THE FIRST HOMOLOGY GROUP WITH TWISTED COEFFICIENTS FOR THE MAPPING CLASS GROUP OF A NON–ORIENTABLE SURFACE WITH BOUNDARY

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ABSTRACT. We determine the first homology group with coefficients in $H_1(N;\mathbb{Z})$ for various mapping class groups of a non–orientable surface N with punctures and/or boundary.

1. Introduction

Let $N_{g,s}^n$ be a smooth, non–orientable, compact surface of genus g with s boundary components and n punctures. If s and/or n is zero, then we omit it from the notation. If we do not want to emphasise the numbers g, s, n, we simply write N for a surface $N_{g,s}^n$. Recall that N_g is a connected sum of g projective planes and $N_{g,s}^n$ is obtained from N_g by removing s open discs and specifying a set $\Sigma = \{z_1, \ldots, z_n\}$ of n distinguished points in the interior of N.

Let $\text{Diff}(N)$ be the group of all diffeomorphisms $h\colon N \to N$ such that h is the identity on each boundary component and $h(\Sigma) = \Sigma$. By $\mathcal{M}(N)$ we denote the quotient group of $\text{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of N.

The mapping class group $\mathcal{M}(S_{g,s}^n)$ of an orientable surface is defined analogously, but we consider only orientation preserving maps.

For any $0 \leq k \leq n$, let $\mathcal{PM}^k(N)$ be the subgroup of $\mathcal{M}(N)$ consisting of elements which fix Σ pointwise and preserve a local orientation around the punctures $\{z_1, \ldots, z_k\}$. For $k = 0$, we obtain so–called *pure mapping class group* $\mathcal{PM}(N)$, and for $k = n$ we get the group $\mathcal{PM}^+(N)$ consisting of maps that preserve local orientation around all the punctures.

1.1. **Background.** Homological computations play a prominent role in the theory of mapping class groups. In the orientable case, Mumford [14] observed that $H_1(\mathcal{M}(S_g))$ is a quotient of $\mathbb{Z}_{10}$. Then Birman [1, 2] showed that if $g \geq 3$, then $H_1(\mathcal{M}(S_g))$ is a quotient of $\mathbb{Z}_2$, and Powell [17] showed that in fact $H_1(\mathcal{M}(S_g))$ is trivial if $g \geq 3$. As for higher homology groups,

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Harer [4, 5] computed $H_i(\mathcal{M}(S_g))$ for $i = 2, 3$ and Madsen and Weiss [10] determined the rational cohomology ring of the stable mapping class group.

In the non–orientable case, Korkmaz [7, 8] computed $H_1(\mathcal{M}(N_g))$ for a closed surface N_g (possibly with marked points). This computation was later [22] extended to the case of a surface with boundary. As for higher homology groups, Wahl [27] identified the stable rational cohomology of $\mathcal{M}(N)$ and Randal-Williams [18] (among other results) extended this identification to $\mathbb{Z}_2$ coefficients.

As for twisted coefficients, Morita in a series of papers [11–13] obtained several fundamental results, in particual he proved that

$$\begin{aligned} H_1(\mathcal{M}(S_g); H_1(S_g;\mathbb{Z})) &\cong \mathbb{Z}_{2g-2}, \quad \text{for } g \geq 2,\\ H^1(\mathcal{M}(S_g); H^1(S_g;\mathbb{Z})) &\cong 0, \quad \text{for } g \geq 1,\\ H^1(\mathcal{M}(S_g^1); H^1(S_g^1;\mathbb{Z})) &\cong \mathbb{Z}, \quad \text{for } g \geq 2,\\ H^1(\mathcal{M}(S_{g,1}); \Lambda^3 H_1(S_g;\mathbb{Z})) &\cong \mathbb{Z} \oplus Z, \quad \text{for } g \geq 3. \end{aligned}$$

We showed in [23] that if $N_{g,s}$ is a non–orientable surface of genus $g \geq 3$ with $s \leq 1$ boundary components, then

$$H_1(\mathcal{M}(N_{g,s}); H_1(N_{g,s};\mathbb{Z})) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \in \{3,4,5,6\},\\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \geq 7. \end{cases} \tag{1.1}$$

There are also similar computations for the hyperelliptic mapping class groups $\mathcal{M}^h(S_g)$. Tanaka [26] showed that $H_1(\mathcal{M}^h(S_g); H_1(S_g;\mathbb{Z})) \cong \mathbb{Z}_2$ for $g \geq 2$, and in the non–orientable case we showed in [24] that

$$H_1(\mathcal{M}^h(N_g); H_1(N_g;\mathbb{Z})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \text{for } g \geq 3.$$

There is also a lot of interesting results concerning the stable twisted (co)homology groups of mapping class groups – see [6, 9, 19, 20] and references there.

1.2. **Main results.** The purpose of this paper is to extendthe the formula (1.1) to the case of surfaces with punctures and/or boundary. We prove the following theorems.

Theorem 1.1. *If $N_{g,s}^n$ is a non–orientable surface of genus $g \geq 3$ with s boundary components and n punctures, then*

$$H_1(\mathcal{PM}^k(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z})) \cong \begin{cases} \mathbb{Z}_2^{3+n} & \text{if } g=3 \text{ and } s=k=0,\\ \mathbb{Z}_2^{1+n+k} & \text{if } g=3,\ s=0 \text{ and } k>0,\\ \mathbb{Z}_2^{n+3s+k} & \text{if } g=3 \text{ and } s>0,\\ \mathbb{Z}_2^{3+n-k} & \text{if } g=4 \text{ and } s=0,\\ \mathbb{Z}_2^{2+n+s-k} & \text{if } g=4 \text{ and } s>0,\\ \mathbb{Z}_2^{3+n-k} & \text{if } g=5 \text{ or } g=6,\\ \mathbb{Z}_2^{2+n-k} & \text{if } g \geq 7. \end{cases}$$

Theorem 1.2. *If $N^n_{g,s}$ is a non–orientable surface of genus $g \geq 3$ with s boundary components and $n \geq 2$ punctures, then*

$$H_1(\mathcal{M}(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z})) \cong \begin{cases} \mathbb{Z}_2^5 & \text{if } g \in \{3,4\} \text{ and } s=0,\\ \mathbb{Z}_2^{3s+2} & \text{if } g=3 \text{ and } s>0,\\ \mathbb{Z}_2^{4+s} & \text{if } g=4 \text{ and } s>0,\\ \mathbb{Z}_2^5 & \text{if } g=5 \text{ or } g=6,\\ \mathbb{Z}_2^4 & \text{if } g\geq 7.\end{cases}$$

Note that we obtained the formula (1.1) from the full presentation for the mapping class group $\mathcal{M}(N_{g,s})$, where $g+s \geq 3$ and $s \in \{0,1\}$, obtained by Paris and Szepietowski [15]. However, we do not have full presentations for the groups $\mathcal{PM}^k(N^n_{g,s})$ and $\mathcal{M}(N^n_{g,s})$, which makes our computation less straightforward.

The starting point for this computation is a simplification of known generating sets for the groups $\mathcal{PM}^k(N^n_{g,s})$ and $\mathcal{M}(N^n_{g,s})$ – see Theorems 4.4, 4.5 and 4.6 in Section 4. Then, in Sections 6, 7 and 8 we perform a detailed analysis of possible relations between these generators in order to obtain a minimal set of generators for the first homology group – see Propositions 6.1, 6.2, 7.1, 7.2 and 8.1. The proofs that these sets of generators are indeed linearly independent occupy Sections 9 and 10. One essential ingredient in these two sections is our recent computation [16] of the homology group

$$H_1(\mathcal{PM}^+(N_3^2); H_1(N_3^2;\mathbb{Z})) \cong \mathbb{Z}_2^5.$$

Section 3 is devoted to the technical details of the action of the mapping class group $\mathcal{M}(N)$ on the first homology group $H_1(N;\mathbb{Z})$. This analysis is continued in Section 5, where we set up a technical background for the computations of the twisted first homology group of various mapping class groups – see Propositions 5.1, 5.2 and 5.3.

2. Preliminaries

2.1. **Non–orientable surfaces.** Represent the surface $N^n_{g,s}$ as a sphere with g crosscaps $\mu_1,\ldots,\mu_g$, n marked points $z_1,\ldots,z_n$, and s boundary components (Figure 1). Let

$$\alpha_1,\ldots,\alpha_{g-1},\beta_1,\ldots,\beta_{\lfloor\frac{g-2}{2}\rfloor},\overline{\beta}_0,\ldots,\overline{\beta}_{\lfloor\frac{g-2}{2}\rfloor},\delta_1,\ldots,\delta_s,\varepsilon_1,\ldots,\varepsilon_{s+n}$$

be two–sided circles indicated in Figures 1, 2 and 3. Small arrows in these figures indicate directions of Dehn twists

$$a_1,\ldots,a_{g-1},b_1,\ldots,b_{\lfloor\frac{g-2}{2}\rfloor},\overline{b}_0,\ldots,\overline{b}_{\lfloor\frac{g-2}{2}\rfloor},d_1,\ldots,d_s,e_1,\ldots,e_{s+n}$$

associated with these circles. We also define: $\varepsilon_0=\alpha_1$ and $e_0=a_1$.

For any two consecutive crosscaps μ_i,μ_{i+1} we define a *crosscap transposition* u_i to be the map which interchanges these two crosscaps (see Figure 4). Similarly, for any two consecutive punctures z_i,z_{i+1} we define *elementary braid* s_i (Figure 4).

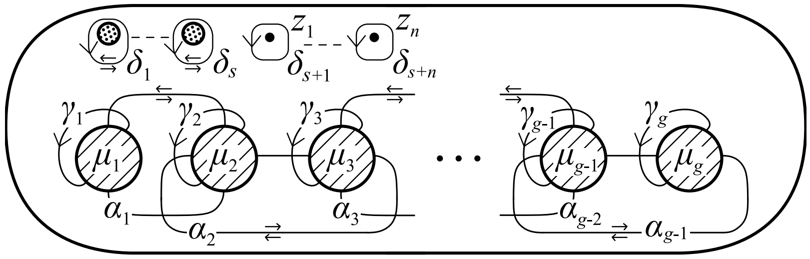

FIGURE 1. Surface $N_{g,s}^n$ as a sphere with crosscaps.

