FINITENESS THEOREM OF MEROMORPHIC FUNCTIONS ON A COMPLETE KÄHLER MANIFOLD

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Abstract. In this article, we will prove a finiteness theorem for nonconstant meromorphic functions satisfying the condition (C_ρ) on a complete Kähler manifold which share four distinct values.

1. Introduction

In 1926, R. Nevanlinna [5] showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f \equiv g$ and if they have the same inverse images for four distinct values then they must be linked by a Möbius transformation. In 1998, H. Fujimoto [1] showed a finiteness theorem for meromorphic functions on $\mathbb{C}$ which share four distinct values with multiplicities truncated by 2. Later, D. D. Thai and T. V. Tan [12], S. D. Quang [7], [8] generalized the result of H. Fujimoto to the case where the values are replaced by small functions. These theorems are called finiteness theorems. We state by S. D. Quang's [7] result as follows.

Theorem A Let f^1, f^2, f^3 be three nonconstant meromorphic functions on $\mathbb{C}$. Let $a_1, ..., a_4$ be distinct small (with respect to f^i, $\forall 1 \leq i \leq 3$) functions on $\mathbb{C}$. Assume that

(i) $\min\{\nu_{f^1 - a_i}, 1\} = \min\{\nu_{f^2 - a_i}, 1\} = \min\{\nu_{f^3 - a_i}, 1\}\ \forall 1 \leq i \leq 3.$

(ii) $\min\{\nu_{f^1 - a_4}, 2\} = \min\{\nu_{f^2 - a_4}, 1\} = \min\{\nu_{f^3 - a_4}, 1\}.$

Then $f^1 = f^2$ or $f^2 = f^3$ or $f^1 = f^3$.

In this paper, we will show a finiteness theorem similar to the above theorem but here we will consider the general case where M is a m- dimensional complete connected Kähler manifold, whose universal covering is biholomorphic to a ball $B^m(R_0) = \{z \in \mathbb{C}^m : \ ||z|| < R_0\}$ (R_0 may be $+\infty$) and $f : M \to \mathbb{C}$ is a nonconstant meromorphic function satisfying the condition (C_ρ).

Here, noting that we consider each function f as a meromorphic mapping from M into $\mathbb{P}^1(\mathbb{C})$ and f satisfies the condition (C_ρ) if there exists a nonzero bounded continuous

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real-valued function h on M such that

$$\rho\Omega_f + dd^c \log h^2 \geq \text{Ric}\omega, \quad \text{for } \rho \geq 0,$$

where Ω_f is the full-back of the Fubini-Study form Ω on $\mathbb{P}^1(\mathbb{C})$ by f, $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}} dz_i \wedge d\overline{z}_j$ is a Kähler form on the Kähler manifold M, $\text{Ric}\omega = dd^c \log(det(h_{i\bar{j}}))$, $d = \partial + \overline{\partial}$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$.

Usually, almost all authors are used to using Cartan's auxialiary functions in order to study the finiteness problem of meromorphic mappings on $\mathbb{C}^m$, and comparing the counting functions of these auxialiary functions with the characteristic functions of the mappings. However, we do not have any concepts of the counting functions and the characteristic functions on a Kähler manifold in the general case. In order to overcome this difficulty, we have to use the notion of the functions of small integration and the functions of bounded integration with respect to a family of meromorphic functions on a Kähler manifold due to S. D. Quang's papers ([9] and [10]) to prove the following theorem. Our result is stated as follows.

Theorem 1.1. *Let M be an m-dimensional connected Kähler manifold whose universal covering is biholomorphic to $\mathbb{C}^m$ or the unit ball $B^m(1)$ of $\mathbb{C}^m$, and let f^1, f^2, f^3 be three nonconstant meromorphic functions on M, satisfying the condition (C_ρ). Let $a_1, ...a_4$ be distinct values on M. Assume that*

$$(i) \min\{\nu^0_{(f^1,a_i)}, 1\} = \min\{\nu^0_{(f^2,a_i)}, 1\} = \min\{\nu^0_{(f^3,a_i)}, 1\}, \ \forall\ 1 \leq i \leq 3,$$
$$(ii) \min\{\nu^0_{(f^1,a_4)}, 2\} = \min\{\nu^0_{(f^2,a_4)}, 2\} = \min\{\nu^0_{(f^3,a_4)}, 2\}.$$

If $\rho < \frac{2}{33}$ then $f^1 = f^2$ or $f^2 = f^3$ or $f^1 = f^3$.

In the case $M = \mathbb{C}$, we may choose $\rho = 0$ and get the result of Theorem A for the special case of distinct values.

2. Basic notions and auxiliary results from Nevanlinna theory

We will recall some basic notions in R. Nevanlinna theory due to [11].

(a) Counting function. We set $||z|| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : ||z|| = r\}\ (0 < r \leq \infty),$$

where $B(\infty) = \mathbb{C}^m$ and $S(\infty) = \emptyset$.

