ON BOREL'S STABLE RANGE FOR THE TWISTED COHOMOLOGY OF $\mathrm{GL}(n,\mathbb{Z})$

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Abstract. Borel's stability and vanishing theorem gives the stable cohomology of $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations. We compute the improved stable range that Borel suggested. In order to further improve Borel's stable range, we adapt the method of Kupers–Miller–Patzt to any algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations.

1. Introduction

Borel proved the stability of the rational cohomology of $\mathrm{GL}(n,\mathbb{Z})$ and computed the stable cohomology [1]. He also proved the vanishing of the stable cohomology of $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in non-trivial algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations [2]. He gave constants for the stable ranges and suggested improved stable ranges, but he did not compute these stable ranges explicitly except for a few families of representations.

Li and Sun [12] improved Borel's stable ranges and obtained stable ranges that are independent of coefficients. For coefficients in polynomial $\mathrm{GL}(n,\mathbb{Z})$-representations, Kupers, Miller and Patzt [11] improved the stable ranges by using arguments on polynomial VIC-modules.

In this paper, we compute the improved stable range that Borel suggested. We also adapt Kupers, Miller and Patzt's argument to coefficients in algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations indexed by *bipartitions*, i.e., pairs of partitions. Our results are weaker than Li and Sun's. However, the methods are very different and we think that it is still worth publishing these results.

1.1. Stable range for the cohomology of $\mathrm{GL}(n,\mathbb{Z})$.

The improved stable range given by Li and Sun [12] is as follows.

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Theorem 1.1 (Borel [1, 2], Li–Sun [12]). *(1) For each integer $n \geq 1$, the algebra map*

$$H^*(\mathrm{GL}(n+1,\mathbb{Z}),\mathbb{Q}) \to H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q})$$

induced by the inclusion $\mathrm{GL}(n,\mathbb{Z}) \hookrightarrow \mathrm{GL}(n+1,\mathbb{Z})$ is an isomorphism for $* \leq n-2$. Moreover, we have an algebra isomorphism

$$\varprojlim_n H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}(x_1,x_2,\ldots), \quad \deg x_i = 4i+1$$

in degrees $* \leq n-2$.

(2) Let V be an algebraic $\mathrm{GL}(n,\mathbb{Q})$-representation such that $V^{\mathrm{GL}(n,\mathbb{Q})} = 0$. Then we have

$$H^p(\mathrm{GL}(n,\mathbb{Z}),V) = 0 \quad \text{for } p \leq n-2.$$

Kupers, Miller and Patzt's stable range [11] for the rational cohomology is wider by 1 than that of Li and Sun.

Theorem 1.2 (Borel [1, 2], Kupers–Miller–Patzt [11]). *We have*

$$H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}(x_1,x_2,\ldots), \quad \deg x_i = 4i+1$$

in degree $* \leq n-1$.

We now state our main result. For a bipartition $\underline{\lambda} = (\lambda,\lambda')$, let $V_{\underline{\lambda}}$ denote the (irreducible or zero) algebraic $\mathrm{GL}(n,\mathbb{Z})$-representation corresponding to $\underline{\lambda}$ (see Section 2).

Theorem 1.3 (Corollary 3.4 and Theorem 4.20, weaker than Theorem 1.1). *Let $\underline{\lambda} \neq (0,0)$ be a bipartition. Then we have*

$$H^p(\mathrm{GL}(n,\mathbb{Z}),V_{\underline{\lambda}}) = 0$$

for $n \geq n_0(\underline{\lambda},p)$.

Here the constant $n_0(\underline{\lambda},p)$ is defined as follows. For a partition λ, let $|\lambda|$ and $l(\lambda)$ denote the size and length of λ, respectively. Let $\underline{\lambda} = (\lambda,\lambda')$ be a bipartition and p a non-negative integer. Let $|\underline{\lambda}| = |\lambda| + |\lambda'|$ and $\deg \underline{\lambda} = |\lambda| - |\lambda'|$, and set

$$n_0(\underline{\lambda},p) = \min\{n_{\mathrm{KMP}}(\underline{\lambda},p), n_{\mathrm{B}}(\underline{\lambda},p)\},$$

where

$$n_{\mathrm{KMP}}(\underline{\lambda},p) = \begin{cases} p+1+|\underline{\lambda}| & \text{if } \lambda = 0 \text{ or } \lambda' = 0 \\ p+1+2|\underline{\lambda}| & \text{otherwise} \end{cases}$$

and

$$n_{\mathrm{B}}(\underline{\lambda},p) = \max\{2p+2, 2|\deg \underline{\lambda}|+1, 2l(\lambda), 2l(\lambda')\}.$$

Remark 1.4. Let us compare the value of $n_{\mathrm{KMP}}(\underline{\lambda}, p)$ and $n_{\mathrm{B}}(\underline{\lambda}, p)$. For a fixed $\underline{\lambda}$, we have $n_{\mathrm{KMP}}(\underline{\lambda}, p) < n_{\mathrm{B}}(\underline{\lambda}, p)$ for all but finitely many p. If p is relatively small with respect to $\underline{\lambda}$ then we sometimes have $n_{\mathrm{B}}(\underline{\lambda}, p) < n_{\mathrm{KMP}}(\underline{\lambda}, p)$. For example, we have $n_{\mathrm{KMP}}((4, 4), 1) = 18$ and $n_{\mathrm{B}}((4, 4), 1) = 4$. Note that Theorem 1.1 gives a better bound 3 in this case.

Remark 1.5. This paper stemmed from the first version of [7] (with a different title), which included the two approaches to improve Borel's stable ranges described in this paper. In [7], we combined the improved version of Borel's theorem with the Hochschild–Serre spectral sequence associated to the short exact sequence of groups

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to 1,$$

where $\mathrm{Aut}(F_n)$ is the automorphism group of a free group F_n of rank n, and IA_n is the IA-automorphism group of F_n, in order to study the stable cohomology of $\mathrm{Aut}(F_n)$ and IA_n possibly with twisted coefficients. After the first version of [7] appeared on the arXiv, Oscar Randal-Williams informed us of the result of Li and Sun about the improvement of the Borel theorem [12]. Since our results for Borel's stable range turned out to be weaker than Li and Sun's, we have decided to remove these results from [7] and to rely on Li and Sun's result there.

1.2. **Organization of the paper.** The rest of this paper is organized as follows. In Section 2, we recall some facts about the representation theory of $\mathrm{GL}(n, \mathbb{Q})$. In Section 3, we recall Borel's stability and vanishing theorem for $\mathrm{GL}(n, \mathbb{Z})$ and compute the improved stable range that Borel suggested for irreducible algebraic representations. In Section 4, we improve the stable range by using the arguments of Kupers, Miller and Patzt [11].

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2. Algebraic $\mathrm{GL}(n, \mathbb{Z})$-representations

Let $n \geq 1$ be an integer. A *polynomial* $\mathrm{GL}(n, \mathbb{Q})$*-representation* is a finite-dimensional $\mathbb{Q}[\mathrm{GL}(n, \mathbb{Q})]$-module V such that after choosing a basis for V, the $(\dim V)^2$ coordinate functions are polynomial in the n^2 variables. A $\mathrm{GL}(n,\mathbb{Q})$-representation is called *algebraic* if the coordinate functions are rational functions. See [6] for some facts from representation theory.

As is well known, irreducible polynomial $\mathrm{GL}(n,\mathbb{Q})$-representations are classified by partitions with at most n parts. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a weakly decreasing sequence of non-negative integers. The *length* $l(\lambda)$ of λ is defined by $l(\lambda) = \max\{\{0\} \cup \{i \mid \lambda_i > 0\}\}$ and the *size* $|\lambda|$ of λ is defined by $|\lambda| = \lambda_1 + \cdots + \lambda_{l(\lambda)}$.

We denote by $H = H(n) = \mathbb{Q}^n$ the standard representation of $\mathrm{GL}(n,\mathbb{Q})$. In the following, we usually omit (n). For a partition λ, the Specht module S^λ for λ is an irreducible representation of $\mathfrak{S}_{|\lambda|}$ defined by using the Young symmetrizer associated to λ. Define the $\mathrm{GL}(n,\mathbb{Q})$-representation

$$V_\lambda = V_\lambda(n) = H^{\otimes|\lambda|} \otimes_{\mathbb{Q}[\mathfrak{S}_{|\lambda|}]} S^\lambda.$$

If $l(\lambda) \le n$, then V_λ is an irreducible polynomial $\mathrm{GL}(n,\mathbb{Q})$-representation. Otherwise, we have $V_\lambda = 0$.

