MAXIMAL STRICT SOLUTIONS FOR SOME QUASILINEAR PARABOLIC SYSTEM OF HONEYCOMB CONSTRUCTION MODEL

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Abstract. This paper treats the initial-boundary value problem for a quasilinear parabolic system in a two-dimensional region presented by Belić, Škarka, Deneubourg and Lax in order to describe the construction process of parallel honeycombs in a beehive. After constructing the local strict solutions by using the theory of abstract parabolic equations, we will define the maximal strict solutions. Unfortunately, we cannot give any general sufficient conditions on the parameters or initial functions for global existence, but we can investigate asymptotic behaviors of the maximal solutions as $t \to T_{\max}$. From numerical computations, we already have a number of examples which suggest the blowup of maximal solutions (i.e., $T_{\max} < \infty$); at the end of the paper, we shall present one such numerical example.

Mathematical Subject Classifications: 35K90, 92C15.

1. Introduction

We consider the initial-boundary value problem for a quasilinear parabolic system

$$
(1.1) \qquad \begin{cases}
\frac{\partial u}{\partial t} = a\Delta u - \mu u \frac{\partial^2 \rho}{\partial x^2} + cuv(u-v) + d - fu, & \Omega \times (0,\infty), \\
\frac{\partial v}{\partial t} = a\Delta v - \mu v \frac{\partial^2 \rho}{\partial y^2} + cuv(v-u) + d - fv, & \Omega \times (0,\infty), \\
\frac{\partial \rho}{\partial t} = b(u+v)\Delta \rho + \nu(u+v) - g\rho, & \Omega \times (0,\infty), \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial \rho}{\partial n} = 0, & \partial\Omega \times (0,\infty), \\
u(x,y,0) = u_0(x,y), \ \ v(x,y,0) = v_0(x,y), \ \ \rho(x,y,0) = \rho_0(x,y), & \Omega,
\end{cases}
$$

in a two-dimensional bounded domain Ω.

This system was presented by Belić-Deneubourg-Lax-Škarka in the papers [1, 6] in order to describe the initial stage of the honeycomb construction of *Apis mellifera*. It is well known that honeybees have a strong tendency to construct parallel and equidistant combs in a beehive. In order to understand this remarkable phenomenon theoretically, the authors of [1, 6] introduced the parabolic system (1.1). Their modeling focuses on the "self-organization" of the social insects (see [2, 4]). In their work, two principle mechanisms were assumed to be active. The first one is cooperative interaction between bees and wax. The worker bees are attracted to the already deposited wax. However, some deposits grow faster than others, some are abandoned, and some fluctuations become amplified to form elongated oval deposits. The second mechanism is competitive interaction among worker bees. The worker bees are divided into groups. Worker bees belonging to the same group orient themselves in the same direction, and deposit or bore wax cooperatively. Contrarily, there is competition between differently oriented groups of bees. For simplicity, it was supposed in [1, 6] that the worker bees were divided into only two major groups, one being bees parallel to the $x0z$-plane and the other being those parallel to the $y0z$-plane.

In the model (1.1), the domain Ω in the (x, y)-plane represents the base of a beehive under which honeybees construct their combs. (In (1.1), the z-axis is pointing downward.) The unknown functions $u = u(x, y, t)$ and $v = v(x, y, t)$ give the average density of bees parallel to the $x0z$-plane and the average density of bees parallel to $y0z$-plane, respectively, at $(x, y, t) \in \Omega \times [0, \infty)$; $\rho = \rho(x, y, t)$ denotes the quantity of wax deposited by the "waxer" bees at $(x, y, t) \in \Omega \times [0, \infty)$. The terms $cuv(u - v)$ and $cuv(v - u)$ represent the competition between the two major groups of bees. The constant term d denotes the flux of differently oriented bees which come into the considered group, and the terms fu and fv correspond to the losses of bees due to leaving and changing orientation, respectively. The medium interaction terms $-\mu u \frac{\partial^2 \rho}{\partial x^2}$ and $-\mu v \frac{\partial^2 \rho}{\partial y^2}$ describe the attraction of bees to the wax and the term $\nu(u + v)$ describes the deposition of wax by bees. The term $-g\rho$ describes the removal and the fall of wax. Finally, the Laplacian terms $a\Delta u$ and $a\Delta v$ represent the "diffusive" imitation of bees, capturing the bees' tendency to take the same orientation as that of the bees nearby; the Laplacian term $b(u + v)\Delta\rho$ represents the deposit of wax due to imitation.

For further details of the model and related experimental results, see the original two papers [1, 6] and the references therein.

In this paper, we assume that $\Omega \subset \mathbb{R}^2$ is either a rectangle $(0, \ell_x) \times (0, \ell_y)$ $(0 < \ell_x,\ \ell_y < \infty)$ or a bounded $\mathcal{C}^3$ domain. All the parameters a, b, c, d, f, g, μ and ν in (1.1) are positive constants. We impose on the unknown functions u, v and ρ the homogeneous Neumann boundary conditions, i.e., $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial \rho}{\partial n} = 0$ on $\partial\Omega$, $n = n(x, y)$ being the outer normal vector at boundary point $(x, y) \in \partial\Omega$. For the initial functions, we assume the following conditions:

$$u_0,\ v_0,\ \rho_0 \in H^{1+\sigma}(\Omega) \subset \mathcal{C}(\overline{\Omega}), \tag{1.2}$$

where $\sigma > 0$ is any positive exponent. In what follows, we will fix σ such that $0 < \sigma < \frac{1}{2}$. In addition, u_0, v_0 and ρ_0 satisfy the following positivity conditions:

$$\min_{(x,y)\in\overline{\Omega}} u_0(x, y) > 0, \quad \min_{(x,y)\in\overline{\Omega}} v_0(x, y) > 0 \quad \text{and} \quad \min_{(x,y)\in\overline{\Omega}} \rho_0(x, y) \geq 0. \tag{1.3}$$

The first objective of this paper is to construct the local strict solution to (1.1) for the initial functions satisfying (1.2)-(1.3). As explained above, the diffusion coefficient for the equation ρ is given by $b(u + v)$, which means that the coefficient depends on the unknown functions u and v. Meanwhile, the equations for u and v include the interaction terms $-\mu u \frac{\partial^2 \rho}{\partial x^2}$ and $-\mu v \frac{\partial^2 \rho}{\partial y^2}$, respectively, for the deposited wax, which means that these equations include nonlinear terms that depend on the second-order partial derivatives with respect to the unknown function ρ. Then, the equations of (1.1) represent a strongly coupled diffusion system which is classified as a quasilinear parabolic system in the theory of nonlinear partial differential equations. Thus, even constructing the local solutions is not a so easy task. However, we can appeal to the theory of abstract parabolic evolution equations (see [7, 8]). More precisely, we shall use [8, Theorem 5.6], in which the existence and uniqueness results are proved for abstract parabolic equations under a general framework. Under suitable settings, it is possible to verify that this theorem is actually applicable to the problem (1.1).

We are next interested in the question of when (1.1) possesses a global strict solution. Unfortunately, we do not yet know any general conditions on the parameters a, b, c, d, f, g, μ and ν or the initial functions u_0, v_0 and ρ_0 which can guarantee the global existence of solutions. On the contrary, we have found many numerical examples which suggest the local strict solutions blow up. One such example will be presented in the last section of this paper.

In view of (1.2)-(1.3), we know that, if the local strict solution blows up at some time $T_{\max} < \infty$, then at least one of the following phenomena has occurred:

$$\overline{\lim_{t \to T_{\max}}} [\|u(t)\|_{H^{1+\sigma}} + \|v(t)\|_{H^{1+\sigma}} + \|\rho(t)\|_{H^{1+\sigma}}] = \infty,$$

$$\underline{\lim_{t \to T_{\max}}} \min_{(x,y)\in\overline{\Omega}} u(x,y,t) = 0 \quad \text{or} \quad \underline{\lim_{t \to T_{\max}}} \min_{(x,y)\in\overline{\Omega}} v(x,y,t) = 0.$$

Actually, we can give more precise information on the behavior of $u(x,y,t)$, $v(x,y,t)$ and $\rho(x,y,t)$ as $t \to T_{\max}$. The second objective of this paper is then to investigate the asymptotic behavior of the maximal strict solution of (1.1) as $t \to T_{\max} < \infty$.

As explained, we cannot expect in general that (1.1) admits a global strict solution, but this does not at all mean that the model (1.1) does not give a description of honeycomb patterns. On the contrary, we have a number of numerical examples which re-create honeycomb patterns during the time interval $(0, T_{\max})$. Some of these were already included previously in [9], but a full paper on these examples will be published elsewhere.

2. Notion and Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a rectangle $(0, \ell_x) \times (0, \ell_y)$ or a bounded $\mathcal{C}^3$ domain. This section is devoted to listing the basic materials of Sobolev spaces in Ω and the basic properties of sectorial operators in $L_2(\Omega)$ which will be needed in this paper. For some of these which may not be so familiar, proofs will be given.

Sobolev Spaces. For $1 \le p \le \infty$, $L_p(\Omega)$ denotes the usual complex L_p-space equipped with the L_p-norm $\|\cdot\|_{L_p}$.

For $1 < p < \infty$ and $m = 0, 1, 2, \ldots$, $H_p^m(\Omega)$ denotes the space of functions $u \in L_p(\Omega)$ whose partial derivatives $\frac{\partial^{i+j} u}{\partial x^i \partial y^j}$ for all the orders $0 \le i + j \le m$ belong to $L_p(\Omega)$, $H_p^m(\Omega)$ being equipped with the norm

$$\|u\|_{H_p^m} = \left(\sum_{0 \le i+j \le m} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\|_{L_p}^p \right)^{\frac{1}{p}}.$$

These definitions are extended for the fractional exponents s, namely, for $1 < p < \infty$ and $0 \le s < \infty$, $H_p^s(\Omega)$ is defined in a reasonable way; see [8, Section 1.11]. For each $1 < p < \infty$, the family $H_p^s(\Omega)$, $(0 \le s < \infty)$ enjoys the interpolation property

$$[H_p^{s_0}(\Omega), H_p^{s_1}(\Omega)]_\theta = H_p^s(\Omega) \qquad \text{(with norm equivalence)} \tag{2.1}$$

for $0 \le s_0 < s < s_1 < \infty$ and $s = (1-\theta)s_0 + \theta s_1$. When $1 < p < \infty$, $p \ne 2$, $H_p^s(\Omega)$ are Banach spaces. When $p = 2$, $H_2^s(\Omega)$ are Hilbert spaces. The spaces $H_2^s(\Omega)$ are simply denoted by $H^s(\Omega)$.

