L^p ESTIMATES FOR THE RIESZ TRANSFORMS ASSOCIATED WITH PARABOLIC SCHRÖDINGER TYPE OPERATORS

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ABSTRACT. Let $\mathcal{P}_2 = \frac{\partial}{\partial t} + (-\Delta)^2 + V^2$ be the parabolic Schrödinger type operator on $\mathbb{R}^{n+1}(n \geq 5)$, where the nonnegative potential V is independent of t and belongs to the reverse Hölder class $RH_s(s \geq n/2)$ and the Gaussian class associated with $(-\Delta)^2$. In this paper, we will establish the L^p estimates for the Riesz transforms $V^{2\alpha}\nabla^j \mathcal{P}_2^{-\beta}$, where $j = 0, 1, 2, 3,\ 0 < \alpha \leq 1 - j/4,\ j/4 < \beta \leq 1$, and $\beta - \alpha \geq j/4$.

1. Introduction and results

For $1 < s < \infty$, a nonnegative locally L^s-integrable function V is said to belong to the reverse Hölder class RH_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^s dy\right)^{1/s} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$.

Clearly, if V belongs to RH_s with $s > 1$, then V is a Muckenhoupt A_∞ weight; see [10]. From weight theory, we know that $V(x)dx$ is a doubling measure, and the class RH_s has self-improvement property [14]; that is, if $V \in RH_s$ for some $s > 1$, then there exists $\epsilon > 0$ such that $V \in RH_{s+\epsilon}$.

Let $\mathcal{H}_2 = (-\Delta)^2 + V^2$ be the Schrödinger type operator on $\mathbb{R}^n$, where $n \geq 5$. In [16], it is stated that a locally integrable function V belongs to the Gaussian class associated with $(-\Delta)^2$, denoted by $V \in G((-\Delta)^2)$, if the operator $\mathcal{H}_2$ generates a C_0 semigroup $\{e^{-t\mathcal{H}_2}\}_{t>0}$ on $L^2(\mathbb{R}^n)$ whose kernel $e^{-t\mathcal{H}_2}(x, y)$ satisfies the Gaussian upper bound. Specifically, there exist constants C and c such that

$$|e^{-t\mathcal{H}_2}(x, y)| \leq Ct^{-\frac{n}{4}} exp\left\{ - c\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}} \right\}$$

holds for all $x, y \in \mathbb{R}^n$ and $t > 0$.

It is a non-trivial fact that the class $G((-\Delta)^2)$ is not empty, and an example is given in [2].

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Consider the parabolic Schrödinger type operator

$$\mathcal{P}_k = \frac{\partial}{\partial t} + (-\Delta)^k + V^k, \quad k = 1, 2$$

on $\mathbb{R}^{n+1}(n \geq 2k+1)$, where V is a nonnegative potential and independent of the variable t. When $V \in RH_s$ for $s \geq n/2$, Gao and Jiang in [7] showed the L^p boundedness of $V\mathcal{P}_1^{-1}$. Based on this result, they obtained the L^p boundedness of operator $\nabla^2\mathcal{P}_1^{-1}$. Carbonaro et al. in [4] improved the results in [7] by the potential V with the variables x, t, which is essentially the generalization to $\mathbb{R}^{n+1}$ of the condition of [7]. Suppose $V \in RH_s$ with $n/2 \leq s < n$, Tang and Han [15] established the L^p boundedness of $\nabla\mathcal{P}_1^{-\frac{1}{2}}$; they also obtained the L^p boundedness of $V^{\frac{1}{2}}\nabla\mathcal{P}_1^{-1}$. As for the parabolic Schrödinger type operator $\mathcal{P}_2$, Liu et al. established estimate for the fundamental solution of $\frac{\partial u}{\partial t} + (-\Delta)^2 u + V^2 u + \lambda u = 0$ in [9], and obtained the L^p estimate for $V^{2\alpha}\mathcal{P}_2^{-\alpha}, 0 < \alpha \leq 1$. Using this result, they got the L^p boundedness of operator $\nabla^4\mathcal{P}_2^{-1}$.

