THE NUCLEUS OF A BLOCK OF A P-SOLVABLE GROUP

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Abstract

Let G be a finite p-solvable group. For a given p-block B of G, we define a canonical pair (K, A), refered to as a nucleus for B, where K is a subgroup of G and A is a block of K of maximal defect, defined uniquely by B up to G-conjugacy. The irreducible characters (ordinary or modular) associated with B are closely related to those associated with A. Also, not surprisingly, (K, A) is just (G, B) in case B is of maximal defect.

Given a normal subgroup N of G and a block b of N, we show that there exist a nucleus $(\widehat{N}, \widehat{b})$ for b and a subgroup $\widehat{G}$ of G containing $\widehat{N}$ as a normal subgroup such that the blocks of G covering b behave quite analogously to those of $\widehat{G}$ covering $\widehat{b}$.

1. Introduction

Fix a prime p and let G be a finite group. Next, let $\mathrm{Bl}(G)$ be the set of p-blocks of G. Recall that a block $B \in \mathrm{Bl}(G)$ is said to be of maximal defect if it has a Sylow p-subgroup of G as a defect group. Since the principal block of G is of maximal defect, then it is clear that a block B of G is of maximal defect if and only if B is of maximal defect in $\mathrm{Bl}(G)$. It is well known that, in general, a block is not of maximal defect, and in some situations blocks of maximal defect are more amenable to proving the validity of certain statements about blocks.

Assume now that G is p-solvable and let B be a block of G. In this paper, we associate with B a canonical pair (K, A), where K is a subgroup of G and A is a block of K of maximal defect, defined uniquely by B up to G-conjugacy such that A^G is defined and equals B and any Sylow p-subgroup of K is a defect group for B. As might be expected, (K, A) is just (G, B) in case B is of maximal defect. Any such pair (K, A) is said to be a *nucleus* for B, and we refer to A as a *nucleus block* for B. (We should mention that we have borrowed the term nucleus from [6], where G. Navarro constructed, in a somewhat similar manner, the (normal) nucleus of an irreducible character of a p-solvable group.)

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If A is a nucleus block for B, then the irreducible characters (ordinary or modular) associated with B are closely related to those associated with A. In fact, character induction defines height-preserving bijections from $\mathrm{Irr}(A)$ onto $\mathrm{Irr}(B)$ and from $\mathrm{IBr}(A)$ onto $\mathrm{IBr}(B)$, where, as is customary, $\mathrm{Irr}(A)$ (resp. $\mathrm{IBr}(A)$) is the set of ordinary (resp. p-Brauer) irreducible characters belonging to the block A.

We continue to assume that G is p-solvable. Let now N be a normal subgroup of G and b a block of N. We denote by $\mathrm{Bl}(G|b)$ the set of blocks of G that cover b. A block B of G covering b is said to be of maximal defect in $\mathrm{Bl}(G|b)$ if the defect of B is greatest among the defects of all the members of $\mathrm{Bl}(G|b)$. Next, given $B \in \mathrm{Bl}(G|b)$ and $\mu \in \mathrm{Irr}(b)$, we write $\mathrm{Irr}(B|\mu)$ for the set of all characters in $\mathrm{Irr}(B)$ that lie over μ. The first of our main results shows that there exist a nucleus $(\widehat{N}, \widehat{b})$ for b and a subgroup $\widehat{G}$ of G containing $\widehat{N}$ as a normal subgroup such that the blocks in $\mathrm{Bl}(\widehat{G}|\widehat{b})$ behave quite analogously to those in $\mathrm{Bl}(G|b)$.

Theorem A. *Let $N \lhd G$, where G is p-solvable and let b be a block of N. Then there exist a nucleus $(\widehat{N}, \widehat{b})$ for b and a subgroup $\widehat{G}$ of G containing $\widehat{N}$ as a normal subgroup such that the following statements hold.*

(a) $\widehat{G} \cap N = \widehat{N}$.

(b) $\widehat{b}$ *is $\widehat{G}$-stable.*

(c) *Block induction defines a bijection from* $\mathrm{Bl}(\widehat{G}|\widehat{b})$ *onto* $\mathrm{Bl}(G|b)$.

(d) *Suppose $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$. Then any defect group of $\widehat{B}$ is one for $\widehat{B}^G$. Furthermore, $\widehat{B}$ is of maximal defect in* $\mathrm{Bl}(\widehat{G})$ *if and only if $\widehat{B}^G$ is of maximal defect in* $\mathrm{Bl}(G|b)$.

(e) *If $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$, then induction defines height-preserving bijections from* $\mathrm{Irr}(\widehat{B})$ *onto* $\mathrm{Irr}(\widehat{B}^G)$ *and from* $\mathrm{IBr}(\widehat{B})$ *onto* $\mathrm{IBr}(\widehat{B}^G)$. *Moreover, if* $\theta \in \mathrm{Irr}(\widehat{B})$ *and* $\varphi \in \mathrm{IBr}(\widehat{B})$, *then it holds that* $d_{\theta\varphi} = d_{\theta^G \varphi^G}$.

(f) *Given $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$ and $\mu \in \mathrm{Irr}(b)$, it holds that*

$$\mathrm{Irr}(\widehat{B}^G|\mu) = \{\theta^G : \theta \in \mathrm{Irr}(\widehat{B}|\widehat{\mu})\},$$

where $\widehat{\mu}$ is the unique character in $\mathrm{Irr}(\widehat{b})$ *such that* $\widehat{\mu}^N = \mu$.

Let now B be an arbitrary block in $\mathrm{Bl}(G|b)$. Then, using Theorem A, we can show that there exist a nucleus $(\widehat{N}, \widehat{b})$ for b, a subgroup G_0 of G containing $\widehat{N}$ as a normal subgroup and a block $B_0 \in \mathrm{Bl}(G_0|\widehat{b})$ of maximal defect in $\mathrm{Bl}(G_0)$ such that the character theory of B "is similar" to that of B_0.

Theorem B. *Let $N \lhd G$, where G is p-solvable and let B and b be blocks of G and N, respectively, such that B covers b. Then there exist a nucleus $(\widehat{N}, \widehat{b})$ for b, a subgroup G_0 of G containing $\widehat{N}$ as a normal subgroup and a block B_0 of G_0 of maximal defect such that the following statements hold.*

(a) $G_0 \cap N = \widehat{N}$.

