TWISTED ALEXANDER POLYNOMIALS ON MANGUM-SHANAHAN CURVES

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Abstract. Mangum and Shanahan construct a complex curve of irreducible $\mathrm{SL}_3(\mathbb{C})$-representations of the fundamental group of a once-punctured torus bundle over the circle. In this paper, we provide an explicit formula for the twisted Alexander polynomial of a once-punctured torus bundle with tunnel number one associated with the Mangum-Shanahan representations. As a corollary, we exhibit the interesting phenomenon that the Reidemeister torsion of the complement of the figure-eight knot is constant on the curve.

1. Introduction

The twisted Alexander polynomial, originally introduced by Lin [7] for a knot in the 3-sphere S^3, and by Wada [12] for finitely presentable groups, is a generalization of the classical Alexander polynomial. For the past thirty years, it has been widely investigated, and has lots of applications in low-dimensional topology, and in particular, knot theory. For example, it is known that the twisted Alexander polynomials associated with representations onto finite groups detect the Thurston norm and fiberedness of 3-dimensional manifolds. However, not much is known about the explicit formula for the twisted Alexander polynomial except in the cases of 2-dimensional representations and the adjoint representation. We refer to the survey papers [2], [9] for other properties and applications.

In [8], Mangum and Shanahan construct a complex curve ρ_τ ($\tau \in \mathbb{C} \setminus \{0\}$) of $\mathrm{SL}_3(\mathbb{C})$-representations of the fundamental group of a once-punctured torus bundle over the circle. It is irreducible for all but finitely many values of the parameter τ. In particular, they show that these representations are different from those obtained by composing representations in $\mathrm{SL}_2(\mathbb{C})$ with the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SL}_3(\mathbb{C})$. Moreover, infinitely many of these representations are conjugate to $\mathrm{SU}(3)$-representations. They also provide an explicit computation of the curve in the case that the bundle is the figure-eight knot complement in S^3, and show that for infinitely many Dehn surgeries on the knot, there is a representation from this curve that

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descends to a representation of the fundamental group of the surgered 3-manifold.

The purpose of this paper is to give an explicit formula for the twisted Alexander polynomial of a once-punctured torus bundle with tunnel number one associated with the Mangum-Shanahan representations ρ_τ. Up to homeomorphism, the once-punctured torus bundles with tunnel number one form an infinite family of 3-manifolds, denoted by $\{M_n\}_{n\in\mathbb{Z}}$, which are finite volume hyperbolic 3-manifolds with exactly one cusp when $|n| > 2$. Moreover, their fundamental groups admit presentations $\pi_1(M_n) = \langle \alpha, \beta \,|\, \beta^n = \omega \rangle$ where ω is a simple word in the generators α, β independent of n, which enables us to calculate the twisted Alexander polynomial explicitly.

Now, our main theorem of this paper is the following. Hereafter, for technical reasons, we assume $\tau^3 + 1 \neq 0$.

Theorem 1.1. *The twisted Alexander polynomial $\Delta_{M_n,\rho_\tau}(t)$ of a once-punctured torus bundle M_n with tunnel number one associated with the Mangum-Shanahan representation ρ_τ is provided by the following formula:*

(1) *If n is odd, then $\Delta_{M_n,\rho_\tau}(t) = 1 + \zeta_n(\tau)t - \zeta_n(\tau^{-1})t^2 - t^3$, where*

$$\zeta_n(\tau) = \frac{2\tau^{3n} - (n+1)\tau^3 + (n-1)\tau^{-3}}{\tau^3 + \tau^{-3} + 2}\tau^{-n}.$$

(2) *If n is even, then $\Delta_{M_n,\rho_\tau}(t) = 1 + \eta_n(\tau)t - \eta_n(\tau^{-1})t^2 - t^3$, where*

$$\eta_n(\tau) = \frac{2\tau^{3n} + (n+1)\tau^3 - (n-1)\tau^{-3}}{\tau^3 + \tau^{-3} + 2}\tau^{-n}.$$

It can be shown that $\zeta_n(\tau)$ and $\eta_n(\tau)$ are actually Laurent polynomials in τ (see Proposition 3.3). We remark that, in the case of $n = -3$ (resp. $n = -1$), which corresponds to the complement of the figure-eight knot (resp. trefoil knot) in S^3, the above formula coincides with one obtained by the first named author in [10, Appendix A]. Baker and Petersen [1, Theorem 11.1] provide an explicit formula for the twisted Alexander polynomial of M_n associated with $\mathrm{SL}_2(\mathbb{C})$-representations. Further, the second named author and Yamaguchi [11] calculate the adjoint Reidemeister torsion of M_n and investigate its vanishing identity.

