ODD WRITHES AND $2k$-MOVES FOR VIRTUAL KNOTS

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Abstract. A $2k$-move is a local deformation on a knot diagram adding or removing $2k$ half-twists, where k is a positive integer. We show that if two virtual knots are related by a finite sequence of $2k$-moves, then their odd writhes are congruent modulo $2k$. Moreover, we provide a necessary and sufficient condition for two virtual knots to have the same congruence class of odd writhes modulo $2k$.

1. Introduction

Let k be a positive integer. A $2k$-*move* on a knot diagram is a local deformation adding or removing $2k$ half-twists as shown in Figure 1.1. A 2-move is equivalent to a crossing change; that is, a 2-move is realized by a crossing change, and vice versa. In this sense a $2k$-move can be considered as a generalization of a crossing change. The $2k$-moves form an important family of local moves in classical knot theory. In fact, they have been well studied by means of many invariants of classical knots and links in the 3-sphere; for example, Alexander polynomials [15], Jones, HOMFLYPT and Kauffman polynomials [21], Burnside groups [5, 6], Milnor invariants [18] and quandles [10].

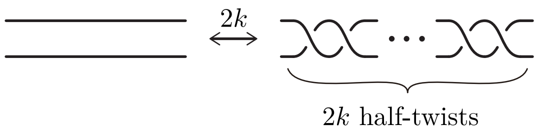

Figure 1.1. A $2k$-move

This paper studies $2k$-moves in the setting of virtual knots, which are a generalization of classical knots discovered by Kauffman [12]. Roughly speaking, a *virtual knot* is an equivalence class of generalized knot diagrams called *virtual knot diagrams* under seven types of local deformations. We say that two virtual knots are related by a $2k$-move if a diagram of one is a result of a $2k$-move on a diagram of the other.

For a virtual knot K, Kauffman [13] introduced an integer-valued invariant $J(K)$ called the *odd writhe*. Satoh and Taniguchi [22] generalized it to a sequence of integer-valued invariants $J_n(K)$ of K called

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the *n-writhes* ($n \in \mathbb{Z} \setminus \{0\}$). This sequence $\{J_n(K)\}_{n\neq 0}$ gives rise to a polynomial invariant $P_K(t)$ of K known as the *affine index polynomial* due to Kauffman [14] as follows:

$$P_K(t) = \sum_{n\neq 0} J_{-n}(K)(t^n - 1),$$

which is essentially equivalent to the *writhe polynomial* due to Cheng and Gao [4]. Refer to [3] for a good survey of virtual knot invariants derived from chord index, including the invariants $J(K)$, $J_n(K)$ and $P_K(t)$.

Recently, Jeong, Choi and Kim [11] established a necessary condition for two virtual knots to be equivalent under $2k$-moves using their affine index polynomials as follows:

Theorem 1.1 ([11, Theorem 2.3])**.** *If two virtual knots K and K' are related by a finite sequence of $2k$-moves, then $P_K(t)$ and $P_{K'}(t)$ are congruent modulo k; that is, $J_n(K)$ and $J_n(K')$ are congruent modulo k for any nonzero integer n.*

Examining their proof of this theorem given in [11], we can find another necessary condition in terms of odd writhes, which states that if two virtual knots K and K' are related by a finite sequence of $2k$-moves, then $J(K)$ and $J(K')$ are congruent modulo $2k$ (Proposition 2.2).

A Ξ-*move* on a virtual knot diagram is a local deformation exchanging the positions of c_1 and c_3 of three consecutive real crossings c_1, c_2 and c_3 as shown in Figure 1.2, where we omit the over/under information of every crossing c_i ($i = 1, 2, 3$). The Ξ-move arises naturally as a diagrammatic characterization of virtual knots having the same odd writhe. In fact, Satoh and Taniguchi [22] showed the following theorem.

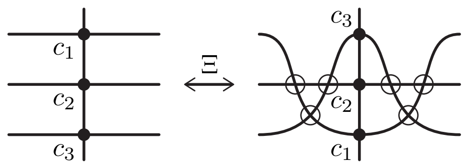

FIGURE 1.2. A Ξ-move

Theorem 1.2 ([22, Theorem 1.7])**.** *For two virtual knots K and K', the following are equivalent:*

(i) *$J(K)$ and $J(K')$ are equal.*
(ii) *K and K' are related by a finite sequence of Ξ-moves.*

Inspired by this theorem, we use Ξ-moves together with $2k$-moves to characterize virtual knots having the same congruence class of odd writhes modulo $2k$. The following theorem is our main result.