(b) *B_0 covers $\widehat{b}$.*

(c) *${B_0}^G$ is defined and equals B.*

(d) *Every defect group of B_0 is one for B.*

(e) *Induction defines height-preserving bijections from* $\mathrm{Irr}(B_0)$ *onto* $\mathrm{Irr}(B)$ *and from* $\mathrm{IBr}(B_0)$ *onto* $\mathrm{IBr}(B)$. *Furthermore, if $\theta \in \mathrm{Irr}(B_0)$ and $\varphi \in \mathrm{IBr}(B_0)$, then it holds that $d_{\theta\varphi} = d_{\theta^G \varphi^G}$.*

2. The nucleus of a block

Throughout the remainder of this paper we fix a p-solvable group G, where p is a prime number. We denote by $\mathcal{P}$ the set of all pairs (H, b), where H is a subgroup of G and b is a block of H. Then G acts by conjugation on $\mathcal{P}$ by $(H, b)^g = (H^g, b^g)$. Now given a block B of G, we let $\mathcal{N}_B$ the set of all those pairs (H, b) in $\mathcal{P}$ such that $H \lhd G$, b is of maximal defect and B covers b. It is clear that $(\langle 1 \rangle, b_{\langle 1 \rangle}) \in \mathcal{N}_B$, where $b_{\langle 1 \rangle}$ is the unique block of the identity subgroup $\langle 1 \rangle$. Also, notice that $\mathcal{N}_B$ is closed under the conjugation action of G. Next, for pairs (H, b) and (H', b') in $\mathcal{N}_B$, we write $(H, b) \leq (H', b')$, if $H \subseteq H'$ and b' covers b. This clearly defines a partial order on $\mathcal{N}_B$.

Lemma 2.1. *Let B be a block of G. Then there is a unique conjugacy class of maximal pairs in $\mathcal{N}_B$.*

Proof. Suppose (M_1, b_1) and (M_2, b_2) are maximal pairs in $\mathcal{N}_B$. Since, clearly, any G-conjugate of a maximal pair is also maximal, it suffices to show that (M_1, b_1) and (M_2, b_2) are conjugate in G.

Since both M_1 and M_2 are normal in G, then so is $M_1 M_2$. Now choose a block b of $M_1 M_2$ covered by B and covering b_1. Next, as B covers both b and b_2, it follows that b covers some G-conjugate of b_2. There is, thus, no loss in assuming that b covers b_2.

Let D be a defect group of B. It follows by [4, Theorem 5.5.16(ii)] that $D \cap M_1$ and $D \cap M_2$ are Sylow p-subgroups of M_1 and M_2, respectively. We claim that $D \cap (M_1 M_2)$ is a Sylow p-subgroup of $M_1 M_2$.

We have

$$|(DM_1)M_2|_p = \frac{|DM_1|_p|M_2|_p}{|(DM_1)\cap M_2|_p} = \frac{|D||M_1|_p|M_2|_p}{|D\cap M_1||(DM_1)\cap M_2|_p} = \frac{|D||M_2|_p}{|(DM_1)\cap M_2|_p}.$$

Now, since $D\cap M_2 \subseteq (DM_1)\cap M_2 \subseteq M_2$, and $D\cap M_2$ is a Sylow p-subgroup of M_2, we must have $|(DM_1)\cap M_2|_p = |M_2|_p$. It follows that $|(DM_1)M_2|_p = |D|$. On the other hand,

$$|(DM_1)M_2|_p = |D(M_1M_2)|_p = \frac{|D||M_1M_2|_p}{|D\cap(M_1M_2)|}.$$

Hence $|D\cap(M_1M_2)| = |M_1M_2|_p$, which proves our claim.

In view of [4, Theorem 5.5.16(ii)], we conclude that $(M_1M_2, b) \in \mathcal{N}_B$. By the maximality of (M_1, b_1) and (M_2, b_2), we are then forced to have $(M_1, b_1) = (M_1M_2, b) = (M_2, b_2)$, which clearly completes the proof of the lemma. $\square$

Lemma 2.2. *Let B be a block of G and choose a maximal pair (M, b) in $\mathcal{N}_B$. If b is invariant in G, then $(M, b) = (G, B)$.*

Proof. Suppose, on the contrary, that $M < G$. Let L/M be a chief factor of G, and observe that since G is p-solvable, L/M is either a p-group or a p'-group. (See [2, p. 5].)

Suppose L/M is a p-group. By Corollary 5.5.6 in [4], there is a unique block b' of L covering b. Since B covers b, then b' must be covered by B. Now let P and R be defect groups of b and b', respectively. Since L stabilizes b, we have that $|R| = |P||L : M|_p$ by [4, Theorem 5.5.16(i)]. It follows that R is a Sylow p-subgroup of L as P is a Sylow p-subgroup of M. Then $(L, b') \in \mathcal{N}_B$, which clearly contradicts the maximality of (M, b). Hence L/M must be a p'-group.

Choose a block b' of L covered by B and covering b. If P is a defect group of b, then P is contained in some defect group R of b' by [4, Theorem 5.5.16(ii)]. Next, since P is a Sylow p-subgroup of M and L/M is a p'-group, then P must be a Sylow p-subgroup of L. This forces $R = P$. Therefore $(L, b') \in \mathcal{N}_B$, contradicting the choice of (M, b). This clearly ends the proof of the lemma. $\square$

Let B be a block of G. A *nucleus* for B is any pair $(K, A) \in \mathcal{P}$ defined inductively as follows. If B is of maximal defect, we let $(K, A) = (G, B)$. Now if B is not of maximal defect, choose a maximal pair (M, b) in $\mathcal{N}_B$, and note that $M < G$. Let T be the inertial group of b in G, and let B' be the Fong-Reynolds correspondent of B over b. Since $M < G$, Lemma 2.2 implies that $T < G$. By the inductive hypothesis, a nucleus for B' is well defined. Take (K, A) to be any nucleus of B'.

Next, we record the following couple of facts about nuclei for future use.

Corollary 2.3. *Let B be a block of G. Then*

(a) *The set of nuclei of B consists of a single G-conjugacy class of pairs.*

(b) *If (K, A) is a nucleus for B, then A is of maximal defect.*

(c) *If $(H, \beta) \in \mathcal{N}_B$, then there exists a nucleus (K, A) of B such that $H \subseteq K$ and β is covered by A.*

Proof. We proceed by induction on $|G|$. Suppose first that B is of maximal defect. Then the unique nucleus of B is (G, B), and there is nothing to prove. Assume now that B is not of maximal defect.