As for the acyclicity of the Mangum-Shanahan representations, that is, the vanishing or non-vanishing of the twisted homology group $H_*(M_n, \rho_\tau)$, we can show the following corollaries:

Corollary 1.2. *Suppose*

$$\begin{cases} (\tau^n + 1)(\tau^n - \tau^{3-2n}) \neq 0, & \text{if } n \text{ is odd, and} \\ (\tau^n - 1)(\tau^n + \tau^{3-2n}) \neq 0, & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

Then, the Mangum-Shanahan representation $\rho_\tau : \pi_1(M_n) \to \mathrm{SL}_3(\mathbb{C})$ is acyclic if and only if $\zeta_n(\tau) \neq \zeta_n(\tau^{-1})$ if n is odd, $\eta_n(\tau) \neq \eta_n(\tau^{-1})$

if n is even. Under these conditions, the Reidemeister torsion of M_n associated with ρ_τ is given by

$$\operatorname{tor}(M_n, \rho_\tau) = \begin{cases} \zeta_n(\tau) - \zeta_n(\tau^{-1}), & \text{if } n \text{ is odd,} \\ \eta_n(\tau) - \eta_n(\tau^{-1}), & \text{if } n \text{ is even.} \end{cases}$$

Of course, with only a few exceptions, a generic point of the Mangum-Shanahan curve ρ_τ is an acyclic representation, namely $H_*(M_n, \rho_\tau) = 0$.

Corollary 1.3. *Under the same assumption as in Corollary 1.2, the following two conditions on M_n are equivalent:*

(1) $\rho_\tau : \pi_1(M_n) \to \mathrm{SL}_3(\mathbb{C})$ *is non-acyclic for any τ satisfying (1.1).*
(2) $n = 0, \pm 1, \pm 3$.

In particular, by definition, $\operatorname{tor}(M_n, \rho_\tau) = 0$ *for* $n = 0, \pm 1, \pm 3$.

As mentioned above, the figure-eight knot complement corresponds to $n = -3$, and then a result of Kitano (see [4]) implies that the Reidemeister torsion of the figure-eight knot for $\mathrm{SL}_2(\mathbb{C})$-representations has continuous variations on the one-dimensional character variety. On the other hand, by Corollary 1.3, the Reidemeister torsion $\operatorname{tor}(M_{-3}, \rho_\tau)$ is constant on the Mangum-Shanahan curve ρ_τ.

This paper is organized as follows. In Section 2, we review the definition of twisted Alexander polynomial, presentation of the fundamental group of a once-punctured torus bundle with tunnel number one, and the construction of the Mangum-Shanahan representation. In Section 3, we give proof of Theorem 1.1, and Corollaries 1.2 and 1.3.

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2. Preliminaries

2.1. **Twisted Alexander polynomials.**

Wada introduced the twisted Alexander polynomial for finitely presentable groups (see [12]). We review its definition for 3-dimensional representations.

Let G be a finitely presentable group with an epimorphism $\varphi : G \to \mathbb{Z} = \langle t \rangle$. We assume that G admits a deficiency one presentation: $G = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle$.

For a given representation $\rho : G \to \mathrm{SL}_3(\mathbb{C})$, we extend the group homomorphism $\varphi \otimes \rho : G \to \mathrm{GL}(3, \mathbb{C}[t^{\pm 1}])$, which is defined by $(\varphi \otimes \rho)(x) = \varphi(x)\rho(x)$ for $x \in G$, to a ring homomorphism $\mathbb{Z}[G] \to M(3, \mathbb{C}[t^{\pm 1}])$, where the target is the matrix algebra over $\mathbb{C}[t^{\pm 1}]$. Let F_n denote the free group $\langle x_1, \ldots, x_n \rangle$ and $\Phi : \mathbb{Z}[F_n] \to M(3, \mathbb{C}[t^{\pm 1}])$ the composition of the surjection $\mathbb{Z}[F_n] \to \mathbb{Z}[G]$ induced by the presentation of G and the ring homomorphism $\mathbb{Z}[G] \to M(3, \mathbb{C}[t^{\pm 1}])$.

