ON THE JONES POLYNOMIAL OF QUASI-ALTERNATING LINKS, II

KHALED QAZAQZEH, AHMAD AL-RHAYYEL, AND NAFAA CHBILI

Abstract. We extend a result of Thistlethwaite [17, Theorem 1(iv)] on the structure of the Jones polynomial of alternating links to the wider class of quasi-alternating links. In particular, we prove that the Jones polynomial of any prime quasi-alternating link that is not a $(2, n)$-torus link has no gap. As an application, we show that the differential grading of the Khovanov homology of any prime quasi-alternating link that is not a $(2, n)$-torus link has no gap. Also, we show that the determinant is an upper bound for the breadth of the Jones polynomial for any quasi-alternating link. Finally, we prove that the Jones polynomial of any non-prime quasi-alternating link L has more than one gap if and only if L is a connected sum of Hopf links.

1. Introduction

Let $f(t) = \sum_{k=n}^{m} a_k t^k$ be a Laurent polynomial with real coefficients such that $a_n \neq 0$ and $a_m \neq 0$. The nonnegative integer $m - n$ is called the *breadth* of f. We say that $f(t)$ has a *gap* of length s if there exists $n \leq i_0 < m$ such that $a_{i_0} \neq 0$ and $a_{i_0+s+1} \neq 0$, while $a_j = 0$ for all $i_0 < j \leq i_0 + s$. Given $g(t) = \sum_{k=n'}^{m'} b_k t^k$ a Laurent polynomial with real coefficients such that $a_{n'} \neq 0$ and $a_{m'} \neq 0$. If $n' > m + 1$, then we say that there is a gap of length $n' - m - 1$ between f and g.

The study of the Jones polynomial of alternating links led to the proof of longstanding conjectures in knot theory [8, 9, 17]. In particular, the independent work of Thistlethwaite [17], Kauffman [8] and Murasugi [9], shows that the breadth of the Jones polynomial of any link is a lower bound of its crossing number and that the equality holds if and only if the link is alternating. If we combine this with the well-known fact that the determinant of any non-split alternating link is bigger than or equal to its crossing number [1], then we conclude that the breadth of the Jones polynomial of any non-split alternating link is smaller than or equal to its determinant. It is worth mentioning here that Thistlethwaite also proved that the Jones polynomial of any prime alternating link that is not a $(2, n)$-torus link has no gap and that the coefficients of this polynomial alternate in sign, [17, Theorem 1].

The class of alternating links has been generalized in several directions. A particularly interesting generalization has been obtained in [11]. Indeed, Ozsváth and Szabó proved that Heegaard Floer homological properties of double branched covers of alternating links extend to a wider class of links that they call quasi-alternating. Unlike alternating links which admit a simple diagrammatic definition, quasi-alternating links have been defined recursively as follows:

Date: 03/08/2024.

2020 *Mathematics Subject Classification.* 57K10, 57K14.

Key words and phrases. quasi-alternating, determinant, breadth, gap, Jones polynomial.

Definition 1.1. The set $\mathcal{Q}$ of quasi-alternating links is the smallest set satisfying the following properties:

- The unknot belongs to $\mathcal{Q}$.
- If L is a link with a diagram D containing a crossing c such that
  1. both smoothings of the diagram D at the crossing c, L_0 and L_1 as in Figure 1 belong to $\mathcal{Q}$, and
  2. $\det(L_0), \det(L_1) \geq 1$,
  3. $\det(L) = \det(L_0) + \det(L_1)$;

  then L is in $\mathcal{Q}$ and in this case we say L is quasi-alternating at the crossing c with quasi-alternating diagram D.

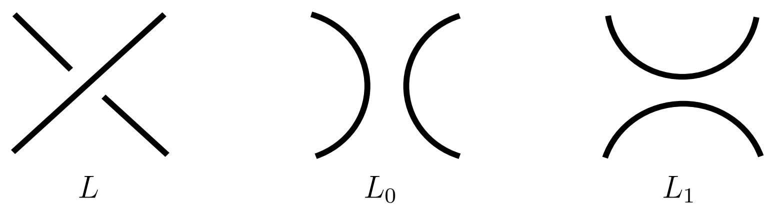

FIGURE 1. The link diagram L at the crossing c and its smoothings L_0 and L_1 respectively.