By choosing a maximal pair (M, b) in $\mathcal{N}_B$, we have $M < G$. Then, in light of Lemma 2.2, the inertial group T of b is a proper subgroup of G. Let B' be the Fong-Reynolds correspondent of B over b. Then the inductive hypothesis ensures that the set of nuclei of B' consists of a single T-conjugacy class of pairs. Next, by Lemma 2.1, any other maximal pair in $\mathcal{N}_B$ is of the form (M, b^g) for some $g \in G$. Then the block B'^g of T^g is the Fong-Reynolds correspondent of B over b^g, and by the construction of the nucleus of a block, it follows that $\{(K, A)^g : (K, A) \text{ is a nucleus for } B'\}$ is precisely the set of nuclei of B'^g. This takes care of (a).

For (b), suppose (K, A) is a nucleus for B. Then there exists a maximal pair (M, b) in $\mathcal{N}_B$ such that (K, A) is a nucleus for the Fong-Reynolds correspondent B' of B over b. Since B' is a block of the inertial group T of b in G and $T < G$, the inductive hypothesis guarantees that A is of maximal defect.

Finally, we prove (c). Suppose that $(H, \beta) \in \mathcal{N}_B$. Then there exists a maximal pair (M, b) in $\mathcal{N}_B$ such that $(H, \beta) \leq (M, b)$. It follows that $(H, \beta) \in \mathcal{N}_{B'}$, where B' is the Fong-Reynolds correspondent of B over b. By the inductive hypothesis, there is a nucleus (K, A) for B' such that $H \subseteq K$ and β is covered by A. Since (K, A) is a nucleus for B, part (c) follows. $\square$

Our next result establishes some connections between a block and its nuclei. Toward that end, the following easy and quite general lemma is needed.

Lemma 2.4. *Let L be a subgroup of an arbitrary finite group H. Suppose B and B' are blocks of H and L, respectively, such that induction defines bijections from* $\mathrm{Irr}(B')$ *onto* $\mathrm{Irr}(B)$ *and from* $\mathrm{IBr}(B')$ *onto* $\mathrm{IBr}(B)$. *Then* $d_{\theta\varphi} = d_{\theta^H \varphi^H}$ *for* $\theta \in \mathrm{Irr}(B')$ *and* $\varphi \in \mathrm{IBr}(B')$.

Proof. Let $\theta \in \mathrm{Irr}(B')$. Then

$$(\theta^H)^0 = (\theta^0)^H = (\sum_{\varphi \in \mathrm{IBr}(B')} d_{\theta\varphi}\varphi)^H = \sum_{\varphi \in \mathrm{IBr}(B')} d_{\theta\varphi}\varphi^H,$$

where $(\theta^H)^0$ (resp. θ^0) is the restriction of θ^H (resp. θ) to the set of p-regular elements of H (resp. L). It follows that $d_{\theta^H \varphi^H} = d_{\theta\varphi}$ for every $\varphi \in \mathrm{IBr}(B')$, as needed to be shown. □

Theorem 2.5. *Let B be a block of G with nucleus (K, A). Then*

(a) *A^G is defined and equals B.*

(b) *A and B have a common defect group.*

(c) *Induction defines height-preserving bijections of* $\mathrm{Irr}(A)$ *onto* $\mathrm{Irr}(B)$ *and of* $\mathrm{IBr}(A)$ *onto* $\mathrm{IBr}(B)$. *Consequently,* $d_{\theta\varphi} = d_{\theta^G \varphi^G}$ *for all* $\theta \in \mathrm{Irr}(A)$ *and all* $\varphi \in \mathrm{IBr}(A)$.

Proof. We proceed by induction on $|G|$. Suppose first that B is of maximal defect. Then $A = B$ and, in this case, all assertions are immediate. We may now assume that B is not of maximal defect. Then there is a maximal pair (M, b) in $\mathcal{N}_B$ such that if T is the inertial group of b in G and B' is the Fong-Reynolds correspondent of B over b, then (K, A) is a nucleus for B'. Since $T < G$, the inductive hypothesis guarantees that all three assertions hold with B' in place of B.

By [4, Theorem 5.5.10(i)], we have that B'^G is defined and equals B. Since A^T is defined and equals B', it follows by Lemma 5.3.4 in [4] that A^G is defined and equals B. This proves (a). Next, the blocks A and B' have a common defect group. Also, by [4, Theorem 5.5.10(iv)], any defect group of B' is one for B. Part (b) is then immediate.

For (c), the inductive hypothesis ensures that the map $\xi \longmapsto \xi^T$ induces a height-preserving bijection from $\mathrm{Irr}(A)$ onto $\mathrm{Irr}(B')$. Then, using [4, Theorem 5.5.10(ii)] and [5, Theorem 9.14(d)], we conclude that the correspondence $\xi \longmapsto \xi^G$ defines a height-preserving bijection from $\mathrm{Irr}(A)$ onto $\mathrm{Irr}(B)$. The proof of the analogous statement for $\mathrm{IBr}(A)$ and $\mathrm{IBr}(B)$ is similar. Finally, the last assertion of (c) is immediate from Lemma 2.4. □

We have accumulated enough information to be able to describe the nuclei of a block of a p-nilpotent group. Suppose, then, that G is p-nilpotent and let $B \in \mathrm{Bl}(G)$. Choose a block β of $\mathrm{O}_{p'}(G)$ covered by B. Next, let J be the inertial group of β in G, and let $\widetilde{B} \in \mathrm{Bl}(J)$ be the Fong-Reynolds correspondent of B over β. We shall see that $(J, \widetilde{B})$ is a nucleus for B.

Since $\mathrm{O}_{p'}(J) = \mathrm{O}_{p'}(G)$ and J stabilizes β, then [5, Theorem 10.20] says that $\widetilde{B}$ is the unique block of J covering β, and that the Sylow p-subgroups of J are precisely the defect groups of $\widetilde{B}$. In particular, we can then write $J = D\mathrm{O}_{p'}(G)$ for some defect group D of $\widetilde{B}$.

Next, since $(\mathrm{O}_{p'}(G), \beta) \in \mathcal{N}_B$, then in view of Corollary 2.3(c), there is a nucleus (K, A) for B such that $\mathrm{O}_{p'}(G) \subseteq K$ and A covers β. Let P be a Sylow p-subgroup of K. Then $K = P\mathrm{O}_{p'}(G)$, and by Corollary 2.3(b), we have that P is a defect group for A.

We argue now that $K \subseteq J$. It is clear that $K \cap J$ is the inertial group of β in K. Now, let $\widetilde{A}$ be the Fong-Reynolds correspondent of A over β. Then, by [4, Theorem 5.5.10(iv)], there is $k \in K$ such that P^k is a defect group for $\widetilde{A}$. Since $P^k\mathrm{O}_{p'}(G) \subseteq K \cap J \subseteq K = P\mathrm{O}_{p'}(G)$, and $\left|P^k\right| = |P|$, it follows that $K \cap J = K$. This clearly proves that $K \subseteq J$, as claimed.

