}}{\sqrt{||h_3(t_2)||}}),$$

(see Section 2.4). Thus, $m(T_{t_1}, T_{t_3}) < m(T_{t_1}, T_{t_2}) + m(T_{t_2}, T_{t_3})$. Hence the family $\{T_t\}$ does not form a geodesic. $\square$

5 The Finsler Structure on $\mathfrak{AT}_A$

Let us first recall the notion of a Finsler structure on a differentiable manifold. We work in the global Finsler setting, adopted e.g. in [19] (that is, without the tensor apparatus). We shall prove that the metric space $(\mathfrak{AT}_A, m)$, or, equivalently, $(\mathfrak{AT}_A, L)$, is Finsler, that is, it is a length metric associated with a Finsler structure. We shall briefly recall the bases of this setting. We start with the definition of a *weak norm*.

Definition 13. Let V be a real vector space. A weak norm on V is a function $V \to [0, \infty)$, $v \mapsto ||v||$ such that the following properties hold for every v and w in V:

1. $||v|| = 0$ if and only if $v = 0$,
2. $||tv|| = t||v||$ for every $t > 0$,
3. $||tv + (1-t)w|| \leq t||v|| + (1-t)||w||$ for every $t \in [0, 1]$.

Let M be a differentiable manifold and TM be the tangent bundle of M.

Definition 14. A Finsler structure on M is a function $F : TM \to [0, \infty)$ such that

1. F is continuous,
2. for each $x \in M$, $F|_{T_xM}$ is a weak norm.

Let F be a Finsler structure on a manifold M. For each C^1 curve $c : [a, b] \to M$, we define

$$l(c) = l_F(c) = \int_a^b F(\dot{c}(t))dt.$$

Definition 15. A metric d on a differentiable manifold M is called Finsler if it is the length metric associated to a Finsler structure, that is, if there exists a Finsler structure F on M such that for every $x, y \in M$ we have

$$d(x, y) = \inf\{l_F(c)\}$$

where c ranges over all piecewise C^1 curves such that $c(0) = x$ and $c(1) = y$.

Now we show that the metric m on $\mathfrak{AT}_A$ is Finsler. Recall that we identified $\mathfrak{AT}_A$ with the submanifold in $(\mathbb{R}_+^*)^3$:

$$\mathfrak{AT}_A = \{(a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3 : a_1^2 + a_2^2 - a_3^2 > 0, a_2^2 + a_3^2 - a_1^2 > 0,\\ a_3^2 + a_1^2 - a_2^2 > 0, \mathrm{Area}(a_1, a_2, a_3) = A\}.$$

Let F be the following function on $\mathfrak{AT}_A$:

$$(a_1, a_2, a_3, v_1, v_2, v_3) \mapsto \max_i \{\frac{|v_i|}{a_i}\}.$$

Here, we have identified the tangent space of the manifold $\mathfrak{AT}_A$ at a point (a_1, a_2, a_3) with the Euclidean subspace of $(\mathbb{R}_+^*)^3$ spanned by the set $\mathfrak{AT}_A$ itself, and (v_1, v_2, v_3) are the coordinates of a tangent vector in the tangent space at the point (a_1, a_2, a_3), where the coordinates are relative to the standard basis of $\mathbb{R}^3$.

It is not difficult now to see that F is a Finsler structure on $\mathfrak{AT}_A$:

Theorem 16. *The metric m on $\mathfrak{AT}_A$ is Finsler. More precisely, the metric m is the length metric associated with the Finsler structure F.*

Proof. Let $a' = (a_1', a_2', a_3')$ and $a'' = (a_1'', a_2'', a_3'')$ be in $\mathfrak{AT}_A$. As in Section 4, we may assume that

$$\exp(m(a', a'')) = \frac{a_3''}{a_3'}$$

and therefore that there exists a C^1 function $g : [0,1] \to \mathfrak{AT}_A$, $g(t) = (a_1(t), a_2(t), a_3(t))$ such that $g(0) = a', g(1) = a''$ and

$$\exp(m(c(t_1), c(t_2))) = \frac{a_3(t_2)}{a_3(t_1)}$$

for all $t_1, t_2 \in [0,1], t_1 \leq t_2$. Note that this implies that $a_3(t)$ is increasing.