We recall some related results of the Schrödinger operator $\mathcal{H}_1 = -\Delta + V$ and the Schrödinger type operator $\mathcal{H}_2$. In [12], Sugano researched the (L^p, L^q) boundedness of operator $V^\alpha\nabla^j\mathcal{H}_1^{-\beta}$ for $j = 0, 1, 0 < \alpha \leq \beta \leq 1$. Liu in [8] extended the result of [12] to the stratified Lie group. Recently, Wang in [18] obtained the (L^p, L^q) boundedness of operator $V^{2\alpha}\mathcal{H}_2^{-\beta}$ for $0 < \alpha \leq \beta \leq 1$, the author of this article in [17] obtained the (L^p, L^q) boundedness of operators $V^{2\alpha}\nabla^j\mathcal{H}_2^{-\beta}$ for $j = 1, 2, 3,\ 0 < \alpha \leq 1 - j/4,\ j/4 < \beta \leq 1$, and $\beta - \alpha \geq j/4$.

Inspired by the above results, in this paper, we concentrate on the (L^p, L^q) boundedness of Riesz transforms

$$T_j(f)(x,t) = V^{2\alpha}\nabla^j\mathcal{P}_2^{-\beta}(f)(x,t), \quad j = 0, 1, 2, 3$$

and their adjoint operators T_j^*. The following (L^p, L^q) estimates are established.

Theorem 1.1. *For $j = 0, 1, 2, 3$, let $0 < \alpha \leq 1 - j/4, j/4 < \beta \leq 1$, and $\beta - \alpha \geq j/4$. Suppose $V \in G((-\Delta)^2) \bigcap RH_s$ with $s \geq \frac{2n}{4-j}$.*

(i) *If $1 < p \leq \left(\frac{2\alpha}{s} + \frac{4(\beta-\alpha)-j}{n+4}\right)^{-1}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$, then*

$$\|T_j(f)\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{n+1})};$$

(ii) *If $\left(\frac{s}{2\alpha}\right)' \leq p < \frac{n+4}{4(\beta-\alpha)-j}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$, then*

$$\|T_j^*(f)\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

Theorem 1.2. *For $j = 1, 2, 3$, let $0 < \alpha \le 1 - j/4, j/4 < \beta \le 1$, and $\beta - \alpha \ge j/4$. Suppose $V \in G((-\Delta)^2) \bigcap RH_s$ with $\frac{n}{2} \le s < \frac{2n}{4-j}$.*

(i) *If $1 < p \le \left(\frac{1}{p_\alpha} + \frac{4(\beta-\alpha)-j}{n+4}\right)^{-1}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$, then*

$$\|T_j(f)\|_{L^q(\mathbb{R}^{n+1})} \le C\|f\|_{L^p(\mathbb{R}^{n+1})};$$

(ii) *If $p'_\alpha \le p < \frac{n+4}{4(\beta-\alpha)-j}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$, then*

$$\|T_j^*(f)\|_{L^q(\mathbb{R}^{n+1})} \le C\|f\|_{L^p(\mathbb{R}^{n+1})},$$

where $\frac{1}{p_\alpha} = \frac{2\alpha+2}{s} - \frac{4-j}{n}$.

We collect the notation to be used throughout this paper:

$$\nabla_x = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}), \quad B = B(x_0, r) = \{y \in \mathbb{R}^n : \ |y - x_0| < r\},$$
$$Q = Q_r(x_0, t_0) = \{(x,t) \in \mathbb{R}^{n+1} : |x - x_0| < r, \ t_0 - r^4 < t \le t_0\}.$$

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. We also use C_N to represent a constant related to a positive integer N.

2. Preliminaries

Throughout this section we always assume $V \in RH_s$ with $s \ge n/2$.

We recall the definition of the auxiliary function (see [11])

$$\frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le 1\right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < m(x,V) < \infty$ for any $x \in \mathbb{R}^n$.

Some important properties concerning the auxiliary function $m(x,V)$ have been proved by Shen in [11].

Lemma 2.1. *There exists constant $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim (1 + rm(x,V))^{l_0}.$$

Lemma 2.2. *For $0 < r < R < \infty$, we have*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim \left(\frac{R}{r}\right)^{n/s-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.3. *There exist $k_0 \ge 1$ such that*

$$(1 + |x-y|m(x,V))^{-k_0} \lesssim \frac{m(x,V)}{m(y,V)} \lesssim (1 + |x-y|m(x,V))^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

A ball $B(x,\frac{1}{m(x,V)})$ is called critical. For $x\in B(x_0,\frac{1}{m(x_0,V)})$, we have $m(x,V)\sim m(y,V)$, if $|x-y|<\frac{C}{m(x,V)}$.

We endow the space $\mathbb{R}^{n+1}$ with the following parabolic metric which is different from the usual Euclidean metric:

$$d((x,t),(y,s))=\max\{|x-y|,|t-s|^{1/4}\}.$$

It is easy to get the following results from Lemma 2.3.

