ON INSTABILITY OF F-YANG-MILLS CONNECTIONS OVER IRREDUCIBLE SYMMETRIC R-SPACES

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Abstract. The notion of F-Yang-Mills connections gives a generalization of Yang-Mills connections, p-Yang-Mills connections and exponential Yang-Mills connections. Here, F is a strictly increasing C^2-function. In this paper, we study an instability for F-Yang-Mills connections on principal fiber bundles over irreducible symmetric R-spaces. In classical Yang-Mills theory, Simons showed that the non-existence theorem for non-flat, weakly stable Yang-Mills connections over the standard sphere with dimension more than four. Recently, a Simons type instability theorem for F-Yang-Mills connections over the standard sphere was given by Baba-Shintani. The purpose of this paper is to prove that the converse of this theorem does not hold in general. In fact, we give a concrete example of F-Yang-Mills instable, irreducible symmetric R-spaces except for the standard sphere. For this, we first give a sufficient condition for an irreducible symmetric R-space to be F-Yang-Mills instable. Next, by classifying the irreducible symmetric R-spaces satisfying this condition, we find that the standard sphere and the Cayley projective plane are only such irreducible symmetric R-spaces. In particular, the Cayley projective plane is F-Yang-Mills instable.

1. Introduction

In this paper, we will consider *F-Yang-Mills connections*, which are known as a generalization of Yang-Mills connections, p-Yang-Mills connections ([4]) and exponential Yang-Mills connections ([14]). Here, F is a strictly increasing C^2-function defined on the interval $[0, T)$, $0 < T \leq \infty$. An F-Yang-Mills connection is defined by a critical point of the F-Yang-Mills functional defined on the space of connections for a principal fiber bundle over a Riemannian manifold. The study of such connections has progressed by extending the results on the usual Yang-Mills connections such as instability theorems (Simons [17], Kobayashi-Ohnita-Takeuchi [13]) and some types of vanishing theorems (Bourguignon-Lawson [3], Kobayashi-Ohnita-Takeuchi [13]).

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Now, we explain the study of the instability for F-Yang-Mills connections. We first recall the instability theorem for Yang-Mills connections over standard spheres due to Simons ([17]).

Theorem 1.1 ([17], see also [3] for the proof). *For $n > 4$, any non-flat, Yang-Mills connection over the standard sphere S^n is instable.*

In other words, this result gives the non-existence theorem for non-flat, weakly stable Yang-Mills connections over the standard sphere S^n with $n > 4$. On the other hand, self-dual connections and anti-self-dual connections provide us weakly stable Yang-Mills connections over the 4-sphere S^4. Bourguignon-Lawson [3, Theorem B] proved that in the case when the structure group is a specific unitary group, any weakly stable Yang-Mills connection over S^4 is either self-dual or anti-self-dual.

After Simons, Baba-Shintani ([1]) gave a Simons type instability theorem for F-Yang-Mills connections on the standard sphere S^n. In the process, they introduced the notion of a degree for the differential F' (Definition 3.1), which we write $d_{F'}$. Roughly speaking, $d_{F'}$ represents the rate of increase of the ratio of F' and F''. They also clarified that the finiteness of $d_{F'}$ is necessary for deriving the Simons type instability theorem for F-Yang-Mills connections. Then, Theorem 1.1 is naturally generalized as follows.

Theorem 1.2 ([1, Corollary 4.12]). *For $n > 4d_{F'} + 4$, any non-flat, F-Yang-Mills connection over S^n is instable.*

Our motivation for studying F-Yang-Mills connections and their instabilities comes from the topology of closed Riemannian manifolds. A characteristic class of a principal fiber bundle P represents elements in the cohomology groups $H^{2k}(M)$ for the base space M of P by means of the curvatures of connections. Kobayashi-Ohnita-Takeuchi ([13, (2.17) Theorem]) proved that if any non-flat, Yang-Mills connection on P is instable for any principal fiber bundle P over a compact Riemannian manifold M, then the second Betti number of M vanishes. Indeed, by representing the first Chern class of a principal $U(1)$-bundle in terms of the curvature of a Yang-Mills connection, the instability of the Yang-Mills connection plays an important role in their proof. We expect to obtain similar results for the topological vanishing theorem by extending the class of connections from Yang-Mills connections to F-Yang-Mills connections. It is a fundamental problem to characterize the standard sphere in terms of appropriate F-Yang-Mills connections.

Our concern is to study a connected, closed Riemannian manifold M such that for every principal fiber bundle over M, any non-flat F-Yang-Mills connection over M is instable. We call this property the *F-Yang-Mills instability* (Definition 3.2), which is a generalization of the Yang-Mills instability due to Kobayashi-Ohnita-Takeuchi ([13, p. 165]). By Theorem 1.2, the standard sphere S^n is F-Yang-Mills instable if $d_{F'} < (n-4)/4$. We will classify F-Yang-Mills instability Riemannian manifolds. In the case when $F(t) = t$, Kobayashi-Ohnita-Takeuchi gave an example of a Yang-Mills instable Riemannian manifold which is not the standard sphere by studying the Yang-Mills instability of isotropy irreducible Riemannian symmetric spaces of compact type ([13, (7.11) Theorem]). In fact, they gave a sufficient condition for M to be Yang-Mills instable ([13, (7.10) Theorem]) and classified such M ([13, (7.11) Theorem]).

For further progress, we will study a generalization of the results in [13] from Yang-Mills connections to F-Yang-Mills connections. After Kobayashi-Ohnita-Takeuchi, an extension of [13, (7.10) Theorem] was studied by Kawagoe [11] for p-Yang-Mills connections, and by Shintani [16] for F-Yang-Mills connections. However, in both extensions, the result corresponding to Theorem [13, (7.11) Theorem] is an open problem.

In this paper, we find an irreducible symmetric R-space which is F-Yang-Mills instable. For this, we first give an sufficient condition for an irreducible symmetric R-space to be F-Yang-Mills instable (Theorem 4.2). Here, we note that an irreducible symmetric R-space is a kind of a (not necessarily isotropy irreducible) Riemannian symmetric space of compact type and has a nice geometrical characterization as explained later. Based on the classification of irreducible symmetric R-spaces, we obtain the following theorem in terms of Theorem 4.2 and the characterization.

Theorem 1.3 (Corollary 4.15)**.** *Let $F : [0, T) \to \mathbb{R}$ be a strictly increasing C^2-function with $0 \leq d_{F'} < 1/6$. Then, the Cayley projective plane $F_4/Spin(9)$ is F-Yang-Mills instable.*

We emphasize that the irreducible symmetric R-spaces satisfying the sufficient condition given in Theorem 4.2 are only the standard sphere as in Theorem 1.2 and the Cayley projective plane (Theorem 4.14). On the other hand, we can verify that the other irreducible symmetric R-spaces do not satisfy the sufficient condition by using Lemma 4.3.

The organization of this paper is as follows: In Section 2, we review the basics of F-Yang-Mills connections, which are related to the present paper. We recall the definition of the F-Yang-Mills functional (Definition 2.1) and its Euler-Lagrange equation

(Corollary 2.2). Furthermore, we give the second variation formula for the functional (Theorem 2.3). We also recall the notions of F-harmonic forms (Definition 2.2) and their indices (Definition 2.4), which were introduced in [1]. In Section 3, we study the instability of F-Yang-Mills connections over submanifolds. In Subsection 3.1, we follow the method given in [1] for the determination of the instability of F-Yang-Mills connections over submanifolds of Euclidean spaces. Our argument is based on Proposition 3.3. The inequality (3.4) in this proposition gives a sufficient condition for an F-Yang-Mills connection to be instable, which yields Theorem 1.2 (Corollary 3.4). Motivated by Theorem 1.2, we introduce the notion of F-Yang-Mills instability for connected, closed Riemannian manifolds (Definition 3.2). In Subsection 3.2, we study the F-Yang-Mills instability of minimal submanifolds M of the standard sphere S. The above method can be applied to this study. We give a sufficient condition for M to be an F-Yang-Mills instable (Theorem 3.7), which gives a reformulation of [16]. Our result is an extension of Kobayashi-Ohnita-Takeuchi's one [13, (6.9) Theorem] and Kawagoe's one [11, Corollary 6.2] to an F-Yang-Mills version. Here, we note that our sufficient condition (3.11) in Theorem 3.7 is described by not only intrinsic curvatures of M, but also the extrinsic curvature γ of $M \subset S$ defined in Definition 3.3. However, it is difficult to determine the exact value of γ in general. In Section 4, we utilize a nice geometrical characterization of irreducible symmetric R-spaces to overcome this difficulty. We first review Takeuchi-Kobayashi's result ([20]), which states that any irreducible symmetric R-space can be immersed into a specified standard sphere as a minimal submanifold. Next, we apply Theorem 3.7 to irreducible symmetric R-spaces M (Theorem 4.2). We observe that Theorem 4.2 is an extension of [13, (7.10) Theorem] and [11, Corollary 6.2] to F-Yang-Mills version in the case when M is isotropy irreducible (Example 4.1). Using a similar argument, it can be verified that Theorem 4.2 coincides with [16, Theorem 25]. Based on the fact that any irreducible symmetric R-space can be realized as an orbit of the isotropy representation of some Riemannian symmetric space L/K of noncompact type, we can give a formula for determining γ for each orbit of this representation (Proposition 4.8). This formula is derived by means of the polarity of the isotropy representation ([5]) and the restricted root system of L/K. Finally, based on the classification due to Kobayashi-Nagano [12], we determine whether each irreducible symmetric R-spaces satisfies the sufficient condition given in Theorem 4.2 or not (Theorem 4.14). As a corollary of Theorem 4.14, we have Theorem 1.3 (Corollary 4.15). We give a brief review of the restricted root system of L/K in Appendix A.

