THURSTON UNIT BALL OF A FAMILY OF n-CHAINED LINKS AND THEIR FIBERED FACES

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ABSTRACT. We determine the Thurston unit ball of a family of n-chained links with p half-twists on one component, where the direction of the twists depends on the sign of p. These links are denoted by $C(n, p)$. For $p \geq 0$, we compute the unit Thurston ball precisely: it is an n-dimensional cocube (the dual of the n-dimensional cube) when $p \geq 1$ and it it the union of a cocube and two simplices when $p = 0$. When $p < 0$, we instead give a conjecture, supported by some computational evidence, on the shape of the Thurston unit ball. Moreover, we are able to identify at least one fibered face for each $C(n, p)$. Finally, we explicitely compute the Teichmüller polynomial for a fibered face of the Thurston unit ball of $C(n, -2)$, for arbitrary $n \geq 3$.

1. Introduction

Let M be a 3-dimensional manifold and suppose, for simplicity, that that M has tori boundaries. In one of his many seminal works [20], W. Thurston introduced a notion of a semi-norm on the second homology vector spaces of M. More precisely, let $[a] \in H_2(M, \partial M; \mathbb{Z})$ be an integral second homology class. Then $[a]$ can be represented by a disjoint union of properly embedded surfaces S_i. The Thurston norm of $[a]$ is then defined to be

$$x(a) := \min\{\sum_i \max\{0, -\chi(S_i)\}\}$$

where the minimum is taken over all possible ways to represent $[a]$ as a disjoint union of properly embedded surfaces. If M is irreducible and atoroidal, this then extends to a norm on $H_2(M, \mathbb{R})$. We sometimes use $||\cdot||$ to denote the Thurston norm. In the same paper, he proves that the unit ball with respect to that norm, that we will call Thurston unit ball, is always a polytope. Even though this concept has had

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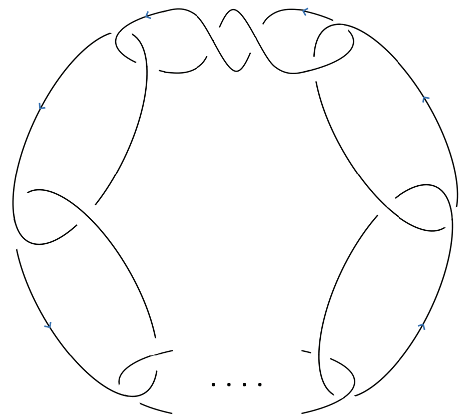

FIGURE 1. The links $C(n, -2)$

huge theoretical consequences, it seems that there are very few cases for which unit Thurston balls are computed explicitely. An interesting question in that regard is the following.

Question. Which polytope can appear as the Thurston unit ball of some 3-manifold?

This question was already posed by Kitayama in [14]. It has been generalized in terms of groups and their first homology by Friedl, Lück and Tillmann [5]. In [18], Pacheco-Tallaj, Schreve and Vlamis investigate the shape of the Thurston unit ball for tunnel number-one manifolds. We refer to Kitayama's survery [14] for more information on recent research about the Thurston norm.

In this article we show that the Thurston unit ball of a 3-dimensional manifold M can contain highly symmetric polytopes of arbitrary high dimensions. We will do so by determining the Thurston unit ball for a family of complements of links. This family will be denoted by $C(n, p)$, for two integers n and p with n positive, and the complements of small enough neighborhoods of $C(n, p)$ in S^3 will be denoted by $M(n, p)$. Briefly speaking, $C(n, p)$ is an n-chained link with p positive half-twist on the first component if p is positive or p negative half-twist on the first component if p is negative (see Figure 1 for an example). In [17], Neumann and Reid prove that $M(n, p)$ with $n \geq 3$ is hyperbolic if and only if $\{|n + p|, |p|\} \not\subseteq \{0, 1, 2\}$. The complements of these links are in some sense generalizations of the magic manifold, which is the complement of $C(3, 0)$. The magic manifold and its properties are thus good examples to keep in mind.

In a previous article [2], the two authors together with Harry Baik and Changsub Kim studied the relation between the minimal (topological) entropy of pseudo-Anosov maps on a surface S and the action of these maps on $H_1(S)$. Here the topological entropy of a pseudo-Anosov map is equal to $\log \lambda$, where λ is the expanding factor of the given map. In order to do so, the use of the complements of $C(n, -2)$ was crucial.

Here are the main results of this article.

Theorem A. *Let $M(n,p)$ be the complement of the link $C(n,p)$ with $n \geq 3$ and B be the Thurston unit ball of $M(n,p)$. Suppose $M(n,p)$ is hyperbolic. Then*

- *If $p \geq 1$, B is an n-dimensional cocube with vertices $(\pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$. (Corollary 5.2)*
- *If $p = 0$, B is the union of an n-dimensional cocube and two simplices. (Theorem 4.1)*

Remark that in [17], the authors show that $M(n,p)$ is hyperbolic if and only if $\{|n+p|, |p|\} \not\subseteq \{0, 1, 2\}$.

A complete answer for the case of $p < 0$ is out of our reach for now. We nonetheless find a set $V(n,p)$ of points in the Thurston unit ball and conjecture that their convex hull, denoted by $B(n,p)$, is the whole Thurston unit ball. We refer to the end for Section 6 for the precise statement of the conjecture. This conjecture is partially supported by computational data, obtained using the program Tnorm [21]. The data is gathered in Appendix B.

When $-n < p < 0$, the link $C(n,p)$ is fibered, as shown by Leininger [15]. For $n \geq 4$, we compute an explicit fiber using an operation that we call 'squeezing' one of the link components (see Definition 7.5).

Theorem B (Theorem 7.6)**.** *Let $C(n,p)$ be a hyperbolic negatively twisted n-chained link and let S be any surface obtained by performing the Seifert algorithm to the diagram obtained after squeezing one of the link components. Then, the cone of $B(n,p)$ containing $[S] \in H_2(M(n,p), \partial M(n,p))$ is fibered.*

When $p = -2$, we also compute the Teichmüller polynomial for every value of n.

Theorem C. *(Theorem 8.1) Suppose $n \geq 5$. Let $\mathcal{C}$ be the fibered cone of $M(n,-2)$ which contains the point $[S_n] \in H_2(M(n,-2), \partial M(n,-2))$, where S_n is the surface depicted in Figure 15. The Teichmüller polynomial P for the fibered cone $\mathcal{C}$ is*

$$P(x_1, \cdots, x_{n-1}, u) := A - \sum_{k=1}^{n} u a_k A_k$$

where $a_1 = 1, a_2 = x_1^{-1}, \cdots, a_n = (x_1 \cdots x_{n-1})^{-1}$, $A := (a_1-u)\cdots(a_n-u)$ *and* $A_k = \dfrac{A}{(a_k-u)(a_{k-1}-u)}$, *where* $a_{n+1} = a_1$.

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2. Preliminary

For a hyperbolic 3-manifold M, possibly with boundary ∂M, Thurston [20] defined a norm $||\cdot||$ on $H_2(M, \partial M; \mathbb{R})$. It turns out the unit norm ball B with respect to the Thurston norm is always a finite-sided polytope. Let $\mathcal{F}$ be a top-dimensional face of B and let $\mathcal{C} = \mathbb{R}\cdot\mathcal{F}$ be the open cone over $\mathcal{F}$. Thurston showed that if M is a fibered 3-manifold, then either all integral points in $\mathcal{C}$ are fibered or none of them are fibered. Here a point of $H_2(M, \partial M; \mathbb{Z})$ is fibered if it admits a representative that is a fiber surface. In the former case, we call $\mathcal{F}$ a *fibered cone* and the associated face $\mathcal{F}$ a *fibered face*. The goal of this paper is to compute this norm ball B and some fibered faces of some hyperbolic manifolds obtained as complements of chain links.

This section contains the essential tools that will be used in the rest of the paper. In this paper, surfaces will be denoted by S or S_i, for some positive integer i, except for spheres that are denoted by S^n where n is the dimension of the sphere.

2.1. Murasugi sums.

David Gabai ([8], [9]) proved theorems related to the fiberedness of embedded surfaces and, whenever they are in fact fibered, about their monodromy map. A key construction in his work is a geometric operation called "Murasugi sum". We begin with the definition of this operation.

Definition 2.1 (Murasugi sum, [8]). The oriented surface $S \subset S^3$ is a *Murasugi sum* of two different oriented surfaces S_1 and S_2 if

1. $S = S_1 \cup S_2$ and $S_1 \cap S_2 = D$, where D is a $2n$-gon,
2. The intersection of S_i, $i = 1, 2$, with D is a disjoint union of n arcs,
3. There is a partition of S^3 into two 3-balls B_1, B_2 satisfying that

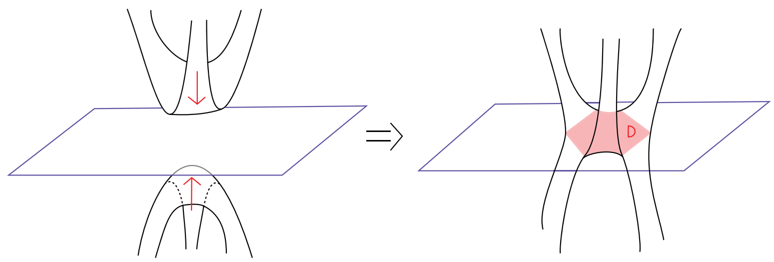

FIGURE 2. Murasugi sum of two surfaces, where D is a hexagon

- $S_i \subset B_i$ for $i = 1, 2$.
- $B_1 \cap B_2 = S^2$ and $S_i \cap S^2 = D$ for $i = 1, 2$.