Lemma 2.4. *There exist $k_0\geq 1$ such that*

$$\Big(1+d\big((x,t),(y,s)\big)m(x,V)\Big)^{-k_0}\lesssim\frac{m(x,V)}{m(y,V)}\lesssim\Big(1+d\big((x,t),(y,s)\big)m(x,V)\Big)^{\frac{k_0}{1+k_0}}$$

for all $(x,t),(y,s)\in\mathbb{R}^{n+1}$.

By Lemma 2.4 we get

Lemma 2.5. *There exist $k_0\geq 1$ such that*

$$\begin{aligned}\Big(1+d\big((x,t),(y,s)\big)\ m(y,V)\Big)^{1/(1+k_0)}&\lesssim 1+d\big((x,t),(y,s)\big)m(x,V)\\&\lesssim\Big(1+d\big((x,t),(y,s)\big)m(y,V)\Big)^{1+k_0}\end{aligned}$$

for all $(x,t),(y,s)\in\mathbb{R}^{n+1}$.

Let $\Gamma_{\mathcal{P}_2}(x,y;y,s;\lambda)$ be the fundamental solution of operator $\mathcal{P}_2=\frac{\partial}{\partial t}+(-\Delta)^2+V^2+\lambda$. By Proposition 4.6 in [16], we have

Lemma 2.6. *Suppose $V\in G((-\Delta)^2)\bigcap RH_s$ with $s\geq n/2$. Then for any positive integer N, there exists a constant C_N such that*

$$\begin{aligned}&\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)\\&\leq\frac{C_N}{\big(1+\lambda^{\frac14}d((x,t),(y,s))\big)^N\big(1+m(x,V)d((x,t),(y,s))\big)^N}\frac{e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}}}{(t-s)^{n/4}}.\end{aligned}$$

It is follows from Lemma 2.5 that Lemma 2.6 is also valid if replace $m(x,V)$ with $m(y,V)$.

Remark 2.7. From the proof of Corollary 2 in [9] and the explanation in [2], it can be seen that the condition $V\in G((-\Delta)^2)$ in Lemma 2.6 is indispensable.

Remark 2.8. It should been noted that $V\in RH_s$ can not guarantee $V\in G((-\Delta)^2)$. For example, although the heat kernel of the bi-Laplacian $(-\Delta)^2$

satisfies

$$|e^{-t(-\Delta)^2}(x,y)| \le Ct^{-\frac{n}{4}} exp\Big\{-c\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big\},$$

but the dimension of the space has a restriction $n < 4$ (see (5.2.1) in [5]), while we working in this article is in the case $n \ge 5$. For more details, please refer to [1]-[3].

Let $\Gamma_0(x,t,y,s,\lambda)$ be the fundamental solution of $\frac{\partial}{\partial t} + (-\Delta)^2 + \lambda$. It follows from Lemma 2.6 that

$$\Gamma_0(x,t;y,s;\lambda) \le \frac{C_N}{\big(1+\lambda^{\frac{1}{4}} d((x,t),(y,s))\big)^N} \frac{e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}}}{(t-s)^{n/4}}.$$

Let $f \in L^\sigma_{loc}(\mathbb{R}^{n+1})$. Denote $|Q|$ by the Lebesgue measure of $Q \subset \mathbb{R}^{n+1}$. The fractional Hardy-Littlewood maximal function $M_{\sigma,\gamma}$ is defined by

$$M_{\sigma,\gamma}(f)(x,t) = \sup_{Q \ni (x,t)} \left(\frac{1}{|Q|^{1-\frac{\sigma\gamma}{n+4}}} \int_Q |f(y,s)|^\sigma dyds \right)^{1/\sigma}.$$

Lemma 2.9. *Suppose* $1 < \sigma < p < \frac{n+4}{\gamma}$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n+4}$. *Then*

$$\|M_{\sigma,\gamma} f\|_{L^q(\mathbb{R}^{n+1})} \le C\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

3. The proof of Theorem 1.1

We denote the kernel functions of the operator $\mathcal{P}_2^{-\beta}$ as $K_{0,\beta}(x,t;y,s)$. We first give the estimates of $K_{0,\beta}(x,t;y,s)$.