Since P is a defect group for A, then, by Theorem 2.5(b), we have that P is a defect group for B. Next, recall that D is a defect group for $\widetilde{B}$. Then D is a defect group for B by [4, Theorem 5.5.10(iv)]. Therefore, P and D are conjugate in G, and consequently $|J| = |D|\,|\mathrm{O}_{p'}(G)| = |P|\,|\mathrm{O}_{p'}(G)| = |K|$. Now, as $K \subseteq J$, we conclude that $K = J$. Since A covers β, and $\widetilde{B}$ is the unique block of J covering β, it follows that $A = \widetilde{B}$. Thus $(J, \widetilde{B})$ is a nucleus for B, as needed.

3. Nuclei and normal subgroups

We begin this section by fixing some notation. Let H be a finite group and let N be a normal subgroup of H. Let b be a block of N and suppose $\mu \in \mathrm{Irr}(N)$. The set of irreducible characters of H lying over μ is denoted by $\mathrm{Irr}(H|\mu)$. Also, we write $\mathrm{Bl}(H|b)$ for the set of blocks of H that cover b. Next, suppose that $\mu \in \mathrm{Irr}(b)$ and $B \in \mathrm{Bl}(H|b)$. We denote by $\mathrm{Irr}(B|\mu)$ the intersection $\mathrm{Irr}(B) \cap \mathrm{Irr}(H|\mu)$.

In order to prove Theorem A, we need a couple of preliminary lemmas.

Lemma 3.1. *Let $N \lhd H$, where H is an arbitrary finite group. Let B and b be blocks of H and N, respectively, such that B covers b.*

(a) *Suppose b is H-stable of maximal defect and B is weakly regular (relative to N). Then B has a Sylow p-subgroup of H as a defect group.*

(b) *Suppose $\mu \in \mathrm{Irr}(b)$ and let B' be the Fong-Reynolds correspondent of B over b. Then character induction defines a bijection of $\mathrm{Irr}(B'|\mu)$ onto $\mathrm{Irr}(B|\mu)$.*

Proof. Part (a) follows immediately from [4, Theorem 5.5.16(i)]. Next, in view of [4, Theorem 5.5.10(ii)], induction defines an injective map from $\mathrm{Irr}(B'|\mu)$ into $\mathrm{Irr}(B|\mu)$. Now suppose $\chi \in \mathrm{Irr}(B|\mu)$. Then there is a unique character $\theta \in \mathrm{Irr}(B')$ such that $\theta^H = \chi$. Since χ lies over μ, then θ lies over μ^h for some $h \in H$. Also, as b is the unique block of N covered by B', notice that $\mu^h \in \mathrm{Irr}(b)$. It follows that $b^h = b$, as $\mu \in \mathrm{Irr}(b)$. Therefore h lies in the inertial group T of b in H, and hence θ must lie over μ. This clearly completes the proof of (b). □

Lemma 3.2. *Let $M \subseteq N$ be normal subgroups of a finite group H. Let β be a block of M with inertial group I in H, and set $J = I \cap N$. Suppose $B \in \mathrm{Bl}(H|\beta)$ and $b \in \mathrm{Bl}(N|\beta)$, and let $B' \in \mathrm{Bl}(I|\beta)$ and $b' \in \mathrm{Bl}(J|\beta)$ be the Fong-Reynolds correspondents of B and b, respectively. Then the following statements hold.*

(a) *Suppose $\nu \in \mathrm{Irr}(\beta)$, $\chi \in \mathrm{Irr}(I|\nu)$ and $\tau \in \mathrm{Irr}(J|\nu)$. Then $\chi^H \in \mathrm{Irr}(H)$ and $\tau^N \in \mathrm{Irr}(N)$. Furthermore, $J \lhd I$, and χ lies over τ if and only if χ^H lies over τ^N.*

(b) *B covers b if and only if B' covers b'.*

(c) *Suppose B covers b and let $\tau \in \mathrm{Irr}(b')$. Then $\tau^N \in \mathrm{Irr}(b)$ and $\mathrm{Irr}(B|\tau^N) = \{\chi^H : \chi \in \mathrm{Irr}(B'|\tau)\}$.*

Proof. Let I_0 be the inertial group of ν in H and let $J_0 = I_0 \cap N$. Then $I_0 \subseteq I$, $J_0 \subseteq J$, $J_0 \lhd I_0$ and $J \lhd I$. Next, let χ_0 and τ_0 be the unique characters in $\mathrm{Irr}(I_0|\nu)$ and $\mathrm{Irr}(J_0|\nu)$, respectively, such that ${\chi_0}^I = \chi$ and ${\tau_0}^J = \tau$. (See [1, Theorem 6.11].) Then, by Lemma 2.6 in [3], we have that χ lies over τ if and only if χ_0 lies over τ_0. Next, by Theorem 6.11 in [1], we have ${\chi_0}^H \in \mathrm{Irr}(H|\nu)$ and ${\tau_0}^N \in \mathrm{Irr}(N|\nu)$. Moreover, by [3, Lemma 2.6], again, it holds that ${\chi_0}^H$ lies over ${\tau_0}^N$ if and only if χ_0 lies over τ_0. Since $\chi^H = {\chi_0}^H$ and $\tau^N = {\tau_0}^N$, the proof of (a) is complete. Next, we prove (b).

Since both B' and b' cover β, we can choose $\nu \in \mathrm{Irr}(\beta)$, $\chi \in \mathrm{Irr}(B'|\nu)$ and $\tau \in \mathrm{Irr}(b'|\nu)$. By part (a), $\chi^H \in \mathrm{Irr}(H)$, $\tau^N \in \mathrm{Irr}(N)$, and χ lies over τ if and only if χ^H lies over τ^N. Also, in light of [4, Theorem 5.5.10(ii)], note that $\chi^H \in \mathrm{Irr}(B)$ and $\tau^N \in \mathrm{Irr}(b)$.

Suppose B' covers b'. Then χ can be chosen to lie over τ. Therefore, χ^H lies over τ^N, and hence B covers b. Conversely, suppose B covers b. Then, there exists a character $\xi \in \mathrm{Irr}(B|\tau^N)$. Since τ^N lies over ν, then so does ξ. It follows by Lemma 3.1(b) that there is $\zeta \in \mathrm{Irr}(B'|\nu)$ such that $\zeta^H = \xi$. We conclude, then, by (a), that ζ lies over τ. Hence B' covers b'. We have thus established (b). Finally, we prove (c).