In simple terms, the Murasugi sum is a way to cut-and-paste two surfaces in an alternating way so that, around the gluing region, it looks like there are $2n$ legs going up and down alternatively (see Figure 2).

The interest of the Murasugi sum is that it preserves the fiberedness and also the monodromies. More precisely, Gabai proved the two following theorems.

Theorem 2.2 ([6]). *Let S be a Murasugi sum of S_1 and S_2. Then S is a fiber surface if and only if both S_1 and S_2 are fiber surfaces.*

Theorem 2.3 ([8], Cor 1.4). *Suppose that S is a Murasugi sum of S_1, S_2 with $\partial S_i = L_i$, where L_i is a fibered link with monodromy f_i fixing pointwise the boundary ∂S_i, resp. Then $L = \partial S$ is a fibered link with fiber S and its monodromy map is $f = f'_2 \circ f'_1$ where f'_i is equal to f_i on the image of S_i in S and is the identity on $S \setminus S_i$.*

Using these two theorems, it is possible to construct fiber surfaces by gluing together smaller fiber surfaces while keeping a nice control on the monodromy maps. A good starting block for this construction is the Hopf link L, which consists of 2 circles that are linked together exactly once. The Hopf band is then a Seifert surface of the Hopf link. It is thus a fiber surface of $S^3 - L$.

Lemma 2.4 (Monodromy of a Hopf band). *The Hopf band is a fiber surface. Moreover, the monodromy of the positive (resp. negative) Hopf band is the right-handed (resp. left-handed) Dehn twist along its core curve.*

In fact, Giroux and Goodman [10] proved that every fibered link in S^3 can be obtained from the unknot by Murasugi summing or desumming along Hopf bands. In that sense, the Hopf bands are building blocks that can be used to construct any fibered link in S^3.

2.2. **Fibers of alternating knots/links.** Suppose that a fibered link L in S^3 is given. In general, it is difficult to find a concrete fiber surface for $S^3 - L$. However, if L is alternating and D is an alternating diagram for L, Gabai [7] showed that the surface obtained using the Seifert algorithm on D will be a fiber surface of minimal genus. We now recall the definition of an alternating link and explain the Seifert algorithm. For more details, we refer to [19].

Let L be an oriented link. A *link diagram* for L is, roughly speaking, the planar graph obtained by projecting L onto a plane. Whenever two edges of this graph cross, a segment of one of the two edges is erased. The choice of which edge to erase depends on which one was met first during the projection. A link diagram is *alternating* if the crossings alternate under and over as one travels along each component of the link. A link is *alternating* if it admits an alternating diagram. See Figure 1 for an example of an alternating link diagram.

Definition 2.5 (Seifert algorithm). Let L be an oriented link. The Seifert algorithm can be described as follows.

(1) For each crossing, cut at the crossing and paste back in such a way that, near the crossing, there are 2 components, as showed in Figure 3.

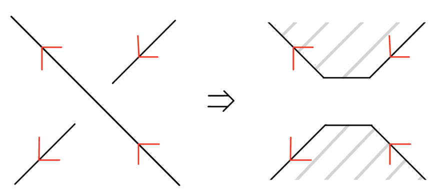

FIGURE 3. Cut and paste at a crossing in Seifert algorithm.

(2) After all these cut-and-paste operations, a disjoint collection of oriented simple closed curves is left. Each curve bounds a disk, unless two or more curves are nested. If some of the curves are nested, we can consider the innermost curve to be lying slightly above the others and repeat this process until there are no more nested curves. We then assign to each region a "+" sign if the region is on the left side of the boundary curve, with respect to its orientation, or "-" sign otherwise. Note that the result is sometimes called a checkerboard coloring.
(3) Finally, reconnect these discs at each crossing with a twisted strip. The direction of the twist is determined by the direction of the original crossing.

The result of this algorithm is a surface S whose (oriented) boundary is L.

The surface obtained from the Seifert algorithm is called a *Seifert surface* for L. The *genus* of a link L is defined to be the minimal genus of a surface in the complement of L whose boundary is L. In [9] Gabai proved that if L is alternating, the genus of L is equal to the genus of any Seifert surface of L.

Theorem 2.6 ([9], Thm 4)**.** *Let L be an oriented link in S^3. If S is a surface obtained by applying Seifert's algorithm to an alternating diagram of L, then S is a surface of minimal genus.*

The following theorem establishes a connection between the genus of a Seifert surface and the possibility of the surface to be a fiber.

Theorem 2.7 (Theorem 4.1.10 in [11])**.** *Let S be a Seifert surface for a fibered link L. Then the following are equivalent.*

(1) *S attains the minimal Seifert genus for L.*
(2) *S is a fiber surface.*

2.3. **Teichmüller polynomial.** The *Teichmüller polynomial* $\theta_{\mathcal{F}}$ for a fibered face $\mathcal{F} \subset H^1(M, \mathbb{R})$ is a polynomial associated to the fibered cone $\mathbb{R}_+ \cdot \mathcal{F}$ that determines the stretch factors of all the monodromies of fibers in the fibered cone. Similarly to the Alexander polynomial, the Teichmüller polynomial has coefficients in the group ring $\mathbb{Z}(G)$ where $G = H_1(M, \mathbb{Z})/\text{torsion}$.

We describe here one way to compute the Teichmüller polynomial. Let $\varphi\colon S \to S$ be a pseudo-Anosov map and let $x = x_1, \cdots, x_{n-1}$ be a multiplicative basis for

$$H = \text{Hom}(H^1(M, \mathbb{Z}^{\varphi}), \mathbb{Z})$$

where $H^1(M, \mathbb{Z})^{\varphi}$ is the φ-invariant cohomology. Remark that we can construct a natural map from $\pi_1(S)$ to H by evaluating cohomology classes on loops. Choose a lift $\tilde{\varphi}\colon \tilde{S} \to \tilde{S}$ of φ to the cover $\tilde{S}$ corresponding to H under the previous map.

Let $M = S \times [0, 1]/(p, 1) \sim (\varphi(p), 0)$ be the mapping torus of φ. Then, we have that

$$G = H_1(M, \mathbb{Z})/\,\text{torsion} = H \oplus \mathbb{Z}$$

Let u denote the generator of the $\mathbb{Z}$ component of G, so that G is generated by $x_1, \cdots, x_{n-1}$ and u. Let V and E be the vertices and the edges of an invariant train track τ on S carrying the pseudo-Anosov map φ. The lifts $\tilde{V}$ and $\tilde{E}$ of V and E to $\tilde{S}$ can respectivily be considered as $\mathbb{Z}(H)$-modules. Therefore, the lift $\tilde{\varphi}$ act as matrices $P_V(x)$ and $P_E(x)$ on these $\mathbb{Z}(H)$-modules. McMullen showed in [16] that the Teichmüller polynomial can then be computed in term of these two matrices.

Theorem 2.8. *The Teichmüller polynomial can be explicitly computed as follows:*

$$\theta_{\mathcal{F}}(x,u) = \frac{\det(uI - P_E(x))}{\det(uI - P_V(x))}$$

3. The n-chained links and their complements

In [12], Eiko Kin analysed in detail a 3-manifold, known as the magic manifold. This manifold has the property that all the faces of its Thurston unit ball are fibered. She was able to precisely determine all the fibered faces and, for each integer point in a fibered face, find the topology of the associated monodromy (i.e: determine its genus and the number of boundary components). In this section, we generalize the technique used for the magic 3-manifold to study sequences of fibers in more general link complements.

An *n-chained link* L is a link with n components that are linked together in a circular fashion. Note that some of the components of L may have self half-twists. A *clasp* of an n-chained link L is the combinatorial structure defined by a pair of crossing of two adjacent link components of L. There are only two different types of clasps, that we will call positive (or $+$) and negative (or $-$) clasps, according to the convention shown in Figure 4.

Let L be an n-chained link. We can always isotope L in such a way that all the half-twists happen in a single component of L. Also, if C_1 and C_2 are two components of L that meet in a $+$ clasp, performing a half twist on C_1 or on C_2 will change the clasp to a $-$ clasp.

Therefore, any n-chained link can isotoped to an n-chained link where all the clasps are $+$ clasps and where all the half-twists happened in a single component. We will denote by $C(n,p)$ the n-chained link L which admits a link diagram in which every clasp is positive and in which there are exactly p half-twists, where the direction of the twists in determined by the sign of the integer p. We will choose the directions of the twists in such a way that the diagram is alternating when p is positive. From now on, whenever we use a link diagram for $C(n,p)$, it will be the one we described here, unless explicitly specified otherwise. Two examples are illustrated in Figure 5. The same Figure also shows how the sign of a clasp can be changed by a half-twist on one of the components of the clasp.

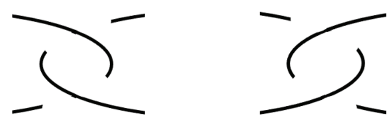

FIGURE 4. The two different kinds of clasps. We will say that the left clasp is positive and the right clasp is negative. Positive and negative clasps will also be referred as $+$ and $-$ clasps, respectively.

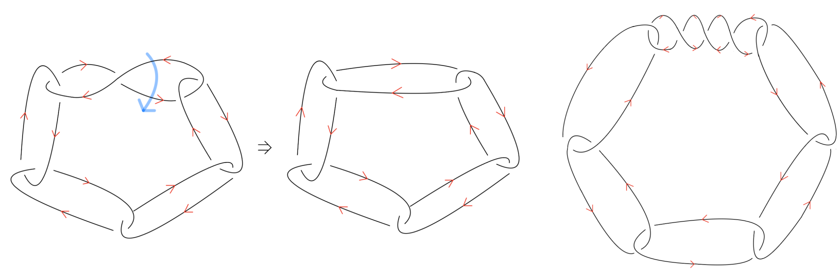

FIGURE 5. $C(5,-1)$ and $C(6,3)$.