Lemma 3.1. *Let* $V \in G((-\Delta)^2) \bigcap RH_s$ *with* $s \ge \frac{n}{2}, 0 < \beta \le 1$. *Then,*

$$|K_{0,\beta}(x,t;y,s)| \lesssim \frac{1}{d((x,t),(y,s))^{n+4-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}.$$

Proof. For $\beta = 1$, by Lemma 2.6 we have

$$|K_{0,\beta}(x,t;y,s)| = |\Gamma_{\mathcal{P}_2}(x,t;y,s;0)|$$
$$\lesssim \frac{1}{\big(1+m(x,V)d((x,t),(y,s))\big)^N} \frac{e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}}}{(t-s)^{n/4}}.$$

Note that

$$\frac{1}{(t-s)^{n/4}} e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}} \lesssim \frac{1}{(d((x,t),(y,s)))^n}. \tag{3.1}$$

Then

$$|K_{0,\beta}(x,t;y,s)| \lesssim \frac{1}{d((x,t),(y,s))^n\big(1+m(x,V)d((x,t),(y,s))\big)^N}.$$

For $0<\beta<1$, by the functional calculus,

$$\mathcal{P}_2^{-\beta}=\frac{\sin\pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}\big(\mathcal{P}_2+\lambda\big)^{-1}d\lambda.$$

Let $f\in C_0^\infty(\mathbb{R}^{n+1})$. As a consequence of

$$\big(\mathcal{P}_2+\lambda\big)^{-1}(f)(x,t)=\int_{\mathbb{R}^{n+1}}\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)f(y,s)dyds,$$

we find that

$$\mathcal{P}_2^{-\beta}(f)(x,t)=\int_{\mathbb{R}^{n+1}}K_{0,\beta}(x,t;y,s)f(y,s)dyds,$$

where

$$K_{0,\beta}(x,t;y,s)=C\int_0^\infty \lambda^{-\beta}\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)d\lambda.$$

From (3.1) and noting

$$\int_0^\infty \frac{\lambda^{-\beta}}{\big(1+\lambda^{\frac14}d((x,t),(y,s))\big)^N}d\lambda\lesssim d((x,t),(y,s))^{4\beta-4}, \tag{3.2}$$

we obtain

$$|K_{0,\beta}(x,t;y,s)|\lesssim\frac{1}{d((x,t),(y,s))^{n+4-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}.$$

□

For $j=1,2,3$, we denote the kernel functions of the operators $\nabla_x^j\mathcal{P}_2^{-\beta}$ as $K_{j,\beta}(x,t;y,s)$. Let us estimate $K_{j,\beta}(x,t;y,s)$.

Lemma 3.2. *For $j=1,2,3$, let $V\in G((-\Delta)^2)\bigcap RH_s$ with $s\geq\frac{2n}{4-j}$, $j/4<\beta\leq 1$. Then*

$$|K_{j,\beta}(x,t;y,s)|\lesssim\frac{1}{d((x,t),(y,s))^{n+4+j-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}.$$

Proof. Similar to the proof of Lemma 3.1, for any $f\in C_0^\infty(\mathbb{R}^{n+1})$, we have

$$\nabla_x^j\mathcal{P}_2^{-\beta}f(x,t)=\int_{\mathbb{R}^{n+1}}K_{j,\beta}(x,t;y,s)f(y,s)dyds,$$

where

$$K_{j,\beta}(x,t;y,s)=\begin{cases}C\int_0^\infty\lambda^{-\beta}\nabla_x^j\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)d\lambda, & \text{if } j/4<\beta<1;\\ \\ \nabla_x^j\Gamma_{\mathcal{P}}(x,t;y,s;0), & \text{if } \beta=1.\end{cases}$$

Suppose $\Gamma_0(x,t;y,s;\lambda)$ is the fundamental solution of $\frac{\partial}{\partial t}+(-\Delta)^2+\lambda$. In accordance with Chapter 9 in [6], for $j=1,2,3$,

$$|\nabla_x^j\Gamma_0(x,t;y,s;\lambda)|\lesssim\frac{1}{(t-s)^{(n+j)/4}}e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}}, \tag{3.3}$$

and by a simple calculate,

$$\frac{1}{(t-s)^{(n+j)/4}}e^{-c\frac{|x-y|^{4/3}}{(t-s)^{1/3}}} \lesssim \frac{1}{(d((x,t),(y,s)))^{n+j}}, \tag{3.4}$$

where c is a positive constant.