Define

$$v_{m-1}(z) := \big(dd^c||z||^2\big)^{m-1} \qquad \text{and}$$
$$\sigma_m(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{m-1} \text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

A divisor ν on a ball $B(R_0)$ is given by a formal sum $\nu = \sum \mu_\lambda X_\lambda$, where $\{X_\lambda\}$ is a locally family of distinct irreducible analytic hypersurfaces in $B(R_0)$ and $\mu_\lambda \in \mathbb{Z}$. We define the support of the divisor ν by setting $\mathrm{Supp}\,(\nu) = \cup_{\mu_\lambda \neq 0} X_\lambda$. Sometimes, we identify the divisor ν with a function $\nu(z)$ from $B(R_0)$ into $\mathbb{Z}$ defined by $\nu(z) := \sum_{X_\lambda \ni z} \mu_\lambda$.

Let M, k be positive integers or $+\infty$. We define the truncated divisors $\nu^{[M]}$ by

$$\nu^{[M]} := \sum_\lambda \min\{\mu_\lambda, M\} X_\lambda,$$

and the truncated counting function to level M of ν by

$$N^{[M]}(r, r_0; \nu) := \int\limits_{r_0}^{r} \frac{n^{[M]}(t, \nu)}{t^{2m-1}} dt \quad (r_0 < r < R_0),$$

where

$$n^{[M]}(t, \nu) := \begin{cases} \int\limits_{\mathrm{Supp}\,(\nu) \cap B(t)} \nu^{[M]} v_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} \nu^{[M]}(z) & \text{if } m = 1. \end{cases}$$

We omit the character $^{[M]}$ if $M = +\infty$.

Let φ be a non-zero meromorphic function on $B(R)$. We denote by ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ. The divisor of φ is defined by

$$\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi.$$

For convenience, we will write $N_\varphi(r, r_0)$ and $N^{[M]}_\varphi(r, r_0)$ for $N(r, r_0; \nu^0_\varphi)$ and $N^{[M]}(r, r_0; \nu^0_\varphi)$ respectively.

(b) Characteristic function. Let $f : B(R_0) \subset \mathbb{C}^m \longrightarrow \mathbb{P}^1(\mathbb{C})$ be a meromorphic mapping. Fix a homogeneous coordinates system $(w_0 : w_1)$ on $\mathbb{P}^1(\mathbb{C})$. We take a reduced representation $f = (f_0 : f_1)$ of f. Set $\|f\| = \big(|f_0|^2 + |f_1|^2\big)^{1/2}$.

The characteristic function of f (with respect to Fubini Study form Ω) is defined by

$$T_f(r, r_0) := \int_{r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} f^*\Omega \wedge v_{m-1}, \qquad 0 < r_0 < r < R_0.$$

By Jensen's formula we have

$$T_f(r, r_0) = \int_{S(r)} \log ||f|| \sigma_m - \int_{S(r_0)} \log ||f|| \sigma_m, \qquad 0 < r_0 < r < R_0.$$

Throughout this paper, we assume that the numbers r_0 and R_0 are fixed with $0 < r_0 < R_0$. By notation "$\|\ P$", we means that the asseartion P hold for all $r \in [r_0, R_0]$ outside a set E of $[0, R_0)$ such that $\int_E dr < \infty$ in case $R_0 = \infty$ and $\int_E \frac{1}{R_0 - r} dr < \infty$ in case $R_0 < \infty$.

(c) Some propositions

Let $f^1, f^2, ..., f^k$ be k meromorphic mappings from the complete Kähler manifold $B^m(1)$ into $\mathbb{P}^1(\mathbb{C})$, which satisfy the condition (C_ρ) for a non-negative number ρ. For each $1 \le u \le k$, we fix a reduced representation $f^u = (f_0^u : f_1^u)$ of f^u and set $||f^u|| = (|f_0^u|^2 + |f_1^u|^2)^{\frac{1}{2}}$.

Definition 2.1. *(Functions of small integration, see* [9], [10]*). A non-negative continuous function g on $B^m(1)$ is said to be of small integration with respective to $f^1, ..., f^k$ at level l_0 if there exist an element $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$ with $|\alpha| \le l_0$, a positive number K, such that for every $0 \le tl_0 < p < 1$,*

$$\int\limits_{S(r)} |z^\alpha g|^t \sigma_m \le K \left(\frac{R^{2m-1}}{R-r} \sum_{u=1}^{k} T_{f^u}(r, r_0) \right)^p,$$

for all r with $0 < r_0 < r < R < 1$, where $z^\alpha = z_1^{\alpha_1} ... z_m^{\alpha_m}$.

We denote by $S(l_0; f^1, ..., f^k)$ the set of all non-negative continuous functions on $B^m(1)$ which are of small integration with respective to $f^1, ..., f^k$ at level l_0. We see that, if g belongs to $S(l_0; f^1, ..., f^k)$ then g is also belongs to $S(l; f^1, ..., f^k)$ for every $l > l_0$. Moreover, if g is a constant function then $g \in S(0; f^1, ..., f^k)$.