We let $V_L(t)$ and $\text{breadth}(L)$ to denote the Jones polynomial and its breadth of the oriented link L, respectively. In addition, we let $c(L)$ and $\det(L)$ to denote the crossing number and the determinant of the link L, respectively. A natural question that arises here is whether the inequalities $\text{breadth}(L) \leq \det(L)$ and $c(L) \leq \det(L)$ hold for any quasi-alternating link L. These inequalities are known to be true in the case of alternating links according to the above discussion. On the other hand, these inequalities were conjectured to hold for quasi-alternating links as well, see [13, Conjecture 3.8] and [15, Conjecture 1.1].

In this paper, we use the spanning tree expansion of the Jones polynomial that has been introduced in [17] to prove that the Jones polynomial of any prime quasi-alternating link that is not a $(2, n)$-torus link has no gap. Hence, we obtain a generalization of [17, Theorem 1(iv)] to the class of quasi-alternating links. Consequently and by using [14, Proposition 3.10], we prove that the differential grading of the Khovanov homology of such a link has no gap. Indeed, this result has been conjectured in [14]. Moreover, we conclude that the determinant of any quasi-alternating link is an upper bound of the breadth of its Jones polynomial, which establishes [13, Conjecture 3.8]. It is noteworthy that these properties of the Jones polynomial provide simple obstructions for a link to be quasi-alternating. Several other obstruction criteria obtained in terms of polynomial invariants and link homology can be found in [6, 10, 11, 14, 16] for instance.

Here is an outline of this paper. In Section 2, we briefly recall the definition of the Jones polynomial and its spanning tree expansion. In Section 3, we prove the main result on the Jones polynomial of quasi-alternating links. Finally, some applications of the main result are discussed in Section 4.

2. Jones Polynomial and the Polynomial Γ_G

This section and to make the paper more self-contained is devoted to recall the definition of the Jones polynomial and review some of its basic properties needed in the sequel. In particular, we shall describe how the Jones polynomial of a link can be calculated using the spanning tree expansion of any Tait graph associated with a link diagram. The reader is refereed to [17] for more details.

Definition 2.1. The Kauffman bracket polynomial is a function from the set of unoriented link diagrams in the oriented plane to the ring of Laurent polynomials with integer coefficients in an indeterminate A. It maps a link L to $\langle L\rangle \in \mathbb{Z}[A^{-1}, A]$ and is uniquely determined by the following relations:

(1) $\langle \bigcirc \rangle = 1$,
(2) $\langle \bigcirc \cup L\rangle = (-A^{-2} - A^{2})\langle L\rangle$,
(3) $\langle L\rangle = A\langle L_0\rangle + A^{-1}\langle L_1\rangle$,

where $\bigcirc$ denotes the unknot and $L, L_0,$ and L_1 represent three unoriented links which are identical except in a small region where they look as in Figure 1.

Given an oriented link diagram L, let $x(L)$ denote the number of negative crossings and let $y(L)$ denote the number of positive crossings in L, see Figure 2. The writhe of L is defined as the integer $w(L) = y(L) - x(L)$.

Definition 2.2. The Jones polynomial $V_L(t)$ of an oriented link L is the Laurent polynomial in $t^{1/2}$ with integer coefficients defined by

$$V_L(t) = ((-A)^{-3w(L)}\langle L\rangle)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where $\langle L\rangle$ denotes the bracket polynomial of the link L with orientation ignored.

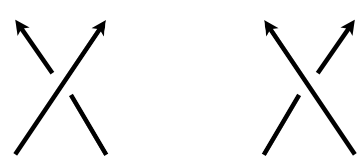

Figure 2. Positive and negative crossings respectively.

Remark 2.3. Let D be a link diagram. A crossing c in D is said to be of type I if the overstrand has positive slope as illustrated by Figure 1. Otherwise, the crossing is said to be of type II. In this paper, our discussions are restricted to the case of type I crossings. Taking the mirror image will enable us to obtain similar results for crossings of type II.