2. Preliminaries

2.1. **Principal fiber bundles and connections.** Let M be an n-dimensional, connected, closed Riemannian manifold and G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\pi : P \to M$ be a principal fiber bundle over M with structure group G. We denote by $r : P \times G \to P$, $(g, p) \mapsto r(g, p) = r_g(p)$ the right action of G on P. We write the adjoint representation of G as $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. A $\mathfrak{g}$-valued differential 1-form A on P is called a *connection* or a connection 1-form, if it satisfies the following two conditions: (1) A is of type Ad, i.e., $r_g^* A = \mathrm{Ad}(g^{-1})A$ holds for all $g \in G$; (2) $A(X^*) = X$ for all $X \in \mathfrak{g}$, where X^* denotes the fundamental vector field on P associated with X. A $\mathfrak{g}$-valued differential k-form ω on P is said to be *horizontal* if for any $p \in P$, $\omega_p(X_1, \ldots, X_k) = 0$ holds whenever at least one of the tangent vectors $X_i \in T_pP$ is vertical ($d\pi_p(X_i) = 0$). We denote by $\Omega^k_{\mathrm{Ad},hor}(P, \mathfrak{g})$ the vector space of horizontal $\mathfrak{g}$-valued k-forms of type Ad on P. For any two connections A, A', the difference $A - A'$ is in $\Omega^1_{\mathrm{Ad},hor}(P, \mathfrak{g})$. Conversely, it is verified that $A + \alpha$ gives another connection on P for all $\alpha \in \Omega^1_{\mathrm{Ad},hor}(P, \mathfrak{g})$ ([7, Proposition 5.13.2, 1]). Hence the set $\mathscr{C}_P$ of connections on P becomes an affine space over the vector space $\Omega^1_{\mathrm{Ad},hor}(P, \mathfrak{g})$. The kernel of a connection A determines a horizontal, right-invariant distribution on P, which we write $\mathcal{H}$. We denote by $\pi_{\mathcal{H}} : TP \to \mathcal{H}$ the natural projection. The *curvature 2-form* R^A of A is defined by $R^A(X_1, X_2) = dA(\pi_{\mathcal{H}}(X_1), \pi_{\mathcal{H}}(X_2))$ for tangent vectors X_1, X_2 of P. Then R^A is an element of $\Omega^2_{\mathrm{Ad},hor}(P, \mathfrak{g})$. It is known that the distribution $\mathcal{H}$ is integrable if and only if R^A vanishes. A connection is said to be *flat*, if its curvature 2-form vanishes.

We make use of a different description of connections on P. Denote by $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ the adjoint bundle of P, that is, the associated vector bundle of P with $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. It follows from [7, Theorem 5.13.4] that $\Omega^k_{\mathrm{Ad},hor}(P, \mathfrak{g})$ is canonically isomorphic with the vector space $\Omega^k(\mathfrak{g}_P) = \Gamma(\Lambda^k T^*M \otimes \mathfrak{g}_P)$ of k-forms on M with values in $\mathfrak{g}_P$. Any connection on P corresponds to a connection on $\mathfrak{g}_P$, i.e., a covariant derivative $\nabla : \Gamma(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$. We also write its curvature 2-form as R^∇. It is shown that the curvature R^∇ of ∇ on $\mathfrak{g}_P$ is in $\Omega^2(\mathfrak{g}_P)$ (cf. [7, Proposition 5.13.2, 2]). In what follows, we identify $\mathscr{C}_P$ with the set of connections on $\mathfrak{g}_P$, which is an affine space over the vector space $\Omega^1(\mathfrak{g}_P)$.

We give a fiber metric on $\mathfrak{g}_P$ which is compatible with connections on $\mathfrak{g}_P$. Such a fiber metric is induced from an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ (cf. [7, Proposition 5.9.7]). In addition, $\langle \cdot, \cdot \rangle$ also induces a pointwise inner product on the vector space $\Omega^k(\mathfrak{g}_P)$, which is denoted by the same symbol $\langle \cdot, \cdot \rangle$. We set $\|\varphi\|^2 = \langle \varphi, \varphi \rangle$

for $\varphi \in \Omega^k(\mathfrak{g}_P)$. Here, we write $\langle \varphi, \psi \rangle$ ($\varphi, \psi \in \Omega^k(\mathfrak{g}_P)$) by means of their components. We take an orthonormal basis $(e_1, \dots, e_n)$ of the tangent space T_xM ($x \in M$) and denote by $(\theta^1, \dots, \theta^n)$ its dual basis. If we write

$$\varphi = \frac{1}{k!} \sum_{i_1,\dots,i_k} \varphi_{e_{i_1},\dots,e_{i_k}} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \,, \quad \psi = \frac{1}{k!} \sum_{i_1,\dots,i_k} \psi_{e_{i_1},\dots,e_{i_k}} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \,,$$

then we obtain

$$\langle \varphi, \psi \rangle = \frac{1}{k!} \sum_{i_1,\dots,i_k} \langle \varphi_{e_{i_1},\dots,e_{i_k}}, \psi_{e_{i_1},\dots,e_{i_k}} \rangle \,.$$

By integrating the pointwise inner product over M, we get an inner product on $\Omega^k(\mathfrak{g}_P)$ as follows:

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle dv, \quad \varphi, \psi \in \Omega^k(\mathfrak{g}_P) \,,$$

where dv denotes the Riemannian volume form on M.

For any connection ∇ on $\mathfrak{g}_P$, the covariant exterior derivative $d^\nabla : \Omega^k(\mathfrak{g}_P) \to \Omega^{k+1}(\mathfrak{g}_P)$ is defined in the natural manner (cf. [7, Definition 5.12.3]). Then it is well-known that the curvature 2-form R^∇ satisfies $d^\nabla R^\nabla = 0$, which is called the *Bianchi identity*. In general, $d^\nabla \circ d^\nabla$ does not vanish. It is verified that if ∇ is flat, then $d^\nabla \circ d^\nabla = 0$ holds. This is an alternative interpretation of flat connections. We denote by δ^∇ the formal adjoint operator of d^∇, that is, $\delta^\nabla : \Omega^k(\mathfrak{g}_P) \to \Omega^{k-1}(\mathfrak{g}_P)$ is defined by $(d^\nabla \psi, \varphi) = (\psi, \delta^\nabla \varphi)$ for $\psi \in \Omega^{k-1}(\mathfrak{g}_P)$ and $\varphi \in \Omega^k(\mathfrak{g}_P)$. Hodge-Laplacian is defined by $\Delta^\nabla = \delta^\nabla \circ d^\nabla + d^\nabla \circ \delta^\nabla$, which gives a differential operator on $\Omega^k(\mathfrak{g}_P)$. A $\mathfrak{g}_P$-valued form φ is called a harmonic form, if $\Delta^\nabla \varphi = 0$ holds. It is verified that $\Delta^\nabla \varphi = 0$ is equivalent to $d^\nabla \varphi = 0$ and $\delta^\nabla \varphi = 0$.

A *Yang-Mills connection* ∇ is defined as a critical point of the Yang-Mills functional

$$\mathscr{YM} : \mathscr{C}_P \to \mathbb{R}; \ \nabla \mapsto \frac{1}{2} \int_M \|R^\nabla\|^2 dv \,.$$

It is shown that its Euler-Lagrange equation is give by $\delta^\nabla R^\nabla = 0$. This equation is called the *Yang-Mills equation*. The Bianchi identity and the Yang-Mills equation imply that the curvature 2-form of any Yang-Mills connection becomes a harmonic form.

2.2. ***F*-Yang-Mills connections and the first variational formula.** In this subsection, we first recall the notion of F-Yang-Mills connections, which is an extension of Yang-Mills connections (Definition 2.1). Second, we recall the notion of F-harmonic forms (Definition 2.2). This notion is an extension of harmonic forms.

Let $0 < T \leq \infty$ and $F : [0, T) \to \mathbb{R}$ be a strictly increasing C^2-function.

Definition 2.1. The *F-Yang-Mills functional* $\mathscr{YM}_F : \mathscr{C}_P \to \mathbb{R}$ is defined by

$$\mathscr{YM}_F(\nabla) = \int_M F(\frac{1}{2}\|R^\nabla\|^2)dv\,.$$

A connection ∇ on $\mathfrak{g}_P$ is called an *F-Yang-Mills connection*, if ∇ is a critical point of $\mathscr{YM}_F$. Then, its curvature 2-form is also called the F-Yang-Mills field of ∇.

For example, if we take $F(t) = t$, then the corresponding F-Yang-Mills functional coincides with the usual Yang-Mills functional $\mathscr{YM}$. Furthermore, we recall two types of F-Yang-Mills connections as follows.

Example 2.1. (1) Let $p \geq 2$. If we put $F_p(t) = (1/p)(2t)^{p/2}$, then the F_p-Yang-Mills functional coincides with the p-Yang-Mills functional (cf. [4]). A critical point of the p-Yang-Mills functional is called a *p-Yang-Mills connection*. It is clear that, for $p = 2$, a 2-Yang-Mills connection is the usual Yang-Mills one. (2) If we put $F_e(t) = e^t$, then the F_e-Yang-Mills functional coincides with the exponential Yang-Mills functional (cf. [14]). A critical point of the exponential Yang-Mills functional is called an *exponential Yang-Mills connection*.

F-Yang-Mills connections are obtained by solving the Euler-Lagrange equation for $\mathscr{YM}_F$. Here, we recall its first variation formula as follows.

Proposition 2.1 ([6, Lemma 3.1], [10, (11)]). *Let ∇^t ($|t| < \varepsilon$) be a C^∞-curve in $\mathscr{C}_P$ with $\nabla^0 = \nabla$. If we put*

$$\alpha = \frac{d}{dt}\bigg|_{t=0} \nabla^t \in \Omega^1(\mathfrak{g}_P)\,,$$

then we have

$$\frac{d}{dt}\bigg|_{t=0} \mathscr{YM}_F(\nabla^t) = \int_M \langle \delta^\nabla(F'(\frac{1}{2}\|R^\nabla\|^2)R^\nabla), \alpha\rangle dv\,.$$

From this proposition, we get:

Corollary 2.2. *A connection ∇ is an F-Yang-Mills connection if and only if ∇ satisfies*

$$\delta^\nabla(F'(\frac{1}{2}\|R^\nabla\|^2)R^\nabla) = 0\,. \tag{2.1}$$

We call the equation (2.1) the *F-Yang-Mills equation*. Clearly, if we take $F(t) = t$, then (2.1) becomes the usual Yang-Mills equation.