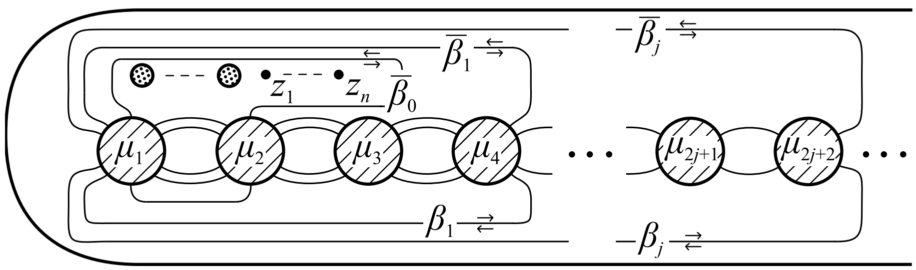

FIGURE 2. Circles $\beta_1, \beta_2, \ldots, \beta_{\lfloor \frac{g-2}{2} \rfloor}$ and $\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{\lfloor \frac{g-2}{2} \rfloor}$.

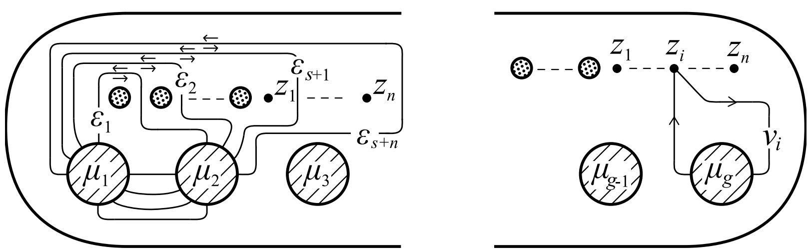

FIGURE 3. Circles $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s+n}$ and paths $\nu_1, \ldots, \nu_n$.

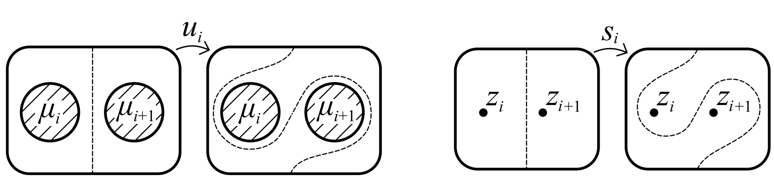

FIGURE 4. Crosscap transposition u_i and elementary braid s_i.

Finally, let v_i, for $i = 1, \ldots, n$ be the puncture slide of z_i along the path ν_i indicated in Figure 3.

2.2. **Homology of groups.** Let us briefly review how to compute the first homology of a group with twisted coefficients – for more details see Section 5 of [24] and references there.

For a given group G and G-module M (that is $\mathbb{Z}G$-module) we define $C_2(G)$ and $C_1(G)$ as the free G-modules generated respectively by symbols $[h_1|h_2]$ and $[h_1]$, where $h_i \in G$. We define also $C_0(G)$ as the free G-module generated by the empty bracket $[\cdot]$. Then the first homology group $H_1(G;M)$ is the first homology group of the complex

$$C_2(G) \otimes_G M \xrightarrow{\partial_2 \otimes_G \mathrm{id}} C_1(G) \otimes_G M \xrightarrow{\partial_1 \otimes_G \mathrm{id}} C_0(G) \otimes_G M \ ,$$

where

$$\begin{aligned} \partial_2([h_1|h_2]) &= h_1[h_2] - [h_1h_2] + [h_1], \\ \partial_1([h]) &= h[\cdot] - [\cdot]. \end{aligned}$$

For simplicity, we write $\otimes_G = \otimes$ and $\partial \otimes \mathrm{id} = \overline{\partial}$ henceforth.

If the group G has a presentation $G = \langle X \mid R \rangle$, then we have an excat sequence

$$1 \longrightarrow N(R) \longrightarrow F(X) \longrightarrow G \longrightarrow 1 \ , \tag{2.1}$$

where $F(X)$ is the free group generated by elements of X (generators) and $N(R)$ is the normal closure in $F(X)$ of the set of relations R. Sequence (2.1) gives the following excat seqence (see for example [3] and references there)

$$N(R)/[N(R), N(R)] \otimes_G M \xrightarrow{i} H_1(F(X); M) \longrightarrow H_1(G; M) \longrightarrow 0 \ ,$$

where $N(R)/[N(R), N(R)]$ is the abelianization of $N(R)$. Hence, we can identify

$$H_1(G; M) = H_1(F(X); M)/\mathrm{Im}(i).$$

Let us now describe how to use the above identification in practice. Let

$$\langle \overline{X} \rangle = \langle [x] \otimes m \mid x \in X, m \in M \rangle \subseteq C_1(F(X)) \otimes M,$$

then $H_1(G;M)$ is a quotient of $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$.

The kernel of this quotient corresponds to relations in G (that is elements of R). To be more precise, if $r \in R$ has the form $x_1 \cdots x_k = y_1 \cdots y_n$ and $m \in M$, then $i(r) \in H_1(F(X); M)$ is equal to

$$\overline{r} \otimes m \colon \ \sum_{i=1}^{k} x_1 \cdots x_{i-1}[x_i] \otimes m - \sum_{i=1}^{n} y_1 \cdots y_{i-1}[y_i] \otimes m. \tag{2.2}$$

Then

$$H_1(G; M) = \langle \overline{X} \rangle \cap \ker \overline{\partial}_1 / \langle \overline{R} \rangle,$$

where

$$\overline{R} = \{ \overline{r} \otimes m \mid r \in R, m \in M \}.$$

3. Action of $\mathcal{M}(N_{g,s}^n)$ on $H_1(N_{g,s}^n;\mathbb{Z})$

Let $\gamma_1,\ldots,\gamma_g,\delta_1,\ldots,\delta_{s+n}$ be circles indicated in Figure 1. Note that $\gamma_1,\ldots,\gamma_g$ are one–sided, $\delta_1,\ldots,\delta_{s+n}$ are two–sided and the $\mathbb{Z}$-module $H_1(N_{g,s}^n;\mathbb{Z})$ is generated by homology classes $[\gamma_1],\ldots,[\gamma_g],[\delta_1],\ldots,[\delta_{s+n-1}]$. These generators are free provided $s+n>0$. In abuse of notation we will not distinguish between the curves $\gamma_1,\ldots,\gamma_g,\delta_1,\ldots,\delta_{s+n}$ and their cycle classes.

The mapping class group $\mathcal{M}(N_{g,s}^n)$ acts on $H_1(N_{g,s}^n;\mathbb{Z})$, hence we have a representation

$$\psi\colon \mathcal{M}(N_{g,s}^n)\to \operatorname{Aut}(H_1(N_{g,s}^n;\mathbb{Z})).$$

It is straightforward to check that

$$\begin{aligned}
\psi(a_j)&=I_{j-1}\oplus\begin{bmatrix}0&1\\-1&2\end{bmatrix}\oplus I_{g-j-1}\oplus I_{s+n-1},\\
\psi(a_j^{-1})&=I_{j-1}\oplus\begin{bmatrix}2&-1\\1&0\end{bmatrix}\oplus I_{g-j-1}\oplus I_{s+n-1},\\
\psi(u_j)=\psi(u_j^{-1})&=I_{j-1}\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus I_{g-j-1}\oplus I_{s+n-1},
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
\psi(b_1)&=\left(I_g+\begin{bmatrix}-1&1&-1&1\\-1&1&-1&1\\-1&1&-1&1\\-1&1&-1&1\end{bmatrix}\oplus 0_{g-4}\right)\oplus I_{s+n-1},\\
\psi(b_1^{-1})&=\left(I_g+\begin{bmatrix}1&-1&1&-1\\1&-1&1&-1\\1&-1&1&-1\\1&-1&1&-1\end{bmatrix}\oplus 0_{g-4}\right)\oplus I_{s+n-1},
\end{aligned}\tag{3.2}$$

(3.3)
$$\psi(e_j)(\xi)=\begin{cases}-\gamma_2-\delta_1-\delta_2+\ldots-\delta_j & \text{if } \xi=\gamma_1 \text{ and } j<s+n,\\ \gamma_1+2\gamma_2+\delta_1+\delta_2+\ldots+\delta_j & \text{if } \xi=\gamma_2 \text{ and } j<s+n,\\ \xi & \text{if } \xi\neq\gamma_1,\ \xi\neq\gamma_2 \text{ and } j<s+n,\end{cases}$$
$$\psi(e_j^{-1})(\xi)=\begin{cases}2\gamma_1+\gamma_2+\delta_1+\delta_2+\ldots+\delta_j & \text{if } \xi=\gamma_1 \text{ and } j<s+n,\\ -\gamma_1-\delta_1-\delta_2+\ldots-\delta_j & \text{if } \xi=\gamma_2 \text{ and } j<s+n,\\ \xi & \text{if } \xi\neq\gamma_1,\ \xi\neq\gamma_2 \text{ and } j<s+n,\end{cases}$$
$$\psi(d_j)=\psi(d_j^{-1})=I_g\oplus I_{s+n-1},$$

(3.4)
$$\psi(s_j)=\psi(s_j^{-1})=I_g\oplus I_{s+j-1}\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus I_{n-j-2},\ \text{if } j<n-1,$$
$$\psi(s_{n-1})(\xi)=\psi(s_{n-1}^{-1})(\xi)=\begin{cases}-(2\gamma_1+\ldots+2\gamma_g-\delta_1+\ldots+\delta_{s+n-1}) & \text{if } \xi=\delta_{s+n-1},\\ \xi & \text{if } \xi\neq\delta_{s+n-1},\end{cases}$$

(3.5)

$$\psi(v_j)(\xi)=\psi(v_j^{-1})(\xi)=\begin{cases}-\delta_{s+j} & \text{if } \xi=\delta_{s+j} \text{ and } j<n,\\ \gamma_g+\delta_{s+j} & \text{if } \xi=\gamma_g \text{ and } j<n,\\ \xi & \text{if } \xi\neq\gamma_g \text{ and } \xi\neq\delta_{s+j} \text{ and } j<n,\end{cases}$$

$$\psi(v_n)(\xi)=\psi(v_n^{-1})(\xi)=\begin{cases}\gamma_g-(2\gamma_1+\ldots+2\gamma_g+\delta_1+\ldots+\delta_{s+n-1}) & \text{if } \xi=\gamma_g,\\ \xi & \text{if } \xi\neq\gamma_g,\end{cases}$$

where I_k is the identity matrix of rank k.