Remark 2.4. (1) If the crossing is positive of type I, then we have $x(L_0) = x(L), y(L_0) = y(L) - 1, x(L_1) = x(L) + e$ and $y(L_1) = y(L) - e - 1$. Therefore, $w(L_0) = w(L) - 1$ and $w(L_1) = w(L) - 2e - 1$, where e denotes the difference between the number of negative crossings in L_1 and the number of negative crossings in L.

(2) If the crossing is negative of type I, then we have $x(L_1) = x(L) - 1, y(L_1) = y(L), x(L_0) = x(L) + e - 1$ and $y(L_0) = y(L) - e$. Therefore, $w(L_1) = w(L) + 1$ and $w(L_0) = w(L) - 2e + 1$, where e denotes the difference between the number of negative crossings in L_0 and the number of negative crossings in L.

Lemma 2.5. *The Jones polynomial of the link L at the crossing c satisfies one of the following skein relations:*

(1) *If c is a positive crossing of type I, then $V_L(t) = -t^{\frac{1}{2}} V_{L_0}(t) - t^{\frac{3e}{2}+1} V_{L_1}(t)$.*
(2) *If c is a negative crossing and of type I, then $V_L(t) = -t^{\frac{3e}{2}-1} V_{L_0}(t) - t^{\frac{-1}{2}} V_{L_1}(t)$.*

where e is as defined in Remark 2.4.

Recall that one can associate a planar signed graph G with any given link diagram. This planar graph, known as the Tait graph, is defined using the checkerboard coloring of the link diagram in the following manner. First, we color the regions of the link diagram in $\mathbb{R}^2$ black and white such that regions that share an arc have different colors. Then, we place a vertex in each black region. The edges of this graph correspond to the crossings of the given link diagram in a way that two vertices are joined by an edge whenever there is a crossing between the two corresponding regions. Moreover, each edge is equipped with a sign according to the scheme in Figure 3. By interchanging black and white regions, we obtain the planar dual graph of G denoted hereafter by G_*. Note here that the signs of the edges of this dual graph are the opposite of their respective dual counterparts in G. It is clear that G is connected if and only if the given link diagram is connected. The discussion in this paper is restricted to connected link diagrams.

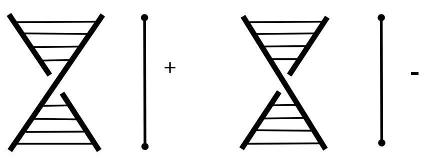

FIGURE 3. Positive and negative edges in the Tait graph.

It is worth mentioning here that the work of Tutte [18] implies that a one-variable polynomial related to the Tutte polynomial χ_G of the Tait graph is equal to the Jones polynomial of the corresponding link up to a phase, namely $V_L(t) = \pm t^r \chi_G(-t, -t^{-1})$ for some half-integer r (see [17]). The work of Thistlewaite [17] implies also that the Jones polynomial can be obtained as a specialization of another polynomial $\Gamma_G(A)$ at $t^{1/2} = A^{-2}$. This polynomial $\Gamma_G(A)$ is defined in terms of the spanning trees of G as follows:

$$\Gamma_G(A) = \sum_{T_i \in \mathbb{T}} w(T_i) = \sum_{T_i \in \mathbb{T}} \left(\prod_{e_j \in G} \mu_{ij} \right),$$

where $\mathbb{T} = \{T_1, T_2, \ldots, T_k\}$ is the set of all spanning trees of G and $e_1, e_2, \ldots, e_m$ are the edges of the graph G with some fixed order. The weight $w(T_i)$ of the spanning tree T_i is the product $\prod_{e_j \in G} \mu_{ij}$ where μ_{ij} denotes the weight corresponding to the state of the edge e_j in the spanning tree T_i. The state of such an edge with respect to the given spanning tree is the internal or external activity of the edge e_j in G with respect to the spanning tree T_i.