Motivated by the F-Yang-Mills equation, Baba-Shintani ([1]) introduced the notion of F-harmonic forms for $\mathfrak{g}_P$-valued forms as follows.

Definition 2.2 ([1, Definition 3.5]). A $\mathfrak{g}_P$-valued form φ is said to be *F-harmonic*, if φ satisfies the following two equations:

$$d^\nabla \varphi = 0\,, \quad \delta^\nabla (F'(\frac{1}{2}\|\varphi\|^2)\varphi) = 0\,. \tag{2.2}$$

For simplicity, φ is said to be *p-harmonic*, if φ is F_p-harmonic, where the function F_p is defined in Example 2.1, (1). Then the corresponding second equation (2.2) is rewritten as $\delta^\nabla(\|\varphi\|^{p-2}\varphi) = 0$.

We note that the curvature 2-form R^∇ of an F-Yang-Mills connection ∇ is F-harmonic.

2.3. **Instability and the second variational formula.** We first recall the notion of a weak stability of an F-Yang-Mills connection.

Definition 2.3. An F-Yang-Mills connection ∇ is said to be weakly stable, if the following inequality holds for any $\alpha \in \Omega^1(\mathfrak{g}_P)$:

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathscr{YM}_F(\nabla^t) \geq 0\,, \quad \alpha = \frac{d}{dt}\bigg|_{t=0} \nabla^t\,.$$

An F-Yang-Mills connection is said to be *instable*, if it is not weakly stable.

For the study of the instability of F-Yang-Mills connections, we give the second variational formula for the F-Yang-Mills functional. For the preparation, we recall the definition of the (first order) Weitzenböck curvature $\mathfrak{R}^\nabla : \Omega^1(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$ for a connection ∇ as follows:

$$\mathfrak{R}^\nabla(\alpha) = \sum_{i,j} [R^\nabla_{ji}, \alpha_j]\theta^i\,, \quad \alpha \in \Omega^1(\mathfrak{g}_P)\,,$$

where α and R^∇ are locally expressed as

$$\alpha = \sum_j \alpha_j \theta^j\,, \quad R^\nabla = \frac{1}{2}\sum_{j,i} R^\nabla_{ji}\theta^j \wedge \theta^i\,.$$

We set

$$[\cdot \wedge \cdot] : \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \to \Omega^2(\mathfrak{g}_P)\,; \quad [\alpha \wedge \beta]_{X,Y} = [\alpha_X, \beta_Y] - [\alpha_Y, \beta_X]\,.$$

By the adjoint invariance of $\langle \cdot, \cdot \rangle$, we have:

$$\langle \mathfrak{R}^\nabla(\alpha), \alpha \rangle = \langle [\alpha \wedge \alpha], R^\nabla \rangle, \quad \alpha \in \Omega^1(\mathfrak{g}_P)\,.$$

In order to describe the second variational formula for the F-Yang-Mills functional, we recall the definition of the index form for an F-harmonic 2-form as follows.

Definition 2.4 ([1, Definition 3.8]). The *index form* of an F-harmonic 2-form $\varphi \in \Omega^2(\mathfrak{g}_P)$ is defined by

$$I_\varphi(\alpha) = \int_M F''(\frac{1}{2}\|\varphi\|^2)\langle d^\nabla \alpha, \varphi\rangle^2 dv + \int_M F'(\frac{1}{2}\|\varphi\|^2)\left\{\langle \mathfrak{R}^\nabla(\alpha), \alpha\rangle + \|d^\nabla \alpha\|^2\right\} dv\,, \quad (2.3)$$

for any $\alpha \in \Omega^1(\mathfrak{g}_P)$.

Then we have the second variational formula as follows.

Theorem 2.3 ([1, Proposition 3.7]). *Let ∇ be an F-Yang-Mills connection and ∇^t ($|t| < \varepsilon$) be a C^∞-curve in $\mathscr{C}_P$ with $\nabla^0 = \nabla$. Then we have:*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathscr{YM}_F(\nabla^t) = I_{R^\nabla}(\alpha)\,,$$

where $\alpha = \left.\dfrac{d}{dt}\right|_{t=0} \nabla^t$.

An alternative expression of the second variation formula is found in [10, (20)]. The difference between them is the integrand of the second term of $I_\varphi(\alpha)$ defined in (2.3) with $\varphi = R^\nabla$. Our formula is more appropriate to determine the instability of an F-Yang-Mills connection.

For the curvature 2-form $R^\nabla \in \Omega^2(\mathfrak{g}_P)$ of a weakly stable F-Yang-Mills connection ∇, Theorem 2.3 yields $I_{R^\nabla}(\alpha) \geq 0$ for any $\alpha \in \Omega^1(\mathfrak{g}_P)$.

By using the second variational formula, we can verify that any flat connection is weakly stable. Indeed, if ∇ is a flat connection, then we have $\langle \mathfrak{R}^\nabla(\alpha), \alpha\rangle = 0$, from which, for any $\alpha \in \Omega^1(\mathfrak{g}_P)$, we obtain

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathscr{YM}_F(\nabla^t) = \int_M F'(0)\|d^\nabla \alpha\|^2 \geq 0\,,$$

where ∇^t is a C^∞-curve in $\mathscr{C}_P$ with $\nabla^0 = \nabla$ and $\alpha = (d/dt)|_{t=0}\nabla^t$.

3. Instability of F-Yang-Mills connections over submanifolds

In this section, we study the instability of F-Yang-Mills connections over submanifolds. In Subsection 3.1, we review the result of [1] for the instability of an F-Yang-Mills connection over a connected, closed Riemannian manifold isometrically immersed in a Euclidean space. We briefly give the derivation of the Simons type instability theorem stated in Theorem 1.2 (Corollary 3.4). Then, motivated by this corollary, we introduce the notion of an F-Yang-Mills instability for a connected, closed Riemannian manifold (Definition 3.2). This notion is a natural extension of the Yang-Mills instability due to Kobayashi-Ohnita-Takeuchi ([13]). In Subsection 3.2, we rewrite the result of [16] in terms of our notion of the F-Yang-Mills instability (Theorem 3.7). This theorem is a natural extension of Kobayashi-Ohnita-Takeuchi's result [13, (6.9) Theorem] to an F-Yang-Mills version.

3.1. **Submanifolds of Euclidean spaces.** Let M be an n-dimensional, connected, closed Riemannian manifold and P be a principal fiber bundle over M with structure group G. Suppose that M is isometrically immersed in an N-dimensional Euclidean space $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$ with $n < N$. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C \leq N\,, \quad 1 \leq i, j, k, l, m \leq n\,, \quad n+1 \leq \mu \leq N\,.$$

Let $(e_1, \dots, e_n)$ be an orthonormal basis of the tangent space T_xM ($x \in M$). We denote by $T_x^\perp M$ the normal space of the submanifold $M \subset \mathbb{R}^N$ and by h the second fundamental form of $M \subset \mathbb{R}^N$. Let $(e_{n+1}, \dots, e_N)$ be an orthonormal basis of $T_x^\perp M$. Let h_{ij}^μ denote the component of $h(e_i, e_j) = \sum_\mu h_{ij}^\mu e_\mu$. We denote by $H = \sum_i h(e_i, e_i)$ the mean curvature vector of $M \subset \mathbb{R}^N$. We set $H^\mu = \langle \sum_i h(e_i, e_i), e_\mu\rangle = \sum_i h_{ii}^\mu$. Let $(E_1, \dots, E_N)$ denote the canonical basis of $\mathbb{R}^N$. We denote by V_A the tangent component of E_A with respect to the orthogonal decomposition $\mathbb{R}^N = T_xM \oplus T_x^\perp M$. For $\varphi \in \Omega^2(\mathfrak{g}_P)$, $\iota_{V_A}\varphi$ gives an element of $\Omega^1(\mathfrak{g}_P)$, where ι denotes the interior product of M.

The following lemma is fundamental in our argument.

Lemma 3.1. *Let ∇ be an F-Yang-Mills connection and $\varphi = R^\nabla$ denote the curvature 2-form of ∇. If the inequality*

$$\sum_A I_\varphi(\iota_{V_A}\varphi) < 0 \tag{3.1}$$

holds, then ∇ is instable.

Proof. We prove this lemma by contraposition. For a weakly stable F-Yang-Mills connection ∇, we have $I_\varphi(\iota_{V_A}\varphi) \geq 0$ for each A, from which $\sum_A I_\varphi(\iota_{V_A}\varphi) \geq 0$ holds. □

Our concern is to find a sufficient condition for the inequality (3.1). For this, we shall calculate the summation $\sum_A I_\varphi(\iota_{V_A}\varphi)$. Let $(\theta^1, \dots, \theta^n)$ be the dual basis of $(e_1, \dots, e_n)$. Kobayashi-Ohnita-Takeuchi ([13, (4.36)]) introduced $R(\varphi, \varphi)$ and $\mathrm{Ric}(\varphi, \varphi)$ for $\varphi \in \Omega^2(\mathfrak{g}_P)$ as follows: If we write $\varphi = (1/2)\sum_{i,j} \varphi_{ij}\theta^i \wedge \theta^j$, then

$$R(\varphi, \varphi) = \sum_{i,j,k,l} R_{ijkl}\langle \varphi_{ij}, \varphi_{kl}\rangle\,, \quad \mathrm{Ric}(\varphi, \varphi) = \sum_{i,j,k,l} R_{ik}\delta_{jl}\langle \varphi_{ij}, \varphi_{kl}\rangle\,,$$

where R_{ijkl} and R_{ik} are the components of the Riemann curvature R and the Ricci curvature Ric on M, respectively, that is, $R(e_k, e_l)e_j = \sum_i R^i_{jkl} e_i = \sum_i R_{ijkl} e_i$ and $R_{ik} = \sum_l R_{lkli}$. By the definition, $R(\varphi, \varphi)$ and $\mathrm{Ric}(\varphi, \varphi)$ are independent of the choice of $(e_1, \ldots, e_n)$. They ([13, (4.36)]) also introduced $H(\varphi, \varphi)$ by

$$H(\varphi, \varphi) = \sum_{i,j,k,l} \sum_{\mu} H^{\mu} h^{\mu}_{ik} \delta_{jl} \langle \varphi_{ij}, \varphi_{kl} \rangle \,.$$

In addition, for the study of the instability of F-Yang-Mills connections, we make use of the following quantity ([1, Definition 4.2]):

$$\boldsymbol{h}_1(\varphi, \varphi) = \sum_{\mu} h^{\mu}_1(\varphi, \varphi) e_{\mu} \,, \quad h^{\mu}_1(\varphi, \varphi) = \sum_{i,j,k,l} h^{\mu}_{ik} \delta_{jl} \langle \varphi_{ij}, \varphi_{kl} \rangle \,.$$

It is verified that $H(\varphi, \varphi)$ and $\boldsymbol{h}_1(\varphi, \varphi)$ are independent of the choice of $(e_1, \ldots, e_n)$ and $(e_{n+1}, \ldots, e_N)$. Furthermore, for each μ, the component $h^{\mu}_1(\varphi, \varphi)$ of $\boldsymbol{h}_1(\varphi, \varphi)$ is also independent of the choice of $(e_1, \ldots, e_n)$. Here, we note that the original definition of $R(\varphi, \varphi)$, $\mathrm{Ric}(\varphi, \varphi)$ and $H(\varphi, \varphi)$ are defined by means of the inner product $(\cdot, \cdot)$ instead of $\langle \cdot, \cdot \rangle$.