4. Generators for the groups $\mathcal{PM}^k(N_{g,s}^n)$ and $\mathcal{M}(N_{g,s}^n)$

The main goal of this section is to obtain simple generating sets for the groups $\mathcal{PM}^k(N_{g,s}^n)$ and $\mathcal{M}(N_{g,s}^n)$ – see Theorems 4.4, 4.5 and 4.6 below. However, we first prove some technical lemmas.

For the rest of this section, let $a_{j,i},\widehat{b}_1,\widehat{b}_{1,i},\widehat{d}_i$ for $j=1,2,3,4$, $i=1,\ldots,s+n$ be Dehn twists about circles $\alpha_{j,i},\widehat{\beta}_1,\widehat{\beta}_{1,i},\widehat{\delta}_i$ shown in Figure 5.

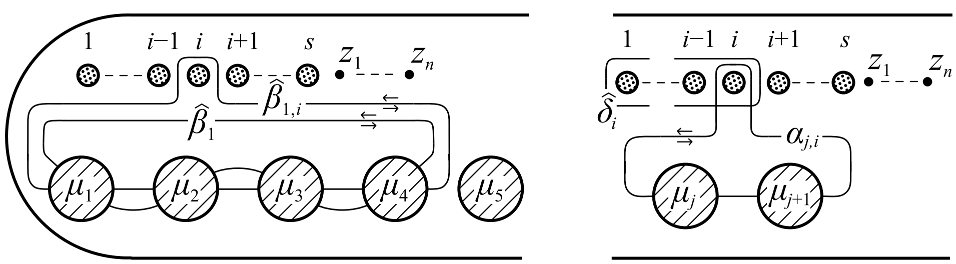

Figure 5. Circles $\alpha_{j,i},\widehat{\beta}_1,\widehat{\beta}_{1,i},\widehat{\delta}_i$.

Lemma 4.1. *Let $g\geq 3$ and $1\leq i\leq s+n$, then $a_{1,i}$ and $a_{2,i}$ are in the subgroup $G\leq\mathcal{M}(N_{g,s}^n)$ generated by*

$$\{u_1,u_2,a_2,e_{i-1},e_i\}.$$

Proof. It is straightforward to check that

$$\alpha_{2,i}=e_ia_2(\varepsilon_{i-1}),$$
$$\alpha_{1,i}=u_2u_1(\alpha_{2,i}).$$

Hence,

$$a_{2,i}=e_ia_2e_{i-1}a_2^{-1}e_i^{-1}\in G,$$
$$a_{1,i}=u_2u_1a_{2,i}u_1^{-1}u_2^{-1}\in G.$$

□

Lemma 4.2. *Let $g\geq 5$ and $1\leq i\leq s$, then d_i is in the subgroup $G\leq\mathcal{M}(N_{g,s}^n)$ generated by*

$$\{u_1,u_2,u_3,u_4,a_1,a_2,a_3,a_4,e_{i-1},e_i,b_1\}.$$

Proof. By Lemma 4.1, $a_{1,i}, a_{2,i} \in G$. Moreover,

$$\begin{aligned}
\alpha_{3,i} &= u_2^{-1}u_3^{-1}(\alpha_{2,i}),\\
\alpha_{4,i} &= u_3^{-1}u_4^{-1}(\alpha_{3,i}),\\
\widehat{\beta}_1 &= u_3^{-1}u_1(\beta_1),\\
\widehat{\beta}_{1,i} &= a_4a_{4,i}^{-1}(\widehat{\beta}_1).
\end{aligned}$$

This proves that

$$a_{3,i}, \widehat{b}_1, \widehat{b}_{1,i} \in G.$$

Moreover, by Lemma 6.12 of [22], there is a lantern relation

$$d_ia_1a_3\widehat{b}_{1,i} = a_{3,i}a_{1,i}\widehat{b}_1.$$

This proves that $d_i \in G$. □

Lemma 4.3. *Let $g \geq 3$ and $s > 0$, then d_s is in the subgroup $G \leq \mathcal{M}(N_{g,s}^n)$ generated by*

$$\{u_1, \ldots, u_{g-1}, a_1, \ldots, a_{g-1}, e_1, \ldots, e_{s+n-1}, d_1, \ldots, d_{s-1}\}.$$

Proof. Let $H \leq G$ be the subgroup of G generated by

$$\{u_1, \ldots, u_{g-1}, a_1, \ldots, a_{g-1}, e_1, \ldots, e_{s+n-1}\}$$

and let $d_{s+1} = d_{s+2} = \ldots = d_{s+n} = 1$.

Note first that,

$$\varepsilon_{s+n} = a_2a_3 \ldots a_{g-1}u_{g-1} \ldots u_3u_2(\alpha_1),$$

and

$$e_{s+n} = (a_2 \ldots a_{g-1}u_{g-1} \ldots u_2)a_1^{-1}(a_2 \ldots a_{g-1}u_{g-1} \ldots u_2)^{-1} \in H. \tag{4.1}$$

We will prove by induction, that for each $k = 1, 2, \ldots, s+n$,

$$d_1d_2 \ldots d_k\left(\widehat{d}_k\right)^{-1} \in H. \tag{4.2}$$

Since $\widehat{d}_1 = d_1$, the statement is true for $k = 1$.

It is straightforward to check that, for each $k = 1, 2, \ldots, s+n$ there is a lantern relation

$$a_1e_{k+1}\widehat{d}_kd_{k+1} = e_ka_{1,k+1}\widehat{d}_{k+1}. \tag{4.3}$$

Assume that

$$\widehat{d}_k = hd_1d_2 \ldots d_k$$

for some $h \in H$. Then, by the formulas (4.1), (4.3) and Lemma 4.1, we have

$$\begin{aligned}
\widehat{d}_kd_{k+1}\widehat{d}_{k+1}^{-1} &= e_{k+1}^{-1}a_1^{-1}e_ka_{1,k+1},\\
hd_1d_2 \ldots d_kd_{k+1}\widehat{d}_{k+1}^{-1} &= e_{k+1}^{-1}a_1^{-1}e_ka_{1,k+1},\\
d_1d_2 \ldots d_kd_{k+1}\widehat{d}_{k+1}^{-1} &= h^{-1}e_{k+1}^{-1}a_1^{-1}e_ka_{1,k+1} \in H.
\end{aligned}$$

This completes the inductive proof of (4.2). In particular,

$$d_1d_2\dots d_s = d_1d_2\dots d_{s+n} = h\widehat{d}_{s+n},$$

for some $h \in H$. Moreover, it is straightforward to check that

$$\widehat{d}_{s+n} = (u_1u_2\dots u_{g-1})^g \in G,$$

hence

$$d_s = d_{s-1}^{-1}\dots d_2^{-1}d_1^{-1}h\widehat{d}_{s+n} \in G.$$

□

Theorem 4.4. *Let $g \geq 3$. Then the mapping class group $\mathcal{PM}^+(N_{g,s}^n) = \mathcal{PM}^n(N_{g,s}^n)$ is generated by*

$$\{a_1,\dots,a_{g-1},u_1,e_1,\dots,e_{s+n-1}\}$$

and additionally

$$\begin{cases}\{d_1,\dots,d_{s-1}\} & \text{if } g=3,\\ \{b_1,d_1,\dots,d_{s-1}\} & \text{if } g=4,\\ \{b_1\} & \text{if } g\geq 5.\end{cases}$$

Proof. Let G be the subgroup of $\mathcal{PM}^+(N_{g,s}^n)$ generated by elements specified in the statement of the theorem. By Theorem 4.1 of [21], $\mathcal{PM}^+(N_{g,s}^n)$ is generated by the crosscap slide $y = a_1u_1$ and $2g+n+2s-4$ twists:

$$\left\{a_1,\dots,a_{g-1},b_1,\dots,b_{\lfloor\frac{g-2}{2}\rfloor},\overline{b}_0,\dots,\overline{b}_{\lfloor\frac{g-2}{2}\rfloor},e_1,\dots,e_{s+n-1},d_1,\dots,d_s\right\}$$

(note that $\overline{b}_{\lfloor\frac{g-2}{2}\rfloor} = b^{-1}_{\lfloor\frac{g-2}{2}\rfloor}$ if g is even). It is enough to show that all these generators are in G.

By Theorem 3.1 of [15],

$$b_{i+1} = (b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6},$$
$$\text{for } i=1,\dots,\left\lfloor\frac{g-2}{2}\right\rfloor-1 \text{ and } b_0=a_0.$$

Hence $b_i \in G$, for $i=2,\dots,\left\lfloor\frac{g-2}{2}\right\rfloor$.