Assume B covers b and let $\tau \in \mathrm{Irr}(b')$. We know that $\tau^N \in \mathrm{Irr}(b)$. Now suppose $\chi \in \mathrm{Irr}(B'|\tau)$. Then $\chi^H \in \mathrm{Irr}(B)$. Moreover, by choosing a character $\nu \in \mathrm{Irr}(\beta)$ under τ, it follows by (a) that $\chi^H \in \mathrm{Irr}(B|\tau^N)$. Next, assume $\xi \in \mathrm{Irr}(B|\tau^N)$. Then, as in the preceeding paragraph, $\xi = \zeta^H$ for some character $\zeta \in \mathrm{Irr}(B'|\tau)$. This proves (c) and concludes the proof of the lemma. □

We are now ready to prove Theorem A.

Proof of Theorem A. Let T be the inertial group of b in G. Suppose first that b is of maximal defect. Then (N, b) is the unique nucleus for b. In this case, we let $(\widehat{N}, \widehat{b}) = (N, b)$ and $\widehat{G} = T$. Now assertions (a) and (b) are clearly satisfied, and (c) is direct from [4, Theorem 5.5.10(i)]. Next, suppose $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$. Then, by Theorem 5.5.10(iv) in [4], any defect group for $\widehat{B}$ is one for $\widehat{B}^G$. It follows by [4, Theorem 5.5.16(i)] that $\widehat{B}^G$ is weakly regular relative to N if and only if $\widehat{B}$ is weakly regular (relative to N), and if such is the case, Lemma 3.1(a) tells us that $\widehat{B}$ has a Sylow p-subgroup of $\widehat{G}$ as a defect group. This completes the proof of (d). Next, by [4, Theorem 5.5.10(ii)] and [5, Theorem 9.14(d)], character induction defines a height-preserving bijection of $\mathrm{Irr}(\widehat{B})$ onto $\mathrm{Irr}(\widehat{B}^G)$. A similar argument establishes the analogous statement for $\mathrm{IBr}(\widehat{B})$ and $\mathrm{IBr}(\widehat{B}^G)$. Now, by Lemma 2.4, it holds that $d_{\theta\varphi} = d_{\theta^G\varphi^G}$ for all $\theta \in \mathrm{Irr}(\widehat{B})$ and all $\varphi \in \mathrm{IBr}(\widehat{B})$. The proof of (e) is then complete. Finally if $\mu \in \mathrm{Irr}(b)$, then $\widehat{\mu} = \mu$ (as $\widehat{b} = b$), and part (f) follows from Lemma 3.1(b). We have thus settled the case, in which b is of maximal defect. We can assume, therefore, that b is not of maximal defect, and we proceed by induction on $|N|$.

Choose a maximal pair (M, β) in $\mathcal{N}_b$. Since M is uniquely determined by b and b is T-stable, note that T normalizes M. Let I be the inertial group of β in T and $J = I \cap N$ (the inertial group of β in N). Next, let b' be the Fong-Reynolds correspondent of b over β. Then b' is a block of J, and as $N \lhd G$, note that $J \lhd I$. Now, since b is not of maximal defect, we have $J < N$ by Lemma 2.2. Then, by the inductive hypothesis, there exist a nucleus $(\widehat{N}, \widehat{b})$

for b' and a subgroup $\widehat{G}$ of I containing $\widehat{N}$ as a normal subgroup such that assertions (a)-(f) all hold with I, J and b' in place of G, N and b.

Now, by definition, $(\widehat{N}, \widehat{b})$ is a nucleus for b, and we need to establish statements (a)-(f) for G, N and b. First, note that (b) is clearly satisfied. Assertion (a) also holds as $\widehat{G} \cap N = \widehat{G} \cap I \cap N = \widehat{G} \cap J = \widehat{N}$, where the last equality holds by the inductive hypothesis. Our next task is to prove (c).

We have that block induction defines a bijection Γ from $\mathrm{Bl}(\widehat{G}|\widehat{b})$ onto $\mathrm{Bl}(I|b')$. Next, we claim that induction also defines a bijection from $\mathrm{Bl}(I|b')$ onto $\mathrm{Bl}(T|b)$.

Recall that I is the inertial group of β in T. Then, Theorem 5.5.10(i) of [4] tells us that block induction defines a bijection Δ from $\mathrm{Bl}(I|\beta)$ onto $\mathrm{Bl}(T|\beta)$. Since both b and b' cover β, it is clear that $\mathrm{Bl}(T|b) \subseteq \mathrm{Bl}(T|\beta)$ and $\mathrm{Bl}(I|b') \subseteq \mathrm{Bl}(I|\beta)$. Then, in view of Lemma 3.2(b), it follows that Δ maps $\mathrm{Bl}(I|b')$ onto $\mathrm{Bl}(T|b)$. This clearly proves our claim.

Next, we have that T is the inertial group of b in G. Then, owing to [4, Theorem 5.5.10(i)], once more, block induction gives us a bijection Θ from $\mathrm{Bl}(T|b)$ onto $\mathrm{Bl}(G|b)$. Now let Λ be the composite $\Gamma\Delta\Theta$. Then Λ is a bijection from $\mathrm{Bl}(\widehat{G}|\widehat{b})$ onto $\mathrm{Bl}(G|b)$. Furthermore, by Lemma 5.3.4 in [4], Λ is just block induction. This proves (c).

Now let $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$. By the inductive hypothesis, any defect group D of $\widehat{B}$ is one for $\widehat{B}^I$. Next, since $\widehat{B}^I$ is the Fong-Reynolds correspondent of $\widehat{B}^T$ over β and $\widehat{B}^T$ is the Fong-Reynolds correspondent of $\widehat{B}^G$ over b, we have, by Theorem 5.5.10(iv) in [4] that D is a defect group for $\widehat{B}^G$. In particular, we conclude that the bijection Λ preserves block defects. It follows, then, that $\widehat{B}$ is of maximal defect in $\mathrm{Bl}(\widehat{G}|\widehat{b})$ if and only if $\widehat{B}^G$ is of maximal defect in $\mathrm{Bl}(G|b)$. Next, since $\widehat{b}$ is $\widehat{G}$-stable of maximal defect, we have, in light of Lemma 3.1(a) and [4, Theorem 5.5.16(i)], that $\widehat{B}$ is of maximal defect in $\mathrm{Bl}(\widehat{G}|\widehat{b})$ if and only if $\widehat{B}$ is of maximal defect in $\mathrm{Bl}(\widehat{G})$. Therefore, $\widehat{B}$ is of maximal defect in $\mathrm{Bl}(\widehat{G})$ if and only if $\widehat{B}^G$ is of maximal defect in $\mathrm{Bl}(G|b)$. We have, thus, completed the proof of (d). Next, we take care of (e).