Let $M(n,p)$ be the complement of a small enough neighborhood $\mathcal{N}(C(n,p))$ of $C(n,p)$. In particular, $M(3,0)$ is the magic 3-manifold. Note that $\partial\mathcal{N}(C(n,p))$ is a disjoint union of n tori. Suppose $M(n,p)$ is hyperbolic. The manifold $M(n,p)$ with $n \geq 3$ is hyperbolic if and only if $\{|n+p|, |p|\} \nsubseteq \{0,1,2\}$, as shown by Neumann and Reid in [17]. Moreover, Leininger [15] shows that, except when $(n,p) = (2,-1)$, the manifold $M(n,p)$ is fibered as long as $n \geq -p \geq 0$. He does so by explicitly computing a fiber surface for $M(n,p)$.

We roughly describe how to obtain such a fiber surface. We can remove a half-twist, at the cost of changing the sign of one of the clasps. Repeat this process until only 2 half-twists remain and then use Seifert's algorithm on the link diagram. The surface S obtained in this way is a horizontal Hopf band Murasugi summed by n vertical Hopf bands. Theorems 2.2 and 2.3 allow us to conlcude that S is indeed a fiber surface for $M(n,p)$.

Now, we focus on the homology of $M(n,p)$. Consider that we draw $C(n,p)$ in such a way that the top link has the p half-twists, as in Figure

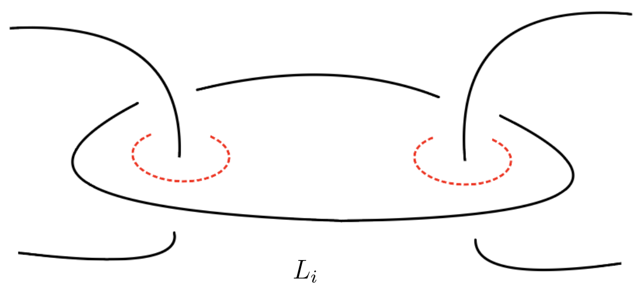

FIGURE 6. The K_i are spheres with 3 boundaries, where one of the boundaries if L_i and the two others are drawn in dotted circles

5. We will denote this top link component as L_1, and we enumerate the other components $L_2, L_3, \cdots L_n$ in a clockwise fashion.

A diagram for a link L is said to be circular if the component $L_1, \cdots, L_n$ of L can be ordered in such a way that L_i forms a clasp exactly with L_{i-1} and L_{i+1}, for every $i = 1, 2, \cdots, n$, where $L_{n+1} = L_1$. A diagram is said to be oriented if each link component is given an orientation. Let D be an oriented circular diagram for $C(n, p)$. There is a standard basis $\{[K_i]_{1 \leq i \leq n}\}$ for $H_2(M, \partial M)$ associated to D, where each K_i is a sphere with three boundaries. Each K_i can be seen as having L_i as one of its boundaries, while the two other boundaries correspond to L_{i-1} and L_{i+1}, as show in Figure 6.

Let $\{[K_i]_{1 \leq i \leq n}\}$ be the standard basis associated to the circular oriented diagram D for $C(n, p)$ that is described in the beginning of this section.

Lemma 3.1 ([2], lemma 4.6). *Suppose $n \geq -p \geq 0$ with $(n, p) \neq (2, -1)$. The fiber S provided by Lemma 4.1 in* [15] *has coordinates $(1, \cdots, 1, -1)$ in the basis $\{[K_i]_{1 \leq i \leq n}\}$. In other words, we have that $[S] = [K_1] + \cdots + [K_{n-1}] - [K_n]$.*

Note that the fiber S is a genus 1 surface with n boundaries, and so its Euler characteristic is equal to n.

4. Thurston unit ball for $C(n, 0)$

4.1. **Thurston unit ball.** We start by stating the main theorem of this section, even though its proof is relegated to the end of the section. The notation used in the statement of the main theorem will nonetheless be used throughout the whole section.

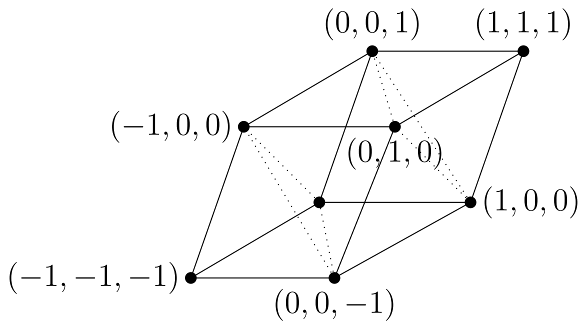

FIGURE 7. The unit Thurston ball B_3 for the link $C(3,0)$. It is the convex hull of $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ and $(1,1,1), (-1,-1,-1)$. The missing edges of the cocube are indicated by dotted edges.

Theorem 4.1. *For $n \geq 3$, the Thurston unit ball B_n of $C(n, 0)$ is the union of:*

(1) *The n-dimensional cocube with vertices $(\pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$, and*
(2) *Two n-simplices: the convex hull of $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$, $\frac{1}{n-2}(1, \cdots, 1)$, and its antipodal image.*

The second homology class represented by $\frac{1}{n}(1, \cdots, 1, -1)$ then lies in the fibered face $\mathcal{F}$, whose vertices are $(1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 1, 0)$, $(0, \cdots, 0, -1)$ and $\frac{1}{n-2}(1, \cdots, 1)$. Moreover, every face of B_n is a fibered face.

Here, the n-dimensional cocube is the dual of the standard cube $[-1, 1]^n$. It can also be seen as an n consecutive suspension of the closed interval $[-1, 1]$.

Note that Theorem 4.1 includes the case of the magic 3-manifold case, which was handled by Thurston in [20]. The Thurston unit ball of the magic manifold is a parallelepiped with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ and $(1, 1, 1), (-1, -1, -1)$, as illustrated by Figure 7.

We observe that

Lemma 4.2. *Suppose that $a_1, \cdots, a_n$ are vertices of a facet F of the Thurston unit ball and let σ be the n-dimensional simplex spanned by these vertices. Then, σ is a subset of F if there exists a point $a \in \sigma$ whose Thurston norm $-\chi(a)$ is equal to 1. In this case, the linear equation of the facet $\mathcal{F}$ is $\sum_{i=1}^{n} x_i/a_i = 1$.*

Proof. The proof is a direct consequence of the fact that the Thurston unit ball is a polytope. □

As in the magic manifold case, we can calculate the Thurston norm of any points in the fibered cone $\mathcal{C} = \mathbb{R}_+ \cdot \mathcal{F}$.

Corollary 4.3. *The convex hull of the points* $e_1, e_2, \cdots, -e_n$ *and* $\frac{1}{n-2}(1, 1, \cdots, 1)$ *is a subset of the facet* $\mathcal{F}$ *of* B_n*. Moreover, for any* $\alpha := (\alpha_1, \cdots, \alpha_n)$ *in the cone* $\mathcal{C} := \mathbb{R}_+ \cdot \mathcal{F}$*, the Thurston norm of* α *is* $\alpha_1 + \cdots + \alpha_{n-1} - \alpha_n$*.*

Proof. Set

$$a_i = \begin{cases} e_i, & 1 \leq i \leq n-1 \\ -e_n, & i = n \end{cases}$$

and $a = \frac{1}{n}(1, \cdots, 1, -1)$. Since we already observed in Lemma 3.1 that na is a fiber and $-\chi(na) = n$, this means that the linear equation $x_1 + \cdots + x_{n-1} - x_n = 1$ is the equation of a supporting hyperplane for the fibered face $\mathcal{F}$. Plugging $(\alpha_1, \cdots, \alpha_n)$ into $x_1 + \cdots + x_{n-1} - x_n$, we get the Euler characteristic for α. ☐

We are now ready to prove Theorem 4.1.

Proof of theorem 4.1. Note that $C(n, 0)$ is circularly symmetric, so the points $p_i := \frac{1}{n}(1, \cdots, 1) - \frac{2}{n}e_i$ for all $1 \leq i \leq n$ is also a fiber. Hence, by Corollary 4.3, the n -dimensional parallelograms P_i of vertices $e_1, \cdots, -e_i, \cdots, e_n$ and $\frac{1}{n-2}(1, 1, \cdots, 1)$ are subsets of the boundary of the Thurston unit ball (each p_i is contained in P_i). However, the union of the P_i forms a closed polytope, so it has to contain the boundary of the Thurston unit ball. ☐

4.2. **Topological type of fibers.** In addition to understanding the Thurston unit ball, we can also get information about each fiber surface in the fiber facet $\mathcal{F}$. To obtain the complete topological type of representatives of given fibered points, we will use a slightly generalized version of the boundary formula proven by Kin and Takasawa, [13].

Lemma 4.4 ([2], lemma 4.9)**.** *Let* $M = M(n, 0)$*,* $\mathcal{F}$ *be the fibered face described in Theorem 4.1 and let* $\mathcal{C} = \mathbb{R}_+ \cdot \mathcal{F}$ *be the associated fibered cone. Suppose* S *is a minimal representative of* $(\alpha_1, \cdots, \alpha_n) \in \mathcal{C}$*. Then the number of boundaries of* S *is equal to* $\sum_{i=1}^{n} \gcd(\alpha_{i-1} + \alpha_{i+1}, \alpha_i)$*, where* $\alpha_{n+1} = \alpha_1$*.*

5. Thurston unit ball for $C(n, p)$ with $p \geq 0$

Since the link $L = C(n, p)$ with $p > 0$ admits an alternating link diagram, its Seifert surface S is a minimal genus surface for L.