Regarding embeddings of $H_p^s(\Omega)$ into $L_q(\Omega)$, the following properties are known. If $0 < s < 1$, then $H^s(\Omega) \subset L_p(\Omega)$ for $p = \frac{2}{1-s}$ with continuous embedding

$$\|u\|_{L_p} \le C_s \|u\|_{H^s}, \qquad u \in H^s(\Omega). \tag{2.2}$$

When $s = 1$, it holds true that $H^1(\Omega) \subset L_p(\Omega)$ for any $2 \le p < \infty$ with the inequality

$$\|u\|_{L_p} \le C_p \|u\|_{H^1}^{1-2/p} \|u\|_{L_2}^{2/p}, \qquad u \in H^1(\Omega). \tag{2.3}$$

If $s > 1$, then $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|u\|_{\mathcal{C}} \le C_s \|u\|_{H^s}, \qquad u \in H^s(\Omega). \tag{2.4}$$

If $2 < p < \infty$, then $H^1_p(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|u\|_{\mathcal{C}} \le C_p \|u\|_{H^1_p}, \qquad u \in H^1_p(\Omega). \tag{2.5}$$

Furthermore, for $s > 1$ (resp. $2 < p < \infty$), the space $H^s(\Omega)$ (resp. $H^1_p(\Omega)$) is verified to be a Banach algebra. Namely, if $s > 1$, then $u, v \in H^s(\Omega)$ implies $uv \in H^s(\Omega)$ with the estimate

$$\|uv\|_{H^s} \le C_s \|u\|_{H^s} \|v\|_{H^s}, \qquad u,\, v \in H^s(\Omega). \tag{2.6}$$

Similarly, if $2 < p < \infty$, then $u, v \in H^1_p(\Omega)$ implies $uv \in H^1_p(\Omega)$ with the estimate

$$\|uv\|_{H^1_p} \le C_p \|u\|_{H^1_p} \|v\|_{H^1_p}, \qquad u,\, v \in H^1_p(\Omega). \tag{2.7}$$

Let $a \in H^s(\Omega)$ with $s > 1$. Then, the multiplication $u \mapsto au$ is a bounded linear operator from $H^1(\Omega)$ into itself with the estimate

$$\|au\|_{H^1} \le C_s \|a\|_{H^s} \|u\|_{H^1}, \qquad u \in H^1(\Omega). \tag{2.8}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise smooth function with $\chi' \in L_\infty(\mathbb{R})$ and $\chi(0) = 0$. Then, $w \mapsto \chi(w)$ is an operator from $H^1(\Omega; \mathbb{R})$ into itself with the property

$$\nabla \chi(w) = \begin{cases} \chi'(w)\nabla w & \text{if} \quad w(x,y) \notin \chi_s, \\ 0 & \text{if} \quad w(x,y) \in \chi_s, \end{cases} \tag{2.9}$$

where χ_s denotes the set of singular points of χ.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then, for $s > 1$, $w \mapsto \chi(w)$ is an operator from $H^s(\Omega; \mathbb{R})$ into itself

(2.10) which is a bounded and locally Lipschitz continuous mapping.

Sectorial Operators. For the Laplace operator $-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ in Ω, let us review how to realize the operator as a linear operator of $L_2(\Omega)$ by equipping it with the homogeneous Neumann boundary conditions on $\partial\Omega$. Consider a sesquilinear form

$$\widetilde{a}(u, v) = \iint_\Omega \nabla u \cdot \nabla \overline{v}\, dxdy, \qquad u,\, v \in H^1(\Omega),$$

on $H^1(\Omega)$. Since for each $u \in H^1(\Omega)$, the correspondence $v \mapsto \widetilde{a}(u, v)$ is a continuous anti-linear functional on $H^1(\Omega)$, there is an element $\widetilde{\Lambda} u \in H^1(\Omega)'$ such that $\widetilde{a}(u, v) = \langle \widetilde{\Lambda} u, v \rangle_{H^{1\prime} \times H^1}$ for all $v \in H^1(\Omega)$, where $H^1(\Omega)'$ is the dual space of $H^1(\Omega)$ and $\langle \cdot, \cdot \rangle_{H^{1\prime} \times H^1}$ is the duality product of $H^1(\Omega)'$ and $H^1(\Omega)$. This relation then defines a bounded linear operator $\widetilde{\Lambda}$ from $H^1(\Omega)$ into $H^1(\Omega)'$.

Identifying $L_2(\Omega)$ and its dual space $L_2(\Omega)'$, we here introduce the triplet

$$H^1(\Omega) \subset L_2(\Omega) \approx L_2(\Omega)' \subset H^1(\Omega)'$$

with dense and continuous embeddings. As is well known, the compatibility property $\langle u, v \rangle_{H^{1\prime} \times H^1} = (u, v)_{L_2}$ holds for $u \in L_2(\Omega)$ and $v \in H^1(\Omega)$. In view of this property, consider for any $\varepsilon > 0$ the sesquilinear form $\widetilde{a}_\varepsilon(u, v) = \widetilde{a}(u, v) + \varepsilon (u, v)_{L_2}$ on $H^1(\Omega)$; then, it can be shown that $\widetilde{a}_\varepsilon(u, v) = \langle (\widetilde{\Lambda} + \varepsilon) u, v \rangle_{H^{1\prime} \times H^1}$ for $u,\, v \in H^1(\Omega)$. Since $\widetilde{a}_\varepsilon(u, v)$ is continuous and coercive on $H^1(\Omega)$, the Lax-Milgram theorem can be applied to $\widetilde{a}_\varepsilon(u,v)$ to conclude that $\widetilde{\Lambda}+\varepsilon$ is actually an isomorphism from $H^1(\Omega)$ onto $H^1(\Omega)'$.

We now define the part Λ of $\widetilde{\Lambda}$ in $L_2(\Omega)$ by $\mathcal{D}(\Lambda) = \{u \in H^1(\Omega);\ \widetilde{\Lambda}u \in L_2(\Omega)\}$ (namely, $u \in \mathcal{D}(\Lambda)$ if and only if $v \mapsto \widetilde{a}(u,v)$ is continuous in v with respect to the L_2-topology) and $\Lambda u = \widetilde{\Lambda}u$, i.e., $\widetilde{a}(u,v) = (\Lambda u, v)_{L_2}$ for $u \in \mathcal{D}(\Lambda)$ and $v \in H^1(\Omega)$. Then, this proves Λ is a positive self-adjoint operator of $L_2(\Omega)$. Furthermore, when Ω is convex or in the class $\mathcal{C}^2$ (which is of course the case under our assumption on Ω), it is known that the domain $\mathcal{D}(\Lambda)$ can be characterized by

$$\mathcal{D}(\Lambda) = H^2_N(\Omega) \equiv \left\{u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\right\}. \tag{2.11}$$

(see [8, Theorems 2.6-2.7]). Clearly, for $u \in H^2_N(\Omega)$, we have $\widetilde{a}(u,v) = (-\Delta u, v)_{L_2}$ and hence $\Lambda u = -\Delta u$ for $u \in \mathcal{D}(\Lambda)$.

In the above sense, the positive self-adjoint operator Λ is considered a realization of $-\Delta$ in $L_2(\Omega)$ under the boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Further, $\widetilde{\Lambda}$ is considered a realization of $-\Delta$ in $H^1(\Omega)'$ under the same boundary conditions on $\partial\Omega$, but in the generalized sense. Thereby, it is reasonable to write $(\nabla u, \nabla v)_{L_2}$ as

$$(\nabla u, \nabla v)_{L_2} = (\Lambda u, u)_{L_2} = (-\Delta u, v)_{L_2}, \qquad u \in H^2_N(\Omega),\ v \in H^1(\Omega), \tag{2.12}$$

$$(\nabla u, \nabla v)_{L_2} = \langle \widetilde{\Lambda}u, v\rangle_{H^{1\prime}\times H^1} = \langle -\Delta u, v\rangle_{H^{1\prime}\times H^1}, \qquad u,\ v \in H^1(\Omega). \tag{2.13}$$

Let $a > 0$ and $f > 0$ both be constants and consider the operator $a\Lambda + f$ in $L_2(\Omega)$. As $a\Lambda + f$ is a positive definite self-adjoint operator having domain $\mathcal{D}(a\Lambda + f) = \mathcal{D}(\Lambda) = H^2_N(\Omega)$, we know that $\mathcal{D}([a\Lambda + f]^\theta) = [L_2(\Omega), H^2_N(\Omega)]_\theta$ for all $0 \le \theta \le 1$. From this, the domains of $[a\Lambda + f]^\theta$ are given by

$$\mathcal{D}([a\Lambda + f]^\theta) = \begin{cases} H^{2\theta}(\Omega) & \text{when } 0 \le \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega) & \text{when } \frac{3}{4} < \theta \le 1, \end{cases} \tag{2.14}$$

with norm equivalence, where $H^{2\theta}_N(\Omega) \equiv \{u \in H^{2\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. For the proof, see [8, Theorems 16.7-16.9].

Consider next a Laplace operator of the form $Bu = \hat{b}(x,y)\Lambda u + gu$ in $L_2(\Omega)$, where $\hat{b}(x,y)$ is a positive function of Ω and g is a positive constant.

Proposition 2.1. *Assume that $\hat{b}(x,y) \in L_\infty(\Omega)$ satisfies $\hat{b}(x,y) \ge \delta$ in Ω for some constant $\delta > 0$. Then, B is a sectorial operator of $L_2(\Omega)$ with domain $\mathcal{D}(B) = \mathcal{D}(\Lambda) = H^2_N(\Omega)$ and with angle $\omega_B < \frac{\pi}{2}$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda \le 0$. Noting that $B - \lambda = \hat{b}[\Lambda + (g-\lambda)\hat{b}^{-1}]$, we will first show that $\Lambda + (g-\lambda)\hat{b}^{-1}$ is an isomorphism from $H^2_N(\Omega)$ onto $L_2(\Omega)$.

To this end, introduce the sesquilinear form

$$\widetilde{b}(u,v) = \iint_\Omega \nabla u \cdot \nabla \overline{v}\, dxdy + (g-\lambda)\iint_\Omega \hat{b}(x,y)^{-1} u\overline{v}\, dxdy, \qquad u,\ v \in H^1(\Omega).$$

It is clear that this form is a continuous and coercive form on $H^1(\Omega)$. As the associated linear operator to this form is $\widetilde{\Lambda} + (g-\lambda)\hat{b}^{-1}$, we can conclude that $\widetilde{\Lambda} + (g-\lambda)\hat{b}^{-1}$ is an isomorphism from $H^1(\Omega)$ onto $H^1(\Omega)'$. Furthermore, by the regularity property of Λ, we know that its part in $L_2(\Omega)$, namely $\Lambda + (g-\lambda)\hat{b}^{-1}$ is an isomorphism from $H^2_N(\Omega)$ onto $L_2(\Omega)$.

As $\mathcal{D}(B) = \mathcal{D}(\Lambda) = H^2_N(\Omega)$, it follows that the operator $B - \lambda$ is an isomorphism from $\mathcal{D}(B)$ onto $L_2(\Omega)$.