Let $Q = (q_{ij})$ be the $3(n-1) \times 3n$ matrix where $q_{ij} = \Phi(\frac{\partial r_i}{\partial x_j})$ and $\frac{\partial}{\partial x_j}$ denotes the free differential by x_j. For $1 \le k \le n$ we denote by Q_k the $3(n-1) \times 3(n-1)$ matrix obtained from Q by removing the kth column. Then the twisted Alexander polynomial $\Delta_{G,\rho}(t)$ associated with $\rho : G \to \mathrm{SL}_3(\mathbb{C})$ is defined by

$$\Delta_{G,\rho}(t) = \frac{\det Q_k}{\det \Phi(x_k - 1)},$$

which is well-defined up to multiplication by $\pm t^i$ ($i \in \mathbb{Z}$). If the group G is the fundamental group of a 3-manifold M, we simply denote $\Delta_{G,\rho}(t) = \Delta_{\pi_1(M),\rho}(t)$ by $\Delta_{M,\rho}(t)$.

2.2. **Once-punctured torus bundles with tunnel number one.** We use the same notations as in the paper [1]. The once-punctured torus bundles over the circle with tunnel number one, up to mirroring, form a one-parameter family $\{M_n\}_{n\in\mathbb{Z}}$. The monodromy ϕ of $M_n = T \times [0,1]/(x,0) \sim (\phi(x),1)$ may be presented as $\phi = D_c D_b^{n+2}$ where b and c are curves on the once-punctured torus fiber T transversally intersecting once and D_a is a right-handed Dehn twist along a curve a.

It is known that the manifold M_n is hyperbolic if and only if $|n| > 2$, contains an essential torus if and only if $|n| = 2$, and is a Seifert fibered space if and only if $|n| \le 1$. It is also known that each manifold M_n may be obtained by $-(n+2)$-Dehn filling of one component of the Whitehead link and is the complement of a certain genus one fibered knot in the lens space $L(n+2,1)$.

Example 2.1. For small n, M_{-1} is the positive trefoil knot complement, M_{-3} is the figure-eight knot complement, and M_3 is the figure-eight sister manifold. The manifold M_{-2} exhibits different behavior than the other manifolds in the family. In fact, as the monodromy of M_{-2} is a single Dehn twist $\phi = D_c$, this curve sweeps out an essential non-separating torus.

Let β and γ be oriented loops on T based at a point $* \in \partial T$ so that β, γ are freely homotopic to b, c respectively. Then, we have $\pi_1(T,*) = \langle \beta, \gamma \rangle \cong F_2$, the free group of rank two, and

$$\begin{aligned}
\pi_1(M_n) &= \langle \beta, \gamma, \mu \,|\, \mu\beta\mu^{-1} = \phi_*(\beta),\ \mu\gamma\mu^{-1} = \phi_*(\gamma) \rangle \\
&= \langle \beta, \gamma, \mu \,|\, \mu\beta\mu^{-1} = \beta\gamma,\ \mu\gamma\mu^{-1} = \gamma(\beta\gamma)^{-(n+2)} \rangle \quad (2.1) \\
&= \langle \alpha, \beta \,|\, \beta^{-n} = \alpha^{-1}\beta\alpha^2\beta\alpha^{-1} \rangle \text{ where } \mu = \beta\alpha. \quad (2.2)
\end{aligned}$$

Here, the abelianization homomorphism $\varphi : \pi_1(M_n) \to \mathbb{Z} = \langle t \rangle$ is given by $\varphi(\beta) = \varphi(\gamma) = 1$ and $\varphi(\mu) = t$. We will use the presentation (2.2) in Section 3 to calculate the twisted Alexander polynomial of M_n.

2.3. **Mangum-Shanahan representations.** In this subsection, we recall the definition of Mangum-Shanahan representations for a once-punctured torus bundle M_n of tunnel number one with monodromy ϕ (see [8] for details). It is a one complex parameter family of representations, denoted by $\rho_\tau : \pi_1(M_n) \to \mathrm{SL}_3(\mathbb{C})$, and is defined to be the composition of two homomorphisms

$$\iota : \pi_1(M_n) \to B_4/Z_4 \quad \text{and} \quad \mathfrak{a} : B_4/Z_4 \to \mathrm{SL}_3(\mathbb{Z}[\tau^{\pm 1}]),$$

where

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, [\sigma_1, \sigma_3] = 1\rangle$$

is the four strand braid group and $Z_4 = \langle (\sigma_1\sigma_2\sigma_3)^4 \rangle$ is its center. The first map is defined by

$$\iota(\beta) = [\sigma_1\sigma_3^{-1}], \ \iota(\gamma) = [\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}], \text{ and } \iota(\mu) = \overline{h}^{-1}(\phi_*),$$

which is induced by an embedding of F_2 into B_4. Here, $\overline{h} : B_4/Z_4 \to \mathrm{Aut}^+(F_2)$ is an isomorphism induced by the conjugate action of B_4 on F_2, namely $h : B_4 \to \mathrm{Aut}(F_2)$, and $\mathrm{Aut}^+(F_2)$ denotes an index two subgroup of $\mathrm{Aut}(F_2)$, which is the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under a natural surjective homomorphism from $\mathrm{Aut}(F_2)$ to $\mathrm{GL}_2(\mathbb{Z})$. The second map $\mathfrak{a}$, which is induced by the reduced Burau representation of B_4, maps each generator of B_4 as follows:

$$\sigma_1 \mapsto \begin{pmatrix} \tau^2 & -\tau^{-1} & 0 \\ 0 & -\tau^{-1} & 0 \\ 0 & 0 & -\tau^{-1} \end{pmatrix}, \ \sigma_2 \mapsto \begin{pmatrix} -\tau^{-1} & 0 & 0 \\ -\tau^2 & \tau^2 & -\tau^{-1} \\ 0 & 0 & -\tau^{-1} \end{pmatrix},$$

$$\text{and} \quad \sigma_3 \mapsto \begin{pmatrix} -\tau^{-1} & 0 & 0 \\ 0 & -\tau^{-1} & 0 \\ 0 & -\tau^2 & \tau^2 \end{pmatrix}.$$

Here, we note that $\mathfrak{a}((\sigma_1\sigma_2\sigma_3)^4) = I$ holds, where I denotes the identity matrix of degree three. Then, the map $\mathfrak{a}$ reduces over the quotient B_4/Z_4. A direct computation shows that

(2.3)
$$\rho_\tau(\beta) = \begin{pmatrix} -\tau^3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -\tau^{-3} \end{pmatrix} \text{ and } \rho_\tau(\gamma) = \begin{pmatrix} 1-\tau^3 & -\tau^{-3} & \tau^{-3} \\ 1-\tau^6 & -\tau^{-3} & 0 \\ 1 & -\tau^{-3} & 0 \end{pmatrix}.$$

In this paper, we call this one complex parameter family of representations $\rho_\tau : \pi_1(M_n) \to \mathrm{SL}_3(\mathbb{C})$ the *Mangum-Shanahan curve* of a once-punctured torus bundle of tunnel number one.

Remark 2.2. (1) If τ is not equal to 0 and a zero of any of the polynomials $1+\tau, 1-\tau, 1+\tau^2, 1+\tau+\tau^2, 1-\tau+\tau^2, 1-\tau^2+\tau^4$, and $1-\tau^3+\tau^6$, the representation ρ_τ is *irreducible* (see (2.1), (2.2) in [8]).

(2) Moreover, ρ_τ is different from the curve of representations one gets by composing a curve of $\mathrm{SL}_2(\mathbb{C})$-representations with the unique irreducible representation $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$.

Now, the next lemma is the key point of this paper.

Lemma 2.3. $\overline{h}^{-1}(\phi_*) = [\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{n+2}]$.

Proof. Let us recall the identification $\pi_1(T, *) = \langle \beta, \gamma \rangle \cong F_2$, and then we can identify β (resp. γ) with $\sigma_1\sigma_3^{-1}$ (resp. $\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$) via the embedding $F_2 \hookrightarrow B_4$. We set $S = \overline{h}(\sigma_2\sigma_1\sigma_3\sigma_1^{-1}\sigma_2^{-1})$ and $R = \overline{h}(\sigma_2\sigma_1\sigma_3)$ as in [8, Remark 4]. Then, we see that the conjugate actions of S and R on F_2 are provided by

$$S(\beta) = \beta\gamma,\ S(\gamma) = \gamma,\ R(\beta) = \gamma,\ R(\gamma) = \beta^{-1}.$$

A straightforward calculation shows that for the word $\nu = RS^{-(n+2)}R^{-1}S^{-1}$, we have

$$\begin{aligned} \nu \circ \phi_*(\beta) &= RS^{-(n+2)}R^{-1}S^{-1}(\beta\gamma) = RS^{-(n+2)}R^{-1}(\beta\gamma^{-1}\gamma) \\ &= RS^{-(n+2)}(\gamma^{-1}) = R(\gamma^{-1}) = \beta. \end{aligned}$$

Similarly, we can obtain $\nu \circ \phi_*(\gamma) = \gamma$. Namely, $\nu \circ \phi_*$ is the identity element in $\mathrm{Aut}^+(F_2)$. Hence, we have

$$\overline{h}^{-1}(\phi_*) = \overline{h}^{-1}(\nu^{-1}) = [SRS^{(n+2)}R^{-1}] = [\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{(n+2)}].$$

This completes the proof of Lemma 2.3. □

3. Proof

In this section, we give proof of Theorem 1.1, and Corollaries 1.2 and 1.3. Let us recall, as stated in the introduction, that we assumed $\tau^3 + 1 \neq 0$.