Proposition 2.2. *(see* [9]*) If $g_i \in S(l_i; f^1, ..., f^k)$ $(1 \le i \le s)$ then $g_1 ... g_s \in S(\sum_{i=1}^s l_i; f^1, ..., f^k)$.*

Definition 2.3. *(Functions of bounded integration, see* [9]*). A meromorphic function h on $B^m(1)$ is said to be of bounded integration with bi-degree (p, l_0) for the family $\{f^1, ..., f^k\}$ if there exists $g \in S(l_0; f^1, ..., f^k)$ satisfying*

$$|h| \le ||f^1||^p ... ||f^k||^p.g,$$

outside a proper analytic subset of $B^m(1)$.

Denote by $B(p, l_0; f^1, ..., f^k)$ the set of all meromorphic functions on $B^m(1)$ which are of bounded integration of bi-degree (p, l_0) for $\{f^1, ..., f^k\}$. We have the following:

* For a meromorphic mapping h, $|h| \in S(l_0; f^1, ..., f^k)$ if $h \in B(0, l_0; f^1, ..., f^k)$.

* $B(p, l_0; f^1, ..., f^k) \subset B(p, l; f^1, ..., f^k)$ for every $0 \le l_0 < l$.

* If $h_i \in B(p_i, l_i; f^1, ..., f^k)$ $(1 \le i \le s)$ then

$$h_1 ... h_s \in B(\sum_{i=1}^s p_i, \sum_{i=1}^s l_i; f^1, ..., f^k).$$

Definition 2.4. *(Cartan's auxiliary function). For meromorphic functions F, G, H on $B^m(R_0)$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, we define the Cartan's auxiliary function as follows:*

$$\Phi^\alpha(F, G, H) = F.G.H. \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ D^\alpha\left(\frac{1}{F}\right) & \mathcal{D}^\alpha\left(\frac{1}{G}\right) & \mathcal{D}^\alpha\left(\frac{1}{H}\right) \end{vmatrix}.$$

Proposition 2.5. *(see [6, Proposition 3.4]). If $\Phi^\alpha(F,G,H)=0$ and $\Phi^\alpha(\frac{1}{F},\frac{1}{G},\frac{1}{H})=0$ for all α with $|\alpha|\le 1$, then one of the following assertions holds :*

(i) $F=G$, $G=H$, or $H=F$,

(ii) $\frac{F}{G}$, $\frac{G}{H}$ and $\frac{H}{F}$ are all constant.

3. Proof of Theorems 1.1

Let f^1,f^2,f^3 be three nonconstant meromorphic functions on $B(R_0)$ satisfying the condition (C_ρ) and $\{a_i\}_{i=1}^4$ be distinct values on $B(R_0)$, we consider each function f^u as meromorphic mappings from $B(R_0)$ into $\mathbb{P}^1(\mathbb{C})$ with a reduced representation $f^u=(f_0^u:f_1^u)$ for each $1\le u\le 3$. We define:

$$+\ (f^u,a_i)=f_0^u-f_1^ua_i=(f^u-a_i)f_1^u$$
$$+\ T(r)=T(r,f^1)+T(r,f^2)+T(r,f^3)$$
$$+\ F_k^{ij}=\frac{(f^k,a_i)}{(f^k,a_j)}\ (1\le k\le 3;\ 1\le i,j\le 4).$$

This easily implies that $\nu^0_{(f^u,a_i)}=\nu^0_{f^u-a_i}$.

We define divisor ν_i by

$$\nu_i(z)=\begin{cases}\nu_{(f^1,a_i)}(z) & \text{if } \nu_{(f^1,a_i)}(z)=\nu_{(f^2,a_i)}(z)=\nu_{(f^3,a_i)}(z)\\ 0 & \text{for otherwise,}\end{cases}$$

and divisor μ_i by

$$\mu_i(z)=\min\{\nu_{(f^1,a_i)}(z);\nu_{(f^2,a_i)}(z);\nu_{(f^3,a_i)}(z)\}.$$

For each i $(1\le i\le 4)$, we take holomorphic functions h_i and φ_i defined on $B(R_0)$ such that $\nu_{h_i}=\nu_i$ and $\nu_{\varphi_i}=\mu_i$.

Without loss of generality, we may assume that $a_i\not\equiv 0$ and $a_i\not\equiv\infty$ for each i $(1\le i\le 4)$. We put $S:=\bigcup_{i=1}^4 Z_{a_i}\cup Z_{\frac{1}{a_i}}$. Here by Z_φ we denote the set of zeros of the meromorphic function φ.

In order to prove Theorem 1.1, we need the following.