Second, we will estimate the operator norm of $\Lambda + (g-\lambda)\hat{b}^{-1}$. To begin with, notice the following inequality holds: $\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\}\|u\|^2_{H^1} \le \operatorname{Re}\widetilde{b}(u,u)$. On the other hand, as

$$\widetilde{b}(u,u) = ([-\Delta + (g-\lambda)\hat{b}^{-1}]u, u)_{L_2}, \qquad u \in H^2_N(\Omega),$$

it follows that $\operatorname{Re}\widetilde{b}(u,u) \le \|[-\Delta + (g-\lambda)\hat{b}^{-1}]u\|_{L_2}\|u\|_{L_2}$. Therefore, we observe that

$$\|u\|^2_{H^1} \le [1/\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\}]\, \|[-\Delta + (g-\lambda)\hat{b}^{-1}]u\|_{L_2}\|u\|_{L_2}. \tag{2.15}$$

Next, we can write

$$\begin{aligned}(\lambda\hat{b}^{-1}u, u)_{L_2} &= ([-\Delta + g\hat{b}^{-1}]u, u)_{L_2} - ([-\Delta + (g-\lambda)\hat{b}^{-1}]u, u)_{L_2} \\ &= (\nabla u, \nabla u)_{L_2} + (g\hat{b}^{-1}u, u)_{L_2} - ([-\Delta + (g-\lambda)\hat{b}^{-1}]u, u)_{L_2}.\end{aligned}$$

Then, since $|\lambda|\|\hat{b}\|^{-1}_{L_\infty}\|u\|^2_{L_2} \le |(\lambda\hat{b}^{-1}u, u)_{L_2}|$, it is seen by (2.15) that

$$\begin{aligned}|\lambda|\|\hat{b}\|^{-1}_{L_\infty}\|u\|^2_{L_2} &\le \max\{1, g\delta^{-1}\}\|u\|^2_{H^1} + \|[-\Delta + (g-\lambda)\hat{b}^{-1}]u\|_{L_2}\|u\|_{L_2} \\ &\le [\max\{1, g\delta^{-1}\}/\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\} + 1]\, \|[-\Delta + (g-\lambda)\hat{b}^{-1}]u\|_{L_2}\|u\|_{L_2},\end{aligned}$$

which yields the estimate

$$\begin{aligned}|\lambda|\|\hat{b}\|^{-1}_{L_\infty}\|u\|_{L_2} &\le [\max\{1, g\delta^{-1}\}/\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\} + 1]\, \|[-\Delta + (g-\lambda)\hat{b}^{-1}]u\|_{L_2} \\ &= [\max\{1, g\delta^{-1}\}/\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\} + 1]\, \|[\Lambda + (g-\lambda)\hat{b}^{-1}]u\|_{L_2}, \qquad u \in H^2_N(\Omega).\end{aligned}$$

Consequently, as $\|[\Lambda + (g-\lambda)\hat{b}^{-1}]u\|_{L_2} \le \delta^{-1}\|(B-\lambda)u\|_{L_2}$ for $u \in \mathcal{D}(B) = \mathcal{D}(\Lambda)$, we have as follows:

$$|\lambda|\|u\|_{L_2} \le \|\hat{b}\|_{L_\infty}\delta^{-1}[\max\{1, g\delta^{-1}\}/\min\{1, g\|\hat{b}\|^{-1}_{L_\infty}\} + 1]\|(B-\lambda)u\|_{L_2}, \quad u \in \mathcal{D}(B),$$

which yields the norm estimate of the resolvent $(B-\lambda)^{-1}$ on $L_2(\Omega)$ for $\operatorname{Re}\lambda \le 0$. □

Smoothness Properties. We shall use the following smoothness properties of the self-adjoint $L_2(\Omega)$ operator Λ.

Proposition 2.2. *Let $\Omega = (0, \ell_x) \times (0, \ell_y)$ and let $0 < \theta < \frac{3}{4}$. If $u \in H^2_N(\Omega)$ satisfies $\Lambda u \in H^{2\theta}(\Omega)$, then $u \in H^{2(\theta+1)}(\Omega)$ and its norm is estimated by $\|u\|_{H^{2(\theta+1)}} \le C_\theta(\|\Lambda u\|_{H^{2\theta}} + \|u\|_{H^{2\theta}})$ for some constant C_θ.*

Proof. As Ω is a non-smooth domain, we cannot obtain this result from the general smoothness properties of elliptic operators in smooth domains. Instead, we have to use the spectral resolution of self-adjoint operators (see [10, Chapter XI, Section 6]). However, ours is a very special case (see [5, Chapitre IV, Section 2]).

Let us utilize the positive definite self-adjoint operator $\Lambda + 1$. As is well known, $\Lambda + 1$ has the eigenvalues $\lambda_{mn} = \left(\frac{m\pi}{\ell_x}\right)^2 + \left(\frac{n\pi}{\ell_y}\right)^2 + 1$ for $m, n = 0, 1, 2, \ldots$, where the eigenfunctions are $\cos\frac{m\pi}{\ell_x}x \cdot \cos\frac{n\pi}{\ell_y}y$. Moreover, the family of these eigenfunctions composes an orthogonal basis of the Fourier series in $L_2(\Omega)$. Therefore, $u \in L_2(\Omega)$ if and only if $\sum_{m,n}|u_{mn}|^2 < \infty$, where u_{mn} are the Fourier coefficients of u, and then u is expanded as

$u = \sum_{m,n} u_{mn} \cos \frac{m\pi}{\ell_x} x \cdot \cos \frac{n\pi}{\ell_y} y$. In addition, $u \in \mathcal{D}(\Lambda)$ if and only if $\sum_{m,n} \lambda_{mn}^2 |u_{mn}|^2 < \infty$, and then $\Lambda u = \sum_{m,n} \lambda_{mn} u_{mn} \cos \frac{m\pi}{\ell_x} x \cdot \cos \frac{n\pi}{\ell_y} y$.

Consider next the square $(\Lambda + 1)^2$ in $L_2(\Omega)$. By definition, $u \in \mathcal{D}((\Lambda+1)^2)$ if and only if $u \in \mathcal{D}(\Lambda+1)$ and $(\Lambda+1)u \in \mathcal{D}(\Lambda+1)$. Then, by using the Fourier coefficients, $u \in \mathcal{D}((\Lambda+1)^2)$ can be characterized by $\sum_{m,n}(\lambda_{mn}^2+1)^2|u_{mn}|^2 < \infty$. Then, we observe that $\mathcal{D}((\Lambda+1)^2) \subset H^4(\Omega)$. Indeed, let $u \in \mathcal{D}((\Lambda+1)^2)$ and consider any integers $0 \le i, j \le 4$, $i + j = 4$. When i, j are even, we have

$$\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = \sum_{m,n} \left(\frac{m\pi}{\ell_x}\right)^i \left(\frac{n\pi}{\ell_y}\right)^j u_{mn} \cos \frac{m\pi}{\ell_x} x \cdot \cos \frac{n\pi}{\ell_y} y \in L_2(\Omega).$$

Similarly, when i, j are odd, we have

$$\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = \sum_{m,n} \left(\frac{m\pi}{\ell_x}\right)^i \left(\frac{n\pi}{\ell_y}\right)^j u_{mn} \sin \frac{m\pi}{\ell_x} x \cdot \sin \frac{n\pi}{\ell_y} y \in L_2(\Omega),$$

for the family $\sin \frac{m\pi}{\ell_x} x \cdot \sin \frac{n\pi}{\ell_y} y$ composing another orthogonal basis of $L_2(\Omega)$. Therefore, it holds true that $u \in H^4(\Omega)$ and satisfies the estimate $\|u\|_{H^4} \le C\|(\Lambda+1)^2 u\|_{L_2}$.

As a consequence, for any $0 < \theta < 1$, the interpolation (2.1) provides the inclusion

$$\mathcal{D}((\Lambda+1)^{2\theta}) = [L_2(\Omega), \mathcal{D}((\Lambda+1)^2)]_\theta \subset [L_2(\Omega), H^4(\Omega)]_\theta = H^{4\theta}(\Omega)$$

with continuous embedding $\|u\|_{H^{4\theta}} \le C\|(\Lambda+1)^{2\theta}u\|_{L_2}$.

We are now ready to verify the proposition. In view of (2.14), the assumption $\Lambda u \in H^{2\theta}(\Omega)$ means that $(\Lambda+1)u$ lies in $\mathcal{D}((\Lambda+1)^\theta)$, i.e., $u \in \mathcal{D}((\Lambda+1)^{\theta+1}) \subset H^{2(\theta+1)}(\Omega)$. Furthermore, the previous inequality, together with (2.14), yields the desired estimate

$$\|u\|_{H^{2(\theta+1)}} \le C\|(\Lambda+1)^{\theta+1}u\|_{L_2} \le C_\theta\|(\Lambda+1)u\|_{H^{2\theta}} \le C_\theta(\|\Lambda u\|_{H^{2\theta}} + \|u\|_{H^{2\theta}}).$$

□

Proposition 2.3. *Let Ω be a $\mathcal{C}^3$ domain and $1 < p < \infty$. If $u \in H_N^2(\Omega)$ satisfies $\Lambda u \in H_p^1(\Omega)$, then $u \in H_p^3(\Omega)$ and its norm is estimated by $\|u\|_{H_p^3} \le C_p(\|\Lambda u\|_{H_p^1} + \|u\|_{H_p^1})$ for some constant C_p.*

Proof. The result can be verified using the general smoothness properties of elliptic operators. For instance, see [3, Theorem 2.5.1.1 and Remark 2.5.1.2]. □

3. Local Solutions

Let the initial functions u_0, v_0, ρ_0 satisfy (1.2)-(1.3). We are now ready to construct a local strict solution to (1.1) by choosing L_2 space as the underlying space to work. We will apply Theorem 5.6 of [8] after some set-up.

First, in view of (1.3), take any small $\delta > 0$ such that

$$\min_{(x,y)\in\overline{\Omega}} [u_0(x,y) + v_0(x,y)] \ge \delta > 0. \tag{3.1}$$

Then, introduce a smooth cutoff function $\chi(w)$ $(-\infty < w < \infty)$ such that

$$\begin{cases} \chi(w) = w \quad \text{when} \quad \frac{\delta}{2} \le w < \infty, \\ \frac{\delta}{4} \le \chi(w) < \frac{\delta}{2} \quad \text{when} \quad 0 \le w < \delta, \\ \chi(w) = \frac{\delta}{4} \quad \text{when} \quad -\infty < w < 0. \end{cases} \tag{3.2}$$

Using this $\chi(w)$, we rewrite (1.1) as

$$\text{(3.3)}\qquad \begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu u\frac{\partial^2\rho}{\partial x^2} + cuv(u-v) + d - fu, & \Omega\times(0,\infty),\\ \frac{\partial v}{\partial t} = a\Delta v - \mu v\frac{\partial^2\rho}{\partial y^2} + cuv(v-u) + d - fv, & \Omega\times(0,\infty),\\ \frac{\partial \rho}{\partial t} = b\chi(\mathrm{Re}[u+v])\Delta\rho + \nu(u+v) - g\rho, & \Omega\times(0,\infty),\\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial \rho}{\partial n} = 0, & \partial\Omega\times(0,\infty),\\ u(x,y,0) = u_0(x,y),\ \ v(x,y,0) = v_0(x,y),\ \ \rho(x,y,0) = \rho_0(x,y), & \Omega. \end{cases}$$

Second, let us formulate (3.3) as an abstract quasilinear evolution equation

$$\text{(3.4)}\qquad \begin{cases} \frac{dU}{dt} + A(U)U = F(U), \qquad 0<t<\infty,\\ U(0) = U_0, \end{cases}$$

in the complex product L_2-space

$$X \equiv \mathbb{L}_2(\Omega) = \left\{ U = \begin{pmatrix} u\\ v\\ \rho \end{pmatrix};\ u,\ v,\ \rho \in L_2(\Omega) \right\}.$$

Here, for any vector $\widetilde{U} \in Z$, $A(\widetilde{U})$ is a linear operator of X, where Z is the complex product space defined by

$$Z \equiv \mathbb{H}^s(\Omega) = \left\{ \widetilde{U} = \begin{pmatrix} \widetilde{u}\\ \widetilde{v}\\ \widetilde{\rho} \end{pmatrix};\ \widetilde{u},\ \widetilde{v},\ \widetilde{\rho} \in H^s(\Omega) \right\}$$

for s fixed such that $1 < s < 1+\sigma$. (Recall that σ is the exponent fixed in (1.2).) Actually, $A(\widetilde{U})$ is defined in the ball $K = \{\widetilde{U} \in Z;\ \|U\|_Z < R\}$ contained in Z, $0 < R < \infty$ being a fixed radius sufficiently large so that ${}^t(u_0, v_0, \rho_0) \in K$. For each $\widetilde{U} \in K$, $A(\widetilde{U})$ is given by

$$A(\widetilde{U})U = \begin{pmatrix} a\Lambda + f & 0 & \mu\widetilde{u}\frac{\partial^2}{\partial x^2}\\ 0 & a\Lambda + f & \mu\widetilde{v}\frac{\partial^2}{\partial y^2}\\ 0 & 0 & b\chi(\mathrm{Re}[\widetilde{u}+\widetilde{v}])\Lambda + g \end{pmatrix} \begin{pmatrix} u\\ v\\ \rho \end{pmatrix},$$

$$\widetilde{U} = \begin{pmatrix} \widetilde{u}\\ \widetilde{v}\\ \widetilde{\rho} \end{pmatrix} \in K,\ \ U = \begin{pmatrix} u\\ v\\ \rho \end{pmatrix}.$$

(Recall that Λ is a realization of $-\Delta$ in $L_2(\Omega)$ under the homogeneous Neumann boundary conditions on $\partial\Omega$.) Meanwhile, $F(U)$ is the nonlinear operator of X given by

$$\text{(3.5)}\qquad F(U) = \begin{pmatrix} cuv(u-v) + d\\ cuv(v-u) + d\\ \nu(u+v) \end{pmatrix}, \qquad U = \begin{pmatrix} u\\ v\\ \rho \end{pmatrix} \in \mathcal{D}(F) \subset \mathbb{L}_6(\Omega).$$

The initial value U_0 of (3.4) is of course taken as $U_0 = {}^t(u_0, v_0, \rho_0) \in K$.