Fix $x_0, y_0 \in \mathbb{R}^n, t_0, s_0 \in \mathbb{R}$, and let $R = \frac{1}{4}d((x_0,t_0),(y_0,s_0))$. Note that $u(x,t) = \Gamma_{\mathcal{P}_2}(x,t;y_0,s_0,\lambda)$ satisfies $\frac{\partial u}{\partial t} + (-\Delta)^2 u + (V^2+\lambda)u = 0$. Then

$$u(x,t)\eta(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma_0(x,t;z,s;\lambda)\left\{\frac{\partial}{\partial s} + (-\Delta)^2 + \lambda\right\}(u\eta)(z,s)dzds,$$

where $\eta(x,t) \in C_0^\infty(Q_{2R}(x_0,t_0))$ such that $\eta \equiv 1$ on $Q_R(x_0,t_0)$. Furthermore, η satisfies the following estimates:

$$\left|\frac{\partial \eta}{\partial s}\right| \lesssim R^{-4}, \qquad |\nabla^b(\triangle^a \eta)| \lesssim R^{-2a-b} \text{ for } 1 \le 2a+b \le 4.$$

Similar to the proof of Lemma 7 in [13] we have

$$\begin{aligned} u(x,t)\eta(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma_0(x,t;z,s;\lambda)\Big\{ &-V^2 u\eta + u\frac{\partial \eta}{\partial s} + 4\Delta(\nabla u \nabla \eta) \\ &+ 2\Delta(u\Delta\eta) - 4\nabla^2 u \nabla^2 \eta - 4\nabla u \nabla(\triangle \eta) - u(\triangle^2\eta)\Big\}dzds, \end{aligned} \tag{3.5}$$

Note

$$\int_{-\infty}^t \frac{e^{-c\frac{|x-z|^{4/3}}{(t-s)^{1/3}}}}{(t-s)^{(n+j)/4}}ds \lesssim \frac{1}{|z-x|^{n-(4-j)}}, \tag{3.6}$$

Then, using (3.3)-(3.6), Lemma 2.6 and by integration by parts we get

$$\begin{aligned} &|\nabla_x^j \Gamma_{\mathcal{P}_2}(x_0,t_0;y_0,s_0;\lambda)| \\ &\lesssim \frac{1}{(1+\lambda^{\frac{1}{4}}R)^N(1+m(x_0,V)R)^N}\left(\frac{1}{R^n}\int_{B(x_0,R)}\frac{V^2(z)dz}{|z-x_0|^{n-(4-j)}} + \frac{1}{R^{n+j}}\right). \end{aligned} \tag{3.7}$$

By the self-improvement of class RH_s we can take some $s > 2n/(4-j)$. Choose $t > 1$ such that $2/s + 1/t = 1$. By Hölder inequality and Lemma 2.1,

$$\begin{aligned} &\int_{B(x_0,R)} \frac{V^2(z)dz}{|z-x_0|^{n-(4-j)}} \\ &\lesssim R^n \left(\frac{1}{R^n}\int_{B(x_0,R)} V(z)^s \mathrm{d}z\right)^{2/s}\left(\frac{1}{R^n}\int_{B(x_0,R)}\frac{\mathrm{d}z}{|z-x_0|^{(n-4+j)t}}\right)^{1/t} \\ &\lesssim R^{n-4}\left(\frac{1}{R^{n-2}}\int_{B(x_0,R)} V(z)\mathrm{d}z\right)^2 R^{-n+4-j} \\ &\lesssim R^{-j}(1+Rm(x_0,V))^{2l_0}. \end{aligned}$$

Then

$$|\nabla_x^j \Gamma_{\mathcal{P}_2}(x_0,t_0;y_0,s_0;\lambda)| \lesssim \frac{1}{(1+\lambda^{\frac{1}{4}}R)^N(1+m(x_0,V)R)^N R^{n+j}}.$$

For $\beta=1$, we have

$$\begin{aligned}|K_{j,\beta}(x,t;y,s)| &= |\nabla_x^j \Gamma_{\mathcal{P}_2}(x,t;y,s;0)| \\ &\lesssim \frac{1}{d((x,t),(y,s))^{n+j}\big(1+m(x,V)d((x,t),(y,s))\big)^N}.\end{aligned}$$

When $j/4<\beta<1$, by (3.2),

$$\begin{aligned}|K_{j,\beta}(x,t;y,s)| &\lesssim \int_0^\infty \lambda^{-\beta}|\nabla_x^j \Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)|d\lambda \\ &\lesssim \frac{1}{d((x,t),(y,s))^{n+4+j-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}.\end{aligned}$$

□

Lemma 3.3. *Assume that $V\in G((-\Delta)^2)\bigcap RH_s$ for $s\geq 2n/(4-j), j=0,1,2,3$. let $0<\alpha\leq 1-j/4, j/4<\beta\leq 1$, and $\beta-\alpha\geq j/4$. Then*

$$|T_j^*(f)(x,t)| \lesssim M_{q_1,\gamma}(f)(x,t)$$

for some $1<q_1<q_0$, where $\frac{1}{q_0}=1-\frac{2\alpha}{s}$ and $\gamma=4(\beta-\alpha)-j$.