Lemma 3.1. *(see* [10]*) Let M be a complete connected Kähler manifold $B^m(1)$. Let $f^1,f^2,...,f^k$ be k nonconstant meromorphic functions on M, which satisfy the condition (C_ρ). Let $a_1,...a_q$ be q distinct values on M. Assume that there exists a non zero holomorphic function $h\in B(p,l_0,f^1,...,f^k)$ such that*

$$\nu_h\ge\lambda\sum_{u=1}^{k}\sum_{i=1}^{q}\nu^{[1]}_{(f^u,a_i)},$$

where p,l_0 are non-negative integers, λ is a positive number. Then we have

$$q\le 2+\rho k+\frac{1}{\lambda}(p+\rho l_0).$$

Lemma 3.2. *With the assumption of Lemma* 3.1*, let* f^1, f^2, f^3 *be three nonconstant meromorphic functions on* M*, which satisfy the condition* (C_ρ)*. Assume that there exist* $i; j \in \{1,2,3,4\}$ $(i \neq j)$ *and* $\alpha \in \mathbb{N}^m$ *with* $|\alpha| = 1$ *such that* $\Phi^\alpha_{ij} \not\equiv 0$*. Then there exists a holomophic function* $g_{ij} \in B(1,1; f^1, f^2, f^3)$ *such that*

$$\nu_{g_{ij}} \geq \sum_{u=1}^{3} \nu_{(f^u,a_j)} + \nu_i + 2 \sum_{s=1, s\neq i,j}^{4} \mu_s.$$

Proof. We have

$$\Phi^\alpha_{ij} = F_1^{ij}.F_2^{ij}.F_3^{ij}.\begin{vmatrix} 1 & 1 & 1 \\ F_1^{ji} & F_2^{ji} & F_3^{ji} \\ \mathcal{D}^\alpha(F_1^{ji}) & \mathcal{D}^\alpha(F_2^{ji}) & \mathcal{D}^\alpha(F_3^{ji}) \end{vmatrix} = \begin{vmatrix} F_1^{ij} & F_2^{ij} & F_3^{ij} \\ 1 & 1 & 1 \\ F_1^{ij}\mathcal{D}^\alpha(F_1^{ji}) & F_2^{ij}\mathcal{D}^\alpha(F_2^{ji}) & F_3^{ij}\mathcal{D}^\alpha(F_3^{ji}) \end{vmatrix}$$
$$= F_1^{ij}(\frac{\mathcal{D}^\alpha(F_3^{ji})}{F_3^{ji}} - \frac{\mathcal{D}^\alpha(F_2^{ji})}{F_2^{ji}}) + F_2^{ij}(\frac{\mathcal{D}^\alpha(F_1^{ji})}{F_1^{ji}} - \frac{\mathcal{D}^\alpha(F_3^{ji})}{F_3^{ji}}) + F_3^{ij}(\frac{\mathcal{D}^\alpha(F_2^{ji})}{F_2^{ji}} - \frac{\mathcal{D}^\alpha(F_1^{ji})}{F_1^{ji}}). \tag{3.3}$$

Put

$$\begin{aligned} g_{ij} =& (f^1,a_i)(f^2,a_j)(f^3,a_j)\left(\frac{\mathcal{D}^\alpha(F_3^{ji})}{F_3^{ji}} - \frac{\mathcal{D}^\alpha(F_2^{ji})}{F_2^{ji}}\right) \\ &+ (f^1,a_j)(f^2,a_i)(f^3,a_j)\left(\frac{\mathcal{D}^\alpha(F_1^{ji})}{F_1^{ji}} - \frac{\mathcal{D}^\alpha(F_3^{ji})}{F_3^{ji}}\right) \\ &+ (f^1,a_j)(f^2,a_j)(f^3,a_i)\left(\frac{\mathcal{D}^\alpha(F_2^{ji})}{F_2^{ji}} - \frac{\mathcal{D}^\alpha(F_1^{ji})}{F_1^{ji}}\right). \end{aligned}$$

Then we have

$$(\prod_{u=1}^{3}(f^u,a_j)).\Phi^\alpha_{ij} = g_{ij}.$$

This implies that

$$|g_{ij}| \leq C.||f^1||.||f^2||.||f^3|| \sum_{u=1}^{3} \left|\frac{D^\alpha(F_u^{ji})}{F_u^{ji}}\right|,$$

where C is a positive constant, and then $g_{ij} \in B(1,1; f^1, f^2, f^3)$. It is clear that

$$\nu_{\Phi^\alpha_{ij}} = -\sum_{u=1}^{3} \nu_{(f^u,a_j)} + \nu_{g_{ij}}. \tag{3.4}$$

For a fixed point $z \in Z_{h_i} \cup \bigcup_{s=1, s\neq i,j}^{4} Z_{(f,a_s)} \setminus S$, we consider the following two cases.