Theorem 1.3. *For two virtual knots K and K', the following are equivalent:*

(i) *$J(K)$ and $J(K')$ are congruent modulo $2k$.*
(ii) *K and K' are related by a finite sequence of $2k$-moves and Ξ-moves.*

In [1, Proposition 25], Carter, Kamada and Saito proved that not every virtual knot can be unknotted by crossing changes, although the crossing change is an unknotting operation for classical knots. Refer also to [9, 23]. This fact justifies the notion of flat virtual knots. A *flat virtual knot* [12], also known as a *virtual string* [23], is an equivalence class of virtual knots up to crossing changes. Equivalently, a flat virtual knot is represented by a virtual knot diagram with all the real crossings replaced by flat crossings, where a *flat crossing* is a transverse double point with no over/under information.

In [2, Lemma 2.2], Cheng showed that the odd writhe for any virtual knot takes values in even integers. Hence any virtual knot K and the trivial one O satisfy $J(K) \equiv J(O) \equiv 0 \pmod 2$. By Theorem 1.3 for $k = 1$, the two knots K and O are related by a finite sequence of 2-moves and Ξ-moves. In other words, we have the following corollary.

Corollary 1.4. *Any flat virtual knot can be deformed into the trivial knot by a finite sequence of flat Ξ-moves; that is, the flat Ξ-move is an unknotting operation for flat virtual knots. Here, a flat Ξ-move is a Ξ-move with all the real crossings replaced by flat ones.* □

For two virtual knots K and K' that are related by a finite sequence of $2k$-moves, we denote by $\mathrm{d}_{2k}(K, K')$ the minimal number of $2k$-moves needed to deform a diagram of K into that of K'. In particular, when $K' = O$ is the trivial knot, we set $\mathrm{u}_{2k}(K) = \mathrm{d}_{2k}(K, O)$.

In [11], Jeong, Choi and Kim provided a lower bound for $\mathrm{d}_{2k}(K, K')$ using the coefficients of the affine index polynomials of K and K' (that is, the n-writhes of K and K'), and demonstrated that their lower bound for $\mathrm{u}_{2k}(K)$ is sharp for some virtual knots K. However, they did not make it clear whether for a pair of nontrivial virtual knots K and K', the lower bound for $\mathrm{d}_{2k}(K, K')$ is sharp. We answer this by proving the following theorem.

Theorem 1.5. *Let p be a positive integer. For any virtual knot K, there is a virtual knot K' with $\mathrm{d}_{2k}(K, K') = p$.*

Moreover we have the following theorem.

Theorem 1.6. *For any positive integer p, there are infinitely many virtual knots K with $\mathrm{u}_{2k}(K) = p$.*

The rest of this paper is organized as follows. In Section 2, we review the definitions of a virtual knot, a Gauss diagram, the n-writhe and the odd writhe, and prove the invariance of the modulo $2k$ reduction of the odd writhe under $2k$-moves. Section 3 is devoted to the proof of Theorem 1.3. Our main tool is the notion of shell-pairs, which are certain pairs of chords of a Gauss diagram introduced in [17]. Finally, in Section 4, we prove Theorems 1.5 and 1.6 using Jeong-Choi-Kim's lower bound for d_{2k} of virtual knots.

2. Odd writhes and $2k$-moves

We begin this section by recalling the definitions of virtual knots and Gauss diagrams from [8, 12]. A *virtual knot diagram* is the image of an immersion of an oriented circle into the plane whose singularities are only transverse double points. Such double points consist of *positive*, *negative* and *virtual crossings* as shown in Figure 2.1. A positive/negative crossing is also called a *real crossing*.