Proof. Let $r=\frac{1}{m(x,V)}, C_k=\{(y,s)\in\mathbb{R}^{n+1}: 2^{k-1}r\leq d((x,t),(y,s))<2^k r\}$. Then by Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned}|T_1^*(f)(x,t)| &= \int_{\mathbb{R}^{n+1}} |K_{j,\beta}(y,s;x,t)|V(y)^{2\alpha}|f(y,s)|\mathrm{d}y\mathrm{d}s \\ &\lesssim \int_{\mathbb{R}^{n+1}} \frac{V(y)^{2\alpha}|f(y,s)|\mathrm{d}y\mathrm{d}s}{d((x,t),(y,s))^{n+4+j-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N} \\ &\lesssim \sum_{k=-\infty}^{\infty}\frac{(2^k r)^{4\beta-j}}{(1+2^k)^N}\frac{1}{(2^k r)^{n+4}}\int_{d((x,t),(y,s))<2^k r} V(y)^{2\alpha}|f(y,s)|\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Choose q_0 such that $\frac{1}{q_0}+\frac{2\alpha}{s}=1$. Then, by Hölder inequality and $V\in RH_s$ we have

$$\begin{aligned}&\frac{1}{(2^k r)^{n+4}}\int_{d((x,t),(y,s))<2^k r} V(y)^{2\alpha}|f(y,s)|\mathrm{d}y\mathrm{d}s \\ &\lesssim \left(\frac{1}{(2^k r)^n}\int_{|y-x|\leq 2^k r} V(y)\mathrm{d}y\right)^{2\alpha}\left(\frac{1}{(2^k r)^{n+4}}\int_{d((x,t),(y,s))<2^k r}|f(y,s)|^{q_0}\mathrm{d}y\mathrm{d}s\right)^{1/q_0}\end{aligned}$$

For $k\geq 1$, by Lemma 2.1,

$$\left(\frac{1}{(2^k r)^n}\int_{|y-x|\leq 2^k r} V(y)\mathrm{d}y\right)^{2\alpha} \lesssim (2^k r)^{-4\alpha}(1+2^k)^{2l_0\alpha} \lesssim (2^k r)^{-4\alpha}2^{2l_0\alpha k}.$$

For $k \le 0$, by Lemma 2.2, we have

$$\begin{aligned}&\left(\frac{1}{(2^k r)^n}\int_{|y-x|\le 2^k r} V(y)\mathrm{d}y\right)^{2\alpha}\\ &\le C(2^k r)^{-4\alpha}\left(\frac{r}{2^k r}\right)^{2\alpha(n/s-2)}\left(\frac{1}{r^{n-2}}\int_{|y-x|\le r} V(y)\mathrm{d}y\right)^{2\alpha}\\ &\le C(2^k r)^{-4\alpha}2^{2\alpha k(2-n/s)}.\end{aligned}$$

Taking $N > 2l_0\alpha$, then

$$\begin{aligned}|T_1^*(f)(x,t)| &\lesssim \left(\sum_{k=-\infty}^{0} 2^{2\alpha k(2-n/s)} + \sum_{k=1}^{\infty} 2^{k(2l_0\alpha - N)}\right)\\ &\quad\times\left(\frac{1}{(2^k r)^{n+4-(4(\beta-\alpha)-j)q_0}}\int_{d((x,t),(y,s))<2^k r} |f(y,s)|^{q_0}\mathrm{d}y\mathrm{d}s\right)^{1/q_0}\\ &\lesssim M_{q_0,\gamma}(f)(x,t),\end{aligned}$$

where $\gamma = 4(\beta-\alpha)-j$, $\frac{1}{q_0} = 1 - \frac{2\alpha}{s}$. By the self-improvement of class RH_s we know that there exists some $1 < q_1 < q_0$ such that

$$|T_1^*(f)(x,t)| \lesssim M_{q_1,\gamma}(|f|)(x,t).$$

□

Proof of Theorem 1.1 By Lemma 3.3 and Lemma 2.9 we get

$$\|T_j^*(f)\|_{L^q(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $(\frac{s}{2\alpha})' = q_0 \le p < \frac{n+4}{4(\beta-\alpha)-j}$, $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$.