The $\mathrm{GL}(n,\mathbb{Q})$-representation H^* dual to H is not polynomial but algebraic since the action of $\mathrm{GL}(n,\mathbb{Q})$ on H^* is given by $A \mapsto ({}^tA)^{-1}$. Let $p, q \ge 0$ be integers. We set $H^{p,q} = H^{\otimes p} \otimes (H^*)^{\otimes q}$. We have an isomorphism $(H^{q,p})^* = (H^{\otimes q} \otimes (H^*)^{\otimes p})^* \cong H^{\otimes p} \otimes (H^*)^{\otimes q} = H^{p,q}$. For a pair $(i,j) \in \{1,\dots,p\} \times \{1,\dots,q\}$, we define the *contraction map*

$$c_{i,j} : H^{p,q} \to H^{p-1,q-1} \tag{2.0.1}$$

by

$$\begin{aligned} &c_{i,j}((v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \otimes f_q)) \\ &\quad = \langle v_i, f_j \rangle (v_1 \otimes \cdots \widehat{v_i} \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \widehat{f_j} \cdots \otimes f_q) \end{aligned}$$

for $v_1, \dots, v_p \in H$ and $f_1, \dots, f_q \in H^*$, where the dual pairing $\langle -, - \rangle : H \otimes H^* \to \mathbb{Q}$ is defined by $\langle v, f \rangle = f(v)$. Note that $\langle -, - \rangle$ is $\mathrm{GL}(n,\mathbb{Q})$-equivariant.

The *traceless part* $H^{\langle p,q \rangle}$ of $H^{p,q}$ is defined by

$$H^{\langle p,q \rangle} = \bigcap_{(i,j) \in \{1,\dots,p\} \times \{1,\dots,q\}} \ker c_{i,j} \subset H^{p,q},$$

which is a $\mathrm{GL}(n,\mathbb{Q})$-subrepresentation of $H^{p,q}$.

A *bipartition* is a pair $\underline{\lambda} = (\lambda, \lambda')$ of two partitions λ and λ'. The *length* $l(\underline{\lambda})$ of the bipartition $\underline{\lambda}$ is defined by $l(\underline{\lambda}) = l(\lambda) + l(\lambda')$. The *degree* of $\underline{\lambda}$ is defined by $\deg \underline{\lambda} = |\lambda| - |\lambda'| \in \mathbb{Z}$, and the *size* of $\underline{\lambda}$ by $|\underline{\lambda}| = |\lambda| + |\lambda'|$. We define the *dual* of $\underline{\lambda}$ by $\underline{\lambda}^* = (\lambda', \lambda)$.

We associate to each bipartition $\underline{\lambda} = (\lambda, \lambda')$ the $\mathrm{GL}(n, \mathbb{Z})$-representation

$$V_{\underline{\lambda}} = V_{\underline{\lambda}}(n) = H^{\langle p,q\rangle} \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} (S^{\lambda} \otimes S^{\lambda'}), \tag{2.0.2}$$

where $p = |\lambda|$ and $q = |\lambda'|$. If $l(\underline{\lambda}) \leq n$, then $V_{\underline{\lambda}}$ is an irreducible algebraic $\mathrm{GL}(n, \mathbb{Q})$-representation with highest weight

$$(\lambda_1, \ldots, \lambda_{l(\lambda)}, 0 \ldots, 0, -\lambda'_{l(\lambda')}, \ldots, -\lambda'_1).$$

Otherwise, we have $V_{\underline{\lambda}} = 0$. It is well known that irreducible algebraic $\mathrm{GL}(n, \mathbb{Q})$-representations are classified by bipartitions $\underline{\lambda}$ with $l(\underline{\lambda}) \leq n$ (see [6, 9]).

The traceless part $H^{\langle p,q\rangle}$ of $H^{p,q}$ admits the following direct-sum decomposition as a $\mathbb{Q}[\mathrm{GL}(n, \mathbb{Q}) \times (\mathfrak{S}_p \times \mathfrak{S}_q)]$-module

$$H^{\langle p,q\rangle} = \bigoplus_{\substack{\underline{\lambda}=(\lambda,\lambda')\text{:bipartition with} \\ l(\underline{\lambda}) \leq n,\ |\lambda|=p,\ |\lambda'|=q}} V_{\underline{\lambda}} \otimes (S^{\lambda} \otimes S^{\lambda'}). \tag{2.0.3}$$

(See [9, Theorem 1.1]. See also Lemma 4.12 for the statement as VIC-modules.)

Note that we have $\mathrm{GL}(n, \mathbb{Q})$-isomorphisms $\det \cong \bigwedge^n V \cong V_{1^n}$, where det denotes the determinant representation, and $1^n = (1, \ldots, 1)$ consists of n copies of 1. For any bipartition $\underline{\lambda} = (\lambda, \lambda')$ with $l(\underline{\lambda}) \leq n$, we have an isomorphism

$$V_{\underline{\lambda}} \cong V_{\mu} \otimes \det^k,$$

for some partition μ with at most n parts and an integer k such that

$$(\lambda_1, \ldots, \lambda_{l(\lambda)}, 0, \ldots, 0, -\lambda'_{l(\lambda')}, \ldots, -\lambda'_1) = (\mu_1 + k, \ldots, \mu_n + k).$$

By an *algebraic* $\mathrm{GL}(n, \mathbb{Z})$*-representation*, we mean the restriction of an algebraic $\mathrm{GL}(n, \mathbb{Q})$-representation to $\mathrm{GL}(n, \mathbb{Z})$. Note that $\det^2$ is trivial as a $\mathrm{GL}(n, \mathbb{Z})$-representation. It follows that any irreducible algebraic $\mathrm{GL}(n, \mathbb{Z})$-representation is obtained from an irreducible polynomial $\mathrm{GL}(n, \mathbb{Q})$-representation by restriction to $\mathrm{GL}(n, \mathbb{Z})$.

3. Borel's improved stable range

In [1, 2], Borel computed the cohomology $H^p(\Gamma, V)$ of an arithmetic group Γ with coefficients in an algebraic Γ-representation V in a stable range

$$p \leq N(\Gamma, V) = \min\{M(\Gamma(\mathbb{R}), V), C(\Gamma(\mathbb{Q}), V)\},$$

where $M(\Gamma(\mathbb{R}), V)$ and $C(\Gamma(\mathbb{Q}), V)$ are constants depending only on Γ and V. For $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, we have $M(\mathrm{SL}(n, \mathbb{R}), V) \geq n - 2$. Borel did not compute the constant $C(\mathrm{SL}(n, \mathbb{Q}), V)$ explicitly except for a few families of representations. Recently, Krannich and Randal-Williams [10] gave an estimate of $C(\mathrm{SL}(n,\mathbb{Q}),V)$.

Borel remarked that one can replace the constant $C(\Gamma(\mathbb{Q}),V)$ by an improved constant $C'(\Gamma(\mathbb{Q}),V) \geq C(\Gamma(\mathbb{Q}),V)$ [2, Remark 3.8]. In this section, we give an estimate of Borel's improved constant for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. The constant $C'(\mathrm{SL}(n,\mathbb{Q}),V)$ depends not only on n but also on V, unlike the cases of $\mathrm{Sp}(2n,\mathbb{Z})$ and $\mathrm{SO}(n,n;\mathbb{Z})$ which were determined by Tshishiku [21].

3.1. **Borel's stable range for the cohomology of $\mathrm{SL}(n,\mathbb{Z})$.** Here we recall Borel's result. Let

$$N'(\mathrm{SL}(n,\mathbb{Z}),V) = \min\{M(\mathrm{SL}(n,\mathbb{R}),V), C'(\mathrm{SL}(n,\mathbb{Q}),V)\},$$

where the constant C' is defined below.

Theorem 3.1 (Borel [1, 2]). *(1) For each integer $n \geq 1$, the algebra map*

$$H^*(\mathrm{SL}(n+1,\mathbb{Z}),\mathbb{Q}) \to H^*(\mathrm{SL}(n,\mathbb{Z}),\mathbb{Q})$$

induced by the inclusion $\mathrm{SL}(n,\mathbb{Z}) \hookrightarrow \mathrm{SL}(n+1,\mathbb{Z})$ is an isomorphism for $* \leq N'(\mathrm{SL}(n,\mathbb{Z}),\mathbb{Q})$. Moreover, we have an algebra isomorphism

$$\varprojlim_n H^*(\mathrm{SL}(n,\mathbb{Z}),\mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}(x_1,x_2,\ldots), \quad \deg x_i = 4i+1$$

in degrees $* \leq N'(\mathrm{SL}(n,\mathbb{Z}),\mathbb{Q})$.

(2) Let V be an algebraic $\mathrm{SL}(n,\mathbb{Q})$-representation such that $V^{\mathrm{SL}(n,\mathbb{Q})} = 0$. Then we have

$$H^p(\mathrm{SL}(n,\mathbb{Z}),V) = 0 \quad \text{for } p \leq N'(\mathrm{SL}(n,\mathbb{Z}),V).$$

3.2. **Preliminaries from representation theory.** Before defining Borel's constant, we recall necessary facts from representation theory. See [6] for details.