By Lemma 3.8 of [15],

$$u_{i+1} = a_ia_{i+1}u_i^{-1}a_{i+1}^{-1}a_i^{-1}, \text{ for } i=1,\dots,g-2,$$

hence $u_2,u_3,\dots,u_{g-1} \in G$. Now it is straightforward to check, that

$$a_{2i+2}\cdots a_{g-2}a_{g-1}u_{g-1}\cdots u_{2i+3}u_{2i+2}(\beta_i) = \overline{\beta}_i, \text{ for } i=0,\dots,\left\lfloor\frac{g-2}{2}\right\rfloor.$$

This shows, that $\overline{b}_i$ is conjugate to b_i by an element of G. Hence $\overline{b}_i \in G$, for $i=0,\dots,\left\lfloor\frac{g-2}{2}\right\rfloor$. This together with Lemma 4.3 complete the proof if $g<5$.

Finally, if $g \geq 5$, then Lemma 4.2 implies that $d_i \in G$, for $i=1,\dots,s$. □

Theorem 4.5. *Let $g \geq 3$ and $0 \leq k \leq n$. Then the mapping class group $\mathcal{PM}^k(N_{g,s}^n)$ is generated by $\mathcal{PM}^+(N_{g,s}^n)$ and $(n-k)$ puncture slides*

$$\{v_{k+1}, \ldots, v_n\}.$$

Proof. The statement follows from the short exact sequence

$$1 \longrightarrow \mathcal{PM}^+(N_{g,s}^n) \xrightarrow{\ i\ } \mathcal{PM}^k(N_{g,s}^n) \xrightarrow{\ p\ } \mathbb{Z}_2^{n-k} \longrightarrow 1$$

and the fact that $\{p(v_{k+1}), \ldots, p(v_n)\}$ generate $\mathbb{Z}_2^{n-k}$. □

Theorem 4.6. *Let $g \geq 3$ and $n \geq 2$. Then the mapping class group $\mathcal{M}(N_{g,s}^n)$ is generated by $\mathcal{PM}^+(N_{g,s}^n)$ and*

$$\{v_n, s_1, \ldots, s_{n-1}\}.$$

Proof. By Theorem 4.5, the pure mapping class group $\mathcal{PM}(N_{g,s}^n) = \mathcal{PM}^0(N_{g,s}^n)$ is generated by $\mathcal{PM}^+(N_{g,s}^n)$ and $\{v_1, \ldots, v_n\}$. Moreover, we have the short exact sequence

$$1 \longrightarrow \mathcal{PM}(N_{g,s}^n) \xrightarrow{\ i\ } \mathcal{M}(N_{g,s}^n) \xrightarrow{\ p\ } S_n \longrightarrow 1,$$

where S_n is the symmetric group on n letters. Now the statement follows from the fact that $p(s_1), \ldots, p(s_{n-1})$ generate S_n and the relation

$$v_{i-1} = s_{j-1}^{-1} v_j s_{j-1}, \qquad \text{for } i = 2, \ldots, n.$$

□

For further reference, let us prove that

Proposition 4.7. *Let $g \geq 3$, $n \geq 2$ and $1 \leq j \leq n-1$. Then*

$$e_{s+j-1} e_{s+j+1} s_j = e_{s+j} s_j^3 e_{s+j}.$$

Proof. It is straightforward to check that ε_{s+j+1} bounds in N a disk with three holes: $\varepsilon_{s+j-1}, \delta_j, \delta_{j+1}$. This implies that there is a lantern relation of the form

$$\begin{aligned} e_{s+j+1} e_{s+j-1} d_j d_{j+1} &= e_{s+j} s_j^2 \left(s_j e_{s+j} s_j^{-1} \right), \\ e_{s+j+1} e_{s+j-1} &= e_{s+j} s_j^3 e_{s+j} s_j^{-1}, \\ e_{s+j+1} e_{s+j-1} s_j &= e_{s+j} s_j^3 e_{s+j}. \end{aligned}$$

□

5. Computing $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$

Let $G = \mathcal{M}(N_{g,s}^n)$, $M = H_1(N_{g,s}^n; \mathbb{Z})$ and assume that $s + n > 0$. Let

$$\xi_i = \begin{cases} \gamma_i & \text{for } i = 1, \ldots, g, \\ \delta_{i-g} & \text{for } i = g+1, \ldots, g+s+n-1. \end{cases}$$

If $h \in G$, then

$$\overline{\partial}_1([h] \otimes \xi_i) = (h-1)[\cdot] \otimes \xi_i = (\psi(h)^{-1} - I_g)\xi_i,$$

where we identified $C_0(G) \otimes M$ with M by the map $[\cdot] \otimes m \mapsto m$.

Let us denote

$$[a_j] \otimes \xi_i,\ [u_j] \otimes \xi_i,\ [b_1] \otimes \xi_i,\ [e_j] \otimes \xi_i,\ [d_j] \otimes \xi_i,\ [s_j] \otimes \xi_i,\ [v_j] \otimes \xi_i$$

respectively by

$$a_{j,i},\ u_{j,i},\ b_{1,i},\ e_{j,i},\ d_{j,i},\ s_{j,i},\ v_{j,i},$$

where $i = 1, \ldots, g+s+n-1$.

Using formulas (3.1)–(3.5), we obtain

$$\overline{\partial}_1(a_{j,i}) = \begin{cases} \gamma_j + \gamma_{j+1} & \text{if } i = j, \\ -\gamma_j - \gamma_{j+1} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{\partial}_1(u_{j,i}) = \begin{cases} -\gamma_j + \gamma_{j+1} & \text{if } i = j, \\ \gamma_j - \gamma_{j+1} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

$$\overline{\partial}_1(b_{1,i}) = \begin{cases} \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 & \text{if } i = 1,3, \\ -\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 & \text{if } i = 2,4, \\ 0 & \text{otherwise,} \end{cases}$$

(5.2)

$$\overline{\partial}_1(e_{j,i}) = \begin{cases} \gamma_1 + \gamma_2 + (\delta_1 + \ldots + \delta_j) & \text{if } i = 1, \\ -\gamma_1 - \gamma_2 - (\delta_1 + \ldots + \delta_j) & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{\partial}_1(d_{j,i}) = 0,$$

$$\text{if } j < n-1, \text{ then } \overline{\partial}_1(s_{j,i}) = \begin{cases} -\delta_{s+j} + \delta_{s+j+1} & \text{if } i = g+s+j, \\ \delta_{s+j} - \delta_{s+j+1} & \text{if } i = g+s+j+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{\partial}_1(s_{n-1,i}) = \begin{cases} -(2\gamma_1 + \ldots + 2\gamma_g) & \\ \quad -(\delta_1 + \ldots + \delta_{s+n-1}) & \\ \quad -\delta_{s+n-1} & \text{if } i = g+s+n-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{if } j < n, \text{ then } \overline{\partial}_1(v_{j,i}) = \begin{cases} \delta_{s+j} & \text{if } i = g, \\ -2\delta_{s+j} & \text{if } i = g+s+j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{\partial}_1(v_{n,i}) = \begin{cases} -(2\gamma_1 + \ldots + 2\gamma_g) & \\ \quad -(\delta_1 + \ldots + \delta_{s+n-1}) & \text{if } i = g, \\ 0 & \text{otherwise.} \end{cases}$$

The above formulas show that all of the following elements are contained in $\ker\overline{\partial}_1$

(K1) $a_{j,i}$ for $j = 1,\dots,g-1$ and $i = 1,\dots,j-1,j+2,\dots,g+s+n-1$,
(K2) $a_{j,j} + a_{j,j+1}$ for $j = 1,\dots,g-1$,
(K3) $u_{1,i}$ for $i = 3,\dots,g+s+n-1$,
(K4) $u_{1,1} + u_{1,2}$,
(K5) $e_{j,i}$ for $j = 1,\dots,s+n-1$ and $i = 3,4,\dots,g+s+n-1$,
(K6) $e_{j,1} + e_{j,2}$ for $j = 1,\dots,s+n-1$,
(K7) $d_{j,i}$ for $j = 1,\dots,s-1$ and $i = 1,\dots,g+s+n-1$,
(K8) $b_{1,i}$ for $i = 5,\dots,g+s+n-1$,
(K9) $b_{1,i} + b_{1,1}$ for $i = 2,4$,
(K10) $b_{1,3} - b_{1,1}$,
(K11) $b_{1,1} - a_{1,1} - a_{3,3}$.

Proposition 5.1. *Let $g \geq 3$, $s+n > 0$ and $G = \mathcal{PM}^+(N_{g,s}^n)$. Then $\langle\overline{X}\rangle \cap \ker\overline{\partial}_1$ is the abelian group generated freely by Generators (K1)–(K6) and additionally*

$$\begin{cases} \text{(K7)} & \text{if } g = 3, \\ \text{(K7), (K8)–(K11)} & \text{if } g = 4, \\ \text{(K8)–(K11)} & \text{if } g \geq 5. \end{cases}$$

Proof. By Theorem 4.4, $\langle\overline{X}\rangle$ is generated freely by $a_{j,i}, u_{1,i}, e_{j,i}$ and

$$\begin{cases} d_{j,i} & \text{if } g = 3, \\ b_{1,i}, d_{j,i} & \text{if } g = 4, \\ b_{1,i} & \text{if } g \geq 5. \end{cases}$$

Suppose that $h \in \langle\overline{X}\rangle \cap \ker\overline{\partial}_1$. We will show that h can be uniquely expressed as a linear combination of generators specified in the statement of the proposition.

We decompose h as follows:

- $h = h_0 = h_1 + h_2$, where h_1 is a combination of Generators (K1)–(K2) and h_2 does not contain $a_{j,i}$ with $i \neq j$;
- $h_2 = h_3 + h_4$, where h_3 is a combination of Generators (K3)–(K4) and h_4 does not contain $u_{1,i}$ with $i \neq 1$;
- $h_4 = h_5 + h_6$, where h_5 is a combination of Generators (K5)–(K6) and h_6 does not contain $e_{j,i}$ for $i > 1$;

If $g = 3$ or $g = 4$, we decompose $h_6 = h_7 + h_8$, where h_7 is a combination of Generators (K7) and h_8 does not contain $d_{j,i}$. If $g \geq 5$, we define $h_7 = 0$ and $h_8 = h_6$.