This state is one of the eight states that will be denoted by an appropriate word in the shorthand symbols. All possible states of the edge e_j with respect to the spanning tree T_i and its corresponding weights are given in Table 2. In this table, L, D, l, d denote internally active, internally inactive, externally active and externally inactive, respectively, if the edge e_j has positive sign. The other entries $\overline{L}, \overline{D}, \overline{l}, \overline{d}$ are used if the edge e_j has negative sign.

State of e_j in the spanning tree T_i	L	D	l	d	$\overline{\mathrm{L}}$	$\overline{\mathrm{D}}$	$\overline{l}$	$\overline{\mathrm{d}}$
μ_{ij}	$-A^{-3}$	A	$-A^{3}$	A^{-1}	$-A^{3}$	A^{-1}	$-A^{-3}$	A

3. The Main Theorem and its Proof

In this section, we shall prove our main result in this paper which is given by the following theorem.

Theorem 3.1. *If L is a prime quasi-alternating link that is not a $(2,n)$-torus link, then the Jones polynomial $V_L(t)$ has no gap.*

For the rest of the paper, we introduce the notion of simple cycle and simple path in any planar graph. A *simple cycle* of a planar graph G is a cycle that encloses exactly one region in the plane. A common path between two simple cycles is called a *simple path*. In general, any spanning tree of a graph G can be described in terms of the simple cycle decomposition of the graph. In particular, if G is decomposed as non-disjoint union of simple cycles $s_1, s_2, \ldots, s_m$, then any spanning tree T is simply equal to $(s_1 - e_1) \cup (s_2 - e_2) \cup \ldots \cup (s_m - e_m)$, where no two distinct edges e_i and e_j belong to the same simple path for any $1 \leq i \neq j \leq m$.

Throughout the rest of this paper, unless otherwise specified, G denotes the Tait graph of the quasi-alternating link diagram of the prime link L. Using the second Reidemeister move, we can assume that each simple path in G consists of edges of the same sign. It can be easily seen that the quasi-alternating diagram can be assumed to be irreducible and connected. Under this assumption and the fact that L is prime, we can suppose that G is non-separable, connected and has no loops or isthmuses as they correspond to removable crossings. Obviously, the Tait graphs of the links L_0 and L_1 can be obtained from the graph G by deleting and contracting the edge which corresponds to the crossing c, respectively. These graphs are denoted hereafter by G_0 and G_1, respectively.

Remark 3.2. According to [17], the polynomial $\Gamma_G(A)$ satisfies the skein relation $\Gamma_G(A) = A^{\epsilon}\Gamma_{G_0}(A) + A^{-\epsilon}\Gamma_{G_1}(A)$, where $\epsilon = \pm 1$ is the sign of the deleted-contracted edge.

Lemma 3.3. *In this settings, there is no cancellation between the terms of $A^{\epsilon}\Gamma_{G_0}(A)$ and the terms of $A^{-\epsilon}\Gamma_{G_1}(A)$ in the skein relation of the polynomial $\Gamma_G(A)$.*

Proof. The proof is straightforward using the following facts:

- The polynomials $A^{\epsilon}\Gamma_{G_0}(A)$ and $A^{-\epsilon}\Gamma_{G_1}(A)$ have only monomials of degrees congruent modulo four.
- The polynomials $A^{\epsilon}\Gamma_{G_0}(A)$, $A^{-\epsilon}\Gamma_{G_1}(A)$ and $\Gamma_G(A)$ are alternating in the sense that if two monomials have degrees congruent modulo eight then their nonzero coefficients are of the same sign.
- $\det(L) = |\Gamma_G(e^{\frac{\pi i}{4}})|$.
- The link L is quasi-alternating at the crossing c, so $\det(L) = \det(L_0) + \det(L_1)$.