Let $n \geq 2$ be an integer. Let $\mathfrak{h} \subset sl_n(\mathbb{C})$ denote the Cartan subalgebra

$$\mathfrak{h} = \{a_1H_1 + \cdots + a_nH_n \mid a_1 + \cdots + a_n = 0\},$$

where H_i is the matrix whose (i,i)-th entry is 1 and other entries are 0. We write the dual vector space $\mathfrak{h}^*$ as

$$\mathfrak{h}^* = \mathbb{C}\{L_1,\ldots,L_n\}/\mathbb{C}(L_1 + \cdots + L_n),$$

where L_i is the linear map from the space of diagonal matrices to $\mathbb{C}$ satisfying $L_i(H_j) = \delta_{i,j}$. The set of *roots* of $sl_n(\mathbb{C})$ is $\{L_i - L_j \mid i \neq j\}$, that of *positive roots* is $\{L_i - L_j \mid i < j\}$ and that of *simple roots* is $\{\alpha_i = L_i - L_{i+1} \mid 1 \leq i \leq n-1\}$.

An element $u = u_1L_1 + \cdots + u_nL_n$ with $\sum u_i = 0$ will be denoted by $[u_1, \ldots, u_n]$. For an element $\phi \in \mathfrak{h}^*$, we write $\phi > 0$ if $\phi = \sum_i c_i\alpha_i$ with $c_i > 0$ for all i. Note that $\phi = [\phi_1, \ldots, \phi_n] \in \mathfrak{h}^*$ satisfies $\phi > 0$ if and only if $\phi_1 + \cdots + \phi_i > 0$ for any $i = 1, \ldots, n-1$.

The *Weyl group* W of $sl_n(\mathbb{C})$ is the symmetric group $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$. The generator s_i permutes L_i and L_{i+1} and fixes the other L_k. The *length* $l(\sigma)$ of an element $\sigma \in W$ is the minimum length of the words in the s_i representing σ. Set $W^q = \{\sigma \in W \mid l(\sigma) = q\}$, which consists of elements that send exactly q positive roots to negative roots. We have $W = \coprod_{q=0}^{l(w_0)} W^q$, where $l(w_0) = \frac{1}{2}n(n-1)$ is the length of the longest element w_0 of $W = \mathfrak{S}_n$.

3.3. **The constant $C'(\mathrm{SL}(n, \mathbb{Q}), V)$.** Here we define Borel's improved constants C'.

For a bipartition $\underline{\lambda}$ with $l(\underline{\lambda}) \leq n$, let

$$\mu_{\underline{\lambda}} = (\mu_1, \ldots, \mu_n) = (\lambda_1, \ldots, \lambda_{l(\lambda)}, 0, \ldots, 0, -\lambda'_{l(\lambda')}, \ldots, -\lambda'_1) \tag{3.3.1}$$

be the highest weight of $V_{\underline{\lambda}}$. Let $\rho \in \mathfrak{h}^*$ be half the sum of the positive roots. Then we have

$$\rho = [\frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \ldots, -\frac{n-1}{2}].$$

Since we have $\mu_1 + \ldots \mu_n = \deg \underline{\lambda}$, it follows that

$$\rho + \mu_{\underline{\lambda}} = [\frac{n-1}{2} - \alpha + \mu_1, \frac{n-3}{2} - \alpha + \mu_2, \ldots, -\frac{n-1}{2} - \alpha + \mu_n],$$

where $\alpha = \frac{1}{n}\deg \underline{\lambda}$. Define

$$C'(\mathrm{SL}(n, \mathbb{Q}), V_{\underline{\lambda}}) = \max\{q \in \{0, \ldots, l(w_0)\} \mid \sigma(\rho + \mu_{\underline{\lambda}}) > 0 \text{ for all } \sigma \in W^q\} \geq 0.$$

Then we can easily check that

$$C'(\mathrm{SL}(n, \mathbb{Q}), V_{\underline{\lambda}}) = C'(\mathrm{SL}(n, \mathbb{Q}), V_{\underline{\lambda}^*}), \tag{3.3.2}$$

where $\underline{\lambda}^*$ is the dual partition of $\underline{\lambda}$.

For an algebraic $\mathrm{SL}(n, \mathbb{Q})$-representation V, we set

$$C'(\mathrm{SL}(n, \mathbb{Q}), V) = \min_{\underline{\lambda}} C'(\mathrm{SL}(n, \mathbb{Q}), V_{\underline{\lambda}}),$$

where $\underline{\lambda}$ runs through all bipartitions such that $V_{\underline{\lambda}}$ is isomorphic to a direct summand of V.

3.4. **Estimate of $C'(\mathrm{SL}(n,\mathbb{Q}),V_{\underline{\lambda}})$.** For each bipartition $\underline{\lambda}$, we define the integer $n_{\mathrm{B}}(\underline{\lambda})\geq 0$ by

$$n_{\mathrm{B}}(\underline{\lambda}):=\max\{2|\deg\underline{\lambda}|+1,2l(\lambda),2l(\lambda')\}.$$

Theorem 3.2. *Let $n\geq 2$. Let $\underline{\lambda}=(\lambda,\lambda')$ be a bipartition. Then for every $n\geq l(\underline{\lambda})$, we have*

$$C'(\mathrm{SL}(n,\mathbb{Q}),V_{\underline{\lambda}})\leq\lfloor n/2\rfloor-1. \tag{3.4.1}$$

Equality holds if $n\geq n_{\mathrm{B}}(\underline{\lambda})$.

Proof. We first prove (3.4.1). If n is odd and $(\rho+\mu_{\underline{\lambda}})_{\frac{n+1}{2}}=-\alpha+\mu_{\frac{n+1}{2}}\geq 0$, then for $\sigma_-=s_{n-1}\cdots s_{\frac{n+1}{2}}\in W$, the coefficient of L_n in $\sigma_-(\rho+\mu_{\underline{\lambda}})$ is $(\rho+\mu_{\underline{\lambda}})_{\frac{n+1}{2}}\geq 0$. If n is odd and $(\rho+\mu_{\underline{\lambda}})_{\frac{n+1}{2}}<0$, then for $\sigma_+=s_1\cdots s_{\frac{n-1}{2}}\in W$, the coefficient of L_1 in $\sigma_+(\rho+\mu_{\underline{\lambda}})$ is $(\rho+\mu_{\underline{\lambda}})_{\frac{n+1}{2}}<0$. If n is even, then we have either $(\rho+\mu_{\underline{\lambda}})_{\frac{n}{2}}=1/2-\alpha+\mu_{\frac{n}{2}}\geq 0$ or $(\rho+\mu_{\underline{\lambda}})_{\frac{n}{2}+1}=-1/2-\alpha+\mu_{\frac{n}{2}+1}\leq 0$. If the former holds, then for $\sigma_-=s_{n-1}\cdots s_{\frac{n}{2}}\in W$, the coefficient of L_n in $\sigma_-(\rho+\mu_{\underline{\lambda}})$ is $(\rho+\mu_{\underline{\lambda}})_{\frac{n}{2}}\geq 0$. If the latter holds, then for $\sigma_+=s_1\cdots s_{\frac{n}{2}}\in W$, the coefficient of L_1 in $\sigma_+(\rho+\mu_{\underline{\lambda}})$ is $(\rho+\mu_{\underline{\lambda}})_{\frac{n}{2}+1}\leq 0$. Therefore, in each case, we have $\sigma_\pm(\rho+\mu_{\underline{\lambda}})\not>0$, which implies (3.4.1).

By (3.3.2), we have only to consider the case where $\alpha\geq 0$, that is, when $|\lambda|\geq|\lambda'|$. Suppose that we have $n\geq n_{\mathrm{B}}(\underline{\lambda})$. Thus, we have $0\leq\alpha<1/2$. We first prove $C'(\mathrm{SL}(n,\mathbb{Q}),V_{\underline{\lambda}})=\lfloor n/2\rfloor-1$ for $2\leq n\leq 4$. For $n=2,3$, this is obvious since we have $\lfloor n/2\rfloor-1=0$. For $n=4$, since we have $l(\lambda),l(\lambda')\leq 2$, it follows that

$$\rho+\mu_{\underline{\lambda}}=[3/2+\lambda_1-\alpha,1/2+\lambda_2-\alpha,-1/2-\lambda'_2-\alpha,-3/2-\lambda'_1-\alpha].$$

Since $0\leq\alpha<1/2$, the first two coefficients are positive and the others are negative. For $\sigma=s_1,s_3\in W^1$, it is easily checked that $\sigma(\rho+\mu_{\underline{\lambda}})>0$. For $\sigma=s_2$, we also have $\sigma(\rho+\mu_{\underline{\lambda}})>0$ since we have $\lambda_1\geq\lambda'_2$ and thus

$$(3/2+\lambda_1-\alpha)+(-1/2-\lambda'_2-\alpha)=1-2\alpha+\lambda_1-\lambda'_2\geq 1-2\alpha>0.$$

Therefore, we have $C'(\mathrm{SL}(n,\mathbb{Q}),V_{\underline{\lambda}})=\lfloor n/2\rfloor-1$ for $n=4$.