Theorem 5.1. *Let* $L = C(n, p)$ *with* $p \geq 1$*. Given arbitrary signs on* $x = (\pm 1, \cdots, \pm 1) \in H_2(M(n, p), \partial \mathcal{N}(L))$*, the Thurston norm of* x *is* n*.*

Proof. We will perform the Seifert algorithm explicitly. Assume first that we arbitrarily fix the signs of each component of x. Note that these signs determine the orientation of each component of the link. Let L be the link with the orientations corresponding to x. As always, the link L is drawn in a circular way so that the twisted component lies at the top (as shown in Figure 5). Label the twisted link L_1 and continue the labeling clockwise.

We begin by applying the algorithm, starting with the crossings involving L_1. Applying the Seifert algorithm to the crossing corresponding to half twists, we get $p-1$ discs. On the contrary, applying the algorithm at the 2 clasps involving L_1, we get arcs on both sides L_1. Now, focus on the right arc and the next link L_2. If the signs of L_1, L_2 agree, then the Seifert algorithm produces one disc, and the arc is still not closed. Otherwise, the Seifert algorithm makes the arc tied and ends up with a disk, and another arc will be created on the right side of L_2.

We proceed inductively until we get $n+p$ disks. Note that the number $n+p$ of circles does not depend on the signs assigned to each component. As the number of crossings in the diagram is equal to $2n+p$, we conclude that the genus of S is

$$\text{Genus of } S = \frac{2+(2n+p)-(n+p)-n}{2} = 1.$$

By Theorem 2.7, the surface S is a minimal genus surface for x. Therefore, S is a minimal representative of x and $||x|| = n$. □

The above Theorem implies that the Thurston unit ball of $M \cong M(n,p)$ is an n dimensional cocube.

Corollary 5.2. *Thurston unit ball of $M(n,p)$ with $p \geq 1$ is an n dimensional cocube with vertices* $(\pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$.

Proof. Let e_i be a canonical basis of $\mathbb{Z}^n \cong H_2(M(n,p), \partial M(n,p))$. Since e_i is represented by a 2 punctured disk, it lies on the Thurston unit ball. By the Theorem 5.1, we know that $(\pm 1/n, \cdots, \pm 1/n)$ is also on the unit ball. For each $(\pm 1/n, \cdots, \pm 1/n)$, it is a convex combination of the canonical basis (with suitable signs). Therefore, we conclude that the convex hull of $\{\pm e_i\}_{1 \leq i \leq n}$ is exactly the Thurston unit ball. □

6. Thurston unit ball for $C(n,p)$ with $p < 0$

In Lemma 4.2 in [15], Leininger proved that $C(n,-p)$ is fibered for $0 \leq p \leq n$ except $(n,p) \neq (2,-1)$. In this section we investigate what is the shape of the Thurston unit ball for the complements of

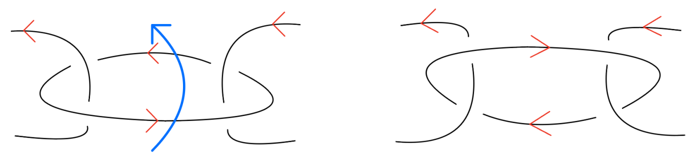

Figure 8. Flip

n-chained links with negative twists. Suppose that we have an n-chain link $C(n,-p)$ and that we have labeled each link component as before. Note that we can untwist all the negative twists. After resolving the negative twists on L_1, the link becomes a chain link with no twists. See Figure 10 for an example. However, the shape of clasps may have changed during this process. We re-assign the orientations of each component L_i in a circular way after resolving all the twists on L_1, and, for each $i = 1, \cdots, n$. Recall that, associated to a circular oriented diagram, there is a standard basis $\{e_i = [K_i]_{1\leq i\leq n}\}$ for $H_2(M, \partial M)$, where each K_i is a sphere with three boundaries, as in Figure 6. Let $\{e_1, \cdots, e_n\}$ be the standard basis associated to the diagram described above.

Each component of the link has 2 clasps, which may now be $+$ clasp or $-$ clasp. Since the $-$ clasp only appears whenever a negative twist is resolved, the number of $-$ clasps in the final diagram is equal to $|p|$. Let D be a circular diagram for $C(n,p)$. We define shape vectors for such diagrams.

Definition 6.1. Suppose $n \geq 4$ and $-\lfloor n/2 \rfloor \leq p < 0$ and D is a circular diagram for $C(n,p)$. The shape vector of D is an n-tuple, whose entries are either $+$ or $-$. The i'th entry records the shape of the clasp formed by L_i and L_{i+1}. For each L_i, we will say that L_i has clasp shape (α, β), with $\alpha, \beta \in \{+,-\}$, if the clasp between L_{i-1} and L_i is an α clasp and the clasp between L_i and L_{i+1} is a β clasp.

Suppose L_i has $-$ shape with L_{i-1} and $+$ shape with L_{i+1}. Here are two isotopic operations that we can perform on such L_i.

(1) A *flip*: we flip L_i so that the $+$ clasp changes to a $-$ clasp and vice versa. Hence a flip exchanges the $(i-1)$'th entry and i'th entry of the shape vector. See Figure 8.
(2) A *full twist*: Cut $M(n,p)$ along K_i. In the slice, there are 2 copies of K_i, say D_1, D_2. Then, glue D_1 and D_2 back, after twisting either D_1 or D_2 by 360 degrees. See Figure 9.

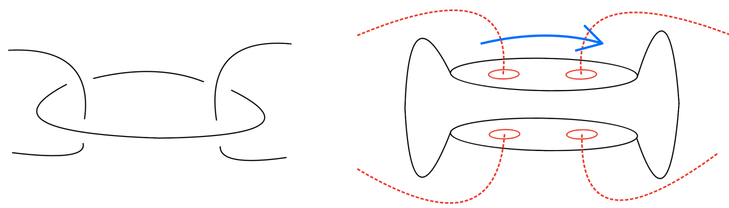

FIGURE 9. Full twist.

Proposition 6.2. *Let $n \geq 4$. Suppose L_i admits $(+,-)$ or $(-,+)$ clasp shape. Then, the homology class $(1,\cdots,0,\cdots,1) \in H_2(M(c,p),\partial M(c,p))$, where the 0 is in the ith entry, admits a sphere with $(n-1)$ boundaries as a representative. Its Thurston norm is thus equal to $n-3$.*

Proof. After performing a full twist on L_i, the now consecutive link components L_{i-1} and L_{i+1} form a clasp, whose shape depends on the direction of the full twist. If we forget about the component L_i, the other link components now form a chain link with $n-1$ components. We can then apply the Seifert algorithm, with all positive orientations, to this new chain link. The Seifert surface S obtained in this way is a sphere with $n-1$ boundaries. Since the surface S does not admit L_i as its boundary component, S is an embedded surface in $M(n,p)$. Since it has no genus, this is the minimal Thurston norm representative of the given homology class. □

We now show how to obtain many other points on the boundary of the Thurston unit ball.

Suppose that L is $C(n,-1)$. After untwisting once, we obtain an n-chained link with shape vector $(-,+,\cdots,+)$. By proposition 6.2, $\frac{1}{n-3}(1,0,1,\cdots,1)$ is a point of Thurston norm equal to one. If we flip L_2, the shape vector changes to $(+,-,+,\cdots,+)$. We can now perform a full twist on L_3 and then, using the same method as in the proof of proposition 6.2, we can deduce that the point $(1,-1,0,1,\cdots,1)$ is also of norm equal to one. Note that we have a -1 on the second entry this time. In conclusion, as the 0 coordinate moves one step on the right, it also introduces a minus sign. Hence, by repeating this process, we obtain a total of $2n$ points on the boundary of the Thurston unit ball. These points are the points of coordinates

$$\frac{1}{n-3}(1,0,1,\cdots,1),\ \frac{1}{n-3}(1,-1,0,1,\cdots,1),\cdots,$$
$$\frac{1}{n-3}(1,-1,\cdots,-1,0) \text{ and } \frac{1}{n-3}(0,-1,\cdots,-1).$$

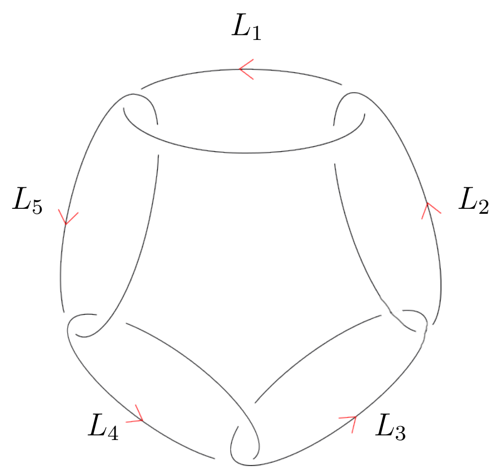

FIGURE 10. $C(5,-2)$. Note that the orientation of each link component is re-assigned in a circular way. Starting from the top link component, we label the components $L_1, L_2, \cdots, L_5$, clockwise.