3.1. **Proof of Theorem 1.1.** We first compute the image of the generator μ in (2.1) under the representation ρ_τ. Since the eigenvalues of $\mathfrak{a}(\sigma_1)$ are $-\tau^{-1}$ and τ^2, using the matrix P below, we obtain the kth-power of $\mathfrak{a}(\sigma_1)$:

(3.1)

$$P = \begin{pmatrix} 0 & \frac{1}{1+\tau^3} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathfrak{a}(\sigma_1^k) = P \begin{pmatrix} (-\tau)^{-k} & 0 & 0 \\ 0 & (-\tau)^{-k} & 0 \\ 0 & 0 & \tau^{2k} \end{pmatrix} P^{-1}.$$

Using Lemma 2.3 and (3.1), a computer-aided calculation provides that

$$\begin{aligned} \rho_\tau(\mu) &= \mathfrak{a} \circ \iota(\mu) = \mathfrak{a} \circ \overline{h}^{-1}(\phi_*) = \mathfrak{a}\big([\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{n+2}]\big) \\ &= \begin{pmatrix} -\tau^{3+2n} & \frac{-(-\tau)^{-n}+\tau^{6+2n}}{\tau^3+\tau^6} & 0 \\ -\tau^{6+2n} & \frac{-(-\tau)^{-n}+\tau^{6+2n}}{1+\tau^3} & (-\tau)^{-n} \\ -\tau^{6+2n} & \frac{(-\tau)^{-n}+\tau^{9+2n}}{\tau^3+\tau^6} & (-\tau)^{-n}(1-\tau^{-3}) \end{pmatrix}. \end{aligned}$$

A similar calculation shows us that ρ_τ maps the elements β^k and β^{-k} to

$$\beta^k \mapsto \begin{pmatrix} (-\tau)^{3k} & \frac{1-(-\tau)^{3k}}{1+\tau^3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1+(-\tau)^{-3k}}{1+\tau^{-3}} & (-\frac{1}{\tau})^{3k} \end{pmatrix},\ \beta^{-k} \mapsto \begin{pmatrix} (-\frac{1}{\tau})^{3k} & \frac{1-(-\tau)^{-3k}}{1+\tau^3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-(-\tau)^{3k}}{1+\tau^{-3}} & (-\tau)^{3k} \end{pmatrix}.$$

Moreover, we have

$$\rho_\tau(\alpha) = \rho_\tau(\beta^{-1}\mu)$$
$$= \begin{cases} \begin{pmatrix} -\tau^{2n}(\tau^3-1) & \frac{\tau^{-n-6}(\tau^3-1)(\tau^{3n+6}+1)}{\tau^3+1} & -\tau^{-n-3} \\ -\tau^{2n+6} & \frac{\tau^{-n}+\tau^{2n+6}}{\tau^3+1} & -\tau^{-n} \\ 0 & \tau^{-n} & -\tau^{-n} \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \\ \begin{pmatrix} -\tau^{2n}(\tau^3-1) & \frac{\tau^{-n-6}(\tau^3-1)(\tau^{3n+6}-1)}{\tau^3+1} & \tau^{-n-3} \\ -\tau^{2n+6} & \frac{-\tau^{-n}+\tau^{2n+6}}{\tau^3+1} & \tau^{-n} \\ 0 & -\tau^{-n} & \tau^{-n} \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$$

Let $r = \beta^{-n} - \alpha^{-1}\beta\alpha^2\beta\alpha^{-1}$ in the presentation (2.2) and assume $n > 0$ (the proof for non-positive n is similar). We recall that the image of the generators under the abelianization $\varphi : \pi_1(M_n) \to \mathbb{Z} = \langle t \rangle$ are $\varphi(\alpha) = t$ and $\varphi(\beta) = 1$. By the free differential calculus, we obtain

$$\begin{aligned} \frac{\partial r}{\partial \beta} &= \big(1 + \beta^{-1} + \cdots + (\beta^{-1})^{(n-1)}\big)\frac{\partial \beta^{-1}}{\partial \beta} - \alpha^{-1} - \alpha^{-1}\beta\alpha^2 \\ &= -\big(1 + \beta^{-1} + \cdots + (\beta^{-1})^{(n-1)}\big)\beta^{-1} - \alpha^{-1} - \alpha^{-1}\beta\alpha^2. \end{aligned}$$

For simplicity, we set $\rho_\tau(\alpha) = A$ and $\rho_\tau(\beta) = B$. Then, by definition, we obtain

$$\Phi\left(\frac{\partial r}{\partial \beta}\right) = -\big(I + B^{-1} + \cdots + (B^{-1})^{n-1}\big)B^{-1} - t^{-1}A^{-1} - tA^{-1}BA^2.$$