In what follows, we will prove that for $n\geq 5$, $\sigma(\rho+\mu_{\underline{\lambda}})>0$ for any $\sigma\in W$ of length $\lfloor n/2\rfloor-1$. Let $k\in\{1,\dots,n-1\}$. By the definition of $l(\sigma)$, we have

$$(\sigma^{-1}(1)-1)+(\sigma^{-1}(2)-2)+\cdots+(\sigma^{-1}(k)-k)\leq\lfloor n/2\rfloor-1.$$

Therefore, the sum of the first k coefficients of $\sigma(\rho + \mu_{\underline{\lambda}})$ is

$$\left(\frac{n-1}{2} - (\sigma^{-1}(1)-1) - \alpha + \mu_{\sigma^{-1}(1)}\right) + \cdots + \left(\frac{n-1}{2} - (\sigma^{-1}(k)-1) - \alpha + \mu_{\sigma^{-1}(k)}\right)$$
$$= \sum_{i=1}^{k}\left(\frac{n-1}{2} - (i-1)\right) - \sum_{i=1}^{k}(\sigma^{-1}(i) - i) + \sum_{i=1}^{k}(\mu_{\sigma^{-1}(i)} - \alpha)$$
$$\geq \frac{(n-k)k}{2} - (\lfloor n/2 \rfloor - 1) + \sum_{i=1}^{k}(\mu_{\sigma^{-1}(i)} - \alpha).$$

Let $T(k)$ denote the right hand side of this inequality. It suffices to show that $T(k) > 0$ for each $k \in \{1, \ldots, n-1\}$. For $k = n-1$, we have

$$\begin{aligned} T(n-1) &\geq 1/2 + \sum_{i=1}^{n-1} \mu_{\sigma^{-1}(i)} - (n-1)\alpha \\ &= 1/2 + (|\lambda| - |\lambda'| - \mu_{\sigma^{-1}(n)}) - (n-1)\alpha \\ &= 1/2 + (n\alpha - \mu_{\sigma^{-1}(n)}) - (n-1)\alpha \\ &= 1/2 + \alpha - \mu_{\sigma^{-1}(n)} \geq 1/2 > 0. \end{aligned}$$

For $1 \leq k \leq n-2$, we have

$$\begin{aligned} T(k) &= \frac{(n-k)k}{2} - (\lfloor n/2 \rfloor - 1) - k\alpha + \sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \\ &> \frac{(n-k)k - n + 2 - k}{2} + \sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \geq \sum_{i=1}^{k} \mu_{\sigma^{-1}(i)}. \end{aligned}$$

Therefore, it suffices to show that $\sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \geq 0$. Let $\mathscr{J} = \{j \in \{1, \ldots, \lfloor n/2 \rfloor\} \mid \sigma(n+1-j) \leq k\}$. If $\mathscr{J} = \emptyset$, then $\sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \geq 0$ follows directly from the definition of $\mathscr{J}$. Otherwise, let $J = \min \mathscr{J}$. Since the length of σ is $\lfloor n/2 \rfloor - 1$, by the hypothesis that $l(\lambda) \leq n/2$ and $l(\lambda') \leq n/2$, we have

$$\sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \geq (\mu_1 + \cdots + \mu_{\lfloor n/2 \rfloor + 1 - J}) + (\mu_{n+1-J} + \cdots + \mu_{n+1-\lfloor n/2 \rfloor}).$$

Let $a = \mu_1 + \cdots + \mu_{\lfloor n/2 \rfloor + 1 - J}$. Then we have $a \geq \frac{\lfloor n/2 \rfloor + 1 - J}{\lfloor n/2 \rfloor}|\lambda|$ since we have

$$|\lambda| = a + (\mu_{\lfloor n/2 \rfloor + 2 - J} + \cdots + \mu_{\lfloor n/2 \rfloor}) \leq a + \frac{J-1}{\lfloor n/2 \rfloor + 1 - J} a = \frac{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor + 1 - J}\, a.$$

Let $b = \mu_{n+1-J} + \cdots + \mu_{n+1-\lfloor n/2 \rfloor}$. In a similar way, we have $b \geq -\frac{\lfloor n/2 \rfloor + 1 - J}{\lfloor n/2 \rfloor}|\lambda'|$. Therefore, we have

$$\sum_{i=1}^{k} \mu_{\sigma^{-1}(i)} \geq a + b \geq \frac{\lfloor n/2 \rfloor + 1 - J}{\lfloor n/2 \rfloor}(|\lambda| - |\lambda'|) = \frac{\lfloor n/2 \rfloor + 1 - J}{\lfloor n/2 \rfloor} n\alpha \geq 0.$$

This completes the proof. □

Note that Theorem 3.2 does not give any information for the value of $C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}})$ if $n < n_{\mathrm{B}}(\underline{\lambda})$. Some computations suggest the following conjecture, which would completely determine $C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}})$.

Conjecture 3.3. *Let $\underline{\lambda}$ be a bipartition and let $n \geq l(\underline{\lambda})$ be an integer. For $i = 1, \ldots, n$, set $a(i) = \frac{n+1}{2} - i - \alpha + \mu_i$, where μ_i is given in* (3.3.1). *Then we have*

$$C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}}) = \min\{i \in \{1, \ldots, n\} \mid a(i) \leq 0 \text{ or } a(n+1-i) \geq 0\} - 2.$$

3.5. **Stable range for the cohomology of** $\mathrm{GL}(n,\mathbb{Z})$**.** Now we regard $V_{\underline{\lambda}}$ as an irreducible algebraic $\mathrm{GL}(n,\mathbb{Z})$-representation. We obtain a stable range for the cohomology of $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in $V_{\underline{\lambda}}$.

For a bipartition $\underline{\lambda}$ and a non-negative integer p, set

$$n_{\mathrm{B}}(\underline{\lambda}, p) = \max\{n_{\mathrm{B}}(\underline{\lambda}), 2p+2\} \geq 2.$$

Borel's result (Theorem 3.1) and the estimate of Borel's constant $C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}})$ (Theorem 3.2) imply the following.

Corollary 3.4 (weaker than Theorem 1.1)**.** *Let $\underline{\lambda}$ be a bipartition, and let $p \geq 0$ and $n \geq n_{\mathrm{B}}(\underline{\lambda}, p)$ be integers. Then we have the following.*

(1) If $\underline{\lambda} = (0,0)$, i.e., $V_{\underline{\lambda}} = \mathbb{Q}$, then we have

$$H^*(\mathrm{SL}(n,\mathbb{Z}), \mathbb{Q}) \cong H^*(\mathrm{GL}(n,\mathbb{Z}), \mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}(x_1, x_2, \ldots), \quad \deg x_i = 4i+1$$

in cohomological degrees $* \leq p$.

(2) If $\underline{\lambda} \neq (0,0)$, then we have

$$H^p(\mathrm{SL}(n,\mathbb{Z}), V_{\underline{\lambda}}) = H^p(\mathrm{GL}(n,\mathbb{Z}), V_{\underline{\lambda}}) = 0.$$

Proof. Since we have $n \geq n_{\mathrm{B}}(\underline{\lambda}, p) \geq n_{\mathrm{B}}(\underline{\lambda}) \geq 2$, by Theorem 3.2, we have $C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}}) = \lfloor n/2 \rfloor - 1$. Therefore, we have

$$C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}}) = \lfloor n/2 \rfloor - 1 \leq n - 2 \leq M(\mathrm{SL}(n,\mathbb{R}), V_{\underline{\lambda}}).$$

Since we have $n \geq n_{\mathrm{B}}(\underline{\lambda}, p) \geq 2p+2$, we have

$$p \leq \lfloor n/2 \rfloor - 1 = C'(\mathrm{SL}(n,\mathbb{Q}), V_{\underline{\lambda}}) = N'(\mathrm{SL}(n,\mathbb{Z}), V_{\underline{\lambda}}).$$

Therefore, the case of $\mathrm{SL}(n,\mathbb{Z})$ follows from Theorem 3.1.

The case of $\mathrm{GL}(n, \mathbb{Z})$ follows from the case for $\mathrm{SL}(n, \mathbb{Z})$ and the Hochschild–Serre spectral sequence for the short exact sequence

$$1 \to \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

□

4. Kupers, Miller and Patzt's method

Kupers, Miller and Patzt [11] improved Borel's original stable range for coefficients in polynomial $\mathrm{GL}(n, \mathbb{Z})$-representations indexed by partitions. Here we adapt their arguments to the case of coefficients in algebraic $\mathrm{GL}(n, \mathbb{Q})$-representations indexed by bipartitions.

4.1. **Polynomial** VIC-**modules.**

There are mutually related theories that can be used in the study of stability of sequence of $\mathrm{GL}(n, \mathbb{Z})$-representations such as coefficient systems [5, 22, 18], representation stability [3], central stability [15] and VIC-modules [16]. The notion of polynomiality was introduced by van der Kallen [22] for coefficient systems, and was generalized by Randal-Williams and Wahl [18]. See also [13, 11]. Their definition is stronger than Djament and Vespa's strong polynomial functors [4].

Here we recall the notions of VIC-modules and polynomial VIC-modules.