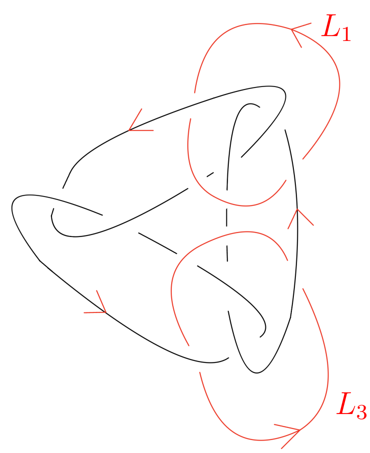

FIGURE 11. After two full twists, one on L_1 and one on L_3, we get the above link. Note that $(0,1,0,1,1)$ is represented by $S_{0,3}$, which is obtained by oriented sum of the 3 disks bounded by L_2, L_4, L_5.

and their antipodal points.

If instead L is $C(n,-2)$, the shape vector contains two negative entries. We can perform a full twist on the two link components L_i and L_j whose clasps on both sides are different, unless L_i and L_j are consecutive link components. In this case, we get 2 zero entries in the new points and hence it represents a sphere with $n-2$ punctures. Therefore, it has Thurston norm $(n-4)$. See Figure 10 and 11 also.

The processes described above generalize to all the links $C(n,p)$, with $p<0$. We therefore obtain the following result.

Corollary 6.3. *By following the process described above, we obtain a set $V := V(n,p)$ of points that lie on the boundary of the Thurston unit ball. Every point in V is obtained by flipping the link components and taking full twists. The flip operation slides the 0 entry to the next coordinate. All points $x=(x_1,\cdots,x_n)$ in V have the following properties.*

(1) $||x||\cdot x_i \in \{-1,0,1\}$ *for all* $i=1,2,\cdots,n$.
(2) *No two consecutive entries are equal to* 0.

Hence, $B=B(n,p)$, the convex hull of $V\cup\{\pm e_i\}$, is contained in the unit Thurston norm ball.

We now give some more details on the shape of the balls $B(n,p)$.

Proposition 6.4. *Let $C(n,p)$ be a negative twisted n-chained link. Choose any $1\le i\le n$ and collect all points in $V(n,p)\cup\{\pm e_i\}$ with $x_i=0$. Then the convex hull of such points forms an $(n-1)$-dimensional polytope and is contained in the union of $B(n-1,p+1)$ and $B(n-1,p)$.*

Proof. After flipping some of the link components, we can suppose that L_i has clasp shape $(-,+)$. Perform a full twist on L_i and forget L_i for the moment. Then, the remaining link components form a new link, which is either $C(n-1,p+1)$ or $C(n-1,p)$, depending on the direction of the twist. More precisely, if the full twist yields a negative shape clasp between L_{i-1} and L_{i+1}, the link $C(n-1,p)$ is obtained. On the other hand, if the full twist yields a positive one, we get $C(n-1,p+1)$. For any points in $V(n-1,p+1)$ or $V(n-1,p)$, if we plug a 0 in the ith tuple, it becomes a point which lies on the boundary of $V(n,p)$. □

We end this section with a question and some remarks.

Question 1. *Is $B(n,p)$ equal to the unit Thurston norm ball of $C(n,p)$ when $p<0$?*

We thanks William Worden and the program Tnorm[21] which helped us to calculate and verify that the question is true for $n\le 6$. We provide the table of all the vertices of the Thurston unit normal ball, calculated by Tnorm, for various $C(n,p)$'s up to $n\le 6$ in the appendix B.

As we already mentioned, $C(n,p)$ is fibered for $0\le -p\le n$. Assigning proper signs, Leininger's fiber surface has coordinates $(1,\cdots,1)$ and has genus 1 and n punctures. Since each vector e_i of the canonical basis represents a twice punctured disk, we can deduce that there is a fibered face $\mathcal{F}$ which contains the standard $(n-1)$ simplex spanned by the $\{e_i\}$'s. Furthermore, using lemma 4.2 and similar methods as in the proof of corollary 4.3, we can get that the Euler characteristic of any primitive points of $(x_1, \cdots, x_n)$ with all positive entries is equal to $\sum_{i=1}^{n} x_i$.

7. Detecting fibered faces

By theorem 4.1 and corollary 5.2, we now understand the shape of the Thurston unit ball of $C(n,p)$ when $p \geq 0$, and some faces when $p < 0$. We now investigate which faces of that unit ball are fibered.

7.1. **p is nonnegative.** Denote by $S(n,p)_x$ the surface obtained from the process in Theorem 5.1, when starting from $x = (\pm 1, \cdots, \pm 1)$. By Theorem 5.1, $S(n,p)_x$ has genus equal to 1 and n boundaries.

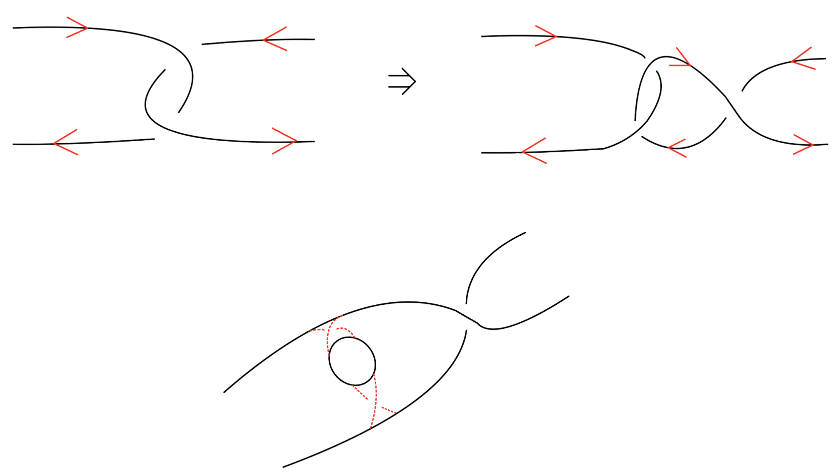

Figure 12. (Murasugi) desum the Hopf bands in the case of different orientations. Observe that the twist of result is compatible with the positive half twist with respect to the clasp shape.

In the language of $H_2(M(n,p), \partial M(n,p))$, the sign change of the given orientation x can be interpreted as a number of half twists after we (Murasugi) desum each vertical Hopf band. In addition, we use the following Lemma, coming from the work of Baader and Graf [1].

Lemma 7.1 (Example 3.1 in [1])**.** *Suppose L is a $(2,2n)$-torus link with a given oriented diagram D such that the Seifert surface obtained from D is a full-twisted annulus. Then L is fibered if and only if $|n| = 1$, and thus the Seifert surface is a positive/negative Hopf band.*

Remark that the $(2, 2n)$-torus link is fibered if the orientation of the two link components is parallel. However, in our case, it cannot happen since their orientations are inherited by the orientation of the link components, so that they must be opposite.

Theorem 7.2 (p is even)**.** *Let p be nonnegative even integer. For a given orientation $x = (\pm 1, \pm 1, \cdots, \pm 1)$, denote by s the number of sign changes. Hence, $s = \sum_{i=1}^{n} \delta_{-1, x_i x_{i+1}}$, where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.*

Then, $S(n, p)_x$ is fibered if and only if $(p, s) = (0, 2), (2, 0)$.

Proof. Choose one clasp. There are two components of the link associated to the clasp. Suppose both have the same orientation in the sense that, after performing the Seifert algorithm ,locally around the clasp, we get a disk and a band at the clasp. It implies that for a Seifert surface of this diagram with given orientations, there is a Hopf band Murasugi summed at the clasp.
If the orientations are different, then, perform a half twist on the one of the component so that it changes the shape of clasp. Performing the Seifert algorithm locally again, we get a disk and a half-twisted band, as in Figure 12. We will thus desum whenever there arises a Murasugi sum of Hopf bands. In the end, a twisted band remains whose boundary is a $(2, p+s)$-torus link. By Lemma 7.1, such a torus link is fibered if and only if $p + s = 2$, which finishes the proof. □

Theorem 7.3 (p is odd)**.** *Let p be a non negative odd integer and let x, s be as in Theorem 7.2. Then, $S(n, p)_x$ is fibered if and only if $(p, s) = (1, 0)$.*

Proof. The only difference compared to the case where p is even is the last desumming process. Since there is an odd number of half-twists, the leftmost and rightmost parts of the top link L_1 do not coincide. Hence, after the desumming process, the remaining part is a twisted band whose boundary is a $(2, p + s + 1)$-torus link. Again, by Lemma 7.1, it is fibered if and only if $p + s + 1 = 2$ and $(p, s) = (1, 0)$ is the only solution. □

Note that $x = \frac{1}{n}(\pm 1, \cdots, \pm 1)$ is the barycenter of the vectors $\pm e_i$. Therefore, together with Lemma 4.2 in [15] we obtain the following corollary.

Corollary 7.4 (Fiberedness of $C(n, p)$)**.** *The link $C(n, p)$ is fibered if and only if $-n - 2 \leq p \leq 2$. Moreover, every face of the Thurston unit ball of $C(n, 0)$ is a fibered face. In contrast, there are only 2 fibered face of $C(n, 1)$ and $C(n, 2)$, one which contain $\frac{1}{n}(1, 1, \cdots, 1)$ and one which contains its antipodal point.*

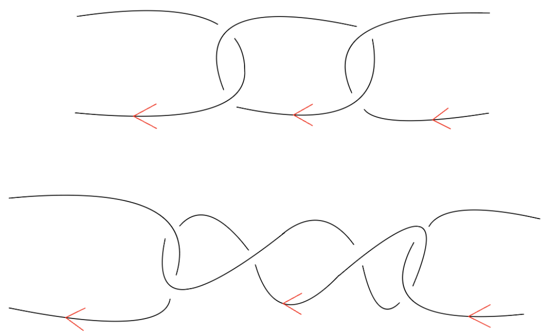

Figure 13. The link component in the middle has a $(-,-)$ shape. After squeezing, the shape of the clasp changes to $(+,+)$ with a squeezed link component.