Lemma 3.1. *The matrix $I + B^{-1} + \cdots + \left(B^{-1}\right)^{n-1}$ is given by*

$$\sum_{k=0}^{n-1}\left(B^{-1}\right)^k = \begin{cases} \begin{pmatrix} \frac{1+\tau^{-3n}}{1+\tau^{-3}} & \frac{-(1+\tau^{-3n})\tau^3+n(1+\tau^3)}{(1+\tau^3)^2} & 0 \\ 0 & n & 0 \\ 0 & \frac{\tau^3(n-1-\tau^{3n}+n\tau^3)}{(1+\tau^3)^2} & \frac{1+\tau^{3n}}{1+\tau^3} \end{pmatrix}, & \textit{if } n \textit{ is odd,} \\ \\ \begin{pmatrix} \frac{1-\tau^{-3n}}{1+\tau^{-3}} & \frac{-(1-\tau^{-3n})\tau^3+n(1+\tau^3)}{(1+\tau^3)^2} & 0 \\ 0 & n & 0 \\ 0 & \frac{\tau^3(n-1+\tau^{3n}+n\tau^3)}{(1+\tau^3)^2} & \frac{1-\tau^{3n}}{1+\tau^3} \end{pmatrix}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. When n is odd, the $(1,1)$-entry of the matrix is given by

$$1 + \left(-\frac{1}{\tau^3}\right) + \cdots + \left(-\frac{1}{\tau^3}\right)^{n-1} = \frac{1\cdot(1-(-\frac{1}{\tau^3})^n)}{1-(-\frac{1}{\tau^3})} = \frac{1+\tau^{-3n}}{1+\tau^{-3}}.$$

Other entries can be calculated by the same way. The proof for the even case is similar. □

Now, let us calculate the denominator of the twisted Alexander polynomial:

$$\det\Phi(\alpha-1)=\det(tA-I)$$

$$(3.2)\qquad =\begin{cases} t^3-\dfrac{-\tau^n+\tau^{3-2n}}{\tau^3+1}t^2+\dfrac{\tau^{2n}-\tau^{3-n}}{\tau^3+1}t-1, & \text{if } n \text{ is odd,} \\[2ex] t^3-\dfrac{\tau^n+\tau^{3-2n}}{\tau^3+1}t^2+\dfrac{\tau^{2n}+\tau^{3-n}}{\tau^3+1}t-1, & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, the numerator of the twisted Alexander polynomial is given by the following: When n is odd, we have

(3.3)

$$\det\Phi\left(\frac{\partial r}{\partial\beta}\right)\doteq -t^6+\lambda_5(\tau)t^5-\lambda_4(\tau)t^4+\lambda_3(\tau)t^3-\lambda_2(\tau)t^2+\lambda_1(\tau)t-1$$

where the coefficients are

$$\lambda_5(\tau)=\frac{1}{(\tau^3+1)^2}\left(-\left((n-1)\tau^6+\tau^3-n\right)\tau^n+\left(\tau^3-1\right)\tau^{3-2n}\right),$$

$$\begin{aligned}\lambda_4(\tau)=&\frac{1}{(\tau^3+1)^2}\left(\left((n+1)\tau^6-n+1\right)\tau^{-n}-2\tau^{3+2n}\right)\\&+\frac{\tau^n-\tau^{3-2n}}{(\tau^3+1)^3}\left(\left((n-1)\tau^6-n-1\right)\tau^n+2\tau^{3-2n}\right)+\frac{\tau^{2n}-\tau^{3-n}}{\tau^3+1},\end{aligned}$$

$$\begin{aligned}\lambda_3(\tau)=&2-\frac{\tau^n-\tau^{3-2n}}{(\tau^3+1)^3}\left(\left((n+1)\tau^6-n+1\right)\tau^{-n}-2\tau^{3+2n}\right)\\&-\frac{\tau^{2n}-\tau^{3-n}}{(\tau^3+1)^3}\left(\left((n-1)\tau^6-n-1\right)\tau^n+2\tau^{3-2n}\right),\end{aligned}$$

$$\begin{aligned}\lambda_2(\tau)=&\frac{\tau^{2n}-\tau^{3-n}}{(\tau^3+1)^3}\left(\left((n+1)\tau^6-n+1\right)\tau^{-n}-2\tau^{3+2n}\right)\\&-\frac{1}{(\tau^3+1)^2}\left(\left((n-1)\tau^6-n-1\right)\tau^n+2\tau^{3-2n}\right)-\frac{\tau^n-\tau^{3-2n}}{\tau^3+1},\end{aligned}$$

$$\lambda_1(\tau)=\frac{1}{(\tau^3+1)^2}\left(\left(n\tau^6-\tau^3-n+1\right)\tau^{-n}+\left(\tau^{-3}-1\right)\tau^{3+2n}\right).$$