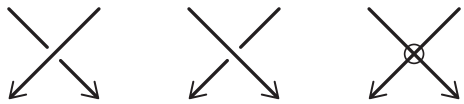

Figure 2.1. Types of double points

Two virtual knot diagrams are said to be *equivalent* if they are related by a finite sequence of *generalized Reidemeister moves* I–VII as shown in Figure 2.2. A *virtual knot* is the equivalence class of a virtual knot diagram. In particular, a classical knot in the 3-sphere can be considered as a virtual knot diagram with no virtual crossings, called a *classical knot diagram*, up to the moves I, II and III. In [8, Theorem 1.B], Goussarov, Polyak and Viro proved that two equivalent classical knot diagrams are related by a finite sequence of moves I, II, and III; that is, the set of virtual knots contains that of classical knots. In this sense, virtual knots are a generalization of classical knots.

A *Gauss diagram* is an oriented circle equipped with a finite number of signed and oriented chords whose endpoints lie disjointly on the circle. In figures the underlying circle and chords of a Gauss diagram will be drawn with thick and thin lines, respectively. Gauss diagrams provide an alternative way of representing virtual knots. For a virtual knot diagram D with n real crossings (and some or no virtual crossings), the *Gauss diagram G_D associated with D* is constructed as follows. It consists of a circle and n chords connecting the preimage of each real crossing of D. Each chord of G_D has the sign of the corresponding real crossing of D, and it is oriented from the overcrossing to the undercrossing. For a virtual knot K, a *Gauss diagram of K* is defined to be a Gauss diagram associated with a virtual knot diagram of K.

A motivation of introducing virtual knot theory comes from the realization of Gauss diagrams. In fact, the construction above defines

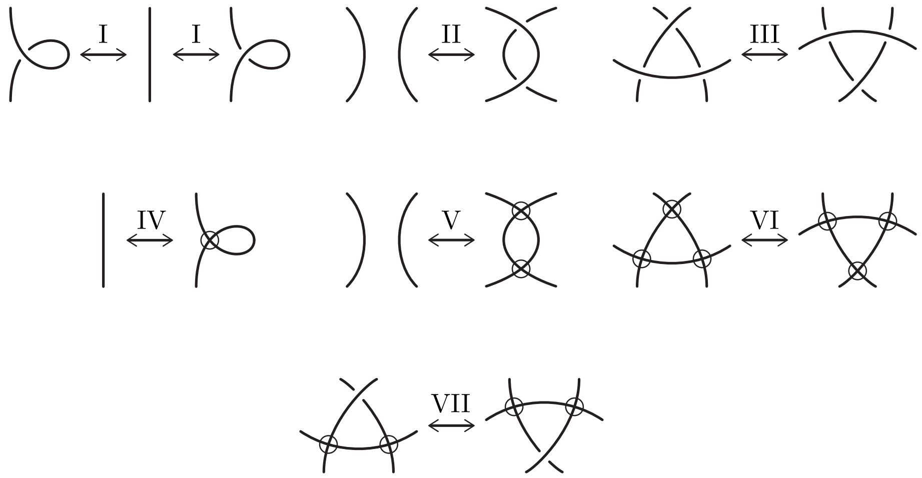

FIGURE 2.2. Generalized Reidemeister moves I–VII

a surjective map from virtual knot diagrams onto Gauss diagrams, although not every Gauss diagram can be realized by a classical knot diagram. Moreover, this map induces a bijection between the set of virtual knots and that of Gauss diagrams modulo Reidemeister moves I, II and III defined in the Gauss diagram level as shown in Figure 2.3 [8, Theorem 1.A]. Refer also to [12, Section 3.2]. Therefore a virtual knot can be regarded as the equivalence class of a Gauss diagram.