Under the above setting, we have the following proposition.

Proposition 3.2 ([1, Theorem 4.3])**.** *For any F-harmonic 2-form $\varphi \in \Omega^2(\mathfrak{g}_P)$, we have:*

$$\sum_A I_{\varphi}(\iota_{V_A}\varphi) = \int_M F''(\frac{1}{2}\|\varphi\|^2) \langle \boldsymbol{h}_1(\varphi, \varphi), \boldsymbol{h}_1(\varphi, \varphi) \rangle dv + \int_M F'(\frac{1}{2}\|\varphi\|^2) \left\{ H(\varphi, \varphi) - 2\mathrm{Ric}(\varphi, \varphi) + R(\varphi, \varphi) \right\} dv \,. \quad (3.2)$$

In order to evaluate the relation between $F'(\|\varphi\|^2/2)$ and $F''(\|\varphi\|^2/2)$ in (3.2), Baba-Shintani ([1]) introduced the notion of a degree for F' as follows.

Definition 3.1 ([1, Definition 4.2]). Let $0 < T \leq \infty$ and $F : [0, T) \to \mathbb{R}$ be a strictly increasing C^2-function defined on $[0, T)$. The *degree* of F' is defined by

$$d_{F'} = \sup_{0<t<T} \frac{tF''(t)}{F'(t)} \,,$$

which may take infinite values.

For example, if we take $F(t) = t$, then we have $d_{F'} = 0$. For the functions F_p $(p \geq 2)$ and F_e defined in Example 2.1, we have $d_{F'_p} = (p-2)/2$ and $d_{F'_e} = \infty$.

Following to the argument in [1, Subsection 4.2], we set $B(\varphi,\varphi)$ for $\varphi \in \Omega^2(\mathfrak{g}_P)$ as follows:

$$B(\varphi,\varphi) = d_{F'}\langle \boldsymbol{h}_1(\varphi,\varphi), \boldsymbol{h}_1(\varphi,\varphi)\rangle + \frac{\|\varphi\|^2}{2}\left\{H(\varphi,\varphi) - 2\mathrm{Ric}(\varphi,\varphi) + R(\varphi,\varphi)\right\}. \quad (3.3)$$

Then, Proposition 3.2 yields the following result.

Proposition 3.3 ([1, Theorem 4.10]). *Let M be a connected, closed Riemannian manifold isometrically immersed in $\mathbb{R}^N$. Assume that the degree $d_{F'}$ is finite. Then, for any non-zero F-harmonic form $\varphi \in \Omega^2(\mathfrak{g}_P)$, if the inequality*

$$B(\varphi,\varphi) < 0 \quad (3.4)$$

holds, then $\sum_A I_\varphi(\iota_{V_A}\varphi) < 0$ holds.

In the case when M is the n-dimensional standard sphere $S^n(r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$ of radius r about the origin, we have $B(\varphi,\varphi) = (1/r^2)(4d_{F'} + 4 - n)\|\varphi\|^4$. Hence, Proposition 3.3 and Lemma 3.1 imply the following corollary.

Corollary 3.4 ([1, Corollary 4.12]). *If the inequality*

$$n > 4d_{F'} + 4 \quad (3.5)$$

holds, then any non-flat, F-Yang-Mils connection over $S^n(r)$ is instable.

This corollary is an extension of the instability theorems of Yang-Mills connections (Simons [17]) and p-Yang-Mills connections (Chen-Zhou [4, Corollary 4.2]). On the other hand, we can find some observations of the instability of F-Yang-Mills connections with $d_{F'} = \infty$ (for example, see [1, Propositions 4.13 and 4.14]).

As shown in Corollary 3.4, the inequality (3.5) is independent of the choice of non-flat, F-Yang-Mills connections over the standard sphere $S^n(r)$. Motivated by such a property of $S^n(r)$, we introduce the following notion.

Definition 3.2. A connected, closed Riemannian manifold M is said to be *F-Yang-Mills instable*, if for any principal fiber bundle P over M, any non-flat, F-Yang-Mills connection on $\mathfrak{g}_P$ over M is instable. For simplicity, M is said to be *p-Yang-Mills instable*, if M is F_p-Yang-Mills instable, where the function F_p is defined in Example 2.1, (1).

This notion is an extension of the Yang-Mills instability in the sense of Kobayashi-Ohnita-Takeuchi ([13, p. 165]). Corollary 3.4 means that $S^n(r)$ satisfying (3.5) is F-Yang-Mills instable. Our concern is the converse of Corollary 3.4, namely, whether an F-Yang-Mills instable, connected, closed Riemannian manifold is isomorphic to the standard sphere in some way. The aim of this paper is to give F-Yang-Mills instable, connected, closed Riemannian manifolds except for the standard sphere.

3.2. **Minimal submanifolds of standard spheres.** In this subsection, we give a sufficient condition for a connected, closed minimal submanifold of the standard sphere to be F-Yang-Mills instable.

Let M be a connected, closed Riemannian manifold isometrically immersed in $\mathbb{R}^N$. Suppose that M is a minimal submanifold of $S^{N-1}(r)$. For any F-harmonic form $\varphi \in \Omega^2(\mathfrak{g}_P)$, we will rewrite the inequality (3.4) by means of some kinds of curvatures of M. For this purpose, we first evaluate $R(\varphi, \varphi)$ and $\mathrm{Ric}(\varphi, \varphi)$ by means of [13, (6.9) Theorem]. Let σ be the Riemann curvature operator of M and ρ be the Ricci curvature operator of M. Here, for each $x \in M$, the two operators $\sigma_x : \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ and $\rho_x : T_xM \to T_xM$ are given as follows:

$$\langle \sigma_x(X \wedge Y), Z \wedge W \rangle = \langle R(X, Y)W, Z \rangle\,, \quad \langle \rho_x(X), Y \rangle = \mathrm{Ric}(X, Y)\,,$$

where the inner product $\langle \cdot, \cdot \rangle$ on $\Lambda^2(T_xM)$ is $\langle X \wedge Y, Z \wedge W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle$ for tangent vectors X, Y, Z, W of M. Then we have $\langle \sigma_x(X \wedge Y), Z \wedge W \rangle = \langle X \wedge Y, \sigma_x(Z \wedge W) \rangle$ and $\langle \rho_x(X), Y \rangle = \langle X, \rho_x(Y) \rangle$, so that σ_x and ρ_x are diagonalizable over $\mathbb{R}$. We denote by s_x the maximum eigenvalue of σ_x and by c_x the minimum eigenvalue of ρ_x. We set

$$s = \sup_{x \in M} s_x\,, \quad c = \inf_{x \in M} c_x\,.$$

Then the following lemma holds.

Lemma 3.5 ([13, (6.7), (6.8)]). *For any $\varphi \in \Omega^2(\mathfrak{g}_P)$, we have $R(\varphi, \varphi) \leq 4s\|\varphi\|^2$ and $\mathrm{Ric}(\varphi, \varphi) \geq 2c\|\varphi\|^2$.*

Next, we calculate $H(\varphi, \varphi)$ by means of the minimality of $M \subset S^{N-1}(r)$. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of T_xM $(x \in M)$ and $(e_{n+1}, \ldots, e_{N-1}, e_N)$ be an orthonormal basis of the normal space $T_x^\perp M$ in $\mathbb{R}^N$ such that e_N is inward normal to $S^{N-1}(r)$. Then we have $h_{ij}^N = (1/r)\delta_{ij}$, from which $H^N = n/r$ holds. In addition, by the minimality of M in $S^{N-1}(r)$, we have $H^\mu = 0$ for $\mu = n+1, \ldots, N-1$. Hence we obtain

$$H(\varphi, \varphi) = \sum_{i,j,k,l} \frac{n}{r^2} \delta_{ik} \delta_{jl} \langle \varphi_{ij}, \varphi_{kl} \rangle = \frac{2n}{r^2} \|\varphi\|^2\,. \tag{3.6}$$

We will evaluate

$$\langle \boldsymbol{h}_1(\varphi,\varphi), \boldsymbol{h}_1(\varphi,\varphi)\rangle = \sum_{\mu=n+1}^{N-1} |h_1^\mu(\varphi,\varphi)|^2 + |h_1^N(\varphi,\varphi)|^2 \,, \tag{3.7}$$

by means of the principal curvatures of $M \subset S^{N-1}(r)$. Let $\tilde{h}$ denote the second fundamental form of $M \subset S^{N-1}(r)$ and $\tilde{A}_\xi$ denote the shape operator of $M \subset S^{N-1}(r)$ associated to $\xi \in \tilde{T}_x^\perp M$. The relation $\langle \tilde{h}(X,Y), \xi\rangle = \langle \tilde{A}_\xi(X), Y\rangle$ holds for $X, Y \in T_xM$ and $\xi \in \tilde{T}_x^\perp M$. In order to calculate the first term of the right hand side of (3.7), we introduce the following nonnegative constant γ.