Next, let us verify that $A(\widetilde{U})$ and F satisfy all the structural assumptions which were made in [8, Theorem 5.6].

For $A(\widetilde{U})$, the first and second diagonal components are each a positive definite self-adjoint $L_2(\Omega)$ operator with the domain $H^2_N(\Omega)$, based on (2.11). In addition, Proposition 2.1 shows that the third diagonal component is a sectorial $L_2(\Omega)$ operator of angle $< \frac{\pi}{2}$

with the same domain $H^2_N(\Omega)$. The other non-zero components of $A(\widetilde{U})$ are bounded operators from $H^2_N(\Omega)$ into $L_2(\Omega)$. It is then possible to utilize [8, Theorem 2.16] to conclude that $A(\widetilde{U})$, $\widetilde{U} \in K$, are sectorial operators of X of angle $\omega_{A(\widetilde{U})} \leq \omega < \frac{\pi}{2}$ and with domain $\mathcal{D}(A(\widetilde{U})) = \mathbb{H}^2_N(\Omega)$. Thereby, the assumptions [8, (5.2)-(5.3)] are satisfied and, as the domains $\mathcal{D}(A(\widetilde{U}))$ are independent of $\widetilde{U} \in K$, [8, (5.4)] is also satisfied for $\nu = 1$. In view of the inequality

$$\begin{aligned}(3.6)\quad &\|A(\widetilde{U}_1)[A(\widetilde{U}_1)^{-1} - A(\widetilde{U}_2)^{-1}]\|_{\mathcal{L}(X)} = \|[A(\widetilde{U}_1) - A(\widetilde{U}_2)]A(\widetilde{U}_2)^{-1}\|_{\mathcal{L}(X)} \\ &\leq C\|A(\widetilde{U}_1) - A(\widetilde{U}_2)\|_{\mathcal{L}(\mathbb{H}^2_N, \mathbb{L}_2)} \leq C[\|\widetilde{u}_1 - \widetilde{u}_2\|_{L_\infty} + \|\widetilde{v}_1 - \widetilde{v}_2\|_{L_\infty}], \qquad \widetilde{U}_1, \widetilde{U}_2 \in K,\end{aligned}$$

we are led to set the space Y as $Y = Z$. Then, due to (2.4), [8, (5.5)] is also satisfied. For [8, (5.6)], we observe the following facts.

Proposition 3.1. *For any exponents* $0 < \theta' < \theta < \theta'' < \frac{3}{4}$, *we have*

$$\mathbb{H}^{2\theta''}(\Omega) \subset \mathcal{D}(A(\widetilde{U})^\theta) \subset \mathbb{H}^{2\theta'}(\Omega), \qquad \widetilde{U} \in K, \tag{3.7}$$

with uniform embeddings.

Proof. We compare $A(\widetilde{U})$ with $A(0)$. As $A(0) = \operatorname{diag}\{a\Lambda + f, a\Lambda + f, b\chi(0)\Lambda + g\}$, $A(0)$ is a positive definite self-adjoint operator of X with the same domain $\mathbb{H}^2_N(\Omega)$ as $A(\widetilde{U})$. Therefore, [8, Theorem 2.25] can be applied to observe that $\mathcal{D}(A(0)^{\theta''}) \subset \mathcal{D}(A(\widetilde{U})^\theta) \subset \mathcal{D}(A(0)^{\theta'})$ with uniformly continuous embeddings. Meanwhile, by (2.14), we know that $\mathcal{D}(A(0)^\theta) = \mathbb{H}^{2\theta}(\Omega)$ for all $0 < \theta < \frac{3}{4}$ with norm equivalence. □

Now, fix any two exponents α and β such that $\frac{s}{2} < \alpha < \beta < \frac{1+\sigma}{2}$. Then, we can see that $\mathcal{D}(A(\widetilde{U})^\beta) \subset \mathcal{D}(A(\widetilde{U})^\alpha) \subset Z = Y$ with uniformly continuous embeddings for $\widetilde{U} \in K$. Thereby, both [8, (5.6)] and [8, (5.7)] are satisfied.

Furthermore, for the nonlinear operator $F(U)$ given in (3.5), we set $\mathcal{D}(F) = Z$. Then, the Lipschitz condition [8, (5.44)] is trivially valid with the space $W = Z$. In addition, fix any exponent η such that $\beta < \eta < 1$ to clear [8, (5.46)-(5.47)].

Then, as an immediate consequence of [8, Theorem 5.6], we obtain the following existence and uniqueness results for (3.4).

Theorem 3.1. *Take a third exponent* γ *such that* $\frac{s}{2} < \alpha < \beta < \gamma < \frac{1+\sigma}{2}$. *Then, for* U_0 *satisfying* (1.2)-(1.3), *there exists a unique local solution to* (3.4) *in the space*

$$U \in \mathcal{C}^{\gamma-\alpha}([0, T_{U_0}]; \mathbb{H}^s(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; \mathbb{L}_2(\Omega)) \cap \mathcal{C}((0, T_{U_0}]; \mathbb{H}^2_N(\Omega)), \tag{3.8}$$

where $T_{U_0} > 0$ *is determined by the magnitude of the norm* $\|U_0\|_{\mathbb{H}^{1+\sigma}}$. *Furthermore,* U *satisfies the estimates*

$$\|U(t) - U(s)\|_{\mathbb{H}^s} \leq C_{U_0}|t - s|^{\gamma-\alpha}, \qquad 0 \leq s, t \leq T_{U_0}, \tag{3.9}$$

$$\|A(U(t))U(t)\|_{\mathbb{L}_2} \leq C_{U_0} t^{\gamma-1}, \qquad 0 < t \leq T_{U_0}, \tag{3.10}$$

where $C_{U_0} > 0$ *is a constant depending on* $\|U_0\|_{\mathbb{H}^{1+\sigma}}$.

Proof. We have already verified that the structural assumptions of [8, Theorem 5.6] are satisfied by the spaces X and $Y = Z = W$ with the exponents $\frac{s}{2} < \alpha < \beta < \eta < 1$.

As $\beta < \gamma < \frac{1+\sigma}{2}$, we see by Proposition 3.1 that $U_0 \in \mathcal{D}(A(U_0)^\gamma)$, namely, U_0 satisfies the assumption [8, (5.30)] made for the initial values. □

Finally, we should remark that the solution $U(t)$ obtained above can be regarded as a local solution to (1.1). In fact, as its complex conjugate $\overline{U(t)}$ is also a local solution to the same problem (3.4), the uniqueness of the solution yields their coincidence $U(t) = \overline{U(t)}$ for every $0 < t \leq T_{U_0}$, namely, $U(t)$ is a real-valued function. Next, as $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ due to (2.4), we can see from (3.8) that $U \in \mathcal{C}^{\gamma-\alpha}([0, T_{U_0}]; \mathcal{C}(\overline{\Omega}))$; in addition, from (3.9), we can see that

$$\|U(t) - U_0\|_{\mathcal{C}} \leq C_{U_0} t^{\gamma-\alpha}, \qquad 0 \leq t \leq T_{U_0}.$$

Therefore, (1.3) implies that there exists a time $\widetilde{T}_{U_0}$, $0 < \widetilde{T}_{U_0} \leq T_{U_0}$ for which the following holds:

$$\min_{(x,y)\in\overline{\Omega}} [u(t) + v(t)] \geq \tfrac{\delta}{2}, \qquad 0 \leq t \leq \widetilde{T}_{U_0}. \tag{3.11}$$

In view of (3.2), this means that $U(t) = {}^t(u(t), v(t), \rho(t))$ satisfies the equations of (1.1). In this sense, under constrains (1.2)-(1.3), we have constructed a unique local strict solution to (1.1) in the space (3.8) on an interval $[0, \widetilde{T}_{U_0}]$. As explained, $\widetilde{T}_{U_0} > 0$ is determined not only by the norm $\|U_0\|_{\mathbb{H}^{1+\sigma}}$ but also by the lower bound $\delta > 0$ in the estimate (3.1).

4. Regularity Properties of Local Solutions

The next two sections are devoted to investigating some properties of the local solution of (1.1) which we constructed above. These properties will be needed in the subsequent arguments.

In this section, we want to prove some regularity properties. Let $U(t)$ be the local solution of (1.1) on $[0, \widetilde{T}_{U_0}]$ for initial functions $U_0 = {}^t(u_0, v_0, \rho_0)$ satisfying (1.2)-(1.3).

Let us begin with showing the temporal regularity of $U(t)$.

Proposition 4.1. *For any $0 < \theta < 1$, the derivative $\frac{dU}{dt}$ belongs to $\mathcal{C}((0, \widetilde{T}_{U_0}]; \mathbb{H}^{2\theta}(\Omega))$.*

Proof. Fix a time $0 < \tau < \widetilde{T}_{U_0}$ arbitrarily and consider $U(t)$ as a solution to the linear evolution equation

$$\begin{cases} \frac{dU}{dt} + A(t)U = F(t), & \tau \leq t \leq \widetilde{T}_{U_0}, \\ U(\tau) = U_\tau, \end{cases} \tag{4.1}$$

where $A(t) = A(U(t))$ for $\tau \leq t \leq \widetilde{T}_{U_0}$, $F(t) = F(U(t))$ for $\tau \leq t \leq \widetilde{T}_{U_0}$, and $U_\tau = U(\tau)$.

Concerning the problem (4.1), we already know from (3.8) that $F \in \mathcal{C}([\tau, \widetilde{T}_{U_0}]; \mathbb{H}^2_N(\Omega))$ and $U_\tau \in \mathbb{H}^2_N(\Omega)$. Then, we are interested in the Hölder condition on $A(t)[A(t)^{-1} - A(s)^{-1}]$, which can be estimated as

$$\begin{aligned} \|A(t)[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(X)} &= \|[A(t) - A(s)]A(s)^{-1}\|_{\mathcal{L}(X)} \\ &\leq C_\tau \|U(t) - U(s)\|_{\mathbb{L}_\infty} \leq C_\tau \|U(t) - U(s)\|_{\mathbb{H}^{1+\sigma}}, \qquad \tau \leq s,\, t \leq \widetilde{T}_{U_0}, \end{aligned}$$

for some constants C_τ depending on τ. In addition, from (3.8), we have

$$\|U(t) - U(s)\|_{\mathbb{H}^{1+\sigma}} \leq C \|U(t) - U(s)\|_{\mathbb{H}^2}^{\frac{1+\sigma}{2}} \|U(t) - U(s)\|_{\mathbb{L}_2}^{\frac{1-\sigma}{2}} \leq C_\tau |t - s|^{\frac{1-\sigma}{2}}.$$

However, this estimate can be improved step by step to reach the optimal estimate.