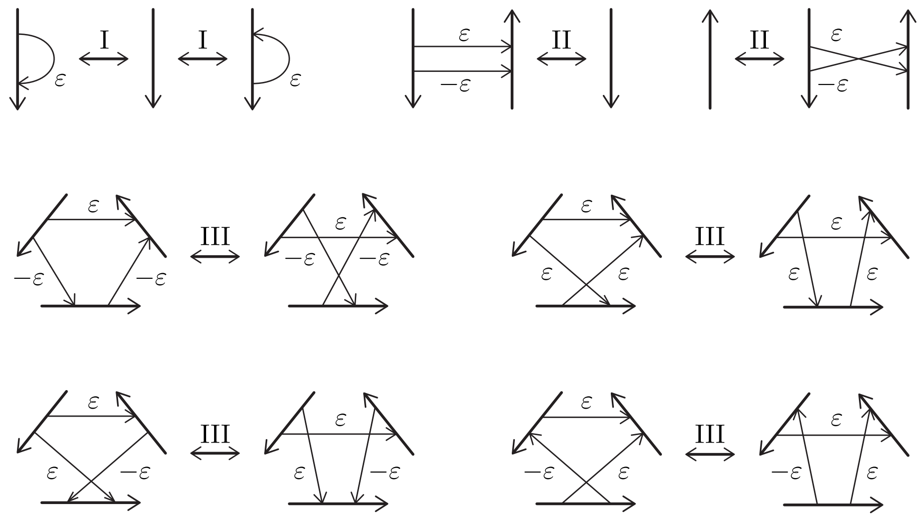

FIGURE 2.3. Reidemeister moves I, II and III on Gauss diagrams ($\varepsilon = \pm 1$)

We will use two deformations on Gauss diagrams as shown in Figure 2.4 as well as the Reidemeister moves I, II and III. These deformations are the counterparts of a $2k$-move and a Ξ-move for Gauss diagrams. More precisely, a *$2k$-move on a Gauss diagram* adds or removes $2k$ chords with the same sign ε whose initial and terminal endpoints

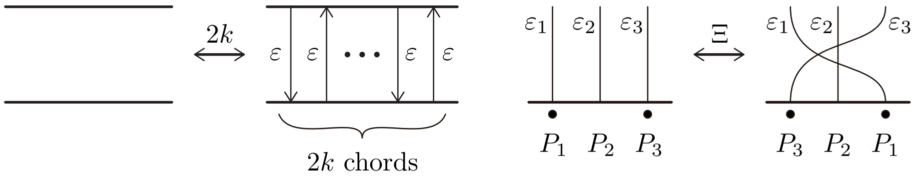

FIGURE 2.4. A $2k$-move and a Ξ-move on Gauss diagrams

appear alternately. Let P_1, P_2 and P_3 be three consecutive endpoints of chords of a Gauss diagram. A Ξ-*move* exchanges the positions of P_1 and P_3, preserving the signs $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and orientations of the chords. In the right of the figure, a pair of dots $\bullet$ marks the two endpoints P_1 and P_3 exchanged by a Ξ-move.

Now we define the n-writhe and the odd writhe of a virtual knot K using Gauss diagrams. For a Gauss diagram G of K, let γ be a chord of G. If γ has a sign ε, then we assign ε and $-\varepsilon$ to the terminal and initial endpoints of γ, respectively. The endpoints of γ divide the underlying circle of G into two oriented arcs. Let α be the arc running from the initial endpoint of γ to the terminal one; see Figure 2.5. The *index* of γ, $\mathrm{ind}(\gamma)$, is the sum of the signs of all the endpoints of chords on α.

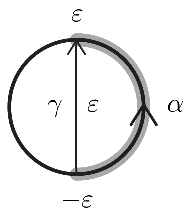

FIGURE 2.5. A chord γ with sign ε and its specified arc α

For an integer n, we denote by $J_n(G)$ the sum of the signs of all the chords with index n. In [22, Lemma 2.3], Satoh and Taniguchi proved that $J_n(G)$ is an invariant of the virtual knot K for any $n \neq 0$; that is, it is independent of the choice of G. This invariant is called the *n-writhe* of K and denoted by $J_n(K)$. The *odd writhe* $J(K)$ of K due to Kauffman [13] can be defined by

$$J(K) = \sum_{n\in\mathbb{Z}} J_{2n-1}(K).$$

Refer to [3, 13, 22] for more details.

The following lemma given in [11] reveals the behavior of $J_n(G)$ of a Gauss diagram G under a $2k$-move on G, and Theorem 1.1 follows from this lemma immediately. We use this lemma to prove the invariance of the modulo $2k$ reduction of the odd writhe under $2k$-moves.

Lemma 2.1 ([11, Lemma 2.2])**.** *If two Gauss diagrams G and G' are related by a single $2k$-move, then there is a unique integer n such that*

$$J_n(G) - J_n(G') = \varepsilon k, \ J_{-n}(G) - J_{-n}(G') = \varepsilon k \text{ and } J_m(G) = J_m(G')$$

for some $\varepsilon = \pm 1$ and any integer $m \neq \pm n$.