If $g \geq 4$, we decompose $h_8 = h_9 + h_{10}$, where h_9 is a combination of Generators (K8)–(K11) and h_{10} does not contain $b_{1,i}$. If $g = 3$ we define $h_9 = 0$ and $h_{10} = h_8$.

Observe also that for each $k = 0,\ldots,8$, h_{k+1} and h_{k+2} are uniquely determined by h_k. Element h_{10} has the form

$$h_{10} = \sum_{j=1}^{g-1} \alpha_j a_{j,j} + \alpha u_{1,1} + \sum_{j=1}^{s+n-1} \beta_j e_{j,1}$$

for some integers $\alpha, \alpha_1, \ldots, \alpha_{g-1}, \beta_1, \ldots, \beta_{s+n-1}$. Hence

$$\begin{aligned} 0 =& \overline{\partial}_1(h_{10}) = \alpha_1(\gamma_1+\gamma_2) + \alpha_2(\gamma_2+\gamma_3) + \ldots + \alpha_{g-1}(\gamma_{g-1}+\gamma_g) \\ &+ \alpha(-\gamma_1+\gamma_2) + \beta_1(\gamma_1+\gamma_2+\delta_1) + \beta_2(\gamma_1+\gamma_2+\delta_1+\delta_2) \\ &+ \ldots + \beta_{s+n-1}(\gamma_1+\gamma_2+\delta_1+\delta_2+\ldots+\delta_{s+n-1}). \end{aligned}$$

This implies that $\beta_{s+n-1} = \ldots = \beta_2 = \beta_1 = 0$, and then $\alpha_{g-1} = \ldots = \alpha_2 = \alpha_1 = \alpha = 0$ and thus $h_{10} = 0$. □

By an analogous argument and Propositions 4.5, 4.6, we get

Proposition 5.2. *Let $g \geq 3$, $s+n > 0$, $0 \leq k \leq n$ and $G = \mathcal{PM}^k(N^n_{g,s})$. Then $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$ is the abelian group generated by generators specified in the statement of Proposition 5.1 and additionally*

(K12) $\widetilde{v}_{j,i}$ *for* $k < j \leq n$ *and* $1 \leq i \leq g+s+n-1$,

where

$$\widetilde{v}_{j,i} = \begin{cases} v_{j,i} & \text{if } k < j \leq n \text{ and } i \notin \{g, g+s+j\}, \\ v_{j,g} + e_{s+j-1,1} - e_{s+j,1} & \text{if } k < j < n \text{ and } i = g, \\ v_{j,g+s+j} - 2e_{s+j-1,1} + 2e_{s+j,1} & \text{if } k < j < n \text{ and } i = g+s+j, \\ v_{n,g} + e_{s+n-1,1} + a_{1,1} - u_{1,1} & \\ \quad +2a_{2,2} + \ldots + 2a_{g-1,g-1} & \text{if } j = n \text{ and } g \text{ is odd}, \\ v_{n,g} + e_{s+n-1,1} + a_{1,1} & \\ \quad 2a_{3,3} + \ldots + 2a_{g-1,g-1} & \text{if } j = n \text{ and } g \text{ is even}. \end{cases}$$

Proposition 5.3. *Let $g \geq 3$, $n \geq 2$ and $G = \mathcal{M}(N^n_{g,s})$. Then $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$ is the abelian group generated by generators specified in the statement of Proposition 5.1, Generators (K12) with $j = n$ specified in the statement of Proposition 5.2, and additionally*

(K13) $s_{j,i}$ *if* $j \leq n-1$ *and* $i \notin \{g+s+j, g+s+j+1\}$,
(K14) $s_{j,g+s+j} + s_{j,g+s+j+1}$ *if* $j < n-1$,
(K15) $s_{j,g+s+j} - e_{s+j-1,1} + 2e_{s+j,1} - e_{s+j+1,1}$ *if* $j < n-1$,
(K16)

$$\begin{cases} s_{n-1,g+s+n-1} + 2e_{s+n-1,1} - e_{s+n-2,1} + a_{1,1} & \\ \quad -u_{1,1} + 2a_{2,2} + \ldots + 2a_{g-1,g-1} & \text{if } g \text{ is odd}, \\ s_{n-1,g+s+n-1} + 2e_{s+n-1,1} - e_{s+n-2,1} + a_{1,1} & \\ \quad +2a_{3,3} + \ldots + 2a_{g-1,g-1} & \text{if } g \text{ is even}. \end{cases}$$

6. Bounding $H_1(\mathcal{PM}^+(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))$ from above

In this section we will use the formula (2.2) to rewrite some relations between generators specified in Theorem 4.4 as relations between homology classes. Our goal is to reduce these generating sets for homology groups to the ones specified in Propositions 6.1 and 6.2 below.

Let

$$i\colon N_{g,1}\to N_{g,s}^n$$

be an embedding of a non–orientable subsurface of genus g with one boundary component such that $N_{g,1}$ is disjoint from $\delta_1,\ldots,\delta_{s+n}$ (the complement of $N_{g,1}$ in $N_{g,s}^n$ is a disk containing $\delta_1,\ldots,\delta_{s+n}$). This embedding induces homomorphisms

$$\begin{array}{ccc}\mathcal{M}(N_{g,1}) & \xrightarrow{i_*} & \mathcal{PM}^+(N_{g,s}^n)\\ \downarrow\varrho & & \downarrow\varrho\\ \mathrm{Aut}(H_1(N_{g,1};\mathbb{Z})) & \xrightarrow{i_*} & \mathrm{Aut}(H_1(N_{g,s}^n;\mathbb{Z}))\end{array}$$

This leads to the following homomorphism

$$H_1(\mathcal{M}(N_{g,1});H_1(N_{g,1};\mathbb{Z}))\xrightarrow{i_*} H_1(\mathcal{PM}^+(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z})).$$

Moreover, some of the generators specified in the statement of Proposition 5.1 are images under this homomorphism of generators used in Proposition 4.2 of [23] to compute $H_1(\mathcal{M}(N_{g,1});H_1(N_{g,1};\mathbb{Z}))$. This allows to transfer (via i_*) some of the relations between these generators obtained in Section 5 of [23]. In particular,

- Generators (K1): $a_{j,i}$ for $j=1,\ldots,g-1$, $i=1,\ldots j-1,j+2,\ldots,g$ generate a cyclic group of order at most 2. They are trivial if $g\geq 7$.
- Generators (K2) generate a cyclic group of order at most 2. They are trivial if $g\geq 4$.
- Generators (K3): $u_{1,i}$ for $i=1,\ldots,g-2$ generate a cyclic group of order at most 2.
- Generator (K4) is trivial.
- Generators (K8): $b_{1,i}$ for $i=5,\ldots,g$ are superfluous (they can be expressed in terms of generators (K1)).
- Generators (K9), (K10) are trivial.
- Generator (K11) has order at most 2.

The formula (2.2) and the relation

$$a_ja_{j+1}a_j=a_{j+1}a_ja_{j+1},\qquad \text{for } j=1,\ldots,g-2,$$

imply that for $i>g$

$$\begin{aligned}0&=([a_j]+a_j[a_{j+1}]+a_ja_{j+1}[a_j]-[a_{j+1}]-a_{j+1}[a_j]-a_{j+1}a_j[a_{j+1}])\otimes\xi_i\\ &=a_{j,i}+a_{j+1,i}+a_{j,i}-a_{j+1,i}-a_{j,i}-a_{j+1,i}=a_{j,i}-a_{j+1,i}.\end{aligned}$$

Hence

$$a_{j,i}=a_{1,i}\quad \text{for } j=1,\ldots,g-1,\ i>g. \tag{6.1}$$

If $s+n\geq 2$, then the relation

$$a_1e_j=e_ja_1 \quad \text{for } j=1,\dots,s+n-1$$

gives

$$\begin{aligned}0&=([a_1]+a_1[e_j]-[e_j]-e_j[a_1])\otimes\xi_i\\&=[a_1]\otimes(I-\psi(e_j^{-1}))\xi_i-[e_j]\otimes(I-\psi(a_1^{-1}))\xi_i\\&=\pm\begin{cases}(a_{1,1}+a_{1,2})+a_{1,g+1}+\ldots+a_{1,g+j}-(e_{j,1}+e_{j,2}) & \text{if } i=1,2,\\ 0 & \text{if } i>2.\end{cases}\end{aligned}$$

This relation implies that Generators (K6) are superfluous

$$e_{j,1}+e_{j,2}=(a_{1,1}+a_{1,2})+a_{1,g+1}+\ldots+a_{1,g+j}. \tag{6.2}$$

The braid relation

$$a_2e_ja_2=e_ja_2e_j \quad \text{for } j=1,\dots,s+n-1$$

gives

$$\begin{aligned}0&=([a_2]+a_2[e_j]+a_2e_j[a_2]-[e_j]-e_j[a_2]-e_ja_2[e_j])\otimes\xi_i\\&=[a_2]\otimes(I+\psi(e_j^{-1}a_2^{-1})-\psi(e_j^{-1}))\gamma_i\\&+[e_j]\otimes(\psi(a_2^{-1})-I-\psi(a_2^{-1}e_j^{-1}))\gamma_i\\&=\begin{cases}a_{2,i}-e_{j,i} & \text{if } i\not\in\{1,2,3\},\\ a_{2,1}-e_{j,3}+a_{2,g+1}+\ldots+a_{2,g+j} & \text{if } i=3,\\ (*)+(a_{2,2}+a_{2,3})+(e_{j,1}+e_{j,2})+e_{j,g+1}+\ldots+e_{j,g+j} & \text{if } i=2,\\ (*) & \text{if } i=1.\end{cases}\end{aligned}$$

In the above formula $(*)$ denotes some expression homologous to 0 by previously obtained relations. As we progress further, we will often perform simplifications based on previously obtained relations, from now on we will use symbol '$\equiv$' in such cases.