* **Case 1**. If z is a zero of the function h_i. Then there exists a neighborhood U of z such that all $\frac{F_k^{ij}}{h_i}$ $(1 \leq k \leq 3)$ are nowhere zero holomorphic functions on U. We rewrite the function Φ^α_{ij} on U as follows

$$\Phi^\alpha_{ij} = h_i \Phi^\alpha\left(\frac{F_1^{ij}}{h_i}, \frac{F_2^{ij}}{h_i}, \frac{F_3^{ij}}{h_i}\right)$$

Then, it yields that

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \nu_{h_i}(z) = \nu_i(z) + 2\sum_{s=1, s\neq i,j}^{4} \mu_s(z).$$

* **Case 2**. If z is a zero of a function $(f, a_t)$with $t \neq \{i, j\}$. We rewrite the function Φ^{α}_{ij} as follows

$$\begin{aligned}
\Phi^{\alpha}_{ij} &= F_1^{ij}.F_2^{ij}.F_3^{ij}.\begin{vmatrix} F_2^{ji} - F_1^{ji} & F_3^{ji} - F_1^{ji} \\ \mathcal{D}^{\alpha}(F_2^{ji} - F_1^{ji}) & \mathcal{D}^{\alpha}(F_3^{ji} - F_1^{ji}) \end{vmatrix} \\
&= F_1^{ij}.F_2^{ij}.F_3^{ij}.\begin{vmatrix} \dfrac{(f^2,a_j)(f^1,a_i)-(f^2,a_i)(f^1,a_j)}{(f^2,a_i)(f^1,a_i)} & \dfrac{(f^3,a_j)(f^1,a_i)-(f^3,a_i)(f^1,a_j)}{(f^3,a_i)(f^1,a_i)} \\ \mathcal{D}^{\alpha}(\dfrac{(f^2,a_j)(f^1,a_i)-(f^2,a_i)(f^1,a_j)}{(f^2,a_i)(f^1,a_i)}) & D^{\alpha}(\dfrac{(f^3,a_j)(f^1,a_i)-(f^3,a_i)(f^1,a_j)}{(f^3,a_i)(f^1,a_i)}) \end{vmatrix} \\
&= h_t^2 F_1^{ij}.F_2^{ij}.F_3^{ij}.\begin{vmatrix} \dfrac{(f^2,a_j)(f^1,a_i)-(f^2,a_i)(f^1,a_j)}{h_t(f^2,a_i)(f^1,a_i)} & \dfrac{(f^3,a_j)(f^1,a_i)-(f^3,a_i)(f^1,a_j)}{h_t(f^3,a_i)(f^1,a_i)} \\ \mathcal{D}^{\alpha}(\dfrac{(f^2,a_j)(f^1,a_i)-(f^2,a_i)(f^1,a_j)}{h_t(f^2,a_i)(f^1,a_i)}) & D^{\alpha}(\dfrac{(f^3,a_j)(f^1,a_i)-(f^3,a_i)(f^1,a_j)}{h_t(f^3,a_i)(f^1,a_i)}) \end{vmatrix}
\end{aligned}$$

We note that all functions $\dfrac{(f^k,a_j)(f^1,a_i)-(f^k,a_i)(f^1,a_j)}{h_t(f^k,a_i)(f^1,a_i)}$ $(k = 2, 3)$ are holomorphic on a neighborhood of z. Therefore, it follows that

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq 2\nu_{h_t}(z) = \nu_i(z) + 2\sum_{s=1, s\neq i,j}^{4} \mu_s(z).$$

From the above two cases, we have

$$\nu_{\Phi^{\alpha}_{ij}}(z) \geq \nu_i(z) + 2\sum_{s=1, s\neq i,j}^{4} \mu_s(z). \tag{3.5}$$

From (3.4) and (3.5), it implies that

$$\nu_{g_{ij}} \geq \sum_{u=1}^{3} \nu_{(f^u,a_j)} + \nu_i + 2\sum_{s=1, s\neq i,j}^{4} \mu_s.$$

□

Lemma 3.6. *With the assumption of Theorem 1.1, if there exist $i, j \in \{1, ..., 4\}$, $i \neq j$ such that $\Phi^{\alpha}_{ij}(F_1^{ij}, F_2^{ij}, F_3^{ij}) = 0$ and $\Phi^{\alpha}_{ji}(F_1^{ji}, F_2^{ji}, F_3^{ji}) = 0$, for every $\alpha = (\alpha_0, ..., \alpha_m) \in \mathbb{N}^m$ with $|\alpha| = 1$ then $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$.*

Proof. Denote by $\{s, t\}$ the set $\{1, ..., 4\}\backslash\{i, j\}$. From Proposition 2.5 we have one of the following two cases:

* **Case 1**. $F_1^{ij} = F_2^{ij}$ or $F_2^{ij} = F_3^{ij}$ or $F_3^{ij} = F_1^{ij}$. It follows that $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$. We have the desired assertion of the lemma.