□

Proposition 3.4. *Let L be a link and c be a crossing of this link consisting of two arcs of two different components such that one of the polynomial $A\langle L_0\rangle$ and $A^{-1}\langle L_1\rangle$ does not consist of only one monomial. Then the gap, if it exists, between the polynomials $A\langle L_0\rangle$ and $A^{-1}\langle L_1\rangle$ is of length three.*

Proof. We assume that c is a crossing between the two components L_1 and L_2 of the link L. We use the second principle of induction on the number of positive crossings between the components L_1 and L_2 for some fixed orientation of the link L. Without loss of generality and by choosing the appropriate orientations on the components L_1 and L_2, we can assume that the crossing c is positive in the link L. The fact that the Jones polynomials of the same link with two different orientations are related by some phase gives us the freedom to choose the orientations on the two components L_1 and L_2 without affecting the length of the gap.

From the induction hypothesis, the result holds for the link K that is obtained from L by switching the crossing c since the number of positive crossings between the components L_1 and L_2 is smaller than the one in the link L and the two polynomials $A^{-1}\langle K_1\rangle$ and $A\langle K_0\rangle$ do not consist of only one monomial as a consequence of the assumption on the link L. In particular, the gap if it exists between the polynomials $A^{-1}\langle K_1\rangle$ and $A\langle K_0\rangle$ is of length at most three. Now the result follows directly if there is no gap between the polynomials $A^{-1}\langle K_1\rangle$ and $A\langle K_0\rangle$ or if $\operatorname{mindeg}(A\langle K_0\rangle) > \operatorname{maxdeg}(A^{-1}\langle K_1\rangle)$ noting that the links K_0 and L_1 are identical and the links K_1 and L_0 are also identical. Thus we can assume that there is a gap of length three between the polynomials $A^{-1}\langle K_1\rangle$ and $A\langle K_0\rangle$ with $\operatorname{mindeg}(A^{-1}\langle K_1\rangle) > \operatorname{maxdeg}(A\langle K_0\rangle)$.

Now we use the skein relation in Lemma 2.5 at the crossing c to evaluate $V_L(t)$ and $V_K(t)$. The assumption $\operatorname{mindeg}(A^{-1}\langle K_1\rangle) > \operatorname{maxdeg}(A\langle K_0\rangle)$ is equivalent to $\operatorname{mindeg}(A^{-1}\langle L_0\rangle) > \operatorname{maxdeg}(A\langle L_1\rangle)$. In this case, the gap between the polynomials $-t^{\frac{1}{2}}V_{L_0}(t)$ and $-t^{\frac{3e}{2}+1}V_{L_1}(t)$ in the link L is of length $(\operatorname{mindeg}(V_{L_0}(t))+\frac{1}{2})-(\operatorname{maxdeg}(V_{L_1}(t))+\frac{3e}{2}+1)-1 = \operatorname{mindeg}(V_{L_0}(t))-\operatorname{maxdeg}(V_{L_1}(t))-\frac{3e}{2}-\frac{3}{2}$ and the gap between the polynomials $-t^{\frac{-1}{2}}V_{L_0}(t)$ and $-t^{\frac{3e}{2}-1}V_{L_1}(t)$ in the link K is of length $(\operatorname{mindeg}(V_{L_0}(t))-\frac{1}{2})-(\operatorname{maxdeg}(V_{L_1}(t))+\frac{3e}{2}-1)-1 = \operatorname{mindeg}(V_{L_0}(t))-\operatorname{maxdeg}(V_{L_1}(t)) - \frac{3e}{2} - \frac{1}{2}$. Thus the result follows since the length of the gap in L is smaller than the length of the gap in the link K.

□

Remark 3.5.
(1) Proposition 3.4 generalizes the result of the first and third authors [14, Prop. 3.15]. It also fixes some typos that appeared in the proof of that proposition.
(2) If we combine the result of Proposition 3.4 with the result of Corollary 3.12 of [14], we conclude that the gap, if it exists, between the polynomials $A\langle L_0\rangle$ and $A^{-1}\langle L_1\rangle$

is of length three or seven. Moreover, a gap of length seven only occurs in the case that the link has breadth of the Jones polynomial equal to two.