The first two cases of this relation and the formula (6.1) imply that Generators (K5)

$$e_{j,i}=\begin{cases}a_{1,i} & \text{if } i\geq 4,\\ a_{2,1}+a_{1,g+1}+\ldots+a_{1,g+j} & \text{if } i=3.\end{cases} \tag{6.3}$$

are superfluous.

The third case together with formulas (6.2) and (6.3) imply that

$$2a_{1,g+j}=0 \quad \text{for } j=1,2,\dots,s+n-1,$$

or equivalently

$$2(e_{j,1}+e_{j,2})=0.$$

The relation

$$a_jd_k=d_ka_j \quad \text{for } j=1,\dots,g-1,\ k=1,\dots,s-1$$

gives

$$\begin{aligned}0&=([a_j]+a_j[d_k]-[d_k]-d_k[a_j])\otimes\xi_i\\&=[a_j]\otimes(I-\psi(d_k^{-1}))\xi_i-[d_k]\otimes(I-\psi(a_j^{-1}))\xi_i\\&=\pm\begin{cases}d_{k,j}+d_{k,j+1}&\text{if } i=j,j+1,\\0&\text{if } i\notin\{j,j+1\}.\end{cases}\end{aligned}$$

This implies that Generators (K7)

$$d_{k,j}=(-1)^{j-1}d_{k,1}\quad\text{for } j=2,\dots,g,\ k=1,\dots,s-1 \tag{6.4}$$

are superfluous.

Similarly, the relation

$$u_1d_k=d_ku_1$$

implies that

$$0=[d_k]\otimes(\psi(u_1^{-1})-I)\gamma_2=d_{k,1}-d_{k,2}=0,\qquad\text{for } k=1,\dots,s-1,$$

which together with the formula (6.4) implies that

$$2d_{k,1}=0\quad\text{for } k=1,\dots,s-1.$$

Relation

$$e_jd_k=d_ke_j$$

implies that

$$0=[d_k]\otimes(\psi(e_j^{-1})-I)\gamma_1\equiv d_{k,g+1}+\ldots+d_{k,g+j}$$
$$\text{for } k=1,\dots,s,\ j=1,\dots,s+n-1.$$

This implies that Generators (K7): $d_{j,i}$ are trivial for $i>g$.

Suppose now that $g=3$ and consider the relation

$$(u_1e_{s+n})^2=\widehat{d}_{s+n}=(a_1a_2)^6, \tag{6.5}$$

where $\widehat{d}_{s+n}$ is defined as in Section 4. The right–hand side of this relation is a chain relation, and the left–hand side is a square of a crosscap slide (see [25], Theorem 7.17, Relation (8)). If $i>3$, and

$$\begin{aligned}M&=I+\psi(a_2^{-1}a_1^{-1})+\psi(a_2^{-1}a_1^{-1})^2+\ldots+\psi(a_2^{-1}a_1^{-1})^5,\\N&=I+\psi(e_{s+n}^{-1}u_1^{-1}),\end{aligned}$$

then Relation (6.5) gives

$$0=[u_1]\otimes N\xi_i+[e_{s+n}]\otimes\psi({u_1}^{-1})N\xi_i-[a_1]\otimes M\xi_i-[a_2]\otimes\psi(a_1^{-1})M\xi_i.$$

If we now assume that $i>g$, then we get

$$\begin{aligned}0&=2[u_1]\otimes\xi_i+2[e_{s+n}]\otimes\xi_i-6[a_1]\otimes\xi_i-6[a_2]\otimes\xi_i\\&=2u_{1,i}+2[e_{s+n}]\otimes\xi_i.\end{aligned} \tag{6.6}$$

As we observed in the formula (4.1),

$$e_{s+n}a_2u_2a_1=a_2u_2,$$

hence if $i > 3$, then

$$
\begin{aligned}
0 &= ([e_{s+n}] + [a_2] + [u_2] + [a_1] - [a_2] - [u_2]) \otimes \xi_i \\
&= [e_{s+n}] \otimes \xi_i + a_{1,i}.
\end{aligned}
\tag{6.7}
$$

By combining formulas (6.6) and (6.7) we get

$$2u_{1,i} = 0, \quad \text{for } i > 3.$$

Note that at this point we proved

Proposition 6.1. *Let $s + n \geq 1$. Then $H_1(\mathcal{PM}^+(N^n_{3,s}); H_1(N^n_{3,s}; \mathbb{Z}))$ is generated by*

$$
\begin{aligned}
\{&a_{1,1} + a_{1,2}, a_{1,3}, a_{1,4}, \ldots, a_{1,2+s+n}, \\
&u_{1,3}, u_{1,4}, \ldots, u_{1,2+s+n}, d_{1,1}, \ldots, d_{s-1,1}\}
\end{aligned}
$$

and each of these generators has order at most 2. □

For the rest of this section assume that $g \geq 4$.

The relation

$$e_j a_3 = a_3 e_j \quad \text{for } j = 1, \ldots, s + n - 1,$$

gives

$$
\begin{aligned}
0 &= [e_j] \otimes (I - \psi(a_3^{-1}))\gamma_1 - [a_3] \otimes (I - \psi(e_j^{-1}))\gamma_1 = \\
&= a_{3,1} + a_{3,2} + a_{3,g+1} + \ldots + a_{3,g+j}.
\end{aligned}
$$

Together with the formula (6.1) this implies that Generators (K1): $a_{j,i}$ are trivial for $i > g$.

Observe that relations

$$u_j u_{j+1} u_j = u_{j+1} u_j u_{j+1}, \quad \text{for } j = 1, 2,$$

easily imply that

$$u_{3,i} = u_{2,i} = u_{1,i}, \quad \text{for } i > g.$$

Hence, the relation

$$e_j u_3 = u_3 e_j, \quad \text{for } j = 1, \ldots, s + n - 1,$$

gives

$$
\begin{aligned}
0 &= [e_j] \otimes (I - \psi(u_3^{-1}))\gamma_1 - [u_3] \otimes (I - \psi(e_j^{-1}))\gamma_1 = \\
&= u_{3,1} + u_{3,2} + u_{3,g+1} + \ldots + u_{3,g+j} = \\
&= u_{3,1} + u_{3,2} + u_{1,g+1} + \ldots + u_{1,g+j}.
\end{aligned}
$$

This implies that Generators (K3): $u_{1,i}$ are trivial for $i > g$.

Relation

$$e_j b_1 = b_1 e_j, \quad \text{for } j = 1, \ldots, s + n - 1,$$

gives

$$
\begin{aligned}
0 &= [e_j] \otimes (I - \psi(b_1^{-1}))(\gamma_1 - \gamma_3) - [b_1] \otimes (I - \psi(e_j^{-1}))(\gamma_1 - \gamma_3) = \\
&= b_{1,1} + b_{1,2} + b_{1,g+1} + \ldots + b_{1,g+j}.
\end{aligned}
$$

This implies that Generators (K8): $b_{1,i}$ are trivial for $i > g$.

At this point we proved

Proposition 6.2. *Let $g > 3$ and $s+n \geq 1$. Then $H_1(\mathcal{PM}^+(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ is generated by*

$$\begin{cases} \{a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, d_{1,1}, \dots, d_{s-1,1}\} & \text{if } g = 4, \\ \{a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}\} & \text{if } g = 5, 6, \\ \{u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}\} & \text{if } g \geq 7, \end{cases}$$

and each of these generators has order at most 2. □

7. Bounding $H_1(\mathcal{PM}^k(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ from above

As in the previous section, we will use the formula (2.2) to reduce the generating set for the group $H_1(\mathcal{PM}^+(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ to the one specified in the statements of Propositions 7.1 and 7.2 below.

By Proposition 5.2, $H_1(\mathcal{PM}^k(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ is generated by generators of the group $H_1(\mathcal{PM}^+(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ and additionally Generators (K12) corresponding to puncture slides $v_{k+1}, \dots, v_n$ (see Proposition 4.5). All the computations from the previous section hold true, hence $H_1(\mathcal{PM}^k(N_{g,s}^n); H_1(N_{g,s}^n;\mathbb{Z}))$ is generated by Generators (K12): $\widetilde{v}_{j,i}$ and elements specified in the statements of Propositions 6.1 and 6.2.

Note that for any $x \in \mathcal{PM}^+(N_{g,s}^n)$ and $k < j \leq n$,

$$y = v_j^{-1} x v_j \in \mathcal{PM}^+(N_{g,s}^n),$$

hence both x and y are products of generators of $\mathcal{PM}^+(N_{g,s}^n)$ (that is these products do not contain puncture slides). Therefore, the relation

$$x v_j = v_j y$$

gives

$$\begin{aligned} 0 &= [x] \otimes \gamma_i + [v_j] \otimes \psi(x^{-1})\gamma_i - [v_j] \otimes \gamma_i - [y] \otimes \psi(v_j^{-1})\gamma_i = \\ &= [v_j] \otimes (\psi(x^{-1}) - I)\gamma_i + A_{j,i}^x = \sum_{r=1}^{g+s+n-1} m_r v_{j,r} + A_{j,i}^x = \\ &= \sum_{r=1}^{g+s+n-1} m_r \widetilde{v}_{j,r} + \widetilde{A}_{j,i}^x, \end{aligned}$$

for some coefficients m_r and expressions $A_{j,i}^x$, $\widetilde{A}_{j,i}^x$ which contain neither $v_{j,r}$ nor $\widetilde{v}_{j,r}$. Moreover, by Proposition 5.2,

$$\sum_{r=1}^{g+s+n-1} m_r \widetilde{v}_{j,r} = -\widetilde{A}_{j,i}^x$$

is an element of the kernel $\ker \overline{\partial}_1$, hence this element is a linear combination of generators specified in the statements of Propositions 6.1 and 6.2.

Now we use the above general analysis to two special cases: $x = a_i$, $i = 1, \dots, g-1$ and $x = e_i$, $i = 1, \dots, s+n-1$.