* **Case 2**. There exist constants $b, c \in \mathbb{C}\backslash\{0,1\}$ with $b \neq c$ such that

$$F_1^{ij} = bF_2^{ij} = cF_3^{ij}. \tag{3.7}$$

Since $\Phi^\alpha_{st}(F_1^{st}, F_2^{st}, F_3^{st}) \neq 0$, $\{z:\ \nu_{(f^k,a_s)} > 0\} \cup \{z:\ \nu_{(f^k,a_t)} > 0\} = \emptyset$. From (3.7), we have

$$f^1 = \frac{(a_i - ba_j)f^2 - (1-b)a_ia_j}{(1-b)f^2 - (a_j - ba_i)} = \frac{(a_i - ca_j)f^3 - (1-c)a_ia_j}{(1-c)f^3 - (a_j - ca_i)}.$$

We set

$$d_t^2 = \frac{(a_i - ba_j)a_t - (1-b)a_ia_j}{(1-b)a_t - (a_j - ba_i)} \text{ and } d_t^3 = \frac{(a_i - ca_j)a_t - (1-c)a_ia_j}{(1-c)a_t - (a_j - ca_i)}.$$

We consider each value $d_t^k = (0, d_t^k),\ k = 2, 3$. Then it is easy to see that

$$\{z:\ \nu_{(f^1,d_t^k)} > 0\} = \{z:\ \nu_{f^1 - d_t^k} > 0\} = \{z:\ \nu_{f^k - d_t^k} > 0\} = \{z:\ \nu_{(f^k,d_t^k)} > 0\} = \emptyset\ (k = 2, 3).$$

Since $b \neq c$ and $b, c \notin \{0, 1\}$, we have $d_t^k \neq a_t$ $(k = 2, 3)$ and $d_t^2 \neq d_t^3$. Then there exists at least one function, for instance it is d_t^2, such that $d_t^2 \neq a_s$.

Applying Lemma 3.1 for the function $1 \in B(0; 0; f^1)$ and three values $\{a_t, a_s, d_t^2\}$, we have

$$1 \leq \rho.$$

This is a contradiction. Hence this case is impossible.

Therefore $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$. □

Proof of Theorem 1.1. Firstly, we assume that $M = B(R_0)$. Suppose contrarily that there exist three distinct nonconstant meromorphic functions f^1, f^2, f^3 on $B(R_0)$, which satisfy the condition (C_ρ).

We set $\mathcal{Q} = \{j:\ \Phi^\alpha_{vj}(F_1^{vj}, F_2^{vj}, F_3^{vj}) = 0, \forall\ v \in \{1, ..., 4\}\}$. We consider the following three cases.

* **Case 1**. $\sharp\mathcal{Q} \geq 2$. There exist $i, j \in \mathcal{Q},\ i \neq j$. Then $\Phi^\alpha_{ij}(F_1^{ij}, F_2^{ij}, F_3^{ij}) = 0$ and $\Phi^\alpha_{ji}(F_1^{ji}, F_2^{ji}, F_3^{ji}) = 0$. By Lemma 3.6 we have $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$. This is a contradiction.

* **Case 2**. $\sharp\mathcal{Q} = 1$. Assume that $\mathcal{Q} = \{j\}$. For $i \neq j$ we have $\Phi^\alpha_{ij}(F_1^{ij}, F_2^{ij}, F_3^{ij}) = 0$, then

$$0 = \Phi^\alpha_{ij} = F_1^{ij}.F_2^{ij}.F_3^{ij}.\begin{vmatrix} F_2^{ji} - F_1^{ji} & F_3^{ji} - F_1^{ji} \\ \mathcal{D}^\alpha(F_2^{ji} - F_1^{ji}) & \mathcal{D}^\alpha(F_3^{ji} - F_1^{ji}) \end{vmatrix}$$

We see that the determinant is a Wronskian. Then $F_k^{ji} - F_1^{ji}$ $(k = 2, 3)$ are linearly dependent. Therefore, there exists a constant $c \in \mathbb{C}$ such that

$$(F_2^{ji} - F_1^{ji}) = c(F_3^{ji} - F_1^{ji}) \Leftrightarrow (1-c)F_1^{ji} - F_2^{ji} + cF_3^{ji} = 0.$$

Since we suppose that f^1, f^2, f^3 are distinct, hence $c \notin \{0,1\}$. Then, for $z \notin S$ it is easy to see that $\nu^{\infty}_{F^{ji}_k}(z) \le \max\{\nu^{\infty}_{F^{ji}_l}(z), \nu^{\infty}_{F^{ji}_t}(z)\}$, i.e.

$$\nu_{(f^k,a_i)}(z) \le \max\{\nu_{(f^l,a_i)}(z), \nu_{(f^t,a_i)}(z)\} \text{ with } \{k,l,t\} = \{1,2,3\}.$$

Hence, if z is a zero of (f^k, a_i) with multiplicity at least 2, then z is a zero of (f^l, a_i) or (f^t, a_i) with multiplicity at least $\nu_{(f^k,a_i)}(z) \ge 2$. Then, it follows that

$$\nu_{(f^k,a_i)}(z) \le 2\left(\min\{\nu_{(f^k,a_i)}(z), \nu_{(f^l,a_i)}(z)\} - 1 + \min\{\nu_{(f^k,a_i)}(z), \nu_{(f^t,a_i)}(z)\} - 1\right)$$