Lemma 3.6. *If the connected sum of two links is quasi-alternating, then each component is also quasi-alternating.*

Proof. Let $L = J\#K$ be a connected sum of two links J and K. Assume that L is quasi-alternating. We shall show that both J and K are quasi-alternating. We apply induction on the determinant of L. It is clear that the result holds if $\det(L) = 1$ since the only quasi-alternating link of determinant one is the unknot. Now, if we smooth L at a crossing where it is quasi-alternating, then we obtain L_0 and L_1 that are quasi-alternating. It is easy to see that either $L_0 = J_0\#K$ and $L_1 = J_1\#K$ or $L_0 = J\#K_0$ and $L_1 = J\#K_1$. Since in both cases $\det(L_0)$ and $\det(L_1)$ are less than $\det(L)$, the result follows by applying the induction hypothesis on L_0 and L_1. □

Remark 3.7. Let $f_1(t)$ and $f_2(t)$ be any two alternating polynomials each of which has only one gap of length one. Then, it can be easily seen that their product $f_1(t)f_2(t)$ has more than one gap of length one only if both of them are of breadth equal to 2. Otherwise, there will be either no gap or just a single gap of length one. Moreover, if one of the polynomials has no gap then the product will have no gap.

Lemma 3.8. *Let L be a quasi-alternating link of* $\text{breadth}(V_L(t)) \leq 3$*, then L is the unknot, the Hopf link or the trefoil knot.*

Proof. We prove this result by induction on the determinant of the link L. It is known that the result holds if the determinant is one since the only quasi-alternating link of determinant one is the unknot. Now we assume that the result holds for any quasi-alternating link of determinant less than the determinant of the link L. This means that the result holds for the quasi-alternating links L_0 and L_1.

It is easy to see that no cancellation occurs when calculating the Jones polynomial using the formulas in Lemma 2.5 if the link L is quasi-alternating at the crossing c. This is a consequence of the facts that the polynomials $V_{L_0}(t)$ and $V_{L_1}(t)$ are alternating, $\det(L) = \det(L_0) + \det(L_1)$ and $\det(L) = |V_L(-1)|$. Thus, according to these facts, we should have $\text{breadth}(V_{L_0}(t)) \leq 3$ and $\text{breadth}(V_{L_1}(t)) \leq 3$. Otherwise, we get $\text{breadth}(V_L(t)) > 3$. Notice that the set of monomials of nonzero coefficients in $V_L(t)$ and $V_{L_0}(t)$ must be either equal, one of them is a subset of the other or disjoint with the maximum difference between their degrees less than or equal to three. By the induction hypothesis, L_0 is either the unknot, the Hopf link or the trefoil knot and the same applies for the link L_1. Therefore, we obtain $\det(L) = \det(L_0) + \det(L_1) \leq 6$. By the classification of quasi-alternating links with small determinants [3, 5], there are only finitely many quasi-alternating links of determinant less than or equal to six. Among these links, only the unknot, the Hopf link and the trefoil knot have breadth less than or equal to three. □

Based on the above discussion, we are now ready to prove Theorem 3.1.

Proof of the Main Theorem. It is easy to see that the powers of A that appear in the polynomial $\Gamma_G(A)$ are congruent modulo 4 as a result that $V_L(t)$ is obtained from $\Gamma_G(A)$ by the normalization $t = A^{-4}$. Consequently, we need to show that the polynomial $\Gamma_G(A)$ has no gap of length bigger than three to prove the claim in Theorem 3.1, whenever L is not a $(2,n)$-torus link. We can assume that G and its dual consist of more than one simple cycle. Otherwise the given link is a $(2,n)$-torus link. Furthermore, we can assume that the breadth of $\Gamma_G(A)$ is bigger than eight, otherwise the result follows directly as a consequence of Lemma 3.8. This last assumption also implies that one of the two polynomials $\Gamma_{G_0}(A)$ and $\Gamma_{G_1}(A)$ consists of more than one monomial.

We apply double induction on the determinant or on the minimal number of deleted edges required to obtain a spanning tree from the given graph G or its dual. In case the determinant is equal to one or the minimal number of deleted edges required to obtain a spanning tree is equal to one, the result follows directly since the link is either the unknot or is a $(2,n)$-torus link.