Let $\mathrm{VIC} = \mathrm{VIC}(\mathbb{Z})$ denote the category of finitely generated free abelian groups and injective morphisms with chosen complements. The Hom-set for a pair of objects M and N is given by

$$\mathrm{Hom}_{\mathrm{VIC}}(M, N) = \{(f, C) \mid f : M \hookrightarrow N,\ C \subset N,\ N = \mathrm{im}(f) \oplus C\}.$$

A VIC-module is a functor from VIC to the category $\mathbf{Vect}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces and linear maps. A *morphism* (also called a VIC-*module map*) $f : V \to V'$ of VIC-modules V and V' is a natural transformation. The VIC-modules and morphisms form an abelian category VIC-mod, as is the case for the category of $\mathcal{C}$-modules for any essentially small category $\mathcal{C}$. The category VIC-mod also has a symmetric monoidal category structure whose tensor product is defined objectwise, i.e., $(V \otimes V')(M) = V(M) \otimes V'(M)$ for $M \in \mathrm{Ob}(\mathrm{VIC})$, and whose monoidal unit is given by the constant functor with value $\mathbb{Q}$.

For each VIC-module V and an integer $n \geq 0$, the vector space $V(\mathbb{Z}^n)$ is naturally equipped with a $\mathrm{GL}(n, \mathbb{Z})$-module structure.

Let V be a VIC-module. Define VIC-modules $\ker V$ and $\operatorname{coker} V$ by

$$\ker V(M) := \ker(V(M) \to V(M \oplus \mathbb{Z})),$$
$$\operatorname{coker} V(M) := \operatorname{coker}(V(M) \to V(M \oplus \mathbb{Z}))$$

for any object M of VIC, where $V(M) \to V(M \oplus \mathbb{Z})$ is induced by the canonical morphism $M \hookrightarrow M \oplus \mathbb{Z}$ of VIC.

Define the polynomiality of VIC-modules inductively as follows.

Definition 4.1. Let $m \geq -1$. We call V *polynomial of degree* -1 *in ranks* $> m$ if $V(M) = 0$ for any object $M \in \mathrm{Ob}(\mathrm{VIC})$ with rank $> m$. For $r \geq 0$, we call V *polynomial of degree* $\leq r$ *in ranks* $> m$ if $\ker V = 0$ in ranks $> m - r - 1$ and if $\operatorname{coker} V$ is polynomial of degree $\leq r - 1$ in ranks $> \max\{m - 1, -1\}$. We call V *polynomial of degree (exactly)* r *in ranks* $> m$ if V is polynomial of degree $\leq r$ in ranks $> m$ and if V is not polynomial of degree $\leq r - 1$ in ranks $> m$. If $m = -1$, then we usually omit "in ranks > -1" and just write polynomial of degree $\leq r$.

For a VIC-module V, the *truncation* $V_{\geq k}$ of V at k is the VIC-submodule of V such that $V_{\geq k}(n) = V(n)$ for $n \geq k$ and $V_{\geq k}(n) = 0$ otherwise.

Example 4.2.
1. The constant functor with value $\mathbb{Q}$ is polynomial of degree 0 in ranks > -1.
2. For $k \geq 0$, let $\mathbb{Q}_k$ denote the VIC-module such that $\mathbb{Q}_k(\mathbb{Z}^k) = \mathbb{Q}$ and $\mathbb{Q}_k(\mathbb{Z}^n) = 0$ for $n \neq k$. Then we have $\ker \mathbb{Q}_k = \mathbb{Q}_k$, $\operatorname{coker} \mathbb{Q}_k = \mathbb{Q}_{k-1}$. Therefore, $\mathbb{Q}_k$ is polynomial of degree 0 in ranks $> k+1$.
3. For the truncation $\mathbb{Q}_{\geq k}$ of $\mathbb{Q}$ at k, we have $\ker \mathbb{Q}_{\geq k} = 0$ and $\operatorname{coker} \mathbb{Q}_{\geq k} = \mathbb{Q}_{k-1}$. Therefore, $\mathbb{Q}_{\geq k}$ is polynomial of degree 0 in ranks $> k$.

Remark 4.3. Our definition of polynomial VIC-modules is slightly stronger than that in [11] for a technical reason. We need this strengthened definition to have Lemma 4.6. In [11], they required the condition that if V is polynomial of degree $\leq r$ in ranks $> m$, then $\ker V = 0$ in ranks $> m$ instead of in ranks $> m-r-1$. Patzt [13] used a still weaker condition in which $\operatorname{coker} V$ is polynomial of degree $\leq r - 1$ in ranks $> m$. If $m = -1$, then all of the three definitions of polynomiality coincide.

Let V be a polynomial VIC-module of degree $\leq r$ in ranks $> m$. The polynomiality of finite-dimensional VIC-modules imply the polynomiality of dimensions, that is, if $\dim(V(\mathbb{Z}^n))$ is finite for each $n > m$, then there is a polynomial $f(x)$ such that we have $\dim(V(\mathbb{Z}^n)) = f(n)$ for any $n > m$. Therefore, we obtain the following lemma.

Lemma 4.4. *Let V be a polynomial* VIC*-module of degree $\leq r$ in ranks $> m$. If there exists a polynomial $P(x)$ of degree r such that for each $n > m$,* $\dim(V(\mathbb{Z}^n)) = P(n)$*, then V is polynomial of degree r in ranks $> m$.*

By the definition of polynomiality, we can always increase ranks m if we fix degrees r. Moreover, we have the following lemma.

Lemma 4.5. *Let $s \geq r$ and $n \geq m + s - r$. If V is a polynomial* VIC*-module of degree $\leq r$ in ranks $> m$, then V is polynomial of degree $\leq s$ in ranks $> n$.*

Proof. If V is polynomial of degree $\leq r$ in ranks $> m$, then $\ker V = 0$ in ranks $> m - r - 1$. Since we have $n - s - 1 \geq m - r - 1$, $\ker V = 0$ in ranks $> n - s - 1$. Also $\operatorname{coker} V$ is polynomial of degree $\leq r - 1$ in ranks $> m - 1$. Since we have $s \geq r$ and $n \geq m + (s - r) \geq m$, $\operatorname{coker} V$ is polynomial of degree $\leq s - 1$ in ranks $> n - 1$. Therefore, V is polynomial of degree $\leq s$ in ranks $> n$. □

It follows from Lemma 4.5 that a polynomial VIC-module V of degree ≤ -1 in ranks $> m$ is polynomial of degree $\leq s$ in ranks $> m + s + 1$ for any $s \geq -1$.

One can see that polynomiality of degree r in ranks $> m$ is not closed under subquotient VIC-modules by Example 4.2. However, we have the following properties of polynomial VIC-modules, in which we adapt Patzt's lemma to our definition of polynomiality.

Lemma 4.6 (Cf. Patzt [13, Lemma 7.3])**.** *(a) Let $m, r \geq -1$ be integers. Let $V' \to V$ and $V \to V''$ be morphisms of* VIC*-modules such that the truncations form the following short exact sequence*

$$0 \to V'_{\geq m-r} \to V_{\geq m-r} \to V''_{\geq m-r} \to 0. \tag{4.1.1}$$

If two of V, V' and V'' are polynomial of degree $\leq r$ in ranks $> m$, then so is the third.

(b) Let V and W be polynomial VIC*-modules of degrees $\leq r$ and $\leq s$, respectively. Then the tensor product $V \otimes W$ has polynomial degree $\leq r + s$.*

Proof. (b) is essentially a special case of [13, Lemma 7.3. (b)].

Let us prove (a) by adapting the proof of [13, Lemma 7.3 (a)]. The case of $r = -1$ is obvious. We use induction on r. Suppose that the case of degree $\leq r-1$ holds. From (4.1.1), by the Snake Lemma, we have an exact sequence

(4.1.2)
$$0 \to \ker V' \to \ker V \to \ker V'' \to \operatorname{coker} V' \to \operatorname{coker} V \to \operatorname{coker} V'' \to 0$$

in ranks $> m-r-1$.

We first show that it suffices to verify that the exact sequence (4.1.2) splits into two short exact sequences in ranks $> m-r-1$

$$0 \to \ker V' \to \ker V \to \ker V'' \to 0, \tag{4.1.3}$$

$$0 \to \operatorname{coker} V' \to \operatorname{coker} V \to \operatorname{coker} V'' \to 0. \tag{4.1.4}$$

Since two of $\ker V'$, $\ker V$ and $\ker V''$ are zero in ranks $> m-r-1$, the other is also zero in ranks $> m-r-1$ by (4.1.3). Since two of $\operatorname{coker} V'$, $\operatorname{coker} V$ and $\operatorname{coker} V''$ are polynomial of degree $\leq r-1$ in ranks $> m-1$, and since we have an exact sequence (4.1.4) in ranks $> m-r-1$, we can use the induction hypothesis, which completes the proof of (a).