By duality, we have

$$\|T_j(f)\|_{L^q(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $\frac{n+4}{n+4-(4(\beta-\alpha)-j)} < q \le \frac{s}{2\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}$. These condition are equivalent to

$$1 < p \le \left(\frac{2\alpha}{s} + \frac{4(\beta-\alpha)-j}{n+4}\right)^{-1}, \text{ and } \frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-j}{n+4}.$$

These complete the proof of Theorem 1.1.

4. The proof of Theorem 1.2

Let us estimate the kernel function $K_{j,\beta}(x,t;y,s)$ when $V \in RH_s$ for $\frac{n}{2} \le s < \frac{2n}{n-j}$.

Lemma 4.1. *For $j=1,2,3$, let $V\in G((-\Delta)^2)\bigcap RH_s$ with $\frac{n}{2}\le s<\frac{2n}{n-j}$ and $\frac{j}{4}<\beta\le 1$. Then,*

$$|K_{j,\beta}(x,t;y,s)|\lesssim \frac{1}{d((x,t),(y,s))^{n+4-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}\times\left(\int_{B(x,|x-y|/4)}\frac{V^2(z)dz}{|z-x|^{n-(4-j)}}+\frac{1}{d((x,t),(y,s))^j}\right).$$

Proof. By (3.7), we have

$$\begin{aligned}&|\nabla_x^j\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)|\\ \lesssim&\frac{1}{(1+\lambda^{\frac14}d((x,t),(y,s)))^N(1+m(x,V)d((x,t),(y,s)))^N d((x,t),(y,s))^n}\\ &\times\left(\int_{B(x,|x-y|/4)}\frac{V^2(z)dz}{|z-x|^{n-(4-j)}}+\frac{1}{d((x,t),(y,s))^j}\right).\end{aligned}$$

Then for $\beta=1$,

$$|K_{j,\beta}(x,t;y,s)|\lesssim \frac{1}{d((x,t),(y,s))^{n}\big(1+m(x,V)d((x,t),(y,s))\big)^N}\times\left(\int_{B(x,|x-y|/4)}\frac{V^2(z)dz}{|z-x|^{n-(4-j)}}+\frac{1}{d((x,t),(y,s))^j}\right).$$

For $j/4<\beta<1$, by (3.2),

$$\begin{aligned}|K_{j,\beta}(x,t;y,s)|&\lesssim\int_0^\infty\lambda^{-\beta}|\nabla_x^j\Gamma_{\mathcal{P}_2}(x,t;y,s;\lambda)|d\lambda\\ &\lesssim\frac{1}{d((x,t),(y,s))^{n+4-4\beta}\big(1+m(x,V)d((x,t),(y,s))\big)^N}\\ &\quad\times\left(\int_{B(x,|x-y|/4)}\frac{V^2(z)dz}{|z-x|^{n-(4-j)}}+\frac{1}{d((x,t),(y,s))^j}\right).\end{aligned}$$

□

Lemma 4.2. *Assume that $V\in G((-\Delta)^2)\bigcap RH_s$ for $\frac{n}{2}\le s<\frac{2n}{n-j}, j=1,2,3$. let $0<\alpha\le 1-j/4, j/4<\beta\le 1$, and $\beta-\alpha\ge j/4$. Then*

$$|T_j^*(f)(x,t)|\lesssim M_{p'_\alpha,\gamma}(f)(x,t)$$

for some $1<q_1<p'_\alpha$, where $\frac{1}{p_\alpha}=\frac{2\alpha+2}{s}-\frac{4-j}{n}$. and $\gamma=4(\beta-\alpha)-j$.

Proof. Let $r=\frac{1}{m(x,V)}, C_k=\{(y,s)\in\mathbb{R}^{n+1}:2^{k-1}r\le d((x,t),(y,s))<2^kr\}$. We choose t such that $\frac1t=\frac2s-\frac{4-j}{n}$. Then $\frac{1}{p'_\alpha}+\frac{2\alpha}{s}+\frac1t=1$. By Hölder inequality,

$$\begin{aligned}|T_j^*(f)(x,t)| = &\int_{\mathbb{R}^{n+1}} |K_{j,\beta}(y,s;x,t)|V(y)^{2\alpha}|f(y,s)|\mathrm{d}y\mathrm{d}s\\ \lesssim &\sum_{k=-\infty}^{\infty}\left(\int_{C_k}|K_{j,\beta}(y,s;x,t)|^t dyds\right)^{1/t}\left(\int_{C_k}V(y)^s dyds\right)^{2\alpha/s}\\ &\times\left(\int_{C_k}|f(y,s)|^{p'_\alpha}dyds\right)^{1/p'_\alpha}.\end{aligned}$$