Lemma 4.1. *Assume that*

$$\|U(t)-U(s)\|_{\mathbb{H}^{1+\sigma}} \le C_\tau |t-s|^\mu, \qquad \tau \le s,\, t \le \widetilde{T}_{U_0}, \tag{4.2}$$

has been obtained for some exponent μ *such that* $\frac{1-\sigma}{2} \le \mu < 1$. *If* $\mu > \frac{1+\sigma}{2}$, *then the estimate* (4.2) *holds with the optimal exponent* $\widetilde{\mu} = 1$ *on an interval* $[\widetilde{\tau}, \widetilde{T}_{U_0}]$ *for any* $\widetilde{\tau} > \tau$. *If* $\mu \le \frac{1+\sigma}{2}$, *then the estimate* (4.2) *can be improved to any exponent* $\widetilde{\mu}$ *such that* $\mu < \widetilde{\mu} < \frac{1-\sigma}{2(1-\mu)}$ *on an interval* $[\widetilde{\tau}, \widetilde{T}_{U_0}]$ *for any* $\widetilde{\tau} > \tau$.

Proof of lemma. Let $U(t,s)$, $\tau \le s \le t \le \widetilde{T}_{U_0}$, be the evolution operator generated by $A(t)$. Then, as a solution of (4.1), $U(t)$ can be expressed by $U(t) = U(t,\tau)U_\tau + \int_\tau^t U(t,s)F(s)ds$. Therefore, its derivative can be written as

$$\frac{dU}{dt}(t) = -A(t)U(t,\tau)U_\tau - \int_\tau^t A(t)U(t,s)F(s)ds + F(t).$$

Furthermore, applying operator $A(t)^\theta$ to this formula, we have

$$A(t)^\theta \frac{dU}{dt}(t) = -A(t)^{1+\theta}U(t,\tau)A(\tau)^{-1}A(\tau)U_\tau \\ - \int_\tau^t A(t)^{1+\theta}U(t,s)A(s)^{-1}A(s)F(s)ds + A(t)^\theta F(t).$$

As the operators $A(t)$ satisfy the Hölder condition [8, (3.30)-(3.31)] with the μ in (4.2) and $\nu = 1$, it can be seen by [8, (3.83)] that for any exponent $\theta < \mu$, the following holds:

$$\|A(t)^{1+\theta}U(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \le C_\tau (t-s)^{-\theta}, \qquad \tau \le s < t \le \widetilde{T}_{U_0}.$$

Therefore, we observe that $\|A(t)^\theta \frac{dU}{dt}(t)\|_{\mathbb{L}_2} \le C_\tau (t-\tau)^{-\theta}$. As $\theta < \mu$ is arbitrary, we can say in view of (3.7) that for any $\theta < \mu$,

$$\|U(t)-U(s)\|_{\mathbb{H}^{2\theta}} \le C_{\widetilde{\tau}}|t-s|, \qquad \widetilde{\tau} \le s,\, t \le \widetilde{T}_{U_0},$$

where $\widetilde{\tau}$ is any fixed time such that $\widetilde{\tau} > \tau$.

Thereby, when $\mu > \frac{1+\sigma}{2}$, θ can be such that $2\theta > 1+\sigma$, which means that (4.2) holds for $\widetilde{\mu} = 1$. Meanwhile, when $\mu \le \frac{1+\sigma}{2}$, we obtain that

$$\|U(t)-U(s)\|_{\mathbb{H}^{1+\sigma}} \le C\|U(t)-U(s)\|_{\mathbb{H}^2}^{\frac{1+\sigma-2\theta}{2(1-\theta)}} \|U(t)-U(s)\|_{\mathbb{H}^{2\theta}}^{\frac{1-\sigma}{2(1-\theta)}} \\ \le C_{\widetilde{\tau}}|t-s|^{\frac{1-\sigma}{2(1-\theta)}}, \qquad \widetilde{\tau} \le s,\, t \le \widetilde{T}_{U_0}.$$

□

By this lemma, we can say that, if $\mu < 1$, then (4.2) holds for exponents which are determined by the recurrence $\mu_0 = \frac{1-\sigma}{2}$ and $\mu_k \le \frac{1+\sigma}{2}$, $\mu_{k+1} = \frac{1-\sigma}{2(1-\mu_k)}$ $(k = 0,1,2,\dots)$ on the intervals $[\widetilde{\tau}_k, \widetilde{T}_{U_0}]$ such that $\tau = \widetilde{\tau}_0 < \widetilde{\tau}_1 < \widetilde{\tau}_2 < \cdots$. However, because $\sigma < \frac{1}{2}$, we can verify that the increasing sequence $\mu_0 < \mu_1 < \mu_2 < \cdots$ must exceed $\frac{1+\sigma}{2}$ for some μ_k. Hence, we conclude that (4.2) must hold for $\mu = 1$ on the interval $[\widetilde{\tau}_{k+1}, \widetilde{T}_{U_0}]$.

As shown, the operators $A(t)$ satisfy [8, (3.30)-(3.31)] for the optimal exponents $\mu = \nu = 1$ on $[\widetilde{\tau}_{k+1}, \widetilde{T}_{U_0}]$. Then, for the same reasons as in the proof of lemma, it is concluded that $\|A(t)^\theta \frac{dU}{dt}(t)\|_{\mathbb{L}_2} \le C_{\widetilde{\tau}_{k+1}}(t-\widetilde{\tau}_{k+1})^{-\theta}$, $\widetilde{\tau}_{k+1} < t \le \widetilde{T}_{U_0}$, for any $\theta < \mu = 1$. Since all of the terms $\widetilde{\tau}_{k+1} > \widetilde{\tau}_k > \cdots > \tau$ can be taken arbitrarily close to $\tau > 0$, which was fixed arbitrarily, we obtain the desired result. □

As a direct consequence of Proposition 4.1, we obtain the following two regularity theorems.

Theorem 4.1. *Let U_0 satisfy* (1.2)-(1.3) *and $U(t)$ be the local solution of* (1.1) *on the interval* $[0, \widetilde{T}_{U_0}]$ *in the space* (3.8). *Then, $U(t)$ lies in* $\mathcal{C}^1((0, \widetilde{T}_{U_0}]; \mathbb{H}^1(\Omega)) \cap \mathcal{C}((0, \widetilde{T}_{U_0}]; \mathbb{H}^3(\Omega))$.

Proof. Clearly it suffices to prove that $U \in \mathcal{C}((0, \widetilde{T}_{U_0}]; \mathbb{H}^3(\Omega))$. From the equation for ρ in (1.1), we have

$$b\Delta\rho(t) = \left\{ \frac{\partial \rho}{\partial t}(t) - \nu[u(t) + v(t)] + g\rho(t) \right\} / [u(t) + v(t)]. \tag{4.3}$$

In addition, Proposition 4.1 implies the regularity of $\rho(t)$ such that $\frac{\partial \rho}{\partial t} \in \mathcal{C}((0, \widetilde{T}_{U_0}]; H^{2\theta}(\Omega))$ for any $0 < \theta < 1$. Then, by (2.10) and (3.8), we observe that all the functions in the right-hand side of (4.3) belong to $H^1(\Omega)$. Hence, using Proposition 2.2 ($\theta = \frac{1}{2}$) with Ω as a rectangle and Proposition 2.3 ($p = 2$) with Ω belonging to $\mathcal{C}^3$, we can verify that $\rho(t)$ lies in $H^3(\Omega)$ for any $0 < t \le \widetilde{T}_{U_0}$.

Furthermore, since

$$a\Delta u(t) = \frac{\partial u}{\partial t}(t) + \mu u(t) \frac{\partial^2 \rho}{\partial x^2}(t) - cu(t)v(t)[u(t) - v(t)] - d + fu(t), \tag{4.4}$$

$$a\Delta v(t) = \frac{\partial v}{\partial t}(t) + \mu v(t) \frac{\partial^2 \rho}{\partial y^2}(t) - cu(t)v(t)[v(t) - u(t)] - d + fv(t), \tag{4.5}$$

we can observe by (2.8), (3.8), Proposition 4.1, and Theorem 4.1 that $\Delta u(t)$ and $\Delta v(t)$ belong to $H^1(\Omega)$. Hence, for the same reasons as the case of $\rho(t)$, we verify that $u(t)$, $v(t) \in H^3(\Omega)$ for any $0 < t \le \widetilde{T}_{U_0}$. □

Theorem 4.2. *Let U_0 satisfy* (1.2)-(1.3) *and $U(t)$ be the local solution of* (1.1) *on the interval* $[0, \widetilde{T}_{U_0}]$ *in the space* (3.8). *Then, $U(t)$ is in* $\mathcal{C}^1((0, \widetilde{T}_{U_0}]; [\mathcal{C}(\overline{\Omega})]^3) \cap \mathcal{C}((0, \widetilde{T}_{U_0}]; [\mathcal{C}^2(\overline{\Omega})]^3)$.

Proof. We can utilize the results obtained by Proposition 4.1 with $\frac{1}{2} < \theta < 1$.

First, by (2.4), we observe that $U \in \mathcal{C}^1([0, \widetilde{T}_{U_0}]; [\mathcal{C}(\overline{\Omega})]^3)$.

Next, we observe from (4.3) that $\Delta\rho(t) \in H^{2\theta}(\Omega)$. Then, when Ω is a rectangle, Proposition 2.2 provides the spatial regularity $\rho(t) \in H^{2(\theta+1)}(\Omega)$. Since $2(\theta + 1) > 3$, we can verify using (2.4) that $\rho(t) \in \mathcal{C}^2(\overline{\Omega})$ for any $0 < t \le \widetilde{T}_{U_0}$. Furthermore, when Ω belongs to $\mathcal{C}^3$, as $H^{2\theta}(\Omega) \subset H^1_p(\Omega)$, from (2.2) with $p = \frac{1}{1-\theta}$, Proposition 2.3 provides the regularity $\rho(t) \in H^3_p(\Omega)$. As $p > 2$, we can verify using (2.5) that $\rho(t) \in \mathcal{C}^2(\overline{\Omega})$ for any $0 < t \le \widetilde{T}_{U_0}$.

Similarly, when Ω is a rectangle, since $\frac{\partial^2 \rho}{\partial x^2}(t) \in H^{2\theta}(\Omega)$ and (2.6) imply in (4.4) that $\Delta u(t) \in H^{2\theta}(\Omega)$, we can verify that $u(t) \in H^{2(\theta+1)}(\Omega) \subset \mathcal{C}^2(\overline{\Omega})$ for any $0 < t \le \widetilde{T}_{U_0}$. Furthermore, when Ω belongs to $\mathcal{C}^3$, since $\frac{\partial^2 \rho}{\partial x^2}(t) \in H^1_p(\Omega)$ and (2.7) imply in (4.4) that $\Delta u(t) \in H^1_p(\Omega)$ for $p > 2$, we have verified that $u(t) \in H^3_p(\Omega) \subset \mathcal{C}^2(\overline{\Omega})$ for any $0 < t \le \widetilde{T}_{U_0}$.

The above argument also holds for $v(t)$ if we instead use (4.5). □

Theorem 4.2 obviously means that the strict solution $U(t)$ of (1.1) on $[0, \widetilde{T}_{U_0}]$ that we obtained for U_0 gives a classical solution to (1.1) for $0 < t \le \widetilde{T}_{U_0}$.