Similarly, when n is even, we have

$$(3.4)\quad \det\Phi\left(\frac{\partial r}{\partial\beta}\right)\doteq -t^6+\delta_5(\tau)t^5-\delta_4(\tau)t^4+\delta_3(\tau)t^3-\delta_2(\tau)t^2+\delta_1(\tau)t-1$$

where the coefficients are

$$\delta_5(\tau)=\frac{1}{(\tau^3+1)^2}\left(\left((n-1)\tau^6+\tau^3-n\right)\tau^n+\left(\tau^3-1\right)\tau^{3-2n}\right),$$

$$\delta_4(\tau) = \frac{1}{(\tau^3+1)^2}\left(-\left((n+1)\tau^6 - n + 1\right)\tau^{-n} - 2\tau^{3+2n}\right)$$
$$+ \frac{\tau^n + \tau^{3-2n}}{(\tau^3+1)^3}\left(\left((n-1)\tau^6 - n - 1\right)\tau^n - 2\tau^{3-2n}\right) + \frac{\tau^{2n} + \tau^{3-n}}{\tau^3+1},$$
$$\delta_3(\tau) = 2 - \frac{\tau^n + \tau^{3-2n}}{(\tau^3+1)^3}\left(\left((n+1)\tau^6 - n + 1\right)\tau^{-n} + 2\tau^{3+2n}\right)$$
$$+ \frac{\tau^{2n} + \tau^{3-n}}{(\tau^3+1)^3}\left(\left((n-1)\tau^6 - n - 1\right)\tau^n - 2\tau^{3-2n}\right),$$
$$\delta_2(\tau) = \frac{\tau^{2n} + \tau^{3-n}}{(\tau^3+1)^3}\left(-\left((n+1)\tau^6 - n + 1\right)\tau^{-n} - 2\tau^{3+2n}\right)$$
$$+ \frac{1}{(\tau^3+1)^2}\left(\left((n-1)\tau^6 - n - 1\right)\tau^n - 2\tau^{3-2n}\right) + \frac{\tau^n + \tau^{3-2n}}{\tau^3+1},$$
$$\delta_1(\tau) = \frac{1}{(\tau^3+1)^2}\left(-\left(n\tau^6 - \tau^3 - n + 1\right)\tau^{-n} + \left(\tau^{-3} - 1\right)\tau^{3+2n}\right).$$

Finally, a computer-aided calculation using (3.2), (3.3), and (3.4) gives us the formulas in Theorem 1.1. This completes the proof.

3.2. **Proof of Corollary 1.2.** The assumption (1.1) of the corollary guarantees that the denominator of the twisted Alxeander polynomial $\det(tA - I)$ evaluated at $t = 1$ is non-zero. Accordingly, the Mangum-Shanahan representation $\rho_\tau : \pi_1(M_n) \to \mathrm{SL}_3(\mathbb{C})$ is acyclic if and only if $\Delta_{M_n,\rho_\tau}(1) \neq 0$ (see [5], [3]). Under these conditions, the Reidemeister torsion $\mathrm{tor}(M_n, \rho_\tau)$ coincides with $\Delta_{M_n,\rho_\tau}(1)$. If ρ_τ is non-acyclic, then $\mathrm{tor}(M_n, \rho_\tau)$ is defined to be zero. Hence, the assertion follows immediately from the formulas in Theorem 1.1.

3.3. **Proof of Corollary 1.3.** Assuming the condition (1.1), we obtain $\det(A - I) \neq 0$ as in the proof of Corollary 1.2.

(2) $\Rightarrow$ (1): Note that $\eta_0(\tau) = 1$. When $n = 0$, we obtain $\eta_n(\tau) = 1 = \eta_n(\tau^{-1})$. Thus, ρ_τ is non-acyclic for any τ satisfying (1.1) by Corollary 1.2.

Note that $\zeta_1(\tau) = 0 = \zeta_{-1}(\tau)$ and $\zeta_3(\tau) = \frac{2\tau^6 - 4 + 2\tau^{-6}}{\tau^3 + \tau^{-3} + 2} = \zeta_{-3}(\tau)$. When $n = \pm 1$ or $n = \pm 3$, we obtain $\zeta_n(\tau) = \zeta_n(\tau^{-1})$. Thus ρ_τ is non-acyclic for any τ satisfying (1.1) by Corollary 1.2.