Proposition 2.2. *If two virtual knots K and K' are related by a finite sequence of $2k$-moves, then $J(K)$ and $J(K')$ are congruent modulo $2k$.*

Proof. Assume that K and K' are related by a single $2k$-move, and let G and G' be Gauss diagrams of K and K', respectively. Then G and G' are related by a finite sequence of a single $2k$-move and several Reidemeister moves. By Lemma 2.1 and [22, Lemma 2.3], there is a unique integer n such that

$$J_n(G) - J_n(G') = \varepsilon k, \ J_{-n}(G) - J_{-n}(G') = \varepsilon k \text{ and } J_m(G) = J_m(G')$$

for some $\varepsilon = \pm 1$ and any integer $m \neq \pm n$. Therefore the difference $J(K) - J(K')$ equals $2\varepsilon k$ for n odd and 0 for n even. □

In [7], Fox introduced the notion of *congruence classes modulo* (n, q) of classical knots for nonnegative integers n and q, and asked whether the set of congruence classes of a classical knot determines the knot type. More precisely, his question is: if two classical knots are congruent modulo (n, q) for all $n \geq 1$ and $q \geq 0$, then are they the same type? It is known [7, 16, 20] that the Alexander and Jones polynomials of classical knots provide information about their congruence classes. For example, in [16, Corollary 2.4], Lackenby proved that if two classical knots are congruent modulo $(n, 2)$ for all $n \geq 1$, then they have the same Jones polynomial.

The notion of Fox's congruence classes can be extended to virtual knots by a diagrammatic way as shown in [16, Figure 1]. We can see that if two virtual knots are related by a finite sequence of $2k$-moves, then they are congruent modulo $(k, 2)$. Therefore it would be interesting to know whether the set of $2k$-move equivalence classes of a virtual knot determines the knot type. As a consequence of Theorem 1.2 and Proposition 2.2, we show the following proposition related to this question, which states that the set of $2k$-move equivalence classes of a virtual knot K determines the Ξ-move equivalence class of K.

Proposition 2.3. *If two virtual knots K and K' are related by a finite sequence of $2k$-moves for all $k \geq 1$, then $J(K)$ and $J(K')$ are equal. Equivalently, if two virtual knots are related by a finite sequence of $2k$-moves for all $k \geq 1$, then they are related by a finite sequence of Ξ-moves.*

Proof. By assuming that K and K' have different odd writhes, there is a positive integer k such that $J(K) \not\equiv J(K') \pmod{2k}$. By Proposition 2.2, this contradicts that K and K' are related by a finite sequence of $2k$-moves for all $k \geq 1$. Thus we have $J(K) = J(K')$. Equivalently by Theorem 1.2, K and K' are related by a finite sequence of Ξ-moves. □

3. Proof of Theorem 1.3

In our proof of Theorem 1.3, the main tool is the notion of a shell-pair, which is a certain pair of chords of a Gauss diagram developed in [17] for classifying 2-component virtual links up to Ξ-moves. It is defined as follows.

Let P_1, P_2 and P_3 be three consecutive endpoints of chords of a Gauss diagram G. We say that a chord of G is a *shell* if it connects P_1 and P_3; see the left of Figure 3.1. Note that the orientation of a shell can be reversed by a Ξ-move exchanging the positions of P_1 and P_3. A *positive shell-pair* (or *negative shell-pair*) consists of a pair of positive shells (or *negative shells*) whose four endpoints are consecutive; see the right of the figure, where we omit the orientations of shells.

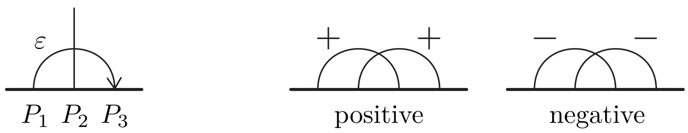

Figure 3.1. A shell and a positive/negative shell-pair

We prepare three results (Lemmas 3.1, 3.2 and Proposition 3.3) to give the proof of Theorem 1.3. The first and second results are used to prove the third one.

The following lemma was shown in [17, 22].