Proof. For $p = 0$, by theorem 7.2 $S(n,0)_x$ is fibered if and only if x has only one entry -1 and the others are all 1 or its antipodal points. By corollary 4.3, each $S(n,0)_{x_i}$ is in the distinct fibered cone, hence every face of Thurston unit ball for $C(n,0)$ is fibered.

Suppose $p = 1$ or $p = 2$. By theorem 7.2 and 7.3, $S(n,p)_x$ is fibered if and only if $x = (1, \cdots, 1)$ or $(-1, \cdots, -1)$. By corollary 5.2, there are only 2 faces whose supporting planes are $\sum_{i=1}^{n} x_i = \pm 1$. □

7.2. p **is negative.** Some faces of the polytope B are actually faces of the Thurston unit ball. We introduce another isotopic operation for link components which have the same clasp shape on both sides.

Definition 7.5 (Squeezing). Suppose L_i has a clasp shape $(+,+)$ or $(-,-)$. Perform a half twists on both sides so that each clasp alters its shape. We will call this operation squeezing the link L_i.

Theorem 7.6. *Let $n \geq 4$ and $C(n,p)$ be a twisted n-chained link with $-n-2 \leq p \leq 0$. Let S be any surface obtained by performing the Seifert algorithm to the diagram obtained after squeezing one of the link components. Then, the cone of $B(n,p)$ containing $[S] \in H_2(M(n,p), \partial M(n,p))$ is fibered.*

Proof. We will proceed by induction. In this proof, every full twist will be performed such that the clasp has a $+$ shape after the operation.

(1) $p = -1$. Choose any point q in V and any link component L_i which has a clasp shape $(+,+)$. There is exactly one 0 entry in q. Let k be its index. By Proposition 6.4, the slice of the unit Thurston norm ball of $C(n,p)$ at $x_k = 0$ must contain the union of $B(n-1,0)$ and $B(n-1,-1)$. Choose one face

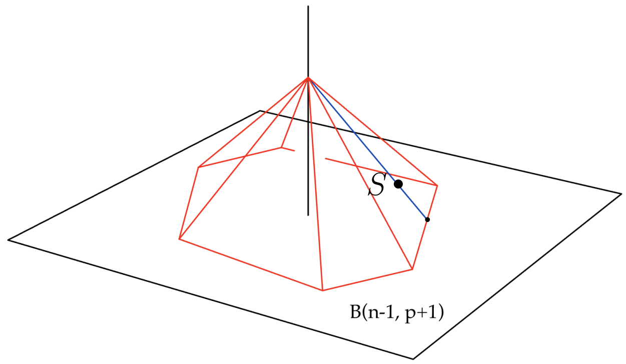

FIGURE 14. The vertical axis is x_k coordinates, orthogonal to $\mathbb{R}^{n-1}$. The convex hull of $V(n-1,p+1)$ lies at the bottom and taking the cone with the apex $x_k = 1$. The point labeled S is a fiber obtained by squeezing.

in $B(n-1,0)$. Since its shape vector is all $+$ (or $-$), any link component of L_k has $(+,+)$ shape (or $(-,-)$). We choose L_i except $i = k-1, k, k+1$ and squeeze it. Taking the inverse orientation of L_i, $(1, \cdots, \underbrace{-1}_{i\text{th}}, \cdots, \underbrace{0}_{k\text{th}}, \cdots, 1)$ is represented by one horizontal Hopf band Murasugi summed by $n-1$ vertical Hopf bands. *i.e.*, $x = \frac{1}{n-1}(1, \cdots, \underbrace{-1}_{i\text{th}}, \cdots, \underbrace{0}_{k\text{th}}, \cdots, 1)$ is in the unit sphere of $C(n,-1)$.
Now the convex sum $\mathbf{x} := \frac{n-1}{n} \times x + \frac{1}{n} \times e_k$ is $\frac{1}{n}(1, \cdots, \underbrace{-1}_{i\text{th}}, \cdots, 1)$.
This is still a fiber, since we choose i carefully so that the squeezing still works even if we undo the full twist. Note that since this point is in the convex hull of $n+1$ vertices, the face containing $\mathbf{x}$ is fibered.

(2) $\lfloor n/2 \rfloor \leq p \leq -2$. By induction, we already have squeezing fibers on the face of $C(n-1,p+1)$. See the figure 14. So it remains to show that such squeezing still works after we undo the full twists. But since $|p+1|$ is strictly smaller than $\lfloor n/2 \rfloor$, there always exists a link component of shape $(+,+)$ or $(-,-)$. Hence by undoing full twists except near the link component, we get the fibered face which contains a squeezing fiber.

□

Theorem 7.6 implies that most of the faces in $B(n,p)$ are actually fibered faces of the Thurston unit norm ball. We provide some computations of the vertices of the Thurston unit ball for the $p<0$ cases in the appendix B. In the remaining section, we will cover the special case of $C(n,-2)$, in which case more explicit calculations can be made.

8. Teichmüller Polynomial for one fibered face of $C(n,-2)$, for $n \geq 5$

In this section we compute explicitly the Teichmüller polynomials for one fibered face of $C(n,-2)$, when $n \geq 5$, so that $M(n,-2)$ is hyperbolic. Let M_n be the exterior complement of the link $C(n,-2)$. We denote by S_n the surface obtained by performing the Seifert algorithm to the link diagram of $C(n,-2)$ shown in Figure 10. We will sometimes omit the subscript n if it is not important in the context. Since M_n is the complement of $C(n,-2)$, the second homology group $H_2 = H_2(M_n, \partial M_n)$ is a free abelian group of rank n, with a canonical basis given by the meridians of the link components. With that in mind, we remark that S_n is a surface of genus one with n boundaries and its coordinates in H_2 are $(1,1,\cdots,1)$. Since S_n is a Murasugi sum of one horizontal Hopf bands with n vertical Hopf bands, it is a fiber. By Theorem 2.3, the monodromy φ_n of this fibering is the composition of the Dehn twists along the cores of the Hopf bands.

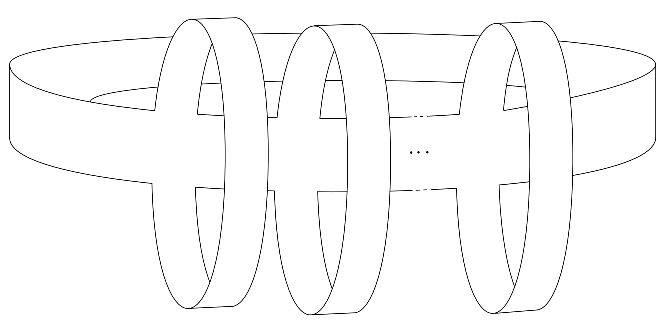

Figure 15. The surface S_n for $C(n,-2)$, the horizontal band is a positive Hopf band and each vertical band is a negative Hopf band. Here we omit the full twists which are supposed to be at each band, as they have no role in the remainder of the calculations.

Thus, if S_n is placed as suggested in figure 15, the monodromy φ_n is the composition of the multi-twists composed of the Dehn twists, all directed downward, around the vertical bands followed by the left Dehn twist along the core of horizontal band. Following the methods in [3] and [16], we compute the Teichmüller polynomial corresponding to the fibered cone $\mathbb{R}_+ \cdot \mathcal{F}$ of the Thurston unit ball which contains the point $(1, \cdots, 1) \in H_2(M_n, \partial M_n)$.

As explained in section 2.3, we first need to compute $H = \mathrm{Hom}\,(H^1(S, \mathbb{Z})^{\varphi}, \mathbb{Z})$ and then understand how the lift $\tilde{\varphi}_n$ of φ_n acts on the cover $\tilde{S}_n$ of S_n which has H as a deck transform group. In this case, as noted in [3], the group H is equal to the φ_n invariant first homology $H_1(S_n : \mathbb{Z})^{\varphi_n}$. We choose $c_0, \cdots, c_n$ as a basis for $H_1(S_n; \mathbb{Z})$, where c_0 is the curve corresponding to the core of the horizontal band and $c_1, \cdots, c_n$ are the curves corresponding to the cores of the vertical bands, c_1 being the leftmost one and c_n the rightmost one. Then, $H_1(S_n : \mathbb{Z})^{\varphi_n}$ is the subspace of $H_1(S_n; \mathbb{Z})$ generated by the column vectors of

$$B_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

We still need to figure out what the cover $\tilde{S}_n$ is, and how $\tilde{\varphi}_n$ acts on it. Once again, the details are all given in [3]. Instead of repeating them here, we give some graphical explanations for the simplest non trivial example, which is $M_3 = C(3, -2)$. In this case, the cover $\tilde{S}_n$ is explicitly drawn in figure 17.

Let T be the matrix representing the H-module action of $\tilde{\varphi}_n$ on $\tilde{S}_n$. Since the monodromy φ_n is the composition of one horizontal Dehn twist and n vertical ones, we can decompose the matrix T into T_V and T_H. These matrices represent the action of the lifts of the vertical multi-twist and the horizontal Dehn twist, respectively, on $\tilde{S}$. Note that the entries of these matrices are in $\mathbb{Z}[G]$, where G is the deck transformation group of $\widetilde{S}$, and hence is isomorphic to $\mathbb{Z}^{n-1}$.