In the first case we get

$$0 = [v_j] \otimes (\psi(a_i^{-1}) - I)\gamma_i + A_{j,i}^{a_i} = \widetilde{v}_{j,i} + \widetilde{v}_{j,i+1} + \widetilde{A}_{j,i}^{a_i}.$$

This implies that generators $\widetilde{v}_{j,2}, \ldots, \widetilde{v}_{j,g}$ are superfluous.

In the second case we get

$$0 = [v_j] \otimes (\psi(e_i^{-1}) - I)\gamma_i + A_{j,i}^{e_i} = \widetilde{v}_{j,1} + \widetilde{v}_{j,2} + \widetilde{v}_{j,g+1} + \ldots + \widetilde{v}_{j,g+i} + \widetilde{A}_{j,i}^{a_i}.$$

This implies that generators $\widetilde{v}_{j,g+1}, \ldots, \widetilde{v}_{j,g+s+n-1}$ are superfluous.

Relations $a_1 v_j = v_j a_1$ and $u_1 v_j = v_j u_1$ give

$$0 = [v_j] \otimes (\psi(a_i^{-1}) - I)\gamma_i - [a_1] \otimes (\psi(v_j^{-1}) - I)\gamma_i = v_{j,1} + v_{j,2},$$
$$0 = [v_j] \otimes (\psi(u_i^{-1}) - I)\gamma_i - [u_1] \otimes (\psi(v_j^{-1}) - I)\gamma_i = v_{j,1} - v_{j,2}.$$

respectively. This implies that $2v_{j,1} = 2\widetilde{v}_{j,1} = 0$ and we proved

Proposition 7.1. *Let $g > 3$, $s + n \geq 1$ and $0 \leq k \leq n$. Then the group $H_1(\mathcal{PM}^k(N_{g,s}^n); H_1(N_{g,s}^n; \mathbb{Z}))$ is generated by*

$$\begin{cases} \{a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, d_{1,1}, \ldots, d_{s-1,1}, v_{k+1,1}, \ldots, v_{n,1}\} & \text{if } g = 4, \\ \{a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, v_{k+1,1}, \ldots, v_{n,1}\} & \text{if } g = 5, 6, \\ \{u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, v_{k+1,1}, \ldots, v_{n,1}\} & \text{if } g \geq 7, \end{cases}$$

and each of these generators has order at most 2. □

For the rest of this section assume that $g = 3$.

If $j < n$, then relations $a_1 v_j = v_j a_1$ and $u_1 v_j = v_j u_1$ give

$$0 = [v_j] \otimes (\psi(a_1^{-1}) - I)\gamma_3 - [a_1] \otimes (\psi(v_j^{-1}) - I)\gamma_3 = -a_{1,3+s+j},$$
$$0 = [v_j] \otimes (\psi(u_1^{-1}) - I)\gamma_3 - [u_1] \otimes (\psi(v_j^{-1}) - I)\gamma_3 = -u_{1,3+s+j},$$

respectively. Hence, $a_{1,i} = u_{1,i} = 0$ if $i > 3 + s + k$.

Finally, if $s + k > 0$, then relations $a_1 v_n = v_n a_1$ and $u_1 v_n = v_n u_1$ give

$$\begin{aligned} 0 &= [v_n] \otimes (\psi(a_1^{-1}) - I)\gamma_3 - [a_1] \otimes (\psi(v_n^{-1}) - I)\gamma_3 = \\ &= 2(a_{1,1} + a_{1,2}) + 2a_{1,3} + a_{1,4} + \ldots + a_{1,2+s+n} = \\ &= a_{1,4} + \ldots + a_{1,3+s+k}, \\ 0 &= [v_n] \otimes (\psi(u_1^{-1}) - I)\gamma_3 - [u_1] \otimes (\psi(v_n^{-1}) - I)\gamma_3 = \\ &= 2(u_{1,1} + u_{1,2}) + 2u_{1,3} + u_{1,4} + \ldots + u_{1,2+s+n} = \\ &= u_{1,4} + \ldots + u_{1,3+s+k}, \end{aligned}$$

respectively. Hence, $a_{1,3+s+k}$ and $u_{1,3+s+k}$ are superfluous provided $s + k > 0$. This proves

Proposition 7.2. *Let* $s+n \geq 1$ *and* $0 \leq k \leq n$. *Then* $H_1(\mathcal{PM}^k(N^n_{3,s}); H_1(N^n_{3,s};\mathbb{Z}))$ *is generated by*

$$\begin{cases} \{a_{1,1}+a_{1,2}, a_{1,3}, u_{1,3}, v_{k+1,1}, \ldots, v_{n,1}\} & \text{if } s+k=0, \\ \{a_{1,1}+a_{1,2}, a_{1,3}, a_{1,4}, \ldots, a_{1,2+s+k}, \\ \quad u_{1,3}, u_{1,4}, \ldots, u_{1,2+s+k}, \\ \quad d_{1,1}, \ldots, d_{s-1,1}, v_{k+1,1}, \ldots, v_{n,1}\} & \text{if } s+k>0, \end{cases}$$

and each of these generators has order at most 2. □

8. Bounding $H_1(\mathcal{M}(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z}))$ from above

As in the previous two sections, we will use the formula (2.2) to reduce the generating set for the group $H_1(\mathcal{M}(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z}))$ to the one specified in the statement of Proposition 8.1 below.

By Proposition 5.3, $H_1(\mathcal{M}(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z}))$ is generated by generators of the group $H_1(\mathcal{PM}^+(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z}))$, Generators (K12) corresponding to puncture slide v_n and additionally Generators (K13)–(K16) corresponding to elementary braids: $s_1, \ldots, s_{n-1}$. All computations from the previous two sections hold true, hence $H_1(\mathcal{M}(N^n_{g,s}); H_1(N^n_{g,s};\mathbb{Z}))$ is generated by generator (K12): $v_{n,1}$, generators (K13)–(K16), and elements specified in the statements of Propositions 6.1 and 6.2. Moreover if $g=3$ and $i \geq 3+s$, then $a_{1,i}$ and $u_{1,i}$ are superfluous.

If $i \neq s+j$, then the relation

$$e_i s_j = s_j e_i$$

gives

$$\begin{aligned} 0 &= [s_j] \otimes (\psi(e_i^{-1}) - I)\gamma_1 - [e_i] \otimes (\psi(s_j^{-1}) - I)\gamma_1 \\ &= s_{j,1} + s_{j,2} + s_{j,g+1} + \ldots + s_{j,g+i} = A_{j,i}. \end{aligned}$$

In particular, Generator (K14)

$$s_{j,g+s+j} + s_{j,g+s+j+1} = A_{j,s+j+1} - A_{j,s+j-1} = 0$$

is trivial and Generators (K13) of the form

$$s_{j,g+i} = \begin{cases} A_{j,i} - A_{j,i-1} = 0 & \text{if } i > 1, \\ A_{j,1} - (s_{j,1} + s_{j,2}) & \text{if } i = 1 \end{cases}$$

are superfluous.

The relation

$$s_j a_i = a_i s_j \quad \text{for } i < g$$

gives

$$0 = [s_j] \otimes (\psi(a_i^{-1}) - I)\gamma_i - [a_i] \otimes (\psi(s_j^{-1}) - I)\gamma_i = s_{j,i} + s_{j,i+1}. \tag{8.1}$$

Relation $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ gives

$$\begin{aligned} 0 &= ([s_j] + s_j[s_{j+1}] + s_j s_{j+1}[s_j] - [s_{j+1}] - s_{j+1}[s_j] - s_{j+1}s_j[s_{j+1}]) \otimes \gamma_1 \\ &= s_{j,i} + s_{j+1,i} + s_{j,i} - s_{j+1,i} - s_{j,i} - s_{j+1,i} = s_{j,i} - s_{j+1,i}. \end{aligned}$$

This together with the formula (8.1) implies that Generators (K13) generate a cyclic group. Moreover, the relation

$$s_j u_1 = u_1 s_j$$

implies that

$$0 = [s_j] \otimes (\psi(u_1^{-1}) - I)\gamma_1 - [u_1] \otimes (\psi(s_j^{-1}) - I)\gamma_1 = s_{j,1} - s_{j,2},$$

which together with the formula (8.1) implies that the cyclic group generated by generators (K13) has order at most two.

By Proposition 4.7,

$$e_{s+j-1}e_{s+j+1}s_j = e_{s+j}s_j^3 e_{s+j},$$

and this relation gives

(8.2)
$$\begin{aligned}
0 =& ([e_{s+j-1}] + e_{s+j-1}[e_{s+j+1}] + e_{s+j-1}e_{s+j+1}[s_j]) \otimes \gamma_1\\
&- ([e_{s+j}] + e_{s+j}(1 + s_j + s_j^2)[s_j] - e_{s+j}s_j^3[e_{s+j}]) \otimes \gamma_1\\
=& e_{s+j-1,1} + [e_{s+j}] \otimes (-I - \psi(s_j^{-3})\psi(e_{s+j}^{-1})\gamma_1 + [e_{s+j+1}] \otimes \psi(e_{s+j-1}^{-1})\gamma_1\\
&+ [s_j] \otimes \Big(\psi(e_{s+j+1}^{-1})\psi(e_{s+j-1}^{-1}) - (I + \psi(s_j^{-1}) + \psi(s_j^{-2}))\psi(e_{s+j}^{-1})\Big)\gamma_1.
\end{aligned}$$

If $j < n-1$ this gives

$$\begin{aligned}
0 =& (s_{j,g+s+j} - e_{s+j-1,1} + 2e_{s+j,1} - e_{s+j+1,1})\\
&+ (e_{s+j,1} + e_{s+j,2}) + (e_{s+j,g+1} + \ldots + e_{s+j,g+s+j-1} + e_{s+j,g+s+j+1})\\
&- (e_{s+j+1,1} + e_{s+j+1,2}) - (e_{s+j+1,g+1} + \ldots + e_{s+j+1,g+s+j-1}) + (*).
\end{aligned}$$