Definition 3.3. We set

$$\gamma = \sup_{x\in M} \gamma_x \,, \quad \gamma_x = \sup\{\|\tilde{A}_\xi\| \mid \xi \in \tilde{T}_x^\perp M,\ \|\xi\| = 1\} \,,$$

where $\|\tilde{A}_\xi\|$ denotes the spectral norm of $\tilde{A}_\xi$.

By the definition, for each $\xi \in \tilde{T}_x^\perp M$ with $\|\xi\| = 1$, the following inequality holds:

$$|\lambda_{\xi,i}| \leq \|\tilde{A}_\xi\| \leq \gamma \,, \quad 1 \leq i \leq n \,, \tag{3.8}$$

where $\lambda_{\xi,1}, \ldots, \lambda_{\xi,n}$ are the eigenvalues of $\tilde{A}_\xi$. Then we have the following lemma.

Lemma 3.6. *For any $\varphi \in \Omega^2(\mathfrak{g}_P)$, we have:*

$$\langle \boldsymbol{h}_1(\varphi,\varphi), \boldsymbol{h}_1(\varphi,\varphi)\rangle \leq 4\left\{(N-n-1)\gamma^2 + \frac{1}{r^2}\right\} \|\varphi\|^4 \,.$$

Proof. A direct calculation shows

$$h_1^N(\varphi,\varphi) = \sum_{i,j,k,l} \frac{1}{r}\delta_{ik}\delta_{jl}\langle \varphi_{ij}, \varphi_{kl}\rangle = \frac{2}{r}\|\varphi\|^2 \,. \tag{3.9}$$

For each $\mu = n+1, \ldots, N-1$, we take an orthonormal basis $(u_1^{(\mu)}, \ldots, u_n^{(\mu)})$ of T_xM which diagonalizes $\tilde{A}_{e_\mu}$, namely, $\tilde{A}_{e_\mu} u_i^{(\mu)} = \lambda_{e_\mu,i} u_i^{(\mu)}$, where $\lambda_{e_\mu,i}$'s are the eigenvalues of $\tilde{A}_\xi$. Then we get $\langle \tilde{h}(u_i^{(\mu)}, u_k^{(\mu)}), e_\mu\rangle = \lambda_{e_\mu,i}\delta_{ik}$. As mentioned before, $h_1^\mu(\varphi,\varphi)$ is independent of the choice of orthonormal bases of T_xM. Hence $h_1^\mu(\varphi,\varphi)$ is calculated by means of $(u_1^{(\mu)}, \ldots, u_n^{(\mu)})$ as follows:

$$\begin{aligned} h_1^\mu(\varphi,\varphi) &= \sum_{i,j,k,l} \langle \tilde{h}(u_i^{(\mu)}, u_k^{(\mu)}), e_\mu\rangle \langle u_j^{(\mu)}, u_l^{(\mu)}\rangle \langle \varphi_{u_i^{(\mu)}, u_j^{(\mu)}}, \varphi_{u_k^{(\mu)}, u_l^{(\mu)}}\rangle \\ &= \sum_{i,j} \lambda_{e_\mu,i} \langle \varphi_{u_i^{(\mu)}, u_j^{(\mu)}}, \varphi_{u_i^{(\mu)}, u_j^{(\mu)}}\rangle \,. \end{aligned}$$

In addition, by (3.8), we obtain

$$|h_1^\mu(\varphi,\varphi)| \leq \gamma \sum_{i,j} \langle \varphi_{u_i^{(\mu)},u_j^{(\mu)}}, \varphi_{u_i^{(\mu)},u_j^{(\mu)}} \rangle = 2\gamma \|\varphi\|^2 . \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.7), we have the assertion. □

From the above argument, we obtain the following theorem, which is a reformulation of [16, Theorem 25] in terms of the notion of the F-Yang-Mills instability.

Theorem 3.7. *Let M be an n-dimensional, connected, closed, immersed minimal submanifold of $S^{N-1}(r)$. Suppose that $d_{F'}$ is nonnegative. Then, if the inequality*

$$4d_{F'} \left\{ (N-n-1)\gamma^2 + \frac{1}{r^2} \right\} + \frac{n}{r^2} - 2c + 2s < 0 \tag{3.11}$$

holds, then we have $\sum_A I_\varphi(\iota_{V_A}\varphi) < 0$ holds for all non-zero F-harmonic form $\varphi \in \Omega^2(\mathfrak{g}_P)$. In particular, (3.11) implies that M is F-Yang-Mills instable.

Proof. It follows from Lemma 3.5 and (3.6) that the following inequality holds:

$$H(\varphi,\varphi) - 2\mathrm{Ric}(\varphi,\varphi) + R(\varphi,\varphi) \leq 2\left(\frac{n}{r^2} - 2c + 2s \right) \|\varphi\|^2 .$$

We also have:

$$d_{F'} \langle \boldsymbol{h}_1(\varphi,\varphi), \boldsymbol{h}_1(\varphi,\varphi) \rangle \leq 4d_{F'} \left\{ (N-n-1)\gamma^2 + \frac{1}{r^2} \right\} \|\varphi\|^4 .$$

Here, we have used that $d_{F'}$ is nonnegative. Substituting the above two inequalities into (3.3), we get:

$$B(\varphi,\varphi) \leq \left[4d_{F'} \left\{ (N-n-1)\gamma^2 + \frac{1}{r^2} \right\} + \frac{n}{r^2} - 2c + 2s \right] \|\varphi\|^4 ,$$

from which the assumption (3.11) yields $B(\varphi,\varphi) < 0$. Thus, by Proposition 3.3, we have the assertion. □

Our concern is to find an connected, closed, minimal submanifold satisfying (3.11). In fact, we give such a submanifold in the next section.

Theorem 3.7 is an extension of Kobayashi-Ohnita-Takeuchi's result [13, (6.9) Theorem] for harmonic forms to F-harmonic forms. Applying Theorem 3.7 to the function $F = F_p$ defined in Example 2.1, (1), the inequality (3.11) is rewritten as follows:

$$2(p-2) \left\{ (N-n-1)\gamma^2 + \frac{1}{r^2} \right\} + \frac{n}{r^2} - 2c + 2s < 0 . \tag{3.12}$$

This inequality gives a sufficient condition for a connected, closed, minimal submanifold of $S^{N-1}(r)$ to be p-Yang-Mills instable. We find an alternative formula for this due to

Kawagoe [11, Theorem 6.1]. There is a slight difference between them in the definition of γ. When our concern in the upper bound on p for (3.12), our result gives a refinement of [11, Theorem 6.1].

4. Instability of F-Yang-Mills connections over irreducible symmetric R-spaces

In this section, we give an irreducible symmetric R-space which is F-Yang-Mills instable in the sense of Definition 3.2. In Subsection 4.1, we first recall the basics of the canonical imbedding f of an irreducible symmetric R-space M. Due to Takeuchi-Kobayashi ([20]), the image $f(M)$ becomes a minimal submanifold of a specified standard sphere (Proposition 4.1). Next, we rewrite (3.11) as (4.2) by applying Theorem 3.7 to irreducible symmetric R-spaces (Theorem 4.1). In Subsection 4.2, we classify irreducible symmetric R-spaces satisfying (4.2) (Theorem 4.14). Then we find that the Cayley projective plane is F-Yang-Mills instable (Corollary 4.15), which gives our proof of Theorem 1.3 stated in Introduction.

4.1. **A sufficient condition for the F-Yang-Mills instability.** Let L be a connected, semisimple Lie group with trivial center and U be a parabolic subgroup of L. The homogeneous space $M = L/U$ is called an R-space. Let $\mathfrak{l}$ and $\mathfrak{u}$ be the Lie algebras of L and U, respectively. Then there exists a hyperbolic element J of $\mathfrak{l}$ (that is, $\mathrm{ad}(J) \in \mathrm{End}(\mathfrak{l})$ is diagonalizable over $\mathbb{R}$) satisfying $\mathfrak{u} = \sum_{\lambda \geq 0} \mathfrak{l}^\lambda$, where the summation ranges over all the nonnegative eigenvalues λ of $\mathrm{ad}(J)$ and $\mathfrak{l}^\lambda (\subset \mathfrak{l})$ denotes the eigenspace of $\mathrm{ad}(J)$ associated to λ. It is shown that there exists a maximal compact subgroup K of L such that J is orthogonal to the Lie algebra $\mathfrak{k}$ of K with respect to the Killing form $(\cdot,\cdot)_{\mathfrak{l}}$ of $\mathfrak{l}$. Then the homogeneous space L/K becomes a Riemannian symmetric space of noncompact type in a natural manner. We have the Cartan decomposition of $\mathfrak{l}$ associated to $\mathfrak{k}$, which we write $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$. Here, $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{l}$ with respect to $(\cdot,\cdot)_{\mathfrak{l}}$. In particular, J is in $\mathfrak{p}$. We denote by $\mathrm{Ad}_L : L \to \mathrm{GL}(\mathfrak{l})$ the adjoint representation of L. Since we have $\mathrm{Ad}_L(k)\mathfrak{p} \subset \mathfrak{p}$ for all $k \in K$, Ad_L induces the adjoint representation of K on $\mathfrak{p}$, which we write $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{p})$. This representation is orthogonal with respect to the inner product defined by the restriction of $(\cdot,\cdot)_{\mathfrak{l}}$ to $\mathfrak{p} \times \mathfrak{p}$. Geometrically, the space $\mathfrak{p}$ is canonically isomorphic to the tangent space at the origin eK of L/K. Under this identification, Ad is equivalent to the isotropy representation of L/K. It is known that K acts transitively on M (cf. [20, Proposition 2.1]), so that M is expressed as $M = K/K_J$, where $K_J = \{k \in K \mid \mathrm{Ad}(k)J = J\}$. In particular,

M is compact. From this expression, we have the K-equivariant map from $M = K/K_J$ to $\mathfrak{p}$ as follows:

$$f(kK_J) = \mathrm{Ad}(k)J \quad (k \in K)\,.$$

This map f is called the *canonical imbedding* of M. Then M is diffeomorphic to the $\mathrm{Ad}(K)$-orbit through J, which we write $\mathrm{Ad}(K)J$. Now, M is called a *symmetric R-space*, if $M = K/K_J$ becomes a symmetric space. A symmetric R-space $M = L/U$ is said to be *irreducible*, if L is simple. The classification of symmetric R-spaces reduces to irreducible ones, which was classified by Kobayashi-Nagano (see [12, p. 895 for classical cases, p. 906 for exceptional cases]. We also find a complete list in [15, p. 41]). From the classification, we observe that an irreducible symmetric R-space is not necessarily isotropy irreducible.