This implies that

$$\begin{aligned}\nu_{(f^k,a_i),>1} \le& 2\left(\min\{\nu_{(f^k,a_i)}(z), \nu_{(f^l,a_i)}(z)\} - \nu^{[1]}_{(f^k,a_i)}(z) + \min\{\nu_{(f^k,a_i)}(z), \nu_{(f^t,a_i)}(z)\} - \nu^{[1]}_{(f^k,a_i)}(z)\right)\\ \le& 2(\nu^{[1]}_{(f^l,a_i)}(z) + \nu^{[1]}_{(f^t,a_i)}(z) - 2\nu^{[1]}_{(f^k,a_i)}(z)).\end{aligned}$$

This yields that

$$\begin{aligned}\nu_i \ge& \nu^{[1]}_{(f^k,a_i)} - \sum_{v=1}^{3} \nu_{(f^v,a_i),>1}\\ \ge& \nu^{[1]}_{(f^k,a_i)}.\end{aligned}$$

Now we denote by $\mathcal{Q}_1 = \{i \in \{1,...,4\} \setminus \{j\} : \ \Phi^{\alpha}_{is}(F^{is}_1, F^{is}_2, F^{is}_3) = 0, \forall s \ne j\}$. If $\sharp \mathcal{Q}_1 \ge 2$, then similar to Case 1 we get $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$. This is a contradiction. Hence there exist at least two elements, denoted by i and s, not in $\mathcal{Q}_1$. By Lemma 3.2, we have

$$\begin{aligned}\nu_{g_{is}} \ge& \sum_{u=1}^{3} \nu_{(f^u,a_s)} + \nu_i + 2\sum_{v=1, v\ne i,s}^{4} \mu_v\\ \ge& \sum_{u=1}^{3} \nu_{(f^u,a_s)} + \nu^{[1]}_{(f^k,a_i)} + 2\sum_{v=1, v\ne i,s}^{4} \nu^{[1]}_{(f^k,a_v)}. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\nu_{g_{is}} \ge \sum_{u=1}^{3} \nu_{(f^u,a_i)} + \nu^{[1]}_{(f^k,a_s)} + 2\sum_{v=1, v\ne s,i}^{4} \nu^{[1]}_{(f^k,a_v)}. \tag{3.9}$$

Summing-up both sides of (3.8) and (3.9), we obtain

$$2\nu_{g_{is}} \ge 4\sum_{v=1}^{4} \nu^{[1]}_{(f^k,a_v)} + \sum_{u=1}^{3} (\nu_{(f^u,a_s)} + \nu_{(f^u,a_i)}) - 3\nu^{[1]}_{(f^k,a_i)} - 3\nu^{[1]}_{(f^k,a_s)}.$$

This implies that

$$\nu^{[1]}_{(f^k,a_i)} + \nu^{[1]}_{(f^k,a_s)} \ge \frac{4}{3}\sum_{v=1}^{4} \nu^{[1]}_{(f^k,a_v)} + \frac{1}{3}\sum_{u=1}^{3} (\nu_{(f^u,a_s)} + \nu_{(f^u,a_i)}) - \frac{2}{3}\nu_{g_{is}}. \tag{3.10}$$

Take $t \in \{1,2,3,4\}\setminus\{j,i,s\}$. If $\Phi^{\alpha}_{jt}(F_1^{jt}, F_2^{jt}, F_3^{jt}) = 0$, then by Lemma 3.6 we have $f^1 = f^2$ or $f^2 = f^3$ or $f^3 = f^1$. This is a contradiction. Hence $\Phi^{\alpha}_{jt}(F_1^{jt}, F_2^{jt}, F_3^{jt}) \neq 0$. By Lemma 3.2 and (3.10), it follows that

$$\begin{aligned}
\nu_{g_{jt}} &\geq \sum_{u=1}^{3} \nu_{(f^u,a_t)} + \nu_j + 2\sum_{v=1,v\neq j,t}^{4} \mu_v \\
&\geq \sum_{u=1}^{3} \nu_{(f^u,a_t)} + \nu^{[1]}_{(f^k,a_j)} + 2\sum_{v=1,v\neq j,t}^{4} \nu^{[1]}_{(f^k,a_v)} \\
&= \sum_{u=1}^{3} \nu_{(f^u,a_t)} + \nu^{[1]}_{(f^k,a_j)} + 2(\nu^{[1]}_{(f^k,a_i)} + \nu^{[1]}_{(f^k,a_s)}) \\
&\geq \sum_{u=1}^{3} \nu_{(f^u,a_t)} + \nu^{[1]}_{(f^k,a_j)} + \frac{8}{3}\sum_{v=1}^{4} \nu^{[1]}_{(f^k,a_v)} + \frac{2}{3}\sum_{u=1}^{3}\big(\nu_{(f^u,a_s)} + \nu_{(f^u,a_i)}\big) - \frac{4}{3}\nu_{g_{is}}.
\end{aligned}$$

Summing-up both sides of the above inequality over all $t \in \{1,2,3,4\}\setminus\{j\}$, we get

$$\sum_{t=1,t\neq j}^{4} \nu_{g_{jt}} \geq \sum_{t=1,t\neq j}^{4}\sum_{u=1}^{3} \nu_{(f^u,a_t)} + \frac{8}{3}\sum_{t=1,t\neq j}^{4}\sum_{u=1}^{3} \nu^{[1]}_{(f^u,a_t)} + 2\sum_{t\neq j}\sum_{u=1}^{3} \nu_{(f^u,a_t)} + 9\nu^{[1]}_{(f^k,a_j)} - 4\nu_{g_{is}}.$$