Now, we assume that the result holds for any quasi-alternating link where the minimal number of deleted edges required to obtain a spanning tree is less than the minimal number of deleted edges required to obtain a spanning tree of the graph G of the link L or the determinant is less than the determinant of the link L. The induction hypothesis implies that the result holds for the quasi-alternating links L_0 and L_1 if they are prime simply since $\det(L_0) < \det(L)$ and $\det(L_1) < \det(L)$. Now we have the following cases to consider:

(1) As a result of [4, Lemma A], we can assume that at least one of the two links L_0 or L_1 is prime. Without loss of generality and by taking the mirror image if required, we can assume that L_1 is prime and L_0 is not prime. Now we consider two subcases:
   (a) If L_1 is not a $(2,n)$-torus link, then as a result of L_0 being not prime, we conclude that L_0 consists of a connected sum of two prime links each of which is quasi-alternating according to Lemma 3.6. In the case that one of these two components is not a $(2,n)$-torus link, then the result follows directly as a consequence of Remark 3.7, Lemma 3.3 and the induction hypothesis on the components of the link L_0 and Remark 3.5(2). In the case of the two components are the $(2,n)$-torus links, then the minimal number of deleted edges required to obtain a spanning tree of the graph G of the link L is two and the fact that the minimal number of deleted edges required to obtain a spanning tree from G_0 is equal to that of G. In this case the result follows since the quasi-alternating link is a 3-strand pretzel link. In such a case, the link satisfies the required property as a result of [2, Theorem 3.10].
   (b) If L_1 is a $(2,n)$-torus link, then the minimal number of deleted edges required to obtain a spanning tree of the graph G of the link L is two and the fact that the minimal number of deleted edges required to obtain a spanning tree from G_1 is equal to that of G minus one. In this case the result follows since the quasi-alternating link is a 3-strand pretzel link. In such a case, the link satisfies the required property as a result of [2, Theorem 3.10].

(2) If L_0 and L_1 are both prime and $\Gamma_{G_0}(A)$ and $\Gamma_{G_1}(A)$ have no gap of length bigger than three, then the result follows directly as a consequence of Remark 3.5(2) and Lemma 3.3.

(3) If L_0 and L_1 are both prime and both $\Gamma_{G_0}(A)$ and $\Gamma_{G_1}(A)$ have a gap of length seven, then both links L_0 and L_1 have minimal number of deleted edges required to obtain a spanning tree from the graphs G_0 and G_1 is one. This implies, if such a case exists, that the minimal number of deleted edges required to obtain a spanning tree from the graph G is at most one and hence the result follows since such a link will be a $(2, n)$-torus link.
(4) If L_0 and L_1 are both prime and one of $\Gamma_{G_0}(A)$ or $\Gamma_{G_1}(A)$ has a gap of length seven but not both. In this case, we can assume that the minimal number of deleted edges required to obtain a spanning tree of the graph G_1 is one and this implies that the minimal number of deleted edges required to obtain a spanning tree of the graph G_0 is two. Therefore, this implies that the minimal number of deleted edges required to obtain a spanning tree of the graph G is two and in this case the result follows since the quasi-alternating link is a 3-strand pretzel link. In such a case, the link satisfies the required property as a result of [2, Theorem 3.10].

□

4. Applications of the Main Theorem

First, we note that Theorem 3.1 is an extension of the well-known result of Thistlethwaite on the Jones polynomial of alternating links [17, Theorem 1(iv)]. It also establishes [2, Conjecture 2.3]. We shall now discuss more applications and consequences of the main result.

In [13], it was conjectured that the breadth of the Jones polynomial of any quasi-alternating link is less than or equal to its determinant. The following corollary shows that this conjecture holds.