(1) $\Rightarrow$ (2): We first show the even case. Let us assume $\eta_n(\tau) = \eta_n(\tau^{-1})$ holds for any τ satisfying (1.1). Namely, we assume the following identity on τ:

$$2\tau^{2n} + (n+1)\tau^{3-n} - (n-1)\tau^{-3-n} \equiv 2\tau^{-2n} + (n+1)\tau^{-3+n} - (n-1)\tau^{3+n}.$$

If we can simplify the left-hand side, at least one of the three equations on the exponent will hold: $2n = 3 - n$, $2n = -3 - n$, and $3 - n = -3 - n$. However, they do not hold because n is even. Next, if the first terms on both sides coincide, $2n = -2n$ must hold. Thus, we obtain $n = 0$.

Then, the above equality holds true. Other combinations also produce $n = 0$. Hence, we have (2) in the even case.

Now, let us consider the odd case. Let us assume $\zeta_n(\tau) = \zeta_n(\tau^{-1})$ holds for any τ satisfying (1.1), namely,

$$2\tau^{2n} + (n-1)\tau^{-3-n} - (n+1)\tau^{3-n} \equiv 2\tau^{-2n} + (n-1)\tau^{3+n} - (n+1)\tau^{-3+n}.$$

If we can simplify the left-hand side, at least one of the three equations on the exponent will hold: $2n = -3-n, 2n = 3-n$, and $-3-n = 3-n$. In this case, we obtain $n = \pm 1$, and the above equality holds true. Next, if the first term on the left-hand side coincides with the second or third term of the right-hand side, then $2n = 3 + n$ or $2n = -3 + n$ must hold. Thus, we obtain $n = \pm 3$, and the above equality holds true. Other combinations also produce $n = \pm 3$. Hence, we have (2) in the odd case.

This completes the proof of Corollary 1.3.

Remark 3.2. Although the coefficients of $\Delta_{M_n,\rho_\tau}(t)$ are Laurent polynomials in τ on the Mangum-Shanahan curve ρ_τ (see Proposition 3.3 below), we see that $\Delta_{M_1,\rho_\tau}(t) = 1-t^3 = \Delta_{M_{-1},\rho_\tau}(t)$ are constant on the curve ρ_τ. This kind of property also holds for the twisted Alexander polynomial of a torus knot for irreducible $\mathrm{SL}_2(\mathbb{C})$-representations (see [6]).

Proposition 3.3. *For $k \in \mathbb{Z}$ we let*

$$f_{2k+1}(a) = \frac{2a^{2k+1} - (2k+2)a + (2k)a^{-1}}{a + a^{-1} + 2},$$
$$g_{2k}(a) = \frac{2a^{2k} + (2k+1)a - (2k-1)a^{-1}}{a + a^{-1} + 2}.$$

Then $f_{2k+1}(a)$ and $g_{2k}(a)$ are in $\mathbb{Z}[a^{\pm 1}]$.

Proof. We prove that $f_{2k+1}(a) \in \mathbb{Z}[a^{\pm 1}]$. Note that $f_1(a) = 0 = f_{-1}(a)$. Moreover

$$f_{-2k-1}(a) = \frac{2a^{-2k-1} + (2k)a - (2k+2)a^{-1}}{a + a^{-1} + 2} = f_{2k+1}(a^{-1}).$$

Hence it suffices to show that $f_{2k+1}(a) \in \mathbb{Z}[a]$ for $k \geq 1$.

For $k \geq 1$, we have

$$\begin{aligned} f_{2k+1}(a) - f_{2k-1}(a) &= \frac{(a^{2k+1} - a^{2k-1}) - (a - a^{-1})}{a + a^{-1} + 2} \\ &= \frac{(a^{2k} - 1)(a - a^{-1})}{a + a^{-1} + 2} \\ &= \frac{(a^{2k} - 1)(a - 1)}{a + 1} \\ &= (a-1)^2 \frac{a^{2k} - 1}{a^2 - 1}. \end{aligned}$$

This implies that $f_{2k+1}(a) - f_1(a) = (a-1)^2 \sum_{i=1}^{k} \frac{a^{2i}-1}{a^2-1}$. Hence

$$\begin{aligned} f_{2k+1}(a) &= f_1(a) + (a-1)^2 \sum_{i=0}^{k} (a^{2i-2} + a^{2i-4} + \cdots + a^2 + 1) \\ &= (a-1)^2 \sum_{i=0}^{k-1} (k-i) a^{2i} \\ &\in \mathbb{Z}[a]. \end{aligned}$$

The proof of $g_{2k}(a) \in \mathbb{Z}[a^{\pm 1}]$ is similar. □

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