Following to Takeuchi-Kobayashi ([20]), we give a K-invariant Riemannian metric on $M = K/K_J$ as follows: Since $-(\cdot,\cdot)_{\mathfrak{l}}$ gives an positive definite inner product on $\mathfrak{k}$, the Lie algebra $\mathfrak{k}_J$ of K_J has the orthogonal complement in $\mathfrak{k}$ with respect to this inner product, which we write $\mathfrak{m} = (\mathfrak{k}_J)^{\perp}$. Under the canonical identification of $\mathfrak{m}$ with T_oM ($o = eK_J$), the adjoint action of K_J on $\mathfrak{m}$ is equivalent to the isotropy representation of K_J on T_oM. Furthermore, by means of this identification, the restriction of $-(\cdot,\cdot)_{\mathfrak{l}}$ to $\mathfrak{m} \times \mathfrak{m}$ gives a K_J-invariant inner product on T_oM, which induces a K-invariant Riemannian metric on $M = K/K_J$ in a natural manner. Then it is shown that the canonical imbedding $f : M \to \mathfrak{p}$ is isometric. We write the Riemannian metric on M as the same symbol $\langle\cdot,\cdot\rangle$ if there is no confusion. In addition, Kobayashi-Takeuchi proved the following proposition.

Proposition 4.1 ([20, Theorem 4.2])**.** *Let M be an irreducible symmetric R-space. We set $n = \dim(M)$ and $N = \dim(\mathfrak{p})$. Let $S = S^{N-1}(\sqrt{2n})$ denote the hypersphere of radius $\sqrt{2n}$ centered at the origin in $\mathfrak{p}$. Then, the image $f(M)$ of the canonical imbedding is a minimal submanifold of S.*

By this proposition it makes sense to determine whether irreducible symmetric R-spaces satisfy the inequality (3.11) stated in Theorem 3.7. By means of Proposition 4.1, Theorem 3.7 is rewritten as follows.

Theorem 4.2. *Let $M = K/K_J$ be an irreducible symmetric R-space associated with a Riemannian symmetric space L/K of noncompact type. We set*

$$B_J = 4d_{F'}\left\{(N-n-1)\gamma^2 + \frac{1}{2n}\right\} + \frac{1}{2} - 2c + 2s\,. \tag{4.1}$$

Suppose that $d_{F'}$ is nonnegative. If the inequality

$$B_J < 0 \tag{4.2}$$

holds, then M is F-Yang-Mills instable.

Example 4.1. We give some observations. (1) If we take $F(t) = t$, then $d_{F'} = 0$ holds, so that the inequality (4.2) is rewritten as

$$\frac{1}{2} - 2c + 2s < 0\,. \tag{4.3}$$

In the case when M is isotropy irreducible, Ohnita [15, Theorem 7] proved that the canonical imbedding is equivalent to the first standard imbedding in the sense of Takahashi [18]. Then, the inequality (4.3) coincides with the inequality stated in [13, (7.10) Theorem]. This means that our inequality (4.2) is an extension of [13, (7.10) Theorem] to an F-Yang-Mills version. We note that the inequality (4.3) is described by means of intrinsic curvatures of M only.

(2) If we take $F = F_p$ ($p \geq 2$) defined in Example 2.1, (1), then the inequality (4.2) is rewritten as

$$2(p-2)\left\{(N-n-1)\gamma^2 + \frac{1}{r^2}\right\} + \frac{1}{2} - 2c + 2s < 0\,.$$

In the case when M is isotropy irreducible, we find an alternative formula for this due to Kawagoe [11, Corollary 6.2]. The difference between them is the same as explained as before. However, [11] did not exhibit concrete examples satisfying the inequality stated in [11, Corollary 6.2].

Under the setting of Theorem 4.2, the first term of the definition (4.1) of B_J is nonnegative. The following lemma shows that (4.3) gives a necessary condition for an irreducible symmetric R-space to satisfy (4.2).

Lemma 4.3. *Fix an irreducible symmetric R-space $M = K/K_J$. Let F_i ($i = 1, 2$) be a strictly increasing C^2-function with $0 \leq d_{F_i'} < \infty$. We use the symbol B_{J,F_i} instead of B_J in order to emphasize the dependence on F_i. Then, $d_{F_1'} \leq d_{F_2'}$ yields $B_{J,F_1} \leq B_{J,F_2}$. In particular, $B_{J,F_2} < 0$ yields $B_{J,F_1} < 0$.*

Kobayashi-Ohnita-Takeuchi [13, (7.11) Theorem] classified isotropy irreducible Riemannian symmetric spaces of compact type satisfying (4.3). This implies that the only isotropy irreducible, irreducible symmetric R-spaces satisfying (4.3) are S^n ($n > 4$) and $F_4/Spin(9)$. Thus, we will determine whether S^n ($n > 4$), $F_4/Spin(9)$ and irreducible symmetric R-spaces which are not isotropy irreducible satisfy the inequality (4.2). For this purpose, we need to determine the value of the constant B_J for these spaces.

4.2. **Determination of the sufficient condition.** Let $M = K/K_J$ be an irreducible symmetric R-space and $f : M \to \mathfrak{p}$ denote the canonical imbedding. The determination of the constant B_J is reduced to those of the constants N, n, c, s and γ. Here, the dimensions of isotropy irreducible Riemannian symmetric spaces are well-known (cf. [8, Table V]), so that we can easily obtain the values of N and n from the isotropy irreducible decomposition of M. In what follows, we focus our attention on the calculation of c, s and γ.

4.2.1. *Determination of the constants c and s.* The Ricci curvatures of irreducible symmetric R-spaces M are determined by Takeuchi ([19, Section 3]). In fact, he determined the Einstein constants of each factor for the locally isometric decomposition of M. From his result we immediately obtain the value of c. On the other hand, Kobayashi-Ohnita-Takeuchi ([13, Table, p. 187]) showed the positive eigenvalues of the Riemann curvature operator for isotropy irreducible Riemannian symmetric spaces with respect to the normal homogeneous Riemannian metric $\langle\cdot,\cdot\rangle'$. In particular, they determined the maximum eigenvalue s' of the Riemann curvature operator of M with respect to $\langle\cdot,\cdot\rangle'$. We note that if there exists $\nu > 0$ satisfying $(\cdot,\cdot)_{\mathfrak{k}} = \nu(\cdot,\cdot)_{\mathfrak{l}}$ on $\mathfrak{k}\times\mathfrak{k}$, then we have $s = \nu s'$. Hence, by applying their result to our setting, we can obtain the value of s for each irreducible symmetric R-space with respect to $\langle\cdot,\cdot\rangle$.

Lemma 4.4. *In the case when $M = S^n$, we have $c = (n-1)/2n$ and $s = 1/2n$.*

Proof. Since S^n is Einstein, c is equal to its Einstein constant. From [19, p. 309] we get $c = (n-1)/2n$. On the other hand, by [13, Table, p. 187], we get $s' = 1/2(n-1)$ for $S^n = SO(n+1)/SO(n)$. In addition, by $(\cdot,\cdot)_{\mathfrak{k}} = ((n-1)/n)(\cdot,\cdot)_{\mathfrak{l}}$ on $\mathfrak{k}\times\mathfrak{k}$, we have:

$$s = \frac{n-1}{n}\cdot\frac{1}{2(n-1)} = \frac{1}{2n}.$$

Thus we have complete the proof. □

Example 4.2. The standard sphere $M = K/K_J = S^n$ is an irreducible symmetric R-space associated with the Riemannian symmetric space $L/K = SO(1,n+1)/SO(n+1)$ of noncompact type. Then we have $N = n+1$, from which we need not to determine γ in order to obtain B_J. In addition, by Lemma 4.4, we have $B_J = (-n + 4d_{F'} + 4)/2n$. It follows from Theorem 4.2 that S^n is F-Yang-Mills instable if $n > 4d_{F'} + 4$. This inequality coincides with (3.5) given in Corollary 3.4.

The following lemma is shown by a similar way to Lemma 4.4. We omit the details for its proof.

Lemma 4.5. *In the case when $M = F_4/Spin(9)$, we have $c = 3/8$ and $s = 1/12$.*

Proposition 4.6. *Let M be an irreducible symmetric R-spaces which is not isotropy irreducible. Then, M does not satisfy the inequality (4.2).*

Proof. By the assumption of this proposition M is locally isometric to $S^1 \times M'$ for some compact connected Einstein symmetric space M' or to $S^{p-1} \times S^{q-1}$ for some p, q with $p \leq q$. In the former case, we have $c = 0$, from which the following holds:

$$\frac{1}{2} - 2c + 2s = \frac{1}{2} + 2s > 0\,.$$

In the latter case, we have $c = (p-2)/2(p+q-2)$ (cf. [19, p. 309, (8)]) and $s = 1/2(p+q-2)$ (cf. [13, Table, p. 187]). Hence we have

$$\frac{1}{2} - 2c + 2s = \frac{q-p+4}{2(p+q-2)} > 0\,.$$

From the above argument M does not satisfy (4.3). From this, the assertion holds. □

The rest task is to obtain the upper bound of $d_{F'}$ for $M = F_4/Spin(9)$ to satisfy the inequality (4.2). For this, we need to obtain the value of γ for the submanifold $f(M) \subset S$.