This implies that

$$\sum_{t=1,t\neq j}^{4} \nu_{g_{jt}} + 4\nu_{g_{is}} \geq \frac{17}{3}\sum_{t=1,t\neq j}^{4}\sum_{u=1}^{3} \nu^{[1]}_{(f^u,a_t)}.$$

So that

$$\nu_{\prod_{t=1,t\neq j}^{4} g_{jt} g_{is}^4} \geq \frac{17}{3}\sum_{t=1,t\neq j}^{4}\sum_{u=1}^{3} \nu^{[1]}_{(f^u,a_t)}.$$

It is clear that $\prod_{t=1,t\neq j}^{4} g_{jt} g_{is}^4 \in B(3,3;f^1,f^2,f^3)$. Then, from Lemma 3.1, we have

$$\frac{2}{15} \leq \rho.$$

This is a contradiction.

* **Case 3**. $\sharp\mathcal{Q} = \emptyset$. Then for all $i \neq j$, by Lemma 3.2 we have

$$\nu_{g_{ij}} \geq \sum_{u=1}^{3} \nu_{(f^u,a_j)} + \nu_i + 2\sum_{s=1,s\neq i,j}^{4} \mu_s.$$

Setting

$$\sigma(1) = 2;\ \sigma(2) = 3;\ \sigma(3) = 4; \sigma(4) = 1,$$

and summing-up both sides of the above inequality over all pairs $(i,\sigma(i))$, we get

$$\begin{aligned}\sum_{i=1}^{4}\nu_{g_{i\sigma(i)}}&\geq\sum_{i=1}^{4}\sum_{u=1}^{3}\nu_{(f^u,a_i)}+\sum_{i=1}^{4}\nu_i+2\sum_{i=1}^{4}\sum_{s=1,s\neq i,\sigma(i)}^{4}\mu_s.\\&\geq\sum_{i=1}^{4}\sum_{u=1}^{3}\nu_{(f^u,a_i)}+\sum_{i=1}^{4}\nu_i+2\sum_{i=1}^{4}\sum_{s=1,s\neq i,\sigma(i)}^{4}\nu^{[1]}_{(f^k,a_s)}.\end{aligned}\tag{3.11}$$

On the other hand, we have

$$\begin{aligned}\nu_{(f^k,a_4),>1}&=\nu^{[2]}_{(f^k,a_4)}-\nu^{[1]}_{(f^k,a_4)}\leq\min\{\nu_{(f^k,a_4)},\nu_{(f^l,a_4)}\}-\nu^{[1]}_{(f^k,a_4)}\\&\leq\nu^{[1]}_{(f^k,a_4)}+\nu^{[1]}_{(f^l,a_4)}-2\nu^{[1]}_{(f^k,a_4)}=\nu^{[1]}_{(f^l,a_4)}-\nu^{[1]}_{(f^k,a_4)}\ (1\leq k,l\leq 3,k\neq l).\end{aligned}$$

Then

$$\nu^{[1]}_{(f^k,a_4)}\leq\nu_4+\sum_{k=1}^{3}\nu_{(f^k,a_4),>1}\leq\nu_4\leq\sum_{i=1}^{4}\nu_i.\tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned}\sum_{i=1}^{4}\nu_{g_{i\sigma(i)}}&\geq\sum_{i=1}^{4}\sum_{u=1}^{3}\nu_{(f^u,a_i)}+\nu^{[1]}_{(f^k,a_4)}+2\sum_{i=1}^{4}\sum_{s=1,s\neq i,\sigma(i)}^{4}\nu^{[1]}_{(f^k,a_s)}\\&\geq\sum_{i=1}^{4}\sum_{u=1}^{3}\nu_{(f^u,a_i)}+\nu^{[1]}_{(f^k,a_4)}+\frac{4}{3}\sum_{i=1}^{4}\sum_{u=1}^{3}\nu^{[1]}_{(f^u,a_i)}\\&\geq\frac{7}{3}\sum_{i=1}^{4}\sum_{u=1}^{3}\nu^{[1]}_{(f^u,a_i)}.\end{aligned}$$

This implies that

$$\nu_{\prod_{i=1}^4 g_{i\sigma(i)}}\geq\frac{7}{3}\sum_{i=1}^{4}\sum_{u=1}^{3}\nu_{(f^u,a_i)}.$$

It is clear that $\prod_{i=1}^4 g_{i\sigma(i)}\in B(4,4;f^1,f^2,f^3)$. Then, from Lemma 3.1, we have

$$\frac{2}{33}\leq\rho.$$

This is a contradiction.

Then, from the above three cases, we see that the supposition is impossible. Hence we must have $f^1=f^2$ or $f^2=f^3$ or $f^3=f^1$. The theorem is proved.

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