By the inductive hypothesis, character induction defines a height-preserving bijection of $\mathrm{Irr}(\widehat{B})$ onto $\mathrm{Irr}(\widehat{B}^I)$. Next, by [4, Theorem 5.5.10(ii)] and [5, Theorem 9.14(d)], induction also defines height-preserving bijections from $\mathrm{Irr}(\widehat{B}^I)$ onto $\mathrm{Irr}(\widehat{B}^T)$ and from $\mathrm{Irr}(\widehat{B}^T)$ onto $\mathrm{Irr}(\widehat{B}^G)$. It follows that character induction yields a height-preserving bijection of $\mathrm{Irr}(\widehat{B})$ onto $\mathrm{Irr}(\widehat{B}^G)$. A similar argument proves the parallel statement for $\mathrm{IBr}(\widehat{B})$ and $\mathrm{IBr}(\widehat{B}^G)$. Then, in light of Lemma 2.4, we conclude that $d_{\theta\varphi} = d_{\theta^G\varphi^G}$ for any $\theta \in \mathrm{Irr}(\widehat{B})$ and any $\varphi \in \mathrm{IBr}(\widehat{B})$. This takes care of assertion (e). Finally, we prove (f).

We have $\widehat{\mu}^J \in \mathrm{Irr}(b')$, and by the inductive hypothesis,

$$\mathrm{Irr}(\widehat{B}^I|\widehat{\mu}^J) = \{\theta^I : \theta \in \mathrm{Irr}(\widehat{B}|\widehat{\mu})\}.$$

Next, recall that the block $\widehat{B}^T$ covers b. Then, in light of Lemma 3.2(c), we have

$$\mathrm{Irr}(\widehat{B}^T|\mu) = \{\chi^T : \chi \in \mathrm{Irr}(\widehat{B}^I|\widehat{\mu}^J)\} = \{\theta^T : \theta \in \mathrm{Irr}(\widehat{B}|\widehat{\mu})\}.$$

Next, since $\widehat{B}^T$ is the Fong-Reynolds correspondent of $\widehat{B}^G$ over b, Lemma 3.1(b) implies that $\mathrm{Irr}(\widehat{B}^G|\mu) = \{\psi^G : \psi \in \mathrm{Irr}(\widehat{B}^T|\mu)\}$. It follows that $\mathrm{Irr}(\widehat{B}^G|\mu) = \{\theta^G : \theta \in \mathrm{Irr}(\widehat{B}|\widehat{\mu})\}$. This proves (f), and completes the proof of the theorem. $\square$

Next, we prove Theorem B.

Proof of Theorem B. Choose a nucleus $(\widehat{N}, \widehat{b})$ for b and a subgroup $\widehat{G}$ of G as in Theorem A. Now by Theorem A(c), there is a unique block $\widehat{B} \in \mathrm{Bl}(\widehat{G}|\widehat{b})$ such that $\widehat{B}^G = B$. Since $(\widehat{N}, \widehat{b}) \in \mathcal{N}_{\widehat{B}}$, Corollary 2.3(c) tells us that there exists a nucleus (G_0, B_0) for $\widehat{B}$ such that $\widehat{N} \subseteq G_0$ and B_0 covers $\widehat{b}$. Observe that the block B_0 is of maximal defect and that $\widehat{N} \lhd G_0$, as $\widehat{G}$ normalizes $\widehat{N}$.

Since $\widehat{G} \cap N = \widehat{N}$ by Theorem A(a) and $\widehat{N} \subseteq G_0 \cap N \subseteq \widehat{G} \cap N$, assertion (a) follows. Next, part (b) is clear.

We have that ${B_0}^{\widehat{G}}$ is defined and equals $\widehat{B}$ by Theorem 2.5(a). Now, as $\widehat{B}^G = B$, Lemma 5.3.4 of [4] implies that ${B_0}^G$ is defined and equals B. This is (c). Next, statement (d) follows by Theorem 2.5(b) and Theorem A(d). Finally, assertion (e) is a consequence of Theorem 2.5(c) and Theorem A(e). This finishes the proof of Theorem B. $\square$

It is natural to seek connections between the nuclei of a block and the Harris-Knörr correspondence. Our final result of this section offers a connection in that direction. In order to prove this result, we need the following general fact about the Harris-Knörr correspondence.

Lemma 3.3. *Let $H \subseteq K$ be normal subgroups of the arbitrary (finite) group Γ, and let β, b and B be blocks of H, K and Γ, respectively, such that B and b cover β. Let Q be a defect group of β, and let $\beta' \in \mathrm{Bl}(\mathrm{N}_H(Q))$ be the Brauer correspondent of β. If $B' \in \mathrm{Bl}(\mathrm{N}_\Gamma(Q)|\beta')$ (resp. $b' \in \mathrm{Bl}(\mathrm{N}_K(Q)|\beta')$)*

is the Harris-Knörr correspondent of B (resp. b), then B covers b if and only if B' covers b'.

Proof. First, assume that B covers b. Since b' covers β' and β' has defect group Q, then by [5, Theorem 9.26], we can choose a defect group P of b' such that $P \cap \mathrm{N}_H(Q) = Q$. Also, in view of [5, Theorem 9.28], note that P is a defect group of b. Now, by [5, Theorem 9.26] again, as β has defect group Q, there exists $x \in K$ such that $P^x \cap H = Q$. Since $Q \subseteq P \cap H$ and $|P \cap H| = |P^x \cap H|$, it follows that $P \cap H = Q$. Then, as $H \lhd \Gamma$, we have $\mathrm{N}_\Gamma(P) \subseteq \mathrm{N}_\Gamma(Q)$.

Let $\widetilde{b} \in \mathrm{Bl}(\mathrm{N}_K(P))$ be the Brauer correspondent of b, and let $\widetilde{B} \in \mathrm{Bl}(\mathrm{N}_\Gamma(P)|\widetilde{b})$ be the Harris-Knörr correspondent of B. Since $\mathrm{N}_K(P) \subseteq \mathrm{N}_K(Q)$, and $\widetilde{b}$ has defect group P, [4, Theorem 5.3.8] says that $\widetilde{b}^{\mathrm{N}_K(Q)}$ is defined and has defect group P. Then, as $\widetilde{b}^K$ is defined and equals b, we conclude by [4, Lemma 5.3.4] that $(\widetilde{b}^{\mathrm{N}_K(Q)})^K$ is defined and equals b.