4.2.2. *Determination of the constants γ and B_J for $F_4/Spin(9)$.* We first give a method to determine the value of γ for a general irreducible symmetric R-space which is realized as the orbit through $J \in \mathfrak{p}$ under the adjoint representation $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{p})$. We note that Ad is a polar representation (cf. [5]). Indeed, any maximal abelian subspace of $\mathfrak{p}$ gives a section of Ad (cf. [2, Theorem 3.2.13]). This fact enable us to construct a method to determine the constant γ. Namely, our method is based on restricted root system theory associated to L/K with respect to a maximal abelian subspace of $\mathfrak{p}$ (see, Appendix A for a brief review of restricted root systems). As shown later in Proposition 4.8, we will derive a formula for γ by means of the restricted root system.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{a}^*$ denote its dual space. We denote by $\Sigma(\subset \mathfrak{a}^* - \{0\})$ the restricted root system of L/K with respect to $\mathfrak{a}$. Without loss of generalities, we may assume that J is an element of $\mathfrak{a}$, since $\mathrm{Ad}(k)\mathfrak{a}$ $(k \in K)$ gives another maximal abelian subspace of $\mathfrak{p}$ and the following relation holds:

$$\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}\,.$$

We describe the restricted root space decomposition of $\mathfrak{k}$ and $\mathfrak{p}$ for Σ as follows: We set $\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [H, X] = 0,\ H \in \mathfrak{a}\}$ and, for each $\lambda \in \Sigma$,

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \mid [H, [H, X]] = \lambda(H)^2 X,\ H \in \mathfrak{a}\}\,,$$
$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} \mid [H, [H, X]] = \lambda(H)^2 X,\ H \in \mathfrak{a}\}\,.$$

Then we have

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda\,, \quad \mathfrak{p} = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{p}_\lambda,$$

where Σ^+ is the set of positive restricted roots of Σ with respect to some ordering. From the above, the tangent space $T_J(\mathrm{Ad}(K)J)$ of the orbit $\mathrm{Ad}(K)J$ is decomposed into

$$T_J(\mathrm{Ad}(K)J) = [\mathfrak{k}, J] = \sum_{\lambda \in \Sigma^+;\, \lambda(J) \neq 0} \mathfrak{p}_\lambda\,.$$

We also have:

$$\tilde{T}^\perp_J(\mathrm{Ad}(K)J) = (J^\perp \cap \mathfrak{a}) \oplus \sum_{\lambda \in \Sigma^+;\, \lambda(J) = 0} \mathfrak{p}_\lambda\,,$$

where $J^\perp \cap \mathfrak{a} = \{X \in \mathfrak{a} \mid \langle J, X \rangle = 0\}$. We set $\mathfrak{p}_J = \{X \in \mathfrak{p} \mid [X, J] = 0\}$ and $\mathfrak{l}_J = \mathfrak{k}_J \oplus \mathfrak{p}_J = \{X \in \mathfrak{l} \mid [X, J] = 0\}$. Then $\mathfrak{l}_J$ is a subalgebra of $\mathfrak{l}$ and $(\mathfrak{l}_J, \mathfrak{k}_J)$ gives an orthogonal symmetric Lie algebra. Since $\mathfrak{a}$ gives a section of $\mathrm{Ad}(K_J)$-action on $\mathfrak{p}_J$, we get:

$$\mathfrak{p}_J = \bigcup_{k \in K_J} \mathrm{Ad}(k)\mathfrak{a}\,.$$

This implies that $\tilde{T}^\perp_J(\mathrm{Ad}(K)J)$ has the following expression:

$$\tilde{T}^\perp_J(\mathrm{Ad}(K)H) = \bigcup_{k \in K_J} \mathrm{Ad}(k)(J^\perp \cap \mathfrak{a})\,.$$

Hence, for any normal vector $\xi \in \tilde{T}^\perp_J(\mathrm{Ad}(K)J)$, there exists $k \in K_J$ satisfying $\mathrm{Ad}(k)\xi \in J^\perp \cap \mathfrak{a}$, from which we have $\tilde{A}_\xi = \mathrm{Ad}(k)^{-1}\tilde{A}_{\mathrm{Ad}(k)\xi}\mathrm{Ad}(k)$. This means that the principal curvatures of $\tilde{A}_\xi$ coincides with those of $\tilde{A}_{\mathrm{Ad}(k)\xi}$ (including their multiplicities). For $\lambda \in \Sigma^+$ with $\lambda(J) \neq 0$ and $\xi \in J^\perp \cap \mathfrak{a}$, we get the following (cf. [2, Example 3.4]):

$$\tilde{A}_\xi|_{\mathfrak{p}_\lambda} = -\frac{\lambda(\xi)}{\lambda(J)}\mathrm{id}_{\mathfrak{p}_\lambda}\,.$$

From the above argument, we conclude:

Lemma 4.7. *For any normal vector $\xi \in \tilde{T}^{\perp}_J(\mathrm{Ad}(K)J)$, there exists $k \in K_J$ with $\mathrm{Ad}(k)\xi \in H^{\perp} \cap \mathfrak{a}$ and the spectrum norm $\|\tilde{A}_\xi\|$ is expressed as follows:*

$$\|\tilde{A}_\xi\| = \max\left\{ \left| -\frac{\lambda(\mathrm{Ad}(k)\xi)}{\lambda(J)} \right| \;\middle|\; \lambda \in \Sigma^+,\, \lambda(J) \neq 0 \right\} .$$

By this lemma, we give our formula for determining $\gamma = \gamma_J$ as follows.

Proposition 4.8. *Under the above setting, we obtain:*

$$\gamma = \max\left\{ \left| -\frac{\lambda(\xi)}{\lambda(J)} \right| \;\middle|\; \lambda \in \Sigma^+,\, \lambda(J) \neq 0,\, \xi \in J^{\perp} \cap \mathfrak{a},\, \|\xi\| = 1 \right\} .$$

Next, we describe $\{\lambda \in \Sigma^+ \mid \lambda(J) \neq 0\}$ and $J^{\perp} \cap \mathfrak{a}$ by means of a fundamental system of Σ as a root system. Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ ($r = \mathrm{rank}(\Sigma)$) be the fundamental system of Σ associated with Σ^+ and $\{H^1, \ldots, H^r\} (\subset \mathfrak{a})$ denote the dual basis of Λ. We write the highest root of Σ associated with Λ as $\tilde{\lambda}$. For a general element J of $\mathfrak{a}$, if $\tilde{\lambda}(J) = 1$ holds, then the R-space K/K_J becomes a symmetric R-space. This implies that, if we express $\tilde{\lambda}$ as

$$\tilde{\lambda} = m_1\lambda_1 + \cdots + m_r\lambda_r \quad (m_1, \ldots, m_r \in \mathbb{Z}_{>0}) ,$$

then, for some i with $m_i = 1$ (if there exists), K/K_{H^i} is a symmetric R-space. Conversely, any irreducible symmetric R-space is constructed in such a way. Hence, without loss of generalities, we may choose $J = H^i$ with $m_i = 1$. From this, we can obtain $\{\lambda \in \Sigma^+ \mid \lambda(J) \neq 0\}$ by using the following lemma.

Lemma 4.9. *Let $J = H^i$ with $m_i = 1$, and $\lambda \in \Sigma^+$ be a positive restricted root. We write $\lambda = l_1\lambda_1 + \cdots + l_r\lambda_r$ for some $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 0}$. Then, λ is in $\{\lambda \in \Sigma^+ \mid \lambda(J) \neq 0\}$ if and only if $l_i = 1$ holds.*

Proof. This lemma immediately from $\lambda(J) = l_i \leq m_i = 1$. □

Corollary 4.10. *Let $J = H^i$ with $m_i = 1$. For any $\lambda \in \Sigma^+$ with $\lambda(J) \neq 0$, we have $\lambda(J) = 1$.*

For each $\lambda \in \Sigma$, we define the restricted root vector $H_\lambda \in \mathfrak{a}$ as follows:

$$\lambda(H) = \langle H_\lambda, H \rangle , \quad H \in \mathfrak{a} .$$

It is shown that $\{H_\lambda \mid \lambda \in \Lambda\}$ gives a basis of $\mathfrak{a}$. Then, we have the following lemma.

Lemma 4.11. *Let $J = H^i$. Then $J^\perp \cap \mathfrak{a}$ has the following description:*

$$J^\perp \cap \mathfrak{a} = \left\{ \sum_{k=1,\,k\neq i}^{r} \xi_k H_{\lambda_k} \;\middle|\; \xi_k \in \mathbb{R},\, 1 \leq k \leq r,\, k \neq i \right\}.$$

Furthermore, by means of $\{H^1, \dots, H^r\}$, $\xi = \sum_{k=1,\,k\neq i}^{r} \xi_k H_{\lambda_k}$ is expressed as follows:

$$\xi = \sum_{l=1}^{r} \tilde{\xi}_l H^l, \quad \tilde{\xi}_l = \sum_{k=1,\,k\neq i}^{r} \xi_k \langle H_{\lambda_l}, H_{\lambda_k} \rangle .$$

Proof. Let $\xi = \sum_{k=1}^{r} \xi_k H_{\lambda_k} \in \mathfrak{a}$. If we denote by C the Cartan matrix of Σ as a root system, that is,

$$C = (C_{kl})_{1\leq k,l\leq r} = \left(\frac{2\langle H_{\lambda_l}, H_{\lambda_k}\rangle}{\langle H_{\lambda_l}, H_{\lambda_l}\rangle} \right)_{1\leq k,l\leq r},$$

then we have

$$\langle J, \xi \rangle = \sum_{k,l=1}^{r} \xi_k \frac{2\langle H_{\lambda_l}, H_{\lambda_k}\rangle}{\langle H_{\lambda_l}, H_{\lambda_l}\rangle} ({}^tC^{-1})_{li} = \sum_{k=1}^{r} \xi_k \left(\sum_{l=1}^{r} {}^tC_{kl} ({}^tC^{-1})_{li} \right) = \sum_{k=1}^{r} \xi_k \delta_{ki} = \xi_i .$$

From this, we have the assertion. □

We are ready to determine the constant γ for $M = F_4/Spin(9)$.