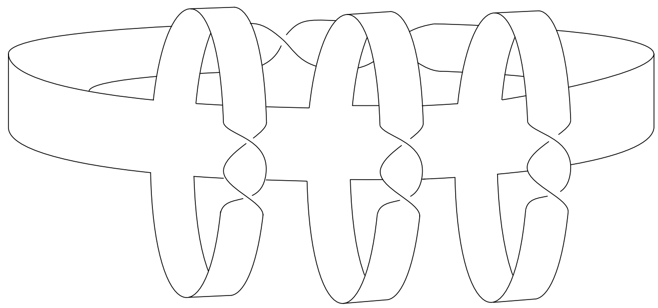

FIGURE 16. The surface S_3, which is the fiber associated to the link $C(3,-2)$

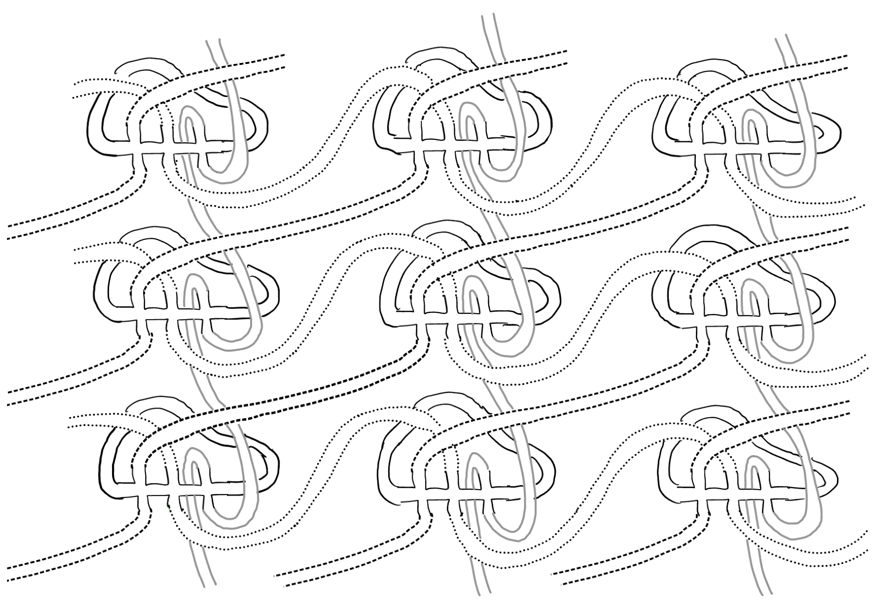

FIGURE 17. The Galois covering $\tilde{S}_3$ of S_3 with deck transform H

Using our conventions, the matrices T_V and T_H are the $2n \times 2n$ matrices shown here.

$$T_V = \begin{bmatrix} (x_1 \cdots x_{n-1})^{-1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (x_1 \cdots x_{n-2})^{-1} & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & x_1^{-1} & \cdots & 0 & 0 & x_1^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (x_1 \cdots x_{n-1})^{-1} & 0 & 0 & \cdots & (x_1 \cdots x_{n-1})^{-1} \end{bmatrix}$$

$$T_H = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

We can consider these matrices as being block matrices with four blocks of size $n \times n$. As such, we get that

$$T_V = \begin{bmatrix} D_s & 0 \\ D & D \end{bmatrix}, \qquad T_H = \begin{bmatrix} I & \mathbf{1} \\ 0 & I \end{bmatrix}$$

where D_s is an $n \times n$ matrix whose diagonal entries are the same as D, but shifted to the right by one, and $\mathbf{1}$ is the $n \times n$ matrix with all entries equal to 1.

By [16], the Teichmüller polynomial can be obtained using the formula

$$P(x_1, \cdots, x_{n-1}, u) := \frac{\det(T_V T_H - uI)}{\det(D - uI)}.$$

The remaining calculations are showed in Appendix A.

Let a_k be the k'th diagonal entry of D. Hence, $a_1 = 1, a_2 = x_1^{-1}, \cdots, a_n = (x_1 \cdots x_{n-1})^{-1}$.

Theorem 8.1 (Teichmüller polynomial)**.** *Let $n \geq 5$. The Teichmüller polynomial P for the fibered cone $\mathcal{C}$ containing the point $[S_n] \in H_2(M_n, \partial M_n)$, where S_n is the surface depicted in Figure 15.*

$$P(x_1, \cdots, x_{n-1}, u) := A - \sum_{k=1}^{n} u a_k A_k$$

where $A := (a_1 - u) \cdots (a_n - u)$ and $A_k = \dfrac{A}{(a_{k-1} - u)(a_k - u)}$, where $a_{n+1} := a_1$.

The manifold M_n can be viewed at the same time as a link complement and has a fibration. Both point of view lead to natural coordinates on $H_2 = H_2(M_n, \partial M_n; \mathbb{Z})$.

It is sometimes more convenient to use the coordinates coming from the link complement point of view for the Teichmüller polynomials. For example, that point of view is more fitted to the computation of the stretch factor of the monodromy of the fiber which has coordinates $(1, 1, \cdots, 1)$ in the basis given by the link components.

The Teichmüller polynomials we computed are using the basis coming from the fibration point of view. We thus need to find the explicit change of coordinates for going from one basis to the other.

Let us fix the notation clearly. The basis Y given by the link complements will be denoted as $y_1, \cdots, y_n$, with y_1 corresponding to the link complement with the self twist. If the monodromy for the fibration of M_n is denoted by φ_n, the corresponding basis X will be $u, x_1, \cdots, x_{n-1}$ where the x_i form a basis for the φ_n invariant cohomology and u corresponds to the suspension flow. We also let $a_0, \cdots, a_{n-1}$ be the canonical basis for $H^1(S_n, \mathbb{Z})$. By the computation above, we already know that $x_i = a_1 - a_{i+1}$. Moreover, as suggested by figure 18, we see that $a_i = y_i - y_{i+1}$, where the indices are taken modulo n as always. Finally, the basis element u corresponding to the suspension flow is simply mapped to y_1.

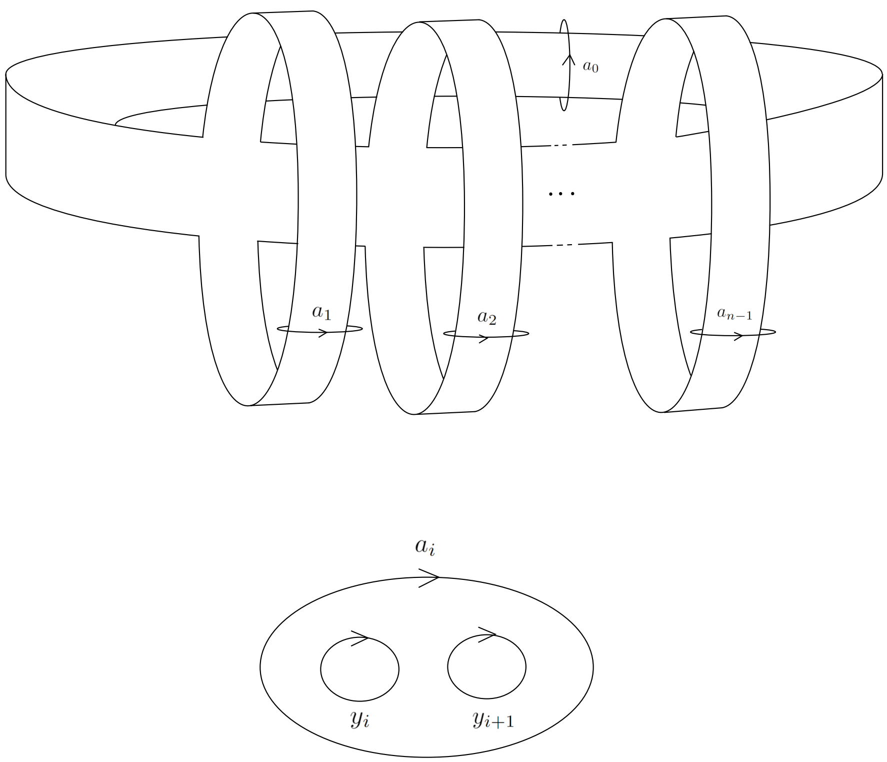

FIGURE 18. The surface S_n and the geometric representation of the a_i,for $i = 0, \cdots, n-1$. On the bottom, we see how each a_i is related to the link components, since a_i, y_i and y_{i+1} always bound a disk in M_n

To sum it up, the change of coordinates is given by

$$\begin{aligned} u &\to y_1 \\ x_1 &\to y_1 - y_3 \\ x_2 &\to y_1 - y_2 + y_3 - y_4 \\ &\vdots \\ x_{n-2} &\to y_1 - y_2 - y_{n-1} + y_n \\ x_{n-1} &\to -y_2 + y_n \end{aligned}$$

Hence the image of the fiber whose coordinates in the basis X are $(1, 1, \cdots, 1)$ has $(0, 0, \cdots, 0, 1)$ as coordinates in the basis Y. The specialization of the Teichmüller polynomial to the point $p = (0, 0, \cdots, 0, 1)$ is then given by

$$(1-u)^n - nu(1-u)^{n-2} = (1-u)^{n-2}(1-(n+2)u+u^2)$$

A simple calculation shows that the largest root of this polynomial is $\frac{n+2+\sqrt{n^2+4n}}{2}$.

Appendix A. Proof of Theorem 8.1

In this appendix we finish the calculations of the Teichmüller polynomial of section 8.

We need to compute the determinant of block matrices, and we make use of the following Lemma to do so.