We claim that

$$F(\dot{g}(t)) = \frac{|d(a_3(t))|}{a_3(t)} = \frac{\dot{a}_3(t)}{a_3(t)} \tag{10}$$

for each $t \in [0,1]$. For otherwise there exists $t_1 \in [0,1]$ and $i \in \{1,2\}$ such that

$$\frac{|\dot{a}_i(t_1)|}{|a_i(t_1)|} > \frac{\dot{a}_3(t_1)}{a_3(t_1)}.$$

Assume that

$$\frac{\dot{a}_i(t_1)}{a_i(t_1)} > \frac{\dot{a}_3(t_1)}{a_3(t_1)}.$$

It follows that $\frac{d}{dt}(\log(\frac{a_i(t)}{a_3(t)})) > 0$ at $t = t_1$ and we may assume that $t_1 < 1$. Therefore there is $t_2 > t_1$ such that

$$\log(\frac{a_i(t_2)}{a_3(t_2)}) > \log(\frac{a_i(t_1)}{a_3(t_1)}).$$

Hence we have

$$\frac{a_i(t_2)}{a_3(t_2)} > \frac{a_i(t_1)}{a_3(t_1)}, \text{ or equivalently } \frac{a_i(t_2)}{a_i(t_1)} > \frac{a_3(t_2)}{a_3(t_1)}.$$

This implies that $\exp(m(g(t_1), g(t_2))) \geq \frac{a_i(t_2)}{a_i(t_1)} > \frac{a_3(t_2)}{a_3(t_1)}$, which is a contradiction. Similarly, we can show that

$$\frac{-\dot{a}_i(t)}{a_i(t)} \leq \frac{\dot{a}_3(t)}{a_3(t)}.$$

Equation (10) implies that

$$l_F(g(t)) = \int_0^1 F(\dot{g}(t)) = \int_0^1 \frac{\dot{a}_3(t)}{a_3(t)} dt$$
$$= \log a_3(1) - \log a_3(0) = \log a_3'' - \log a_3' = m(a', a'').$$

Now we show that for any C^1 curve $c : [0, 1] \to \mathfrak{AT}_A$, $c(t) = (a_1(t), a_2(t), a_3(t))$. we have

$$l_F(c) \geq m(a', a'').$$

We have

$$L_f(c) \geq \int_0^1 \frac{|\dot{a}_i(t)|}{a_i(t)} dt \geq |\log a_i(1) - \log a_i(0)| = |\log a_i'' - \log a_i'|.$$

Hence,

$$L_f(c) \geq \max_i \{|\log a_i'' - \log a_i'|\} = m(a, a').$$

We conclude that

$$m(a', a'') = \inf l_F(c),$$

where $c : [0, 1] \to \mathfrak{AT}_A$ is a C^1 curve such that $c(0) = a'$ and $c(1) = a''$.

□

6 Symmetries of the space $\overline{\mathfrak{AT}_A}$

In this section, we study the isometry group $\mathrm{Isom}(\overline{\mathfrak{AT}_A})$ of $\overline{\mathfrak{AT}_A}$ with respect to the metric m.

The space $\overline{\mathfrak{AT}_A}$ has three boundary components, each of them corresponding to one angle of the triangle becoming right. If we consider the angle model and fix $i \in \{1, 2, 3\}$, then a boundary component is given by

$$\{(\theta_1, \theta_2, \theta_3) : 0 < \theta_1, \theta_2, \theta_3 \leq \frac{\pi}{2}, \theta_1 + \theta_2 + \theta_3 = \pi, \theta_i = \frac{\pi}{2}\}.$$

This shows first that topologically $\overline{\mathfrak{AT}_A}$ is a disc with three punctures on its boundary. A puncture may be regarded as a "triangle" with angles $\frac{\pi}{2}, \frac{\pi}{2}, 0$ and area A. Furthermore these boundary components are geodesics since any injective continuous map from $[0, 1]$ to a boundary component is a geodesic. This can easily be deduced from Theorem 12. We represent the boundary component for which $\theta_i = \frac{\pi}{2}$ by $\mathcal{B}_i$. Note that in the angle model, $\overline{\mathfrak{AT}_A}$ is just a Euclidean triangle with three punctures at its vertices.