Corollary 4.1. *Let L be a quasi-alternating link, then* $\mathrm{breadth}(V_L(t)) \leq \det(L)$. *Moreover, the equality holds only if L is a $(2, n)$-torus link or if it is a connected sum of Hopf links.*

Proof. Let us first assume that L is prime quasi-alternating. In this case and according to Theorem 3.1, the Jones polynomial $V_L(t)$ consists of $k + 1$ distinct consecutive monomials of coefficients $a_1, a_2, \ldots, a_{k+1}$ such that at most one of them is zero. This implies that $\mathrm{breadth}(V_L(t)) = k$. Now the result follows directly since the Jones polynomial is alternating which implies that $\det(L) = |V_L(-1)| = |a_1| + |a_2| + \ldots + |a_{k+1}| \geq k = \mathrm{breadth}(V_L(t))$. If L is not prime, then Lemma 3.6 implies that the components of the connected sum are quasi-alternating. The inequality holds since the breadth of the Jones polynomial is additive while the determinant is multiplicative under the connected sum operation. □

Remark 4.2. Corollary 4.1 implies that there are only finitely many values of the breadth of the Jones polynomial of quasi-alternating links of a given determinant. This result can be also obtained as a consequence of [12, Theorem 1.3]. Also this property in Corollary 4.1, which is known to be true for alternating links, represents a simple obstruction criteria for a link to be quasi-alternating.

In addition, Theorem 3.1 can be used to establish [14, Conjecture 4.12].

Corollary 4.3. *Let L be a prime quasi-alternating link that is not a $(2, n)$-torus link, then the differential grading in Khovanov homology of L has no gap.*

Proof. The claim follows directly from Theorem 3.1 and [14, Proposition 3.10]. □

Corollary 4.4. *Let L be a quasi-alternating link. Then the Jones polynomial of L has more than one gap if and only if L is a connected sum of Hopf links.*

Proof. The result follows directly from the fact that the Jones polynomial is multiplicative under the connected sum together with the results in Theorem 3.1, Lemma 3.6, Lemma 3.8 and Remark 3.7. □

We include the following example that explains how the obstruction obtained from Theorem 3.1 can be used to show that a given link is not quasi-alternating.

Example 4.5. The Jones polynomial of the Kanenobu knot $K(p,q)$ defined in [7] is given by

$$\begin{aligned} V_{K(p,q)}(t) =&(-1)^{p+q}(t^{p+q-4} - 2t^{p+q-3} + 3t^{p+q-2} - 4t^{p+q-1} + 4t^{p+q} - 4t^{p+q+1} + \\ &3t^{p+q+2} - 2t^{p+q+3} + t^{p+q+4}) + 1. \end{aligned}$$

According to [13, Corollary 3.3], all Kanenobu knots with $|p| + |q| \geq 19$ are not quasi-alternating. This result can be sharpened using the obstruction in Theorem 3.1 by stating that $K(p,q)$ are not quasi-alternating whenever $|p| + |q| \geq 19$ or $|p + q| > 6$.

Finally, we enclose this paper with the following conjecture which is motivated by the result in Lemma 3.8 and the fact that there are only finitely many alternating links with a given breadth. This last fact is a consequence of the fact that the breadth of the Jones polynomial is equal to the crossing number of any alternating link [8, 9, 17].

Conjecture 4.6. *There are only finitely many quasi-alternating links with a given breadth.*

A positive solution of this conjecture not only implies a positive solution of Conjecture 3.8 in [3] based on the result of Corollary 4.1, but it also generalizes this property from the class of alternating links to the class of quasi-alternating links.

At the end, it is worth pointing out that Conjecture 1.1 in [15] suggests the crossing number as a lower bound of the determinant for any quasi-alternating link. This lower bound is sharper than the one introduced in Corollary 4.1 as a result of the known fact that the crossing number is an upper bound of the breadth of the Jones polynomial of any link. This conjecture has been verified for many classes of quasi-alternating links, but to the best of our knowledge, the conjecture is still open. It is not too hard to see that such conjecture implies Conjecture 3.8 in [3] and Conjecture 4.6 above.

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Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan, 21163
Email address: qazaqzeh@yu.edu.jo
URL: http://faculty.yu.edu.jo/qazaqzeh

Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan, 21163
Email address: al-rhayyel@yu.edu.jo
URL: https://faculty.yu.edu.jo/Alrhayyel

Department of Mathematical Sciences, College of Science, UAE University, 15551 Al Ain, U.A.E.
Email address: nafaachbili@uaeu.ac.ae
URL: http://faculty.uaeu.ac.ae/nafaachbili