Lemma A.1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *a block matrix, where* A *and* D *are square matrices of same size. If* D *is invertible, then* $\det(M) = \det(A - BD^{-1}C)\det(D)$. *Moreover, if* C *and* D *commute, we get that* $\det(M) = \det(AD - BC)$.

Proof. Suppose that D is invertible. Then M can be factorized as

$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}$$

Taking the determinant on both side, we conclude that the first part of the Lemma holds. If C and D commutes, we get that

$$\det(M) = \det(A - BD^{-1}C)\det(D) = \det(AD - BD^{-1}CD) = \det(AD - BC).$$

Hence, the second part of the Lemma also holds. □

The matrix $T_V T_H - uI$ can be expressed as a block matrix,

$$T_V T_H - uI = \begin{bmatrix} D_s - uI & D_s \cdot \mathbf{1} \\ D & D \cdot \mathbf{1} + D - uI \end{bmatrix}$$

By performing some row reductions, we can simplify this matrix.

$$\begin{bmatrix} I & 0 \\ -DD_s^{-1} & I \end{bmatrix} \times \begin{bmatrix} D_s - uI & D_s \cdot \mathbf{1} \\ D & D \cdot \mathbf{1} + D - uI \end{bmatrix} = \begin{bmatrix} D_s - uI & D_s \cdot \mathbf{1} \\ uDD_s^{-1} & D - uI \end{bmatrix}$$

Such operations do not affect the determinant and now the bottom two block matrices are both diagonals, so they commute. Hence we can apply lemma A.1 to compute the determinant

$$\begin{aligned} \det(T_V T_H - uI) &= \det((D_s - uI)(D - uI) - uD_s \cdot \mathbf{1} \cdot DD_s^{-1}) \\ &= \det(D_s((D_s - uI)(D - uI) - u\mathbf{1} \cdot D)D_s^{-1}) \\ &= \det((D_s - uI)(D - uI) - u\mathbf{1} \cdot D) \end{aligned}$$

Let $B_k := (a_{k-1} - u)(a_k - u)$, where $a_0 = a_n$. Then the matrix $(D_s - uI)(D - uI) - u\mathbf{1} \cdot D$ is

$$\begin{bmatrix} B_1 - ua_1 & -ua_1 & \cdots & -ua_1 \\ -ua_2 & B_2 - ua_2 & \cdots & -ua_2 \\ \vdots & \vdots & \ddots & \vdots \\ -ua_n & -ua_n & \cdots & B_n - ua_n \end{bmatrix}$$

In order to calculate the determinant of this matrix, we will use the following Lemma.

Lemma A.2. *Let A be the following matrix.*

$$A = \begin{bmatrix} c_1 - u & -u & \cdots & -u \\ -u & c_2 - u & \cdots & -u \\ \vdots & \vdots & \ddots & \vdots \\ -u & -u & \cdots & c_n - u \end{bmatrix}$$

Then, $\det(A) = c_1 \cdots c_n - (\sum_{i=1}^n c_1 \cdots \hat{c_i} \cdots c_n)u$, *where* $c_1 \cdots \hat{c_i} \cdots c_n = c_1 \cdots c_{i-1} c_{i+1} \cdots c_n$.

Proof. We proceed by induction. For $n = 2$, the determinant is equal to $(c_1 - u)(c_2 - u) - u^2 = c_1 c_2 - (c_1 + c_2)u$. Suppose now that the Lemma holds for any natural number $n - 1$ and let A be $n \times n$ matrix of the given form. By induction hypothesis, the determinant of the upper-left $(n-1) \times (n-1)$ block of A is $c_1 \cdots c_{n-1} - (\sum_{i=1}^{n-1} c_1 \cdots \hat{c_i} \cdots c_{n-1})u$. We now compute $\det(A)$ using cofactor expansion on the last row of A. Then, we have that

$$\det(A) = \left[c_1 \cdots c_{n-1} - \left(\sum_{i=1}^{n-1} c_1 \cdots \hat{c_i} \cdots c_{n-1} \right) u \right] (c_n - u) + (-u) \times \text{ (other terms)}$$

Each other term is in fact the determinant of a $(n-1) \times (n-1)$ block whose ith column is omitted and $(n-1)$th column has $-u$ on all its entries. If we cyclically permute from the i'th column to the last column, the determinant of this matrix is equal to $-c_1 \cdots \hat{c}_i \cdots c_{n-1} u$, by applying $c_i = 0$. The sign of each cyclic permutation offsets to the alternating sum in the determinant formula. Hence, we get

$$\begin{aligned} \det(A) &= \left[c_1 \cdots c_{n-1} - \left(\sum_{i=1}^{n-1} c_1 \cdots \hat{c}_i \cdots c_{n-1} \right) u \right] (c_n - u) \\ &\quad + \left(\sum_{i=1}^{n-1} c_1 \cdots \hat{c}_i \cdots c_{n-1} \right) u^2 \\ &= c_1 \cdots c_n - \left(\sum_{i=1}^{n} c_1 \cdots \hat{c}_i \cdots c_n \right) u \end{aligned}$$

□

The given matrix can be factorized as follows.

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} \frac{B_1}{a_1} - u & -u & \cdots & -u \\ -u & \frac{B_2}{a_2} - u & \cdots & -u \\ \vdots & \vdots & \ddots & \vdots \\ -u & -u & \cdots & \frac{B_n}{a_n} - u \end{bmatrix}$$

Apply Lemma A.2, the determinant of the given matrix is

$$B_1 \cdots B_n - (a_1 B_2 \cdots B_n + B_1 a_2 B_3 \cdots B_n + \cdots B_1 \cdots B_{n-1} a_n) u$$

The Teichmüller polynomial is obtained by dividing $(a_1 - u) \cdots (a_n - u)$, the determinant of $D - uI$.

Appendix B. Some calculations of $C(n,p)$ with $p < 0$

In this section we give our calculations and tables of vertices for some $C(n,p), p < 0$ cases.

We thank William Worden, as we made extensive use of his paper [4] and the program, called 'Tnorm', that he developed. Tnorm is able to compute the vertices of the Thurston unit ball of given links complements. In the tables in this section, we list the vertices, except for vertices of the form $\pm e_i$'s, together with the topological type of their representatives. The left columns of the tables are the coordinates of the vertices and the right columns are the corresponding surfaces representing them in the second homology groups.

$C(4,-1)$	
$\pm(1,0,1,1)$	$S_{0,3}$
$\pm(1,-1,0,1)$	$S_{0,3}$
$\pm(1,-1,-1,0)$	$S_{0,3}$
$\pm(0,-1,-1,-1)$	$S_{0,3}$

$C(5,-1)$		$C(5,-2)$	
$\pm(1/2,0,1/2,1/2,1/2)$	$\frac{1}{2}S_{0,4}$	$\pm(0,1,0,1,1)$	$S_{0,3}$
$\pm(1/2,-1/2,0,1/2,1/2)$	$\frac{1}{2}S_{0,4}$	$\pm(0,1,-1,0,1)$	$S_{0,3}$
$\pm(1/2,-1/2,-1/2,0,1/2)$	$\frac{1}{2}S_{0,4}$	$\pm(1,0,-1,0,1)$	$S_{0,3}$
$\pm(1/2,-1/2,-1/2,-1/2,0)$	$\frac{1}{2}S_{0,4}$	$\pm(1,0,-1,-1,0)$	$S_{0,3}$
$\pm(0,-1/2,-1/2,-1/2,-1/2)$	$\frac{1}{2}S_{0,4}$	$\pm(1,-1,0,-1,0)$	$S_{0,3}$

$C(6,-1)$	
$\pm(1/3,0,1/3,1/3,1/3,1/3)$	$\frac{1}{3}S_{0,5}$
$\pm(1/3,-1/3,0,1/3,1/3,1/3)$	$\frac{1}{3}S_{0,5}$
$\pm(1/3,-1/3,-1/3,0,1/3,1/3)$	$\frac{1}{3}S_{0,5}$
$\pm(1/3,-1/3,-1/3,-1/3,0,1/3)$	$\frac{1}{3}S_{0,5}$
$\pm(1/3,-1/3,-1/3,-1/3,-1/3,0)$	$\frac{1}{3}S_{0,5}$
$\pm(0,-1/3,-1/3,-1/3,-1/3,-1/3)$	$\frac{1}{3}S_{0,5}$

$C(6,-2)$	
$\pm(0,1/2,0,1/2,1/2,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,0,-1/2,0,1/2,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,-1/2,0,-1/2,0,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,-1/2,1/2,0,-1/2,0)$	$\frac{1}{2}S_{0,4}$
$\pm(0,1/2,-1/2,-1/2,0,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,0,-1/2,-1/2,-1/2,0)$	$\frac{1}{2}S_{0,4}$
$\pm(0,1/2,-1/2,0,1/2,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,0,-1/2,-1/2,0,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,-1/2,0,-1/2,-1/2,0)$	$\frac{1}{2}S_{0,4}$

$C(6,-3)$	
$\pm(0,1/2,1/2,0,1/2,1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,0,-1/2,1/2,0,-1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,-1/2,0,1/2,-1/2,0)$	$\frac{1}{2}S_{0,4}$
$\pm(0,-1/2,1/2,0,1/2,-1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(-1/2,0,1/2,1/2,0,-1/2)$	$\frac{1}{2}S_{0,4}$
$\pm(1/2,1/2,0,-1/2,-1/2,0)$	$\frac{1}{2}S_{0,4}$
$\pm(0,1,0,-1,0,1)$	$S_{0,3}$
$\pm(-1,0,1,0,1,0)$	$S_{0,3}$

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