Lemma 3.1 ([17, Lemmas 4.1 and 4.2], [22, Fig. 13])**.** *Let G, G' and G'' be Gauss diagrams.*

(i) *If G' is obtained from G by a local deformation exchanging the positions of a shell-pair and an endpoint of a chord in G, which preserves the orientations of the chords, as shown in the top of Figure 3.2, then G and G' are related by a finite sequence of Ξ-moves and Reidemeister moves.*

(ii) *If G'' is obtained from G by a local deformation adding or removing two consecutive shell-pairs with opposite signs as shown in the bottom of Figure 3.2, then G and G'' are related by a finite sequence of Ξ-moves and Reidemeister moves.*

Lemma 3.2. *Let G and G' be Gauss diagrams, and k a positive integer. If G' is obtained from G by a local deformation adding or removing k consecutive shell-pairs with the same sign ε as shown in Figure 3.3, then G and G' are related by a finite sequence of $2k$-moves, Ξ-moves and Reidemeister moves.*

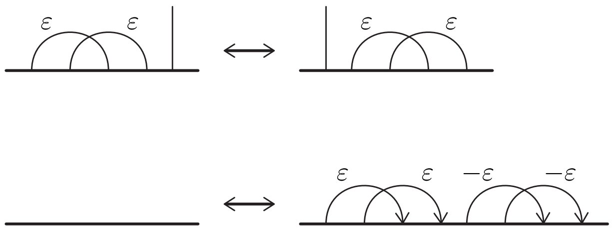

FIGURE 3.2. Local deformations in Lemma 3.1

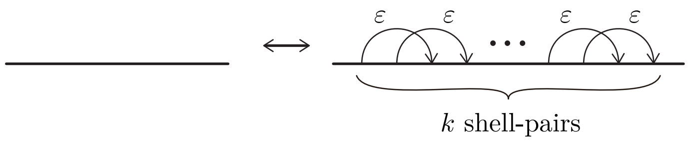

FIGURE 3.3. Adding or removing k consecutive shell-pairs

Proof. We only prove the result for $k = 2$. The other cases are shown similarly.

Assume that G' is obtained from G by adding two consecutive shell-pairs with sign ε. The proof follows from Figure 3.4, which gives a sequence of Gauss diagrams

$$G = G_0, G_1, \ldots, G_6 = G'$$

such that for each $i = 1, 2, \ldots, 6$, G_i is obtained from G_{i-1} by a combination of 4-moves, Ξ-moves and Reidemeister moves. More precisely, we obtain G_1 from $G_0 = G$ by a Reidemeister move I adding a positive chord, G_2 from G_1 by a 4-move adding four chords with sign ε, and G_3 from G_2 by a Ξ-move exchanging the positions of the two endpoints with dots $\bullet$. By Lemma 3.1(i), we can move the resulting shell-pair, preserving the orientations of the chords, to get G_4 from G_3. After deforming G_4 into G_5 by a Ξ-move, we finally obtain $G_6 = G'$ by two Ξ-moves reversing the orientations of shells and a Reidemeister move I removing a positive chord. □

For an integer a, let $G(a)$ be the Gauss diagram in Figure 3.5; that is, it consists of $|a|$ shell-pairs with sign ε, where $\varepsilon = 1$ for $a > 0$ and $\varepsilon = -1$ for $a < 0$. In particular, $G(0)$ is the Gauss diagram with no chords. Denote by $K(a)$ the virtual knot represented by $G(a)$. We remark that $K(a)$ satisfies $J(K(a)) = 2a$.

We give a normal form of an equivalence class of virtual knots under $2k$-moves and Ξ-moves as follows:

Proposition 3.3. *Any virtual knot K is related to $K(a)$ for some $a \in \mathbb{Z}$ with $0 \leq a < k$ by a finite sequence of $2k$-moves and Ξ-moves.*

Proof. By [22, Proposition 7.2], any Gauss diagram G of K can be deformed into $G(a)$ for some $a \in \mathbb{Z}$ by a finite sequence of Ξ-moves