We will show that (4.1.2) splits into (4.1.3) and (4.1.4). If V'' is polynomial of degree $\leq r$ in ranks $> m$, then we have $\ker V'' = 0$ in ranks $> m-r-1$, which implies the splitting of the exact sequence (4.1.2). Suppose V and V' are polynomial of degree $\leq r$ in ranks $> m$. Then $\operatorname{coker} V'$ is polynomial of degree $\leq r-1$ in ranks $> m-1$. Therefore, we have $\ker \operatorname{coker} V' = 0$ in ranks $> m-r-1$. By the following commutative diagram in ranks $> m-r-1$

$$\begin{array}{ccc} \ker V'' & \longrightarrow & \operatorname{coker} V' \\ {\scriptstyle =}\uparrow & & \uparrow \\ \ker^2 V'' & \longrightarrow & \ker \operatorname{coker} V' = 0, \end{array}$$

the exact sequence (4.1.2) splits.

∎

We also need the following lemma.

Lemma 4.7. *Let V and V' be* VIC*-modules such that $V \oplus V'$ is polynomial of degree $\leq r$ in ranks $> m$. Then both V and V' are polynomial of degree $\leq r$ in ranks $> m$.*

Proof.

Since we have a short exact sequence of VIC-modules

$$0 \to V \to V \oplus V' \to V' \to 0,$$

the statement follows from the proof of Lemma 4.6 (a). □

4.2. **Polynomial VIC-module $V^{\langle p,q\rangle}$ of traceless tensors.** For a free abelian group M, let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Associating $M_{\mathbb{Q}}$ to each object M of VIC forms a VIC-module $V^{1,0}$. We have another VIC-module $V^{0,1}$ such that $V^{0,1}(M) = M_{\mathbb{Q}}^*$ for an object M of VIC. For a morphism $(f, C) : M \to N$ of VIC, if we choose a basis $\{x_i\}$ for $M_{\mathbb{Q}}$ and $\{z_j\}$ for $C_{\mathbb{Q}}$, then $\{f(x_i)\} \cup \{z_j\}$ forms a basis for $N_{\mathbb{Q}}$. Then the linear map $V^{0,1}((f, C)) : M_{\mathbb{Q}}^* \to N_{\mathbb{Q}}^*$ sends x^i to y^i, where $\{x^i\}$ is the dual basis for $M_{\mathbb{Q}}^*$ and y^i is dual to $f(x_i)$ for each i. The VIC-modules $V^{1,0}$ and $V^{0,1}$ are polynomial of degree 1 by [17, Definition 3.3]. For $p, q \geq 0$, let

$$V^{p,q} = (V^{1,0})^{\otimes p} \otimes (V^{0,1})^{\otimes q}, \quad M \mapsto M_{\mathbb{Q}}^{p,q} = M_{\mathbb{Q}}^{\otimes p} \otimes (M_{\mathbb{Q}}^*)^{\otimes q}$$

denote the tensor product of copies of VIC-modules $V^{1,0}$ and $V^{0,1}$. By Lemma 4.6, the VIC-module $V^{p,q}$ is polynomial of degree $\leq p+q$, which is actually of degree $p+q$ by Lemma 4.4 since we have $\dim(V^{p,q}(n)) = n^{p+q}$ for $n \geq 0$.

The construction of the $\mathrm{GL}(n, \mathbb{Q})$-module $H^{\langle p,q\rangle}$ in Section 2 can be extended into a VIC-module since there is a VIC-module map

$$c : V^{1,1} \to V^{0,0},$$

such that for each $M \in \mathrm{Ob}(\mathrm{VIC})$, the map $c_M : V^{1,1}(M) \to V^{0,0}(M)$ is the evaluation map $M_{\mathbb{Q}} \otimes M_{\mathbb{Q}}^* \to \mathbb{Q}$, $v \otimes f \mapsto f(v)$. Let $V^{\langle p,q\rangle}$ denote the VIC-module consisting of $H^{\langle p,q\rangle}(n)$, which we call the *traceless part* of $V^{p,q}$.

For $0 \leq l \leq \min\{p, q\}$, let

$$\Lambda_{p,q}(l) = \{((i_1, j_1), \dots, (i_{l+1}, j_{l+1})) \in ([p] \times [q])^{l+1} \mid {\scriptstyle 1 \leq i_1 < i_2 < \cdots < i_{l+1} \leq p, \atop j_1, j_2, \dots, j_{l+1} :\ \text{distinct}}\}.$$

For $I = ((i_1, j_1), \dots, (i_{l+1}, j_{l+1})) \in \Lambda_{p,q}(l)$, let

$$c_I : V^{p,q} \to V^{p-l-1, q-l-1}$$

denote the VIC-module map that is obtained as the composition of contraction maps defined by

$$(v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \otimes f_q)$$

$$\mapsto \left(\prod_{r=1}^{l+1} \langle v_{i_r}, f_{j_r} \rangle \right) (v_1 \otimes \cdots \widehat{v_{i_1}} \cdots \widehat{v_{i_{l+1}}} \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \widehat{f_{j_1}} \cdots \widehat{f_{j_{l+1}}} \cdots \otimes f_q).$$

Let $k = \min\{p, q\}$. Define an increasing filtration $F^* = \{F^l\}_{0 \leq l \leq k}$

$$V^{\langle p,q\rangle} = F^0 \subset F^1 \subset \cdots \subset F^l \subset F^{l+1} \subset \cdots \subset F^k = V^{p,q}$$

of the VIC-module $V^{p,q}$ by

$$F^l = \ker\left(\bigoplus_{I\in\Lambda_{p,q}(l)} c_I \;:\; V^{p,q} \to \bigoplus_{I\in\Lambda_{p,q}(l)} V^{p-l-1,q-l-1}\right). \tag{4.2.1}$$

Lemma 4.8. *For $1 \le l \le k$, consider the following sequence of* VIC-*modules*

$$0 \to F^{l-1} \xrightarrow{i} F^l \xrightarrow{\pi} \bigoplus_{I\in\Lambda_{p,q}(l-1)} V^{\langle p-l,q-l\rangle} \to 0, \tag{4.2.2}$$

where i is the inclusion and π is the restriction of the map $\bigoplus_{I\in\Lambda_{p,q}(l-1)} c_I$ to F^l and $\bigoplus_{I\in\Lambda_{p,q}(l-1)} V^{\langle p-l,q-l\rangle}$. If $n \ge p+q$, then the sequence (4.2.2) *evaluated on $\mathbb{Z}^n$ is exact.*

Proof.

By the definition of F^l, (4.2.2) is exact at F^{l-1} and F^l.

By the definition of $F^l \subset V^{p,q}$, it follows from Lemma 4.9 below that $\pi : F^l \to \bigoplus_{I\in\Lambda_{p,q}(l-1)} V^{\langle p-l,q-l\rangle}$ is surjective when evaluated on $\mathbb{Z}^n$ for $n \ge p+q$. □

Note that the map from F^l in (4.2.2) is not always surjective in small ranks. For example, (4.2.2) is not exact when $l = p = q$ since we have $F^p(0) = 0$ and $V^{\langle 0,0\rangle}(0) = \mathbb{Q}$.

Lemma 4.9. *If $n \ge p+q$, then the image of the* VIC-*module map*

$$\bigoplus_{I\in\Lambda_{p,q}(l-1)} c_I : V^{p,q} \to \bigoplus_{I\in\Lambda_{p,q}(l-1)} V^{p-l,q-l}$$

evaluated on $\mathbb{Z}^n$ contains $\bigoplus_{I\in\Lambda_{p,q}(l-1)} V^{\langle p-l,q-l\rangle}(\mathbb{Z}^n)$.

Proof. Let $\{e_i \mid i = 1,\dots,n\}$ be a basis of $\mathbb{Q}^n = (\mathbb{Z}^n)_{\mathbb{Q}}$ and $\{e_i^*\}$ the dual basis of $(\mathbb{Q}^n)^*$. Since $n \ge p+q \ge (p-l)+(q-l)$, the traceless part $V^{\langle p-l,q-l\rangle}(\mathbb{Z}^n) \subset V^{p-l,q-l}(\mathbb{Z}^n)$ is generated by the element

$$e_{p-l,q-l} := e_1 \otimes e_2 \otimes \dots \otimes e_{p-l} \otimes e_n^* \otimes e_{n-1}^* \otimes \dots \otimes e_{n-(q-l)+1}^* \in V^{p-l,q-l}(\mathbb{Z}^n)$$

as an $\mathrm{GL}(n,\mathbb{Z})$-module (see [8, Lemma 2.2]). Since $n \ge p+q$, for each $I \in \Lambda_{p,q}(l-1)$, we have an element $e_{p-l,q-l;I} \in V^{p,q}(\mathbb{Z}^n)$ such that $c_I(e_{p-l,q-l;I}) = e_{p-l,q-l}$ and $c_{I'}(e_{p-l,q-l;I}) = 0$ for $I' \ne I$. Indeed, $e_{p-l,q-l;I}$ is the permutation of the tensor $e_{p-l,q-l} \otimes (e_{p-l+1} \otimes e_{p-l+1}^*) \otimes \dots \otimes (e_p \otimes e_p^*)$ obtained as follows. For each $k = 1,\dots,l$, the tensors e_{p-l+k}, e_{p-l+k}^* are placed at the positions specified by $(i_k, j_k) \in I$, respectively, and the tensor factors of $e_{p-l,q-l}$ are placed at the other positions in an order-preserving way. For example, if $p = 3, q = 5, l = 2, n = 10$ and $I = \{(1, 2), (3, 5)\}$, then we have $e_{p-l,q-l;I} = e_2 \otimes e_1 \otimes e_3 \otimes e_{10}^* \otimes e_2^* \otimes e_9^* \otimes e_8^* \otimes e_3^*$. □

By using (4.2.2), we can easily check that for $n \geq p + q$

$$\dim(V^{\langle p,q\rangle}(n)) = \sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{p}{i}\binom{q}{i} i!\, n^{p+q-2i}, \tag{4.2.3}$$

which is a monic polynomial in n of degree $p + q$.