Now both b' and $\widetilde{b}^{\mathrm{N}_K(Q)}$ are blocks of $\mathrm{N}_K(Q)$ having P as a defect group. Since $(b')^K = b = (\widetilde{b}^{\mathrm{N}_K(Q)})^K$, it follows by [4, Theorem 5.3.8] that $\widetilde{b}^{\mathrm{N}_K(Q)} = b'$. Consequently, as $\widetilde{b}$ has defect group P and $\mathrm{N}_{\mathrm{N}_K(Q)}(P) = \mathrm{N}_K(P)$, the block $\widetilde{b}$ is the Brauer correspondent of b' in $\mathrm{N}_{\mathrm{N}_K(Q)}(P)$. Next, since $\mathrm{N}_{\mathrm{N}_\Gamma(Q)}(P) = \mathrm{N}_\Gamma(P)$ and $\widetilde{B} \in \mathrm{Bl}(\mathrm{N}_\Gamma(P)|\widetilde{b})$, then [5, Theorem 9.28] implies that $\widetilde{B}^{\mathrm{N}_\Gamma(Q)}$ is defined and covers b'. Thus, in particular, the block $\widetilde{B}^{\mathrm{N}_\Gamma(Q)}$ covers β'. Now, by [5, Theorem 9.28] again, $(\widetilde{B}^{\mathrm{N}_\Gamma(Q)})^\Gamma$ is defined and covers β. Next, in view of [4, Lemma 5.3.4], we have $(\widetilde{B}^{\mathrm{N}_\Gamma(Q)})^\Gamma = \widetilde{B}^\Gamma = B$. Since $B' \in \mathrm{Bl}(\mathrm{N}_\Gamma(Q)|\beta')$ is the Harris-Knörr correspondent of B, we are forced to have $\widetilde{B}^{\mathrm{N}_\Gamma(Q)} = B'$. We conclude, then, that B' covers b'.

Now, assume that B' covers b'. Our goal is to show that B covers b. First, recall that $\widetilde{b}$ is the Brauer correspondent of b' in $\mathrm{N}_{\mathrm{N}_K(Q)}(P)$. Let now $B_0 \in \mathrm{Bl}(\mathrm{N}_\Gamma(P)|\widetilde{b})$ be the Harris-Knörr correspondent of $B' \in \mathrm{Bl}(\mathrm{N}_\Gamma(Q)|b')$ (recall that $\mathrm{N}_{\mathrm{N}_\Gamma(Q)}(P) = \mathrm{N}_\Gamma(P)$). Since $(B_0)^{\mathrm{N}_\Gamma(Q)} = B'$ and $(B')^\Gamma = B$, then by [4, Lemma 5.3.4], $(B_0)^\Gamma$ is defined and equals B. Now, as B_0 covers $\widetilde{b}$, then [5, Theorem 9.28] tells us that B covers b, as needed to be shown. □

Theorem 3.4. *Let $B \in \mathrm{Bl}(G)$ and suppose $(H, \beta) \in \mathcal{N}_B$. Let Q be a defect group of β and let $\beta' \in \mathrm{Bl}(\mathrm{N}_H(Q))$ be the Brauer correspondent of β. Also, let $B' \in \mathrm{Bl}(\mathrm{N}_G(Q)|\beta')$ be the Harris-Knörr correspondent of B. If (K, A) is any nucleus of B with $(H, \beta) \le (K, A)$, then $(\mathrm{N}_K(Q), A')$ is a nucleus of B', where $A' \in \mathrm{Bl}(\mathrm{N}_K(Q)|\beta')$ is the Harris-Knörr correspondent* *of* A.

Proof. First, we prove by induction on $|G|$ that there exists a nucleus (J,C) of B with $(H,\beta) \leq (J,C)$ such that $(\mathrm{N}_J(Q),C')$ is a nucleus of B', where $C' \in \mathrm{Bl}(\mathrm{N}_J(Q)|\beta')$ is the Harris-Knörr correspondent of C.

Choose a maximal pair (M,b) in $\mathcal{N}_B$ with $(H,\beta) \leq (M,b)$, and let $b' \in \mathrm{Bl}(\mathrm{N}_M(Q)|\beta')$ be the Harris-Knörr correspondent of b. Since B covers b, note, by Lemma 3.3, that B' covers b'. Next, in view of [5, Theorem 9.28], the blocks b and b' have a common defect group. Now, as $\mathrm{N}_M(Q) \subseteq M$ and b is of maximal defect, then b' is of maximal defect. We conclude, then, that the pair $(\mathrm{N}_M(Q),b')$ lies in $\mathcal{N}_{B'}$. Now, we claim that, in fact, $(\mathrm{N}_M(Q),b')$ is maximal in $\mathcal{N}_{B'}$.

Suppose, on the contrary, that $(\mathrm{N}_M(Q),b')$ is not maximal in $\mathcal{N}_{B'}$. Then, there is $(U,b_0) \in \mathcal{N}_{B'}$ with $(\mathrm{N}_M(Q),b') \leq (U,b_0)$ and $|\mathrm{N}_M(Q)| < |U|$. Now write $L = HU$. Then,

$$\mathrm{N}_L(Q) = L \cap \mathrm{N}_G(Q) = HU \cap \mathrm{N}_G(Q) = U\mathrm{N}_H(Q) = U,$$

where the last equality holds since $\mathrm{N}_H(Q)$ is contained in U. Next, as Q is a Sylow p-subgroup of the normal subgroup H of G, we have $G = H\mathrm{N}_G(Q)$ by the Frattini argument. Since $U \lhd \mathrm{N}_G(Q)$, it follows that L is normal in G. Also, as $M = H\mathrm{N}_M(Q)$ and $\mathrm{N}_M(Q) \subseteq U$, we have that $M \subseteq L$.

Choose a Sylow p-subgroup P of U. As $Q \lhd U$, we have $Q \subseteq P$. Also, note that P is a defect group for b_0. Now, since

$$|L|_p = \frac{|H|_p\,|U|_p}{|H \cap U|_p} = \frac{|H|_p\,|U|_p}{|H \cap \mathrm{N}_L(Q)|_p} = \frac{|H|_p\,|U|_p}{|\mathrm{N}_H(Q)|_p} = |U|_p\,,$$

we see that P is a Sylow p-subgroup of L. Also, as b_0 covers b' and b' covers β', we observe that b_0 covers β'. Then, by [5, Theorem 9.28], $(b_0)^L$ is a block of L covering β and having P as a defect group. Furthermore, since B' covers b_0, we have that B covers $(b_0)^L$ by Lemma 3.3. We conclude, then, that $(L,(b_0)^L) \in \mathcal{N}_B$. Next, since b_0 covers b', then, again by Lemma 3.3, the block $(b_0)^L$ covers b. Now, by the maximality of the pair (M,b), we are forced to have $L = M$. Then $U = \mathrm{N}_L(Q) = \mathrm{N}_M(Q)$, which contradicts our assumption that $\mathrm{N}_M(Q)$ is proper in U. We have thus proved that $(\mathrm{N}_M(Q),b')$ is maximal in $\mathcal{N}_{B'}$, as claimed.