5. Positivity of Local Solutions

In this section, we want to prove the positivity of local solutions. Let $U(t)$ be the local solution of (1.1) on $[0, \widetilde{T}_{U_0}]$ for initial functions $U_0 = {}^t(u_0, v_0, \rho_0)$ satisfying (1.2)-(1.3). Our goal is then to show that $U(t) = {}^t(u(t), v(t), \rho(t))$ also satisfies the positivity conditions in (1.3) for every $0 < t \leq \widetilde{T}_{U_0}$.

Theorem 5.1. *It holds true that* $\min_{(x,y)\in\overline{\Omega}} u(x, y, t) > 0$ *and* $\min_{(x,y)\in\overline{\Omega}} v(x, y, t) > 0$ *for any* $0 < t \leq \widetilde{T}_{U_0}$.

Proof. By (1.3) and (3.8), there is a time $0 < \tau < \widetilde{T}_{U_0}$ such that $\min_{(x,y)\in\overline{\Omega}} u(x, y, t) > 0$ for every $0 \leq t \leq \tau$. Thus, it suffices to prove the assertion of theorem for $\tau < t \leq \widetilde{T}_{U_0}$.

First, let us prove that $u(t) \geq 0$ for $\tau < t \leq \widetilde{T}_{U_0}$. For this purpose, we use the cutoff function $H(u)$ defined by $H(u) = \frac{u^2}{2}$ for $-\infty < u < 0$ and $H(u) = 0$ for $0 \leq u < \infty$. Then, consider the function

$$\varphi(t) = \iint_\Omega H(u(x, y, t))dxdy, \qquad \tau \leq t \leq \widetilde{T}_{U_0}.$$

Clearly, $\varphi(t)$ is a nonnegative $\mathcal{C}^1$ function with derivative

$$\begin{aligned}\varphi'(t) = \iint_\Omega H'(u(t))\tfrac{\partial u}{\partial t}(t)dxdy = a \iint_\Omega H'(u)\Delta u\, dxdy \\ + \iint_\Omega H'(u)[-\mu\rho_{xx} + cv(u - v)]u\, dxdy + \iint_\Omega H'(u)[-fu + d]dxdy.\end{aligned}$$

(Here, we denoted $\frac{\partial^2\rho}{\partial x^2}$ by the symbol ρ_{xx} for simplicity.) Since $H'(u) \in H^1(\Omega)$, we observe by (2.9) and (2.12) that

$$\iint_\Omega H'(u)\Delta u\, dxdy = -\iint_\Omega \nabla H'(u) \cdot \nabla u\, dxdy = -\iint_\Omega |\nabla H'(u)|^2 dxdy \leq 0.$$

In addition, since $H'(u)u = 2H(u)$ and $H'(u) \leq 0$, it follows that

$$\varphi'(t) \leq (D - f) \iint_\Omega H'(u)u\, dxdy = 2(D - f)\varphi(t),$$

where $D = \max_{\tau\leq t\leq\widetilde{T}_{U_0}} \max_{(x,y)\in\overline{\Omega}} | - \mu\rho_{xx} + cv(u - v)|$ (recall Theorem 3.10). Therefore, $\varphi(t) \leq e^{2(D-f)(t-\tau)}\varphi(\tau)$, but since $\varphi(\tau) = 0$, we conclude that $\varphi(t) = 0$ for any $\tau \leq t \leq \widetilde{T}_{U_0}$, i.e., $u(t) \geq 0$ for any $\tau \leq t \leq \widetilde{T}_{U_0}$.

Second, fix a constant $\delta_\tau = \min_{(x,y)\in\overline{\Omega}} u(x, y, \tau) > 0$ and define the function

$$r(t) = \delta_\tau e^{-(D+f)(t-\tau)} + [d/(D + f)][1 - e^{-(D+f)(t-\tau)}], \qquad \tau \leq t \leq \widetilde{T}_{U_0}.$$

We will prove that $u(t) \geq r(t)$ for any $\tau \leq t \leq \widetilde{T}_{U_0}$.

For this purpose, we consider as before the function

$$\psi(t) = \iint_\Omega H(u(x, y, t) - r(t))dxdy, \qquad \tau \leq t \leq \widetilde{T}_{U_0},$$

the derivative of which is given by

$$\psi'(t) = \iint_\Omega H'\left(u(t) - r(t)\right)\left[\tfrac{\partial u}{\partial t}(t) - r'(t)\right]dxdy.$$

Since $r(t)$ is a solution of the ordinary differential equation $r'(t) = -(D+f)r(t) + d$ and $u(t)$ satisfies the differential inequality $\frac{\partial u}{\partial t}(t) \geq a\Delta u(t) - (D+f)u(t) + d$, we can see that $\frac{\partial u}{\partial t}(t) - r'(t) \geq a\Delta u(t) - (D+f)[(u(t) - r(t)]$. Then, by the same arguments as before (recall that $H'(u-r) \leq 0$), it follows that $\psi'(t) \leq -2(D+f)\psi(t)$; namely, $\psi(t) \leq e^{-2(D+f)(t-\tau)}\psi(\tau)$; therefore, $\psi(\tau) = 0$ yields the desired vanishing $\psi(t) = 0$ for any $\tau \leq t \leq \widetilde{T}_{U_0}$, i.e., $u(t) \geq r(t) > 0$ for any $\tau \leq t \leq \widetilde{T}_{U_0}$.

The same arguments as for $u(t)$ prove the assertion of the theorem for $v(t)$. □

Next, we verify the positivity of $\rho(t)$.

Theorem 5.2. *For the local solution $U(t) = {}^t(u(t), v(t), \rho(t))$ on $[0, \widetilde{T}_{U_0}]$ it holds that $\rho(t) \geq 0$ for all $0 < t \leq \widetilde{T}_{U_0}$.*

Proof. Using the same cutoff function $H(\rho)$ as above for the variable $-\infty < \rho < \infty$, consider the function

$$\varphi(t) = \iint_\Omega H(\rho(x, y, t))dxdy, \qquad 0 \leq t \leq \widetilde{T}_{U_0}.$$

Clearly, $\varphi(t)$ is a nonnegative $\mathcal{C}^1$ function for $0 < t \leq \widetilde{T}_{U_0}$ with derivative

$$\varphi'(t) = \iint_\Omega H'(\rho(t))\tfrac{\partial \rho}{\partial t}(t)dxdy = b\iint_\Omega H'(\rho)(u+v)\Delta\rho\, dxdy + \iint_\Omega H'(\rho)[\nu(u+v) - g\rho]dxdy.$$

Since $H'(\rho) \in H^1(\Omega)$, we can observe by (2.8) and (2.12) that

$$\begin{aligned}\iint_\Omega H'(\rho)(u+v)\Delta\rho\, dxdy &= -\iint_\Omega \nabla[H'(\rho)(u+v)] \cdot \nabla\rho\, dxdy \\ &= -\iint_\Omega (u+v)\nabla H'(\rho) \cdot \nabla\rho\, dxdy - \iint_\Omega H'(\rho)\nabla(u+v) \cdot \nabla\rho\, dxdy.\end{aligned}$$

In view of $\nabla H'(\rho) \cdot \nabla\rho = |\nabla H'(\rho)|^2$ (from (2.9)) and (3.11), we have

$$-\iint_\Omega (u+v)\nabla H'(\rho) \cdot \nabla\rho\, dxdy \leq -(\delta/2)\iint_\Omega |\nabla H'(\rho)|^2 dxdy.$$

Meanwhile, as $H'(\rho)\nabla\rho = H'(\rho)\nabla H'(\rho)$ from (2.9) again,

$$\begin{aligned}-\iint_\Omega H'(\rho)\nabla(u+v) \cdot \nabla\rho\, dxdy &= -\iint_\Omega H'(\rho)\nabla(u+v) \cdot \nabla H'(\rho)dxdy \\ &\leq \|H'(\rho)\|_{L_p}\|\nabla(u+v)\|_{L_q}\|\nabla H'(\rho)\|_{L_2}\end{aligned}$$

for any $2 < p,\ q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Furthermore, applying (2.3) to $\|H'(\rho)\|_{L_p}$, we have

$$\|H'(\rho)\|_{L_p}\|\nabla(u+v)\|_{L_q}\|\nabla H'(\rho)\|_{L_2} \leq C_p\|H'(\rho)\|_{H^1}^{2(1-1/p)}\|H'(u)\|_{L_2}^{2/p}\|\nabla(u+v)\|_{L_q}.$$

Therefore, by the Hölder inequality, we obtain the estimate

$$\|H'(\rho)\|_{L_p}\|\nabla(u+v)\|_{L_q}\|\nabla H'(\rho)\|_{L_2} \leq \varepsilon\|H'(\rho)\|_{H^1}^2 + C_{p,\varepsilon}\|H'(\rho)\|_{L_2}^2\|\nabla(u+v)\|_{L_q}^p$$

for any small $\varepsilon > 0$. In addition, it is clear that $H'(\rho)(u+v) \le 0$ and $-H'(\rho)\rho \le 0$. Therefore, taking $\varepsilon = \frac{\delta}{2}$, we arrive at the differential inequality

$$\varphi'(t) \le C_p(\|\nabla[u(t)+v(t)]\|_{L_q}^p + 1)\|H'(\rho(t))\|_{L_2}^2 \le C_p(\|\nabla[u(t)+v(t)]\|_{L_q}^p + 1)\varphi(t).$$

This differential inequality then implies

$$\varphi(t) \le \varphi(0)e^{C_p\int_0^t(\|\nabla[u(\tau)+v(\tau)]\|_{L_q}^p+1)d\tau}, \qquad 0 \le t \le \widetilde{T}_{U_0}.$$

However, the estimate (3.10) yields with the aid of (2.3) the inequality $\|\nabla[u(\tau)+v(\tau)]\|_{L_q}^p \le C_q\tau^{p(\gamma-1)}$ for any $2 < q < \infty$. Thereby, if p is fixed such that $2 < p < \frac{1}{1-\gamma}$, i.e., $p(\gamma-1) > -1$, then $\|\nabla[u(\tau)+v(\tau)]\|_{L_q}^p$ is integrable on $[0,t]$. Hence, $\varphi(0) = 0$ yields the vanishing $\varphi(t) = 0$ for any $0 < t \le \widetilde{T}_{U_0}$, i.e., $\rho(t) \ge 0$ for any $0 < t \le \widetilde{T}_{U_0}$. □

6. Maximal Solutions

For the initial functions $U_0 = {}^t(u_0, v_0, \rho_0)$ satisfying assumptions (1.2)-(1.3), let us define their maximal strict solution of (1.1) and investigate its asymptotic behavior as $t \to T_{\max}$.

For U_0, we say that $U(t) = {}^t(u(t), v(t), \rho(t))$ is a local solution to (1.1) on an interval $[0, T_U]$ if $U(t)$ is a function lying in

$$U \in \mathcal{C}^{\gamma-\alpha}([0,T_U]; \mathbb{H}^s(\Omega)) \cap \mathcal{C}^1((0,T_U]; \mathbb{L}_2(\Omega)) \cap \mathcal{C}((0,T_U]; \mathbb{H}_N^2(\Omega)),$$

satisfies the positivity conditions in (1.3) for each $0 \le t \le T$, and also satisfies all the equalities, including the initial conditions, in (1.1).