This implies that Generator (K15) is superfluous. If $j = n-1$, then the formula (8.2) yields a more complicated expression, however it is also of the form

$$0 = s_{n-1,g+s+n-1} + (*),$$

where $(*)$ denotes some expression which does not contain $s_{j,i}$. This implies that Generator (K16) is also superfluous, and we proved that

Proposition 8.1. *Let $g \geq 3$ and $n \geq 2$. Then $H_1(\mathcal{M}(N_{g,s}^n); H_1(N_{g,s}^n; \mathbb{Z}))$ is generated by*

$$\begin{cases}
a_{1,1} + a_{1,2}, a_{1,3}, u_{1,3}, v_{n,1}, s_{1,1} & \text{if } g = 3 \text{ and } s = 0,\\
a_{1,1} + a_{1,2}, a_{1,3}, a_{1,4}, \ldots, a_{1,2+s}, & \\
\quad u_{1,3}, u_{1,4}, \ldots, u_{1,2+s}, & \\
\quad d_{1,1}, \ldots, d_{s-1,1}, v_{n,1}, s_{1,1} & \text{if } g = 3 \text{ and } s > 0,\\
a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, d_{1,1}, \ldots, d_{s-1,1}, v_{n,1}, s_{1,1} & \text{if } g = 4,\\
a_{1,3}, u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, v_{n,1}, s_{1,1} & \text{if } g = 5 \text{ or } g = 6,\\
u_{1,3}, b_{1,1} - a_{1,1} - a_{3,3}, v_{n,1}, s_{1,1} & \text{if } g \geq 7,
\end{cases}$$

and each of these generators has order at most 2. □

9. Bounding $H_1(\mathcal{PM}^k(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))$ from below

In this section we use various quotients of $\mathcal{PM}^k(N_{g,s}^n)$ in order to prove that all homology classes specified in Propositions 7.1 and 7.2 are nontrivial. This will complete the proof of Theorem 1.1.

If we glue a disk to each boundary components of $N_{g,s}^n$ and forget about punctures, then we get a closed non–orientable surface N_g of genus g. If

$$i\colon N_{g,s}^n \to N_g$$

is the corresponding inclusion map, then i induces homomorphisms

$$\begin{array}{ccc} \mathcal{PM}^k(N_{g,s}^n) & \xrightarrow{i_*} & \mathcal{M}(N_g) \\ \big\downarrow\psi & & \big\downarrow\psi \\ \mathrm{Aut}(H_1(N_{g,s}^n;\mathbb{Z})) & \xrightarrow{i_*} & \mathrm{Aut}(H_1(N_g;\mathbb{Z})) \end{array}$$

This leads to the following homomorphism

$$H_1(\mathcal{PM}^k(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z})) \xrightarrow{i_*} H_1(\mathcal{M}(N_g);H_1(N_g;\mathbb{Z})).$$

Moreover, by Theorem 1.1 of [23] (see the very last formula in the proof of that theorem), we have

$$\begin{aligned} i_*(a_{1,1}+a_{1,2}) &\neq 0 \quad \text{if } g=3,\\ i_*(u_{1,3}) &\neq 0 \quad \text{if } g\geq 3,\\ i_*(a_{1,3}) &\neq 0 \quad \text{if } g\in\{3,4,5,6\},\\ i_*(b_{1,1}-a_{1,1}-a_{3,3}) &\neq 0 \quad \text{if } g\geq 4 \end{aligned}$$

and all these classes are linearly independent.

In order to prove that homology classes corresponding to puncture slides are nontrivial, fix $k<j\leq n$ and consider the following homomorphisms

$$\begin{aligned} \alpha\colon \mathcal{PM}^+(N_{g,s}^n) &\to \mathbb{Z}_2,\\ \beta\colon H_1(N_{g,s}^n;\mathbb{Z}) &\to \mathbb{Z}_2. \end{aligned}$$

The first homomorphism is defined as follows: $\alpha(f)=1$ if and only if f changes the local orientation around the puncture z_j. The second one is the composition

$$H_1(N_{g,s}^n;\mathbb{Z}) \longrightarrow H_1(N_{g,s}^n;\mathbb{Z}_2) \longrightarrow \langle\gamma_1\rangle.$$

of the reduction to $\mathbb{Z}_2$ coefficients and the projection:

$$\begin{cases} \gamma_1,\gamma_2,\ldots,\gamma_g \longmapsto \gamma_1,\\ \delta_1,\delta_2,\ldots,\delta_{s+n-1} \longmapsto 0. \end{cases}$$

It is straightforward to check that for any $m\in H_1(N_{g,s}^n;\mathbb{Z})$ and $f\in\mathcal{PM}^+(N_{g,s}^n)$

$$\beta(f(m))=\beta(m).$$

Hence, if we regard $\langle\gamma\rangle$ as a trivial $\alpha(\mathcal{PM}^+(N_{g,s}^n))$ module, then (α,β) induce homomorphism

$$H_1(\mathcal{PM}^k(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))\xrightarrow{(\alpha,\beta)}H_1(\mathbb{Z}_2;\mathbb{Z}_2)\cong\mathbb{Z}_2.$$

Moreover, if x is one of the generators specified in the statements of Propositions 7.1 and 7.2, then

$$(\alpha,\beta)(x)\neq 0\iff x=v_{j,1}.$$

This implies that $v_{j,1}$ is nontrivial and independent from other generators.

If $s\geq 2$ and $g\leq 4$, then for any fixed $1\leq j\leq s-1$ there is a homomorphism

$$H_1(\mathcal{PM}^k(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))\xrightarrow{i_*}H_1(\mathcal{M}(N_{g+2});H_1(N_{g+2};\mathbb{Z}))$$

induced by the inclusion $i\colon N_{g,s}^n\to N_{g+2}$, where N_{g+2} is a closed non–orientable surface of genus $g+2$ obtained from $N_{g,s}^n$, by forgetting the punctures, connecting boundary components of numbers $1\leq j\leq s-1$ and s by a cylinder, and gluing a disk to all the remaining boundary components.

Moreover, δ_j becomes a two–sided nonseparating circle in N_{g+2}, hence we can choose generators for $\mathcal{M}(N_{g+2})$ so that

$$i_*(d_{j,1})=a_{1,3}\in H_1(\mathcal{M}(N_{g+2});H_1(N_{g+2};\mathbb{Z})).$$

By Theorem 1.1 of [23], this homology class is nontrivial provided $g+2\leq 6$. This completes the proof of Theorem 1.1 if $g>3$ or $s+k\leq 1$.

Hence, assume that $g=3$ and $s+k\geq 2$. Fix $1\leq j\leq s+k-1$ and and glue a disk with a puncture to each boundary component of $N_{3,s}^n$. Then forget about all the punctures except those with numbers j and $s+k$. As a result we obtain an inclusion

$$i\colon N_{3,s}^n\to N_3^2,$$

which leads to a homomorphism

$$H_1(\mathcal{PM}^k(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))\xrightarrow{i_*}H_1(\mathcal{M}(N_3^2);H_1(N_3^2;\mathbb{Z})).$$

By Theorem 1.2 of [16],

$$i_*(a_{1,3+j})=a_{1,4}\neq 0,$$
$$i_*(u_{1,3+j})=u_{1,4}\neq 0.$$

This implies that generators $a_{1,4},\dots,a_{1,2+s+k}$, $u_{1,4},\dots,u_{1,2+s+k}$ are nontrivial and linearly independent. This concludes the proof of Theorem 1.1.

10. Bounding $H_1(\mathcal{M}(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))$ from below

In this section we will show that all generators specified in the statement of Proposition 8.1 are nontrivial and linearly independent. This will complete the proof of Theorem 1.2.

As in the previous section, we argue that homology classes

$$
\begin{cases}
a_{1,1}+a_{1,2} & \text{if } g=3,\\
a_{1,4},\dots a_{1,2+s} & \text{if } g=3 \text{ and } s>1,\\
u_{1,4},\dots u_{1,2+s} & \text{if } g=3 \text{ and } s>1,\\
a_{1,3} & \text{if } g<7,\\
u_{1,3}, & \\
b_{1,1}-a_{1,1}-a_{3,3} & \text{if } g\geq 4,\\
d_{j,1} & \text{if } g\leq 4 \text{ and } 1\leq j\leq s-1
\end{cases} \tag{10.1}
$$

are nontrivial and independent. Hence, it is enough to show that if

$$0=A+\nu v_{n,1}+\mu s_{1,1},$$

where A is a linear combination of generators (10.1), then $\nu=\mu=0$. Let

$$\beta\colon H_1(N_{g,s}^n;\mathbb{Z})\to\mathbb{Z}_2$$

be defined as in the previous section and define

$$\alpha\colon \mathcal{M}(N_{g,s}^n)\to\mathbb{Z}_2$$

as follows: $\alpha(f)=1$ if and only if f changes the local orientation around an odd number of punctures. Then there is an induced homomorphism

$$H_1(\mathcal{M}(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))\xrightarrow{(\alpha,\beta)}H_1(\mathbb{Z}_2;\mathbb{Z}_2)\cong\mathbb{Z}_2$$

and

$$0=(\alpha,\beta)(A+\nu v_{n,1}+\mu s_{1,1})=\nu.$$

Now define

$$\alpha'\colon \mathcal{M}(N_{g,s}^n)\to\mathbb{Z}_2$$

to be the sign of the permutation

$$(z_1,\dots,z_n)\mapsto(f(z_1),f(z_2),\dots,f(z_n)),$$

that is $\alpha'(f)=1$ if and only if the above permutation is odd. As before, there is an induced homomorphism

$$H_1(\mathcal{M}(N_{g,s}^n);H_1(N_{g,s}^n;\mathbb{Z}))\xrightarrow{(\alpha',\beta)}H_1(\mathbb{Z}_2;\mathbb{Z}_2)\cong\mathbb{Z}_2$$

and

$$0=(\alpha,\beta)(A+\nu v_{n,1}+\mu s_{1,1})=\mu.$$

This concludes the proof of Theorem 1.2.

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