Next, let S be the inertial group of b' in $\mathrm{N}_G(Q)$, and write $T = HS$. Our task now is to show that T is the inertial group of b in G. First, if $t \in T$, then $t = hs$ for some $h \in H$ and $s \in S$. Now,

$$b^t = b^{hs} = b^s = ((b')^M)^s = ((b')^s)^M = (b')^M = b,$$

where the second equality holds, since $h \in H \subseteq M$. This shows that T stabilizes b. It remains to show that, in fact, T is the full stabilizer of b in G. Let then $g \in G$ with $b^g = b$. We can write $g = h'n$, where $h' \in H$ and $n \in \mathrm{N}_G(Q)$. Hence $b^n = b$. Therefore, in particular, b covers the block $\beta^n \in \mathrm{Bl}(H)$. It follows that $\beta^n = \beta^m$ for some $m \in M$. Furthermore, as $M = H\mathrm{N}_M(Q)$, we may assume that $m \in \mathrm{N}_M(Q)$. Now $nm^{-1} \in \mathrm{N}_G(Q)$, and so $(\beta')^{nm^{-1}}$ is a block of $\mathrm{N}_H(Q)$ having Q as a defect group (recall that Q is a defect group of β'). Moreover, we have

$$((\beta')^{nm^{-1}})^H = ((\beta')^H)^{nm^{-1}} = \beta^{nm^{-1}} = \beta.$$

Since β' is the Brauer correspondent of β in $\mathrm{N}_H(Q)$, we are forced to have $(\beta')^{nm^{-1}} = \beta'$. Now, the block $(b')^{nm^{-1}}$ of $\mathrm{N}_M(Q)$ covers β' and

$$((b')^{nm^{-1}})^M = ((b')^M)^{nm^{-1}} = b^{nm^{-1}} = b^{m^{-1}} = b.$$

It follows, by [5, Theorem 9.28], that $(b')^{nm^{-1}} = b'$. Then, as $m \in \mathrm{N}_M(Q)$, we get $(b')^n = (b')^m = b'$. Therefore $n \in S$, and hence $g = h'n \in HS = T$. This proves that T is the inertial group of b in G, as needed to be shown.

Next,

$$\mathrm{N}_T(Q) = T \cap \mathrm{N}_G(Q) = HS \cap \mathrm{N}_G(Q) = S\mathrm{N}_H(Q) = S,$$

where the last equality holds, since $\mathrm{N}_H(Q) \subseteq S$.

Suppose first that $T = G$. Then $S = \mathrm{N}_G(Q)$, and by Lemma 2.2, we conclude that B and B' are both of maximal defect. Therefore, (G, B) (resp. $(\mathrm{N}_G(Q), B')$) is the unique nucleus for B (resp. B'). In this case, (J, C) is precisely (G, B).

Assume now that $T < G$. Since $T = HS$, notice that $S < \mathrm{N}_G(Q)$. Let $\widehat{B} \in \mathrm{Bl}(S)$ be the Fong-Reynolds correspondent of B' over b'. Then $\widehat{B}$ covers β', and hence, by [5, Theorem 9.28], $(\widehat{B})^T$ is defined and covers β. Also, in view of Lemma 3.3, note that $(\widehat{B})^T$ covers b. Next, we have $(\widehat{B})^{\mathrm{N}_G(Q)} = B'$ and $(B')^G = B$. It follows by [4, Lemma 5.3.4] that $(\widehat{B})^G$ is defined and equals B. Then, by [4, Lemma 5.3.4] again, we conclude that $((\widehat{B})^T)^G$ is defined and equals B. It follows that $(\widehat{B})^T$ is the Fong-Reynolds correspondent of B

over b. Now by the inductive hypothesis, there is a nucleus (J, C) of $(\widehat{B})^T$ with $(H, \beta) \leq (J, C)$ and such that $(\mathrm{N}_J(Q), C')$ is a nucleus of $\widehat{B}$, where $C' \in \mathrm{Bl}(\mathrm{N}_J(Q)|\beta')$ is the Harris-Knörr correspondent of C. Since, by the definition of nuclei, (J, C) (resp. $(\mathrm{N}_J(Q), C')$) is a nucleus of B (resp. B'), we are clearly done in this case.

Now to complete the proof of the theorem, we let (K, A) be an arbitrary nucleus of B such that $(H, \beta) \leq (K, A)$. We know that B has a nucleus (J, C) with $(H, \beta) \leq (J, C)$ such that B' has nucleus $(\mathrm{N}_J(Q), C')$, where $C' \in \mathrm{Bl}(\mathrm{N}_J(Q)|\beta')$ is the Harris-Knörr correspondent of C. Then, by Corollary 2.3(a), $(K, A) = (J, C)^x$ for some $x \in G$. Furthermore, as $G = H\mathrm{N}_G(Q)$ and $H \subseteq J$, we may assume that $x \in \mathrm{N}_G(Q)$. Now C covers both blocks β and $\beta^{x^{-1}}$ of H. It follows that $\beta^{x^{-1}} = \beta^y$ for some $y \in J$. Since $J = H\mathrm{N}_J(Q)$, we may assume that $y \in \mathrm{N}_J(Q)$. Now $yx \in \mathrm{N}_G(Q)$, and so $(\beta')^{yx}$ is a block of $\mathrm{N}_H(Q)$ with defect group Q. Moreover,

$$((\beta')^{yx})^H = ((\beta')^H)^{yx} = \beta^{yx} = \beta.$$

Then, we must have $(\beta')^{yx} = \beta'$. Now, since $y \in \mathrm{N}_J(Q)$, $C' \in \mathrm{Bl}(\mathrm{N}_J(Q)|\beta')$ and $J^{yx} = J^x = K$, we get $(C')^x = (C')^{yx} \in \mathrm{Bl}(\mathrm{N}_K(Q)|\beta')$. Also,

$$((C')^x)^K = ((C')^x)^{J^x} = ((C')^J)^x = C^x = A.$$

Therefore, $(C')^x = A'$, the Harris-Knörr correspondent of A. Finally, in view of Corollary 2.3(a), since $x \in \mathrm{N}_G(Q)$ and $(\mathrm{N}_J(Q), C')$ is a nucleus of B', we conclude that $(\mathrm{N}_K(Q), A') = (\mathrm{N}_J(Q), C')^x$ is a nucleus of B'. The proof of the theorem is now complete. $\square$

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