Proposition 4.10. *The* VIC-*module* $V^{\langle p,q\rangle}$ *is polynomial of degree* $p+q$ *in ranks* $> 2(p + q)$.

Proof. By symmetry, we may assume $p \geq q$. We prove that $V^{\langle p,q\rangle}$ is polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$ by induction on q. If $q = 0$, then for any $i \geq 0$, the VIC-module $V^{\langle i,0\rangle} = V^{\otimes i}$ is polynomial of degree $\leq i$ in ranks > -1, hence in ranks $> 2i$. Suppose that $V^{\langle i,j\rangle}$ is polynomial of degree $\leq i + j$ in ranks $> 2(i + j)$ for any $j \leq q - 1$ and $i \geq j$. Then we have a filtration $F^* = \{F^l\}_{0\leq l\leq q}$ of $V^{\langle p,q\rangle}$ defined in (4.2.1). Here we use descending induction on l. For $l = q$, we have $F^l = F^q = V^{p,q}$, which is polynomial of degree $\leq p + q$. Suppose that F^l is polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$. Then from (4.2.2), we have an exact sequence

$$0 \to F^{l-1} \to F^l \to (V^{\langle p-l,q-l\rangle})^{\oplus\binom{p}{l}\binom{q}{l}l!} \to 0$$

in ranks $> p+q-1 = 2(p+q)-(p+q)-1$. By the induction hypothesis, $V^{\langle p-l,q-l\rangle}$ is polynomial of degree $\leq p + q - 2l$ in ranks $> 2(p + q - 2l)$. By Lemma 4.5, $V^{\langle p-l,q-l\rangle}$ is also polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$. Hence, by Lemma 4.6, $(V^{\langle p-l,q-l\rangle})^{\oplus\binom{p}{l}\binom{q}{l}l!}$ is polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$. Since F^l is polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$, by Lemma 4.6, so is F^{l-1}. Therefore, by induction, we see that $V^{\langle p,q\rangle}$ is a polynomial VIC-module of degree $\leq p+q$ in ranks $> 2(p+q)$. By Lemma 4.4 and (4.2.3), the polynomial degree of $V^{\langle p,q\rangle}$ is $p + q$. □

Remark 4.11. (1) The range "$> 2(p+q)$" in the statement of Proposition 4.10 is not optimal. For example, if $p = 0$ or $q = 0$, then we have $V^{\langle p,q\rangle} = V^{p,q}$, which is polynomial of degree $p + q$ in ranks > -1.

(2) Proposition 4.10 holds also with the definitions of polynomiality given in Patzt [13] and Kupers–Miller–Patzt [11], since our definition of polynomiality is stronger than theirs. The range "$> 2(p + q)$" could be improved if we use their definitions. If we use Patzt's definition, we

see that the range is $> p+q-1$ by the same argument as the proof of Proposition 4.10.

4.3. **The polynomial VIC-module $V_{\underline{\lambda}}$.** Here we construct a polynomial VIC-module $V_{\underline{\lambda}}$ for each bipartition $\underline{\lambda}$.

For a bipartition $\underline{\lambda} = (\lambda, \lambda')$, let $p = |\lambda|$ and $q = |\lambda'|$. Define the VIC-module $V_{\underline{\lambda}}$ as

$$V_{\underline{\lambda}}(M) = V^{\langle p,q \rangle}(M) \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} (S^{\lambda} \otimes S^{\lambda'})$$

for $M \in \mathrm{Ob}(\mathrm{VIC})$, where $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$ acts on $V^{\langle p,q \rangle}(M)$ on the right by permutation of tensor factors. Note that the $\mathrm{GL}(n, \mathbb{Z})$-module $V_{\underline{\lambda}}(\mathbb{Z}^n)$ is isomorphic to the $\mathrm{GL}(n, \mathbb{Z})$-module $V_{\underline{\lambda}}(n)$ defined in (2.0.2).

The direct sum decomposition (2.0.3) of $\mathrm{GL}(n, \mathbb{Z})$-modules can be extended to the following.

Lemma 4.12. *We have a direct sum decomposition of $V^{\langle p,q \rangle}$ as a $\mathrm{VIC} \times (\mathfrak{S}_p \times \mathfrak{S}_q)$-module*

$$V^{\langle p,q \rangle} = \bigoplus_{\substack{\underline{\lambda}=(\lambda,\lambda') \\ |\lambda|=p,\ |\lambda'|=q}} V_{\underline{\lambda}} \otimes (S^{\lambda} \otimes S^{\lambda'}). \tag{4.3.1}$$

Proof. In the category of $\mathrm{VIC} \times (\mathfrak{S}_p \times \mathfrak{S}_q)$-modules, we have

$$\begin{aligned}
V^{\langle p,q \rangle} &\cong V^{\langle p,q \rangle} \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} (\mathbb{Q}[\mathfrak{S}_p] \otimes \mathbb{Q}[\mathfrak{S}_q]) \\
&\cong V^{\langle p,q \rangle} \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} \left(\bigoplus_{|\lambda|=p} S^{\lambda} \otimes S^{\lambda} \right) \otimes \left(\bigoplus_{|\lambda'|=q} S^{\lambda'} \otimes S^{\lambda'} \right) \\
&\cong \bigoplus_{|\lambda|=p,\ |\lambda'|=q} \left(V^{\langle p,q \rangle} \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} (S^{\lambda} \otimes S^{\lambda'}) \right) \otimes (S^{\lambda} \otimes S^{\lambda'}) \\
&\cong \bigoplus_{|\lambda|=p,\ |\lambda'|=q} V_{\underline{\lambda}} \otimes (S^{\lambda} \otimes S^{\lambda'}).
\end{aligned}$$

□

The following lemma should be well known, but we sketch a proof here since we could not find a suitable reference.

Lemma 4.13. *For each bipartition $\underline{\lambda}$, there is a polynomial $f_{\underline{\lambda}}(x)$ of degree $|\underline{\lambda}|$ such that $\dim(V_{\underline{\lambda}}(n)) = f_{\underline{\lambda}}(n)$ for $n \geq |\underline{\lambda}|$.*

Proof. Let $n \geq |\underline{\lambda}|$. The dimension of $V_{\lambda,0}(n) \otimes V_{0,\lambda'}(n)$ is polynomial of degree $|\underline{\lambda}|$. We can obtain the lemma by using induction, since we have a decomposition

$$V_{\lambda,0}(n) \otimes V_{0,\lambda'}(n) \cong V_{\lambda,\lambda'}(n) \oplus \bigoplus_{|\mu|<|\lambda|,\ |\mu'|<|\lambda'|} V_{\mu,\mu'}(n)^{\oplus c_{\mu,\mu'}},$$

where the constants $c_{\mu,\mu'}$, *not* depending on n, are determined by the Littlewood–Richardson coefficients (see [9]). □

Proposition 4.14. *Let $\underline{\lambda} = (\lambda, \lambda')$ be a bipartition. If either $\lambda = 0$ or $\lambda' = 0$, then the* VIC*-module $V_{\underline{\lambda}}$ is polynomial of degree $|\underline{\lambda}|$ (in ranks > -1). Otherwise, the* VIC*-module $V_{\underline{\lambda}}$ is polynomial of degree $|\underline{\lambda}|$ in ranks $> 2|\underline{\lambda}|$.*

Proof. Let $p = |\lambda|$ and $q = |\lambda'|$.

Suppose $p = 0$ or $q = 0$. Then $V^{\langle p,q\rangle} = V^{p,q}$ is polynomial of degree $\leq p+q$. Since $V_{\underline{\lambda}}$ is a direct summand of $V^{p,q}$ by Lemma 4.12, it follows from Lemma 4.7 that $V_{\underline{\lambda}}$ is polynomial of degree $\leq p + q = |\underline{\lambda}|$.

Suppose $p, q \neq 0$. By Proposition 4.10, $V^{\langle p,q\rangle}$ is polynomial of degree $\leq p + q$ in ranks $> 2(p + q)$. By Lemma 4.12, $V_{\underline{\lambda}}$ is a direct summand of $V^{\langle p,q\rangle}$. Hence, by Lemma 4.7, it follows that $V_{\underline{\lambda}}$ is a polynomial VIC-modules of degree $\leq p + q$ in ranks $> 2(p + q)$. □

4.4. Irreducibility of $V_{\underline{\lambda}}$ in the stable category of VIC-modules.

In this subsection, we make a digression and observe that the VIC-module $V_{\underline{\lambda}}$ is an irreducible object in the stable category of VIC-modules, which is independently known to Powell [14].