It is not difficult to see that $\overline{\mathfrak{AT}_A}$ is unbounded. Let us take, in this paragraph, $A = \frac{1}{2}$ and let us denote the equilateral triangle having area $\frac{1}{2}$ by T_e. Consider a sequence of isosceles triangles $T_n = (\theta_1(n), \theta_2(n), \theta_3(n))$ such that $\theta_2(n) = \theta_3(n)$ and $\lim_{n\to\infty} \theta_3(n) = \frac{\pi}{2}$. It requires a simple calculation to show that $m(T_e, T_n) \to \infty$ as $n \to \infty$. Now consider a sequence of triangles $T'_n = (\theta_1(n), \frac{\pi}{2}, \theta_3(n))$, where $\theta_1(n) \to \frac{\pi}{2}$ as $n \to \infty$. Let us also consider the sequence $T''_n = (\frac{\pi}{2}, \theta_1(n), \theta_3(n))$. It can be shown that $m(T'_n, T''_n) \to 0$ as $n \to \infty$, see Remark 17. This means that $\overline{\mathfrak{AT}_{\frac{1}{2}}}$ resembles an ideal triangle in the hyperbolic plane: it has three geodesic boundary components and any two of these components is a line which converges from each side to a puncture.

Now we consider the isometry group $\mathrm{Isom}(\overline{\mathfrak{AT}_A})$. The symmetric group $S_3 = Sym\{1, 2, 3\}$ can be regarded as a subgroup of $\mathrm{Isom}(\overline{\mathfrak{AT}_A})$ as follows. Let $\sigma \in S_3$ and consider the length model for $\overline{\mathfrak{AT}_A}$. We identify σ with the map sending (a_1, a_2, a_3) to $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. It is not difficult to see that this map is an isometry of $\overline{\mathfrak{AT}_A}$. Thus we may consider S_3 as a subgroup of $\mathrm{Isom}(\overline{\mathfrak{AT}_A})$. We wish to prove now that there are no other isometries, that is, $S_3 = \overline{\mathfrak{AT}_A}$.

Let $\{i, j, k\} = \{1, 2, 3\}$. The 2-uple $\{T, T'\}$ is called an *i-pair* if $T \in \mathcal{B}_j$, $T' \in \mathcal{B}_k$ and $\theta_k = \theta'_j$. Note that since $\theta_i = \theta'_i$, there is a unique geodesic between T and T' up to parametrization.

Lemma 5. *If $\{T, T'\}$ and $\{T_1, T'_1\}$ are different i-pairs, then $m(T, T') \neq m(T_1, T'_1)$.*

Proof. Without loss of generality, assume that $i = 1, j = 2, k = 3$. Since $\overline{\mathfrak{AT}_A}$ and $\overline{\mathfrak{AT}_{\frac{1}{2}}}$ are isometric, we may suppose that $A = \frac{1}{2}$. This will make the calculations easier. Then, in the edge model,

$$T = (a, \sqrt{a^2 + \frac{1}{a^2}}, \frac{1}{a}), T' = (a, \frac{1}{a}, \sqrt{a^2 + \frac{1}{a^2}}).$$

It follows that

$$m(T, T') = \log \sqrt{a^2 + \frac{1}{a^2}} - \log \frac{1}{a} = \frac{1}{2} \log(1 + a^4),$$

which is an injective function of a. This proves the claim. ☐

Remark 17. As $a \to 0$, we have $m(T, T') = \frac{1}{2} \log(1 + a^4) \to 0$. This means that the distance between two boundary components is arbitrarily small near a puncture.