By definition, if $U(t)$ is a local solution on $[0, T_U]$, then there is a positive constant $\delta_U > 0$ such that $u(t) + v(t) \ge \delta_U$ for any $0 < t \le T_U$. This means that $U(t)$ can always be extended to a larger interval $[0, T_U + \varDelta T]$ ($\varDelta T > 0$). If there exist two local solutions for U_0, then either they coincide or one is an extension of the other. These facts enable us to define the maximal local solution of (1.1) on an open interval $[0, T_{\max})$ (which shall be called simply a maximal solution of (1.1) for U_0). Then, the maximal solution $U(t)$ belongs to the function space

$$U \in \mathcal{C}^{\gamma-\alpha}([0,T_{\max}); \mathbb{H}^s(\Omega)) \cap \mathcal{C}^1((0,T_{\max}); \mathbb{L}_2(\Omega)) \cap \mathcal{C}((0,T_{\max}); \mathbb{H}_N^2(\Omega)), \tag{6.1}$$

and $U(t) = {}^t(u(t), v(t), \rho(t))$ has positivity as follows:

$$\min_{(x,y)\in\overline{\Omega}} u(t) > 0, \quad \min_{(x,y)\in\overline{\Omega}} v(t) > 0 \quad \text{and} \quad \min_{(x,y)\in\overline{\Omega}} \rho(t) \ge 0 \quad \text{for any} \quad 0 \le t < T_{\max}. \tag{6.2}$$

Furthermore, by Theorems 4.1 and 4.2, $U(t)$ belongs to

$$U \in \mathcal{C}^1((0,T_{\max}); \mathbb{H}^1(\Omega)) \cap \mathcal{C}((0,T_{\max}); \mathbb{H}^3(\Omega)), \tag{6.3}$$

$$U \in \mathcal{C}^1((0,T_{\max}); [\mathcal{C}(\overline{\Omega})]^3) \cap \mathcal{C}((0,T_{\max}); [\mathcal{C}^2(\overline{\Omega})]^3). \tag{6.4}$$

When $T_{\max} = \infty$, the maximal solution is a global solution of (1.1). In contrast, when $T_{\max} < \infty$, some blowup must take place in the maximal solution of (1.1). At this point, we know that at least one of the following phenomena occurs:

$$\varliminf_{t\to T_{\max}} \min_{(x,y)\in\overline{\Omega}} u(t) = 0, \quad \varliminf_{t\to T_{\max}} \min_{(x,y)\in\overline{\Omega}} v(t) = 0,$$
$$\text{or} \quad \varlimsup_{t\to T_{\max}} \|U(t)\|_{\mathbb{H}^{1+\sigma}} = \infty \ \ (\text{for any } \sigma > 0).$$

However, we actually know the behavior of $U(t)$ as $t \to T_{\max}$ in some more detail.

Theorem 6.1. *Suppose $T_{\max} < \infty$. Then, at least one of the following holds:*

$$\varliminf_{t\to T_{\max}} \min_{(x,y)\in\overline{\Omega}} [u(t)+v(t)] = 0 \quad \text{or} \quad \varlimsup_{t\to T_{\max}} \max_{(x,y)\in\overline{\Omega}} [u(t)+v(t)] = \infty.$$

Proof. Let us prove the theorem by contradiction. Fixing a time $0 < \tau < T_{\max}$, suppose that there would exist constants $M > \delta > 0$ such that

$$\min_{(x,y)\in\overline{\Omega}} [u(t)+v(t)] \ge \delta \qquad \text{for every} \quad \tau \le t < T_{\max}, \tag{6.5}$$

and

$$\max_{(x,y)\in\overline{\Omega}} [u(t)+v(t)] \le M \qquad \text{for every} \quad \tau \le t < T_{\max}. \tag{6.6}$$

Under the constrains (6.5) and (6.6), we will establish step by step the uniform upper norm estimate $\|U(t)\|_{\mathbb{H}^2} \le C$ for $\tau \le t < T_{\max}$. Throughout the proof, C denotes a universal positive constant determined by Ω, the initial constants a, b, c, d, f, g, μ, ν in (1.1), the norm $\|U(\tau)\|_{\mathbb{H}^2}$, the time $T_{\max}$, and the constants δ, M in (6.5) and (6.6) in some specific way, so C may change from occurrence to occurrence.

Step 1. Integrate the third equation of (1.1) over Ω. This gives

$$\frac{d}{dt}\iint_\Omega \rho\, dxdy + g\iint_\Omega \rho\, dxdy \le bM\iint_\Omega |\Delta\rho| dxdy + \nu M|\Omega|.$$

Separately, multiply the third equation by $\Delta\rho(t)$ and integrate over Ω. This gives

$$\frac{1}{2}\frac{d}{dt}\iint_\Omega |\nabla\rho|^2 dxdy + b\iint_\Omega (u+v)|\Delta\rho|^2 dxdy + g\iint_\Omega |\nabla\rho|^2 dxdy \le \nu M\iint_\Omega |\Delta\rho| dxdy,$$

where we used the formula $\frac{1}{2}\frac{d}{dt}\|\nabla\rho(t)\|_{L_2}^2 = \left(\nabla\frac{\partial\rho}{\partial t}(t), \nabla\rho(t)\right)_{L_2}$ and property (2.12). Now add these two differential inequalities. Since $\iint_\Omega |\Delta\rho| dxdy \le \varepsilon\iint_\Omega |\Delta\rho|^2 dxdy + C_\varepsilon$ for any small $\varepsilon > 0$, we obtain, using (6.5), the differential inequality

$$\frac{d}{dt}\left[\|\rho(t)\|_{L_1} + \frac{1}{2}\|\nabla\rho(t)\|_{L_2}^2\right] + \frac{b\delta}{2}\|\Delta\rho(t)\|_{L_2}^2$$
$$+ g\left[\|\rho(t)\|_{L_1} + \|\nabla\rho(t)\|_{L_2}^2\right] \le C, \qquad \tau \le t < T_{\max}.$$

Solving the above differential inequality, we conclude that

$$\|\rho(t)\|_{L_1} + \|\nabla\rho(t)\|_{L_2}^2 \le C, \qquad \tau \le t < T_{\max}, \tag{6.7}$$

together with

$$\int_\tau^{T_{\max}} \|\Delta\rho(t)\|_{L_2}^2 dt < \infty. \tag{6.8}$$

Step 2. We use Poincaré's inequality

$$\left\|\rho - \frac{1}{|\Omega|}\iint_\Omega \rho\, dxdy\right\|_{L_2} \le C\|\nabla\rho\|_{L_2}, \qquad \rho \in H^1(\Omega),$$

which implies the inequality

$$\|\rho\|_{L_2} \le C(\|\nabla\rho\|_{L_2} + \|\rho\|_{L_1}), \qquad \rho \in H^1(\Omega).$$

Then from (6.7), we obtain that

$$\|\rho(t)\|_{H^1}^2 \le C, \qquad \tau \le t < T_{\max}. \tag{6.9}$$

Meanwhile, (2.14) ($\theta = 1$) yields the inequality $\|\rho\|_{H^2}^2 \le C(\|\Delta\rho\|_{L_2}^2 + \|\rho\|_{L_2}^2)$ for $\rho \in H_N^2(\Omega)$, so (6.8) gives the estimate

$$\int_\tau^{T_{\max}} \|\rho(t)\|_{H^2}^2 dt < \infty. \tag{6.10}$$

Step 3. Multiply the first equation of (1.1) by $\Delta u(t)$ and integrate over Ω. Then, in view of (6.6), we have

$$\frac{1}{2}\frac{d}{dt}\iint_\Omega |\nabla u|^2 dxdy + a\iint_\Omega |\Delta u|^2 dxdy + f\iint_\Omega |\nabla u|^2 dxdy \\ \le \iint_\Omega \left[\mu M|\rho_{xx}| + (cM^3 + d)\right] |\Delta u| dxdy.$$

(As before, ρ_{xx} denotes $\frac{\partial^2\rho}{\partial x^2}$.) After some calculations, we obtain the differential inequality

$$\frac{d}{dt}\|\nabla u(t)\|_{L_2}^2 + a\|\Delta u(t)\|_{L_2}^2 + 2f\|\nabla u(t)\|_{L_2}^2 \le C[\|\rho(t)\|_{H^2}^2 + 1], \qquad \tau \le t < T_{\max}.$$

Using (6.10), we can solve this differential inequality to conclude that $\|\nabla u(t)\|_{L_2}^2 \le C$, i.e., $\|u(t)\|_{H^1}^2 \le C$ for $\tau \le t < T_{\max}$.

The corresponding result holds for $v(t)$. Therefore,

$$\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \le C, \qquad \tau \le t < T_{\max}. \tag{6.11}$$

At the same time, we can observe that

$$\int_\tau^{T_{\max}} \left[\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2\right] dt < \infty. \tag{6.12}$$

Step 4. Let us note that, from Theorem 4.1 and property (2.8), all the terms in the third equation of (1.1) take their values in $H^1(\Omega)$ for $\tau < t < T_{\max}$, so since Δ is a bounded operator from $H^1(\Omega)$ into $H^1(\Omega)'$, we can apply Δ to the equation. After applying Δ, take the duality product $\langle \cdot, \cdot \rangle_{H^{1\prime} \times H^1}$ of the equation and the function $\Delta\rho(t) \in H^1(\Omega)$. Then, since $\frac{1}{2}\frac{d}{dt}\|\Delta\rho(t)\|_{L_2}^2 = \langle \Delta\frac{\partial\rho}{\partial t}(t), \Delta\rho(t)\rangle_{H^{1\prime}\times H^1}$ and Δ has property (2.13), it follows that

$$\frac{1}{2}\frac{d}{dt}\iint_\Omega |\Delta\rho|^2 dxdy + b\iint_\Omega \nabla[(u+v)\Delta\rho] \cdot \nabla\Delta\rho\, dxdy + g\iint_\Omega |\Delta\rho|^2 dxdy \\ = -\nu\iint_\Omega \nabla(u+v) \cdot \nabla\Delta\rho\, dxdy.$$

From (6.5), we see that

$$\frac{1}{2}\frac{d}{dt}\iint_\Omega |\Delta\rho|^2 dxdy + b\delta\iint_\Omega |\nabla\Delta\rho|^2 dxdy + g\iint_\Omega |\Delta\rho|^2 dxdy \\ \le C\iint_\Omega |\nabla(u+v)|(|\Delta\rho| + 1)|\nabla\Delta\rho| dxdy$$

and that

$$\frac{d}{dt}\iint_\Omega |\Delta\rho|^2 dxdy + b\delta\iint_\Omega |\nabla\Delta\rho|^2 dxdy + 2g\iint_\Omega |\Delta\rho|^2 dxdy$$
$$\leq C\iint_\Omega |\nabla(u+v)|^2(|\Delta\rho|^2+1)dxdy.$$

Then using (2.3) and (6.11), we can estimate the integral in the right-hand side as

$$\iint_\Omega |\nabla(u+v)|^2(|\Delta\rho|^2+1)dxdy \leq \|\nabla(u+v)\|_{L_4}^2\|\Delta\rho\|_{L_4}^2 + \|\nabla(u+v)\|_{L_2}^2$$
$$\leq C[\|\nabla(u+v)\|_{L_2}\|\nabla(u+v)\|_{H^1}\|\Delta\rho\|_{L_2}\|\Delta\rho\|_{H^1}+1] \leq C[\|u+v\|_{H^2}\|\Delta\rho\|_{L_2}$$
$$\times(\|\nabla\Delta\rho\|_{L_2}+\|\Delta\rho\|_{L_2})+1] \leq \varepsilon\|\nabla\Delta\rho\|_{L_2}^2 + C_\varepsilon[(\|u+v\|_{H^2}^2+1)\|\Delta\rho\|_{L_2}^2+1],$$

where $\varepsilon>0$ is any small number. Therefore, we arrive at the differential inequality

$$\frac{d}{dt}\|\Delta\rho(t)\|_{L_2}^2 + \frac{b\delta}{2}\|\nabla\Delta\rho(t)\|_{L_2}^2 + 2g\|\Delta\rho(t)\|_{L_2}^2$$
$$\leq C\{[\|u(t)+v(t)\|_{H^2}^2+1]\|\Delta\rho(t)\|_{L_2}^2+1\}, \qquad \tau\leq t<T_{\max}.$$