The functor $V_{\underline{\lambda}}$ is *not* an irreducible object in the category VIC-mod of VIC-modules since any irreducible object V in VIC-mod is concentrated at one rank, i.e., there is an integer $r \geq 0$ such that we have $V(n) = 0$ for all $n \neq r$. Thus the VIC-module $V_{\underline{\lambda}}$ has infinitely many irreducible subquotients.

We here consider the *stable category of* VIC*-modules* defined by Djament and Vespa [4], which is defined as follows. (See also [19, 20] and Remark 4.16 below.) A VIC-module F is called *stably zero* if for any element $x \in F(n)$, $n \geq 0$, there is $N \geq n$ such that we have $F(i_{n,N})(x) = 0$, where $i_{n,N} : \mathbb{Z}^n \hookrightarrow \mathbb{Z}^N$ is the canonical inclusion. Let Sz denote the full subcategory of VIC-mod whose objects are stably zero VIC-module. Then Sz is a Serre subcategory of VIC-mod. The stable category of VIC-modules, St, is the quotient abelian category

$\mathsf{St} = \text{VIC-mod}/\mathsf{Sz}$. Let $\pi : \text{VIC-mod} \to \mathsf{St}$ denote the canonical functor.

Proposition 4.15 (independently known to Powell [14]). *For each bipartition $\underline{\lambda}$, the object $\pi(V_{\underline{\lambda}})$ is irreducible in* St.

Proof. Recall that the $\mathrm{GL}(n,\mathbb{Z})$-module $V_{\underline{\lambda}}(\mathbb{Z}^n)$ given by the VIC-module structure coincides with the irreducible $\mathrm{GL}(n,\mathbb{Z})$-module $V_{\underline{\lambda}}(n)$ defined in (2.0.2). If V is any VIC-submodule of $V_{\underline{\lambda}}$ such that $V \neq 0$, then there is an integer $N \geq l(\underline{\lambda})$ such that we have

$$V(\mathbb{Z}^n) = \begin{cases} 0 & (0 \leq n < N) \\ V_{\underline{\lambda}}(\mathbb{Z}^n) & (N \leq n). \end{cases}$$

Since the quotient VIC-module $V_{\underline{\lambda}}/V$ is stably zero, we have an isomorphism $V \cong V_{\underline{\lambda}}$ in the stable category St. Hence $\pi(V_{\underline{\lambda}})$ is irreducible in St. □

Remark 4.16. Proposition 4.15 could also be proved by adapting Sam and Snowden's results on $\mathrm{VIC}(\mathbb{C})$-modules [19, 20]. In [19], they proved that simple objects of the category $\mathrm{Rep}(\mathrm{GL})$ of algebraic $\mathrm{GL}_\infty(\mathbb{C})$-modules are classified by bipartitions, and in [20] they proved that $\mathrm{Rep}(\mathrm{GL})$ is equivalent to the stable category of algebraic $\mathrm{VIC}(\mathbb{C})$-modules. These results seem to imply that the $\mathrm{VIC}(\mathbb{C})$-variant of the VIC-module $V_{\underline{\lambda}}$ is a simple object in the stable category of $\mathrm{VIC}(\mathbb{C})$-modules.

Remark 4.17. In the stable category St, one can check that $\pi(V^{p,q})$ admits a composition series with composition factors of the form $\pi(V_{\underline{\lambda}})$ for bipartitions $\underline{\lambda}$ by using the exact sequence (4.2.2) and Lemma 4.12.

4.5. **Improvement of Borel's vanishing theorem.** In this subsection, we improve Borel's vanishing range for coefficients in the VIC-modules $V_{\underline{\lambda}}$ by adapting the proof of the following result of Kupers, Miller and Patzt [11].

Theorem 4.18 (Kupers–Miller–Patzt [11, Theorem 7.6]). *Let $\lambda \neq 0$ be a partition. Then we have*

$$H_p(\mathrm{GL}(n,\mathbb{Z}), V_{\lambda,0}) = 0$$

for $p < n - |\lambda|$. For trivial coefficients, we have an isomorphism

$$H_p(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q}) \cong H_p(\mathrm{GL}(n+1,\mathbb{Z}),\mathbb{Q})$$

for $p < n$.

In order to prove Theorem 4.18, Kupers, Miller and Patzt used the homology $H_p(\mathrm{GL}(n,\mathbb{Z}),\mathrm{GL}(n-1,\mathbb{Z});V(\mathbb{Z}^n),V(\mathbb{Z}^{n-1}))$ of the pair $(\mathrm{GL}(n-1,\mathbb{Z}),\mathrm{GL}(n,\mathbb{Z}))$ with coefficients in VIC-modules V defined as follows. (See [5] for details of the construction.) Let R_i be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathrm{GL}(i,\mathbb{Z})]$ for $i=n-1,n$. There is a chain map $R_{n-1}\to R_n$ induced by $\mathrm{id}_{\mathbb{Z}}$, which is unique up to chain homotopy. Then we have a chain map

$$\Phi : R_{n-1}\otimes_{\mathrm{GL}(n-1,\mathbb{Z})} V(\mathbb{Z}^{n-1}) \to R_n \otimes_{\mathrm{GL}(n,\mathbb{Z})} V(\mathbb{Z}^n)$$

induced by the chain map $R_{n-1}\to R_n$ and the structure map $V(\mathbb{Z}^{n-1})\to V(\mathbb{Z}^n)$. The homology $H_p(\mathrm{GL}(n,\mathbb{Z}),\mathrm{GL}(n-1,\mathbb{Z});V(\mathbb{Z}^n),V(\mathbb{Z}^{n-1}))$ is defined as the homology of the mapping cone $C_*(\Phi)$ of the chain map Φ. We have the following.

Theorem 4.19 (Kupers–Miller–Patzt [11, Theorem 7.3])**.** *Let V be a polynomial* VIC*-module of degree r in ranks $> m$ in the sense of* [11]*. Then we have*

$$H_p(\mathrm{GL}(n,\mathbb{Z}),\mathrm{GL}(n-1,\mathbb{Z});V(\mathbb{Z}^n),V(\mathbb{Z}^{n-1}))=0$$

for $p<n-\max\{r,m\}$. Consequently, the inclusion $\mathrm{GL}(n-1,\mathbb{Z})\to\mathrm{GL}(n,\mathbb{Z})$ *induces an isomorphism*

$$H_p(\mathrm{GL}(n-1,\mathbb{Z}),V(\mathbb{Z}^{n-1}))\xrightarrow{\cong} H_p(\mathrm{GL}(n,\mathbb{Z}),V(\mathbb{Z}^n))$$

for $p<n-1-\max\{r,m\}$.

Here we adapt Theorem 4.19 and generalize Theorem 4.18 to the VIC-module $V_{\underline{\lambda}}$. For a bipartition $\underline{\lambda}=(\lambda,\lambda')$ and a non-negative integer p, set

$$n_{\mathrm{KMP}}(\underline{\lambda},p)=\begin{cases} p+1+|\underline{\lambda}| & (\text{if } \lambda=0 \text{ or } \lambda'=0)\\ p+1+2|\underline{\lambda}| & (\text{otherwise}).\end{cases}$$

Theorem 4.20 (weaker than Theorem 1.1)**.** *Let $\underline{\lambda}\neq(0,0)$ be a bipartition. Then we have*

$$H^p(\mathrm{GL}(n,\mathbb{Z}),V_{\underline{\lambda}})=0$$

for $n\geq n_{\mathrm{KMP}}(\underline{\lambda},p)$.

Proof. Recall from Remark 4.3 that our definition of polynomiality is stronger than that in [11]. By Proposition 4.14 and Theorem 4.19, we have

$$H_{p-1}(\mathrm{GL}(n-1,\mathbb{Z}),V_{\underline{\lambda}}(\mathbb{Z}^{n-1}))\cong H_{p-1}(\mathrm{GL}(n,\mathbb{Z}),V_{\underline{\lambda}}(\mathbb{Z}^n))$$

for $p < n - |\underline{\lambda}|$ if $\lambda = 0$ or $\lambda' = 0$, and for $p < n - 2|\underline{\lambda}|$ otherwise. Considering the dual vector space, we obtain $H_{p-1}(\mathrm{GL}(n,\mathbb{Z}), V_{\underline{\lambda}}(\mathbb{Z}^n))^* \cong H^{p-1}(\mathrm{GL}(n,\mathbb{Z}), V_{\underline{\lambda}^*}(\mathbb{Z}^n))$. By Corollary 3.4, we have $H^p(\mathrm{GL}(n,\mathbb{Z}), V_{\underline{\lambda}}) = 0$ for $n \geq p + 1 + |\underline{\lambda}|$ if $\lambda = 0$ or $\lambda' = 0$, and for $n \geq p + 1 + 2|\underline{\lambda}|$ otherwise. □

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