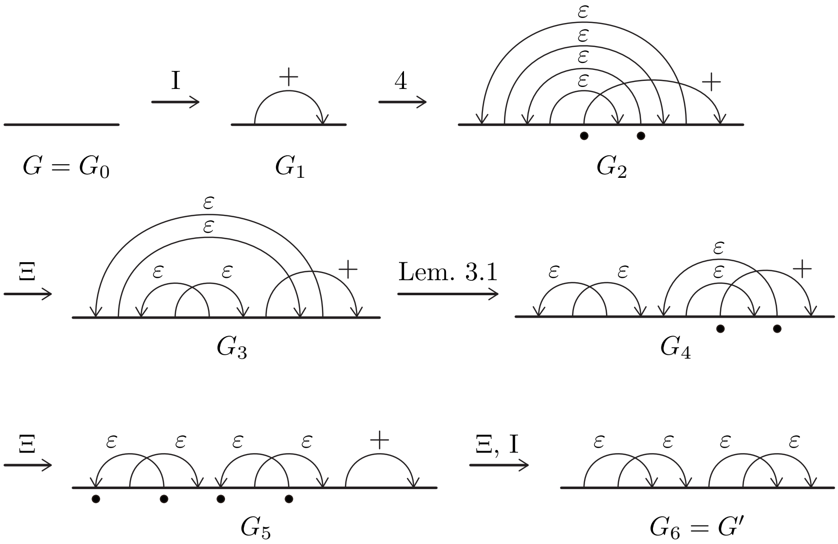

FIGURE 3.4. Proof of Lemma 3.2 for $k = 2$

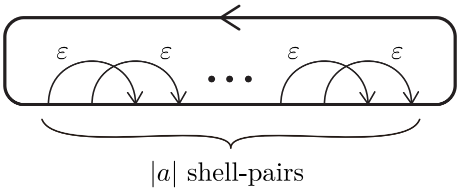

FIGURE 3.5. The Gauss diagram $G(a)$

and Reidemeister moves. If a satisfies $0 \leq a < k$, then we have the conclusion.

For $k \leq a$, there is a unique positive integer p with $0 \leq a - pk < k$. Lemma 3.2 allows us to add pk consecutive negative shell-pairs to $G(a)$. From the resulting Gauss diagram, we can remove pk pairs of shell-pairs with opposite signs by Lemma 3.1(ii) in order to obtain $G(a-pk)$. Thus G is related to $G(a - pk)$ by a finite sequence of $2k$-moves, Ξ-moves and Reidemeister moves.

In the case $a < 0$, let q be the positive integer with $0 \leq a + qk < k$. Using Lemmas 3.1(ii) and 3.2, we add qk consecutive positive shell-pairs to $G(a)$, and then remove qk pairs of shell-pairs with opposite signs. Finally G is related to $G(a + qk)$ by a finite sequence of $2k$-moves, Ξ-moves and Reidemeister moves. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. (i)⇒(ii): By Proposition 3.3, K and K' are related to $K(a)$ and $K(a')$ for some $a, a' \in \mathbb{Z}$ with $0 \leq a, a' < k$, respectively, by a finite sequence of $2k$-moves and Ξ-moves. Then it follows from Theorem 1.2 and Proposition 2.2 that

$$J(K) \equiv J(K(a)) = 2a \pmod{2k}$$

and

$$J(K') \equiv J(K(a')) = 2a' \pmod{2k}.$$

By assumption, we have $2a \equiv 2a' \pmod{2k}$. Since the nonnegative integers a and a' are less than k, we have $a = a'$. Thus $K(a)$ and $K(a')$ coincide.

(ii)⇒(i): This follows from Theorem 1.2 and Proposition 2.2. □

The following corollary is an immediate consequence of the proof of Theorem 1.3.

Corollary 3.4. *A complete system of representatives of the equivalence classes of virtual knots under $2k$-moves and Ξ-moves is given by the set*

$$\{K(a) \mid a \in \mathbb{Z},\ 0 \leq a < k\}.$$

In particular, the number of equivalence classes equals k. □

4. Proofs of Theorems 1.5 and 1.6

For two virtual knots K and K' that are related by a finite sequence of $2k$-moves, Jeong, Choi and Kim [11] provided a lower bound for $\mathrm{d}_{2k}(K, K')$ using the affine index polynomials of K and K', which can be rephrased in terms of the n-writhes as follows:

Theorem 4.1 ([11, Theorem 2.3])**.** *Let K and K' be virtual knots such that they are related by a finite sequence of $2k$-moves. Then we have*

$$\mathrm{d}_{2k}(K, K') \geq \frac{1}{k} \sum_{n>0} |J_n(K) - J_n(K')| = \frac{1}{k} \sum_{n<0} |J_n(K) - J_n(K')|.$$

In particular, when $K' = O$ is the trivial knot, we have

$$\mathrm{u}_{2k}(K) \geq \frac{1}{k} \sum_{n>0} |J_n(K)| = \frac{1}{k} \sum_{n<0} |J_n(K)|.$$

We conclude this paper with the proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. Consider a long virtual knot diagram T whose closure represents the virtual knot K. Let K' be the virtual knot represented by the diagram D in the left of Figure 4.1. The Gauss diagram G_D associated with D is given in the right of this figure, where the boxed part depicts the Gauss diagram G_T corresponding to T.

Removing $2pk$ half-twists from D by $2k$-moves p times, we can deform D into a diagram of K. Thus we have $\mathrm{d}_{2k}(K, K') \leq p$.

The $2pk$ vertical chords in G_D consist of pk positive chords with index 1 and pk positive chords with index -1, and the remaining one

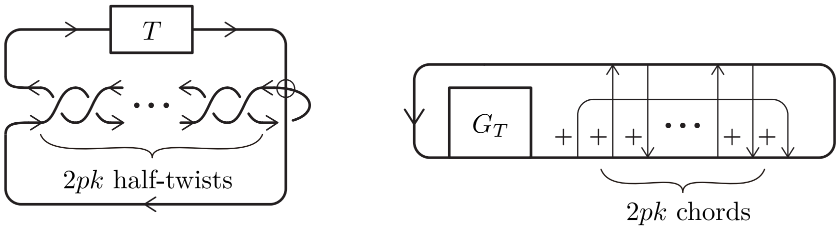

FIGURE 4.1. A diagram D of K' and its Gauss diagram G_D

chord of G_D excluding the chords in G_T has index 0. Therefore it follows from [22, Lemma 4.3] that

$$J_n(K') = \begin{cases} J_1(K) + pk & \text{if } n = 1, \\ J_{-1}(K) + pk & \text{if } n = -1, \\ J_n(K) & \text{if } n \neq 0, \pm 1. \end{cases}$$

By Theorem 4.1, we have

$$\mathrm{d}_{2k}(K, K') \geq \frac{1}{k}|J_1(K) - J_1(K')| = \frac{1}{k}|-pk| = p,$$

and hence $\mathrm{d}_{2k}(K, K') = p$. □

Proof of Theorem 1.6. For a positive integer s, let K_s be the virtual knot represented by the diagram D_s in Figure 4.2. As shown in the proof of [19, Theorem 2.8], the set $\{K_s \mid s \geq 1\}$ forms an infinite family of virtual knots with $\mathrm{u}(K_s) = pk$ for any $s \geq 1$, where $\mathrm{u}(K_s)$ is the minimal number of crossing changes needed to deform a diagram of K_s into that of the trivial knot O.

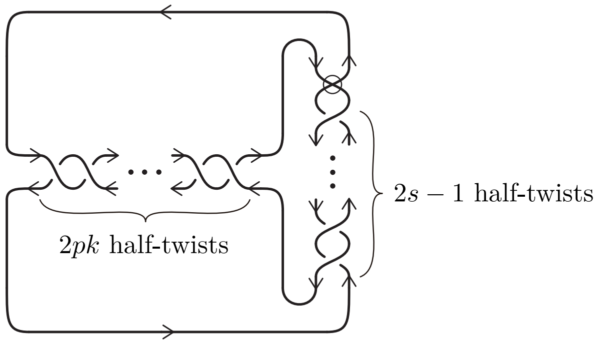

FIGURE 4.2. A virtual knot diagram D_s

Since a $2k$-move is realized by crossing changes k times, we have $\mathrm{u}_{2k}(K_s) \geq \frac{1}{k}\mathrm{u}(K_s) = p$. On the other hand, since D_s can be deformed into a diagram of O by $2k$-moves p times removing $2pk$ half-twists, we have $\mathrm{u}_{2k}(K_s) \leq p$. Thus K_s satisfies $\mathrm{u}_{2k}(K_s) = p$ for any $s \geq 1$. □

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