Now using (6.12), we can solve this differential inequality to conclude that $\|\Delta\rho(t)\|_{L_2}^2 \leq C$ for $\tau\leq t<T_{\max}$, i.e.,

$$\|\rho(t)\|_{H^2}^2 \leq C, \qquad \tau\leq t<T_{\max}. \tag{6.13}$$

At the same time, by virtue of Propositions 2.2 and 2.3, we can observe that

$$\int_\tau^{T_{\max}} \|\rho(t)\|_{H^3}^2 dt < \infty. \tag{6.14}$$

Step 5. We now use the fact that all the terms in the first equation of (1.1) take their values in $H^1(\Omega)$. After applying Δ to the equation, take the duality products $\langle\cdot,\cdot\rangle_{H^{1'}\times H^1}$ with $\Delta u(t)$. Then, for the same reasons as before, it follows that

$$\frac{1}{2}\frac{d}{dt}\iint_\Omega |\Delta u|^2 dxdy + a\iint_\Omega |\nabla\Delta u|^2 dxdy + f\iint_\Omega |\Delta u|^2 dxdy$$
$$= \iint_\Omega \nabla[\mu u\rho_{xx} - cuv(u-v) - d]\cdot\nabla\Delta u\, dxdy,$$

and that

$$\frac{d}{dt}\iint_\Omega |\Delta u|^2 dxdy + a\iint_\Omega |\nabla\Delta u|^2 dxdy + 2f\iint_\Omega |\Delta u|^2 dxdy$$
$$\leq C\iint_\Omega \left(|\nabla[u\rho_{xx}]|^2 + |\nabla[uv(u-v)]|^2\right) dxdy.$$

Here, using (6.11) and (6.13), the first integral in the right-hand side is estimated as

$$\iint_\Omega |\nabla[u\rho_{xx}]|^2 dxdy \leq C\iint \left(|\nabla u|^2|\rho_{xx}|^2 + |\nabla\rho_{xx}|^2\right) dxdy$$
$$\leq C(\|\nabla u\|_{L_4}^2\|\rho_{xx}\|_{L_4}^2 + \|\rho\|_{H^3}^2) \leq C(\|u\|_{H^2}\|\rho\|_{H^3} + \|\rho\|_{H^3}^2) \leq \varepsilon\|\Delta u\|_{L_2}^2 + C_\varepsilon\|\rho\|_{H^3}^2,$$

$\varepsilon > 0$ being any small number. Meanwhile, using (6.6) and (6.11), the second integral is estimated as

$$\iint_\Omega |\nabla[uv(u-v)]|^2 dxdy \le C \iint_\Omega (|\nabla u|^2 + |\nabla v|^2) dxdy \le C.$$

Therefore, we arrive at the differential inequality

$$\frac{d}{dt}\|\Delta u(t)\|_{L_2}^2 + a\|\nabla\Delta u(t)\|_{L_2}^2 + f\|\Delta u(t)\|_{L_2}^2 \le C(\|\rho(t)\|_{H^3}^2 + 1), \quad \tau \le t < T_{\max}.$$

Using (6.14), this inequality together with (6.11) yields the estimate

$$\|u(t)\|_{H^2}^2 \le C, \qquad \tau \le t < T_{\max}. \tag{6.15}$$

At the same time, we have $\int_\tau^{T_{\max}} \|u(t)\|_{H^3}^2 dt < \infty$.

Of course, the corresponding result holds for $v(t)$, namely,

$$\|v(t)\|_{H^2}^2 \le C, \qquad \tau \le t < T_{\max}, \tag{6.16}$$

together with $\int_\tau^{T_{\max}} \|v(t)\|_{H^3}^2 dt < \infty$.

Completion of the proof. Using the estimates (6.13), (6.15) and (6.16) established above, we can now claim that the norm $\|U(t)\|_{H^2}$ is bounded from above by some constant for any $\tau \le t < T_{\max}$. But combining this with assumption (6.5) immediately leads to a contradiction. Indeed, take any time $t_0 \in [\tau, T_{\max})$ and consider the equation (1.1) at initial time t_0 with initial value $U(t_0)$. As $U(t_0)$ satisfies (1.2)-(1.3), there exists a local solution in some interval $[t_0, t_0 + \Delta t]$. Here, as remarked after the proof of Theorem 3.1, the existence time interval $\Delta t > 0$ is determined by the lower bound $\delta > 0$ in (3.1) and the norm $\|U(t_0)\|_{\mathbb{H}^{1+\sigma}}$, which are both uniform for $t_0 \in [\tau, T_{\max})$. This means that Δt must also be uniform with respect to the initial time t_0. Since t_0 can be arbitrarily close to $T_{\max}$, this clearly contradicts the maximality of the solution $U(t)$. □

On the behavior of $\rho(t)$ as $t \to T_{\max}$, we can state the following theorem.

Theorem 6.2. *Suppose $T_{\max} < \infty$. Then $\overline{\lim}_{t\to T_{\max}} \max_{(x,y)\in\overline{\Omega}} [|\frac{\partial^2\rho}{\partial x^2}(t)| + |\frac{\partial^2\rho}{\partial y^2}(t)|] = \infty$.*

Proof. We again apply proof by contradiction. Fixing a time $0 < \tau < T_{\max}$, suppose that there exists $N > 0$ for which the following holds:

$$\max_{(x,y)\in\overline{\Omega}} \left[\left|\frac{\partial^2\rho}{\partial x^2}(t)\right| + \left|\frac{\partial^2\rho}{\partial y^2}(t)\right|\right] \le N \qquad \text{for every} \quad \tau \le t < T_{\max}. \tag{6.17}$$

Under the constrain (6.17), we will show that $u(t)$ and $v(t)$ satisfy the conditions (6.5) and (6.6), which, as we already know, implies a contradiction. Throughout the proof, as before C denotes a universal positive constant determined by Ω, the initial constants a, b, c, d, f, g, μ, ν in (1.1), the norm $\|U(\tau)\|_{\mathbb{H}^2}$, the time $T_{\max}$, and the constant N in (6.17) in some specific way.

In this proof, we use $w = u + v$ rather than u and v separately, so we add the first and second equations of (1.1) to get

$$\frac{\partial w}{\partial t} = a\Delta w - \mu\left[u\frac{\partial^2\rho}{\partial x^2} + v\frac{\partial^2\rho}{\partial y^2}\right] + 2d - fw. \tag{6.18}$$

Verification of (6.5). Put $\delta_\tau = \min_{(x,y)\in\overline{\Omega}} w(\tau) > 0$ and define the function

$$r(t) = \delta_\tau e^{-(\mu N+f)(t-\tau)} + [2d/(\mu N+f)][1 - e^{-(\mu N+f)(t-\tau)}], \qquad \tau \le t < T_{\max}.$$

Then, under constrain (6.17), we can repeat the same arguments as in the proof of Theorem 5.1 for $w(t) - r(t)$ to derive from equation (6.18) for $w(t)$ and the equation $r'(t) = -(\mu N + f)r + 2d$ for $r(t)$ that

$$u(t) + v(t) = w(t) \geq r(t) \qquad \text{for any } \tau \leq t < T_{\max}.$$

Verification of (6.6). Let $2 < p < \infty$ and then multiply (6.18) by pw^{p-1} and integrate over Ω:

$$\frac{d}{dt}\iint_\Omega w^p dxdy + [4a(p-1)/p]\iint_\Omega |\nabla w^{\frac{p}{2}}|^2 dxdy + fp\iint_\Omega w^p dxdy \\ = p\iint_\Omega [\mu(u\rho_{xx} + v\rho_{yy}) + 2d]w^{p-1}dxdy.$$

Here, we used the formula $w^{p-2}|\nabla w|^2 = (4/p^2)|\nabla w^{\frac{p}{2}}|^2$. Then from (6.17), we see that

$$p\iint_\Omega [\mu(u\rho_{xx} + v\rho_{yy}) + 2d]w^{p-1}dxdy \\ \leq Cp\iint_\Omega (Nw^p + w^{p-1})dxdy \leq Cp\iint_\Omega (w^p + 1)dxdy.$$

Therefore, we obtain the differential inequality

$$\frac{d}{dt}\|w(t)\|_{L_p}^p \leq Cp(\|w(t)\|_{L_p}^p + 1), \qquad \tau \leq t < T_{\max}.$$

As this differential inequality yields the estimate $\|w(t)\|_{L_p}^p \leq Cpe^{Cp}$, it is finally verified that

$$\|w(t)\|_{L_\infty} = \lim_{p\to\infty} \|w(t)\|_{L_p} \leq \lim_{p\to\infty} [Cp]^{\frac{1}{p}} e^C = e^C, \qquad \tau \leq t < T_{\max}.$$

□

7. Numerical Example

In this last section, we will present a numerical example which suggests the blowup of the maximal solution.

Set $\Omega = (0, 10) \times (0, 5)$ and the parameters as $a = b = c = f = g = 1$, $d = 1.6$ and $\mu = 5$, $\nu = 1$. Further, take the initial functions as $u_0(x, y) = v_0(x, y) = 1.6 + \zeta$, $\rho_0(x, y) = 0$, where ζ is a random perturbation between -0.01 and 0.01. We used the explicit difference method for choosing the time step and the spatial step as 5.0×10^{-5} and 0.04, respectively.

The graphs obtained for $\chi(t) = \max_{(x,y)\in\overline{\Omega}} [|\frac{\partial^2\rho}{\partial x^2}(t)| + |\frac{\partial^2\rho}{\partial y^2}(t)|]$ are shown in Figure. In both, the horizontal axis denotes the time variable t and the vertical axis denotes $\chi(t)$.

As we can observe in the two graphs in Figure, after $t = 15.0$, the value of $\chi(t)$ increases very rapidly and our numerical computations quickly collapse, meaning the value of $\chi(t)$ exceeds the numerical limit of the computer and the value of $\delta(t) = \min_{(x,y)\in\overline{\Omega}} [u(t)+v(t)]$ changes from positive to negative discontinuously in a single time step.

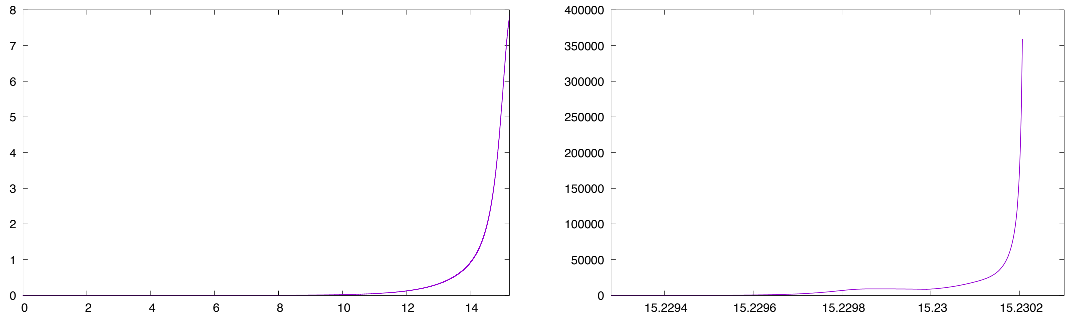

Figure: Graphs of $\chi(t)$

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