Theorem 18. *$Isom(\overline{\mathfrak{AT}_A}) = S_3$.*

Proof. It suffices to prove that an isometry that sends $\mathcal{B}_i$ to itself for all $i \in \{1, 2, 3\}$ is the identity. Let σ be such an isometry. Consider an $i-$pair $\{T, T'\}$. First of all observe that $\{\sigma(T), \sigma(T')\}$ is an i-pair. This is true since two elements in different boundary components are l-pairs for some $l \in \{1, 2, 3\}$ if and only if there is, up to parametrization, a unique geodesic between them. But to be joined by a unique geodesic is an isometry invariant. It follows that $m(T, T') = m(\sigma(T), \sigma(T'))$. Since σ fixes boundary components, Lemma 5 implies that $T = \sigma(T)$ and $T' = \sigma(T')$. Therefore σ gives us an isometry of the geodesic between T and T'. But if we restrict m to this geodesic, we get the usual metric on some interval of the real line.

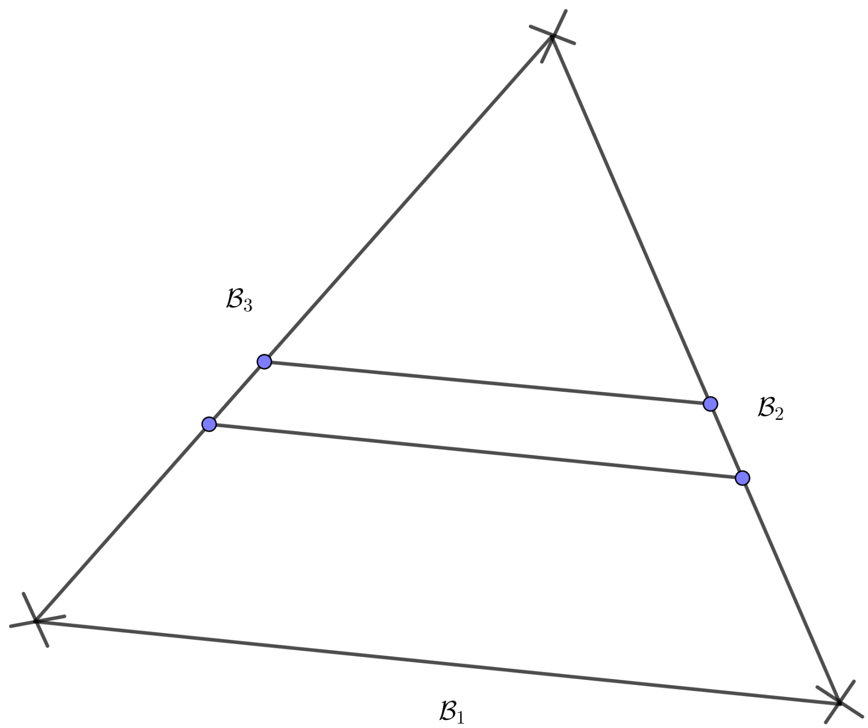

Figure 9: The angle model, see §3.2.

Since an isometry of an interval fixing its endpoint is trivial, it follows that the restriction of σ to such a geodesic is the identity map. Since any point in $\overline{\mathfrak{AT}_A}$ lies in such a geodesic, it follows that σ is the identity map. $\square$

In Figure 9, we give the angle model together with some 1-pairs and geodesics between them. Note that the geodesic between two i-pairs is a straight line segment which is parallel to the boundary component $\mathcal{B}_i$. Let $\mathbb{R}_+^*$ be the set of nonzero positive real numbers. Consider the function $\overline{\mathfrak{AT}_A} \to \mathbb{R}_+^*$ which sends a triangle to its angle at the i-th vertex. Then it follows that the inverse image of any point in $\mathbb{R}_+^*$ is a geodesic segment between two i-pairs.

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