Lemma 4.12. *In the case when $M = F_4/Spin(9)$, we have $\gamma = 1/4\sqrt{6}$.*

Proof. We give a realization of $M = F_4/Spin(9)$ as an orbit of the isotropy representation of the Riemannian symmetric space $L/K = E_6^{-26}/F_4$ of noncompact type. The restricted root system Σ of L/K is of type A_2. If $\Lambda = \{\lambda_1, \lambda_2\}$ is a fundamental system of Σ, then the highest root $\tilde{\lambda}$ of Σ associated with Λ is expressed by $\tilde{\lambda} = \lambda_1 + \lambda_2$. Then we have $K/K_J = F_4/Spin(9)$ with $J = H^1$. From Lemma 4.9, we get $\{\lambda \in \Sigma^+ \mid \lambda(J) \neq 0\} = \{\lambda_1, \lambda_1 + \lambda_2\}$. Here, it is shown that, for any $\lambda \in \Sigma^+$, the length $\|H_\lambda\|$ is given as follows (see, Appendix A for the proof):

$$\|H_\lambda\| = \frac{1}{2\sqrt{6}} . \tag{4.4}$$

In addition, since the angle between H_{λ_1} and H_{λ_2} is equal to $2\pi/3$, we have:

$$(\langle H_{\lambda_k}, H_{\lambda_l}\rangle)_{1\leq k,l\leq 2} = \begin{pmatrix} 1/24 & -1/48 \\ -1/48 & 1/24 \end{pmatrix} .$$

By Lemma 4.11, any vector $\xi = \xi_2 H_{\lambda_2} \in J^\perp \cap \mathfrak{a}$ is rewritten as $\xi = \tilde{\xi}_1 H^1 + \tilde{\xi}_2 H^2$ with $\tilde{\xi}_1 = -\xi_2/48$ and $\tilde{\xi}_2 = \xi_2/24$. Then we obtain $\|\xi\| = (1/2\sqrt{6})|\xi_2|$ and

$$\left| -\frac{\lambda_1(\xi/\|\xi\|)}{\lambda_1(J)} \right| = \frac{|\tilde{\xi}_1|}{\|\xi\|} = \frac{1}{4\sqrt{6}}, \quad \left| -\frac{(\lambda_1+\lambda_2)(\xi/\|\xi\|)}{(\lambda_1+\lambda_2)(J)} \right| = \frac{|\tilde{\xi}_1 + \tilde{\xi}_2|}{\|\xi\|} = \frac{1}{4\sqrt{6}} .$$

Thus, we have the assertion from Proposition 4.8. □

We obtain the constant B_J for $M = F_4/Spin(9)$ by means Lemmas 4.5 and 4.12. Namely, we have the following proposition.

Proposition 4.13. *In the case when $M = F_4/Spin(9)$, we have $B_J = \dfrac{d_{F'}}{2} - \dfrac{1}{12}$.*

From the above argument, we conclude:

Theorem 4.14. *The standard sphere S^n with $0 \leq d_{F'} < (n-4)/4$ and the Cayley projective space $F_4/Spin(9)$ with $0 \leq d_{F'} < 1/6$ satisfy the inequality (4.2). Furthermore, they are the only irreducible symmetric R-spaces satisfying this inequality.*

From Theorems 4.2 and 4.14 we get the following corollary.

Corollary 4.15 (Theorem 1.3)**.** *Let $F : [0, T) \to \mathbb{R}$ be a strictly increasing C^2-function with $0 \leq d_{F'} < 1/6$. Then, the Cayley projective plane $F_4/Spin(9)$ is F-Yang-Mills instable.*

Example 4.3. Applying Corollary 4.15 to the function $F = F_p$ defined in Example 2.1, (1), the Cayley projective plane $F_4/Spin(9)$ is p-Yang-Mills instable for $2 \leq p < 2+1/3$.

Appendix A. Riemannian symmetric spaces of noncompact type and their restricted root systems

Let L/K be a Riemannian symmetric space of noncompact type. Here, K is a maximal compact subgroup of L. Then there exists an involution of L satisfying $L_0^\theta \subset K \subset L^\theta$, where L^θ denotes the fixed-point subgroup of θ in L, and L_0^θ denotes the its identity component. We write the Lie algebras of L and K as $\mathfrak{l}$ and $\mathfrak{k}$, respectively. The differentiation of θ at the identity element in L gives a Cartan involution of $\mathfrak{l}$, which we write the same symbol θ. Then we have $\mathfrak{k} = \mathfrak{l}^\theta$. Let $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{l}$ associated with $\mathfrak{k}$, where $\mathfrak{p} = \mathfrak{l}^{-\theta}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and Σ denote the restricted root system of L/K with respect to $\mathfrak{a}$. The restriction of the Killing form of $\mathfrak{l}$ to $\mathfrak{a} \times \mathfrak{a}$ gives a positive definite inner product on $\mathfrak{a}$, which we write $\langle \cdot, \cdot \rangle$. For $\lambda \in \Sigma$, we denote by $H_\lambda \in \mathfrak{a}$ the restricted root vector of λ, that is, $\lambda(H) = \langle H_\lambda, H \rangle$ for $H \in \mathfrak{a}$. Under the above setting, we give a method to determine the length $\|H_\lambda\| = \sqrt{\langle H_\lambda, H_\lambda \rangle}$ of H_λ ($\lambda \in \Sigma$) and prove the equality (4.4).

We first describe Σ by means of the root system of the complexification $\mathfrak{l}^{\mathbb{C}}$ of $\mathfrak{l}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$ containing $\mathfrak{a}$. Then $\mathfrak{h}$ is θ-invariant, from which we have

$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}$. We define the real vector space $\mathfrak{h}_{\mathbb{R}}$ by $\mathfrak{h}_{\mathbb{R}} = \sqrt{-1}(\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}(\subset \mathfrak{h}^{\mathbb{C}})$. It is shown the the restriction of the Killing form of $\mathfrak{l}^{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}$ gives a positive definite inner product on $\mathfrak{h}_{\mathbb{R}}$, which we write the same symbol $\langle \cdot, \cdot \rangle$ if there is no confusion. We denote by $\tilde{\Sigma}(\subset \mathfrak{h}_{\mathbb{R}}^* - \{0\})$ the root system of $\mathfrak{l}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. If we put $\tilde{\Sigma}_0 = \{\alpha \in \tilde{\Sigma} \mid \alpha(H) = 0\,, H \in \mathfrak{a}\}$, then the following relation holds:

$$\Sigma = \left\{ \alpha|_{\mathfrak{a}} \;\middle|\; \alpha \in \tilde{\Sigma} - \tilde{\Sigma}_0 \right\} .$$

Next, we give a formula to obtain the length $\|H_\lambda\|$ of $\lambda \in \Sigma$ by means of the lengths of root vectors for $\tilde{\Sigma}$. Here, the root vector $H_\alpha \in \mathfrak{h}_{\mathbb{R}}$ for $\alpha \in \tilde{\Sigma}$ is defined by

$$\alpha(H) = \langle H_\alpha, H \rangle\,, \quad H \in \mathfrak{h}_{\mathbb{R}},$$

which is well-defined since α takes real-valued on $\mathfrak{h}_{\mathbb{R}}$. We set $\sigma = -\theta^{\mathbb{C}}|_{\mathfrak{h}_{\mathbb{R}}}$, which gives a permutation on $\tilde{\Sigma}$. We have $\sigma(H_\alpha) = H_{\sigma(\alpha)}$ for $\alpha \in \tilde{\Sigma}$. For any $\lambda \in \Sigma$ there exists $\alpha \in \tilde{\Sigma} - \tilde{\Sigma}_0$ satisfying $\lambda = \alpha|_{\mathfrak{a}}$. Then the vector H_λ for $\lambda \in \Sigma$ coincides with the $\mathfrak{a}$-component of $H_\alpha \in \mathfrak{h}_{\mathbb{R}}$. Namely, we have $H_\lambda = (1/2)(H_\alpha + H_{\sigma(\alpha)})$, from which we get the following lemma.

Lemma A.1. *Let $\lambda \in \Sigma$ and $\alpha \in \tilde{\Sigma} - \tilde{\Sigma}_0$ with $\lambda = \alpha|_{\mathfrak{a}}$. Then we have:*

$$\|H_\lambda\|^2 = \frac{1}{2} \left\{ \langle H_\alpha, H_\alpha \rangle + \langle H_\alpha, H_{\sigma(\alpha)} \rangle \right\} . \tag{A.1}$$

The following relation is useful to determine $H_{\sigma(\alpha)}$ in (A.1):

$$H_{\zeta+\eta} = H_\zeta + H_\eta \quad (\zeta, \eta \in \tilde{\Sigma}\,, \zeta + \eta \in \tilde{\Sigma})\,. \tag{A.2}$$

We are ready to prove (4.4).

Proof of (4.4). Let $L/K = E_6^{-26}/F_4$. We write the root system of $\mathfrak{l}^{\mathbb{C}} = \mathfrak{e}_6^{\mathbb{C}}$ as $\tilde{\Sigma} = E_6$. We can determine the action of σ on $\tilde{\Sigma}$ in terms of Satake diagram ([8, p. 532, TABLE VI]). Indeed, there exists a fundamental system $\tilde{\Lambda} = \{\alpha_1, \ldots, \alpha_6\}$ of $\tilde{\Sigma}$ such that, for each $i = 1, \ldots, 6$, $\sigma(\alpha_i)$ is given by

$$\sigma(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\,, \quad \sigma(\alpha_6) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\,,$$

and $\sigma(\alpha_i) = -\alpha_i$ $(i = 2, 3, 4, 5)$. We note that α_1 is normal to α_j $(j = 2, 4, 5, 6)$. The length $\|H_\lambda\|$ $(\lambda \in \Sigma = A_2)$ is independent of the choice of λ. It is sufficient to show the length of $\|H_{\lambda_1}\|$ with $\lambda_1 = \alpha_1|_{\mathfrak{a}}$ is equal to $1/2\sqrt{6}$. From Lemma A.1 and (A.2), we get:

$$\|H_{\lambda_1}\|^2 = \langle H_{\alpha_1}, H_{\alpha_1} \rangle + \langle H_{\alpha_1}, H_{\alpha_3} \rangle = \frac{1}{24}\,.$$

Here, in the last equality, we have used the result of Yokota ([21, p. 82]) for the length of the root vector H_{α_i} $(i = 1, 3)$. Hence we have completed the proof. □

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