Similar to the estimates in Theorem 1.1,

$$\left(\frac{1}{(2^kr)^{n+4}}\int_{C_k}V(y)^s dyds\right)^{2\alpha/s}\lesssim\begin{cases}(2^kr)^{-4\alpha}2^{2kl_0\alpha}, & \text{if } k\geq 1;\\ (2^kr)^{-4\alpha}2^{2k\alpha(2-n/s)}, & \text{if } k\leq 0.\end{cases}$$

For $(y,s)\in C_k$,

$$\begin{aligned}|K_{j,\beta}(y,s;x,t)| &\lesssim \frac{1}{(2^kr)^{n+4-4\beta}(1+2^k)^N}\left(\int_{|z-y|<2^kr}\frac{V(z)^2dz}{|z-y|^{n-(4-j)}}+\frac{1}{(2^kr)^j}\right)\\ &\lesssim \frac{1}{(2^kr)^{n+4-4\beta}(1+2^k)^N}\left(\mathcal{I}_{4-j}(V^2\chi_{B(x,2^{k+1}r)})(y)+(2^kr)^{-j}\right),\end{aligned}$$

where $\mathcal{I}_\alpha(f)(y)=\int_{\mathbb{R}^n}\frac{f(z)dz}{|z-y|^{n-\alpha}}$. Note that $\frac{1}{t}=\frac{2}{s}-\frac{4-j}{n}$, by the theorem on fractional integral and $V\in RH_s$, we get

$$\begin{aligned}&\left(\frac{1}{(2^kr)^n}\int_{|y-x|\leq 2^kr}\left(\mathcal{I}_{4-j}(V^2\chi_{B(x,2^{k+1}r)})(y)+(2^kr)^{-j}\right)^t dy\right)^{1/t}\\ &\lesssim (2^kr)^{-\frac{n}{t}+\frac{2n}{s}}\left(\frac{1}{(2^kr)^n}\int_{B(x,2^{k+1}r)}V(y)^s dy\right)^{2/s}+(2^kr)^{-j}\\ &\lesssim (2^kr)^{-\frac{n}{t}+\frac{2n}{s}-4}\left(\frac{1}{(2^kr)^{n-2}}\int_{B(x,2^kr)}V(y)dy\right)^2+(2^kr)^{-j}\\ &\lesssim (2^kr)^{-j}(1+2^k)^{2l_0}.\end{aligned}$$

Then

$$\begin{aligned}&(2^kr)^{n+4}\left(\frac{1}{(2^kr)^{n+4}}\int_{C_k}|K_{j,\beta}(y,s;x,t)|^t dyds\right)^{1/t}\\ &\lesssim \frac{1}{(2^jr)^{-4\beta}(1+2^k)^N}\left(\frac{1}{(2^kr)^{n+4}}\int_{C_k}\left(\mathcal{I}_{4-j}(V^2\chi_{B(x,2^{j+1}r)})(y)+(2^kr)^{-j}\right)^t dyds\right)^{1/t}\\ &\lesssim \frac{1}{(2^jr)^{-4\beta}(1+2^k)^N}\left(\frac{1}{(2^kr)^n}\int_{\mathbb{R}^n}\left(\mathcal{I}_{4-j}(V^2\chi_{B(x,2^{j+1}r)})(y)+(2^kr)^{-j}\right)^t dy\right)^{1/t}\\ &\lesssim \frac{1}{(2^jr)^{j-4\beta}(1+2^k)^N}\left(\frac{1}{(2^kr)^{n-2}}\int_{B(x,2^kr)}V(y)dy+1\right)\\ &\lesssim \frac{1}{(2^kr)^{j-4\beta}(1+2^k)^{N'}}.\end{aligned}$$

Therefore, taking $N > 2\alpha l_0$,

$$\begin{aligned} |T_j^* f(x,t)| &\lesssim \sum_{k=-\infty}^{\infty} \int_{C_k} |K_{j,\beta}(y,s;x,t)| V(y)^{2\alpha} |f(y,s)| dyds \\ &\lesssim \left(\sum_{k=-\infty}^{0} 2^{2k\alpha(2-n/q)} + \sum_{k=1}^{\infty} 2^{k(N-2\alpha l_0)} \right) M_{p'_\alpha,\gamma}(x,t) \\ &\lesssim M_{p'_\alpha,\gamma}(f)(x,t). \end{aligned}$$

By the self-improvement of class RH_s we know that there exists some $1 < q_1 < p'_\alpha$ such that

$$|T_j^* f(x,t)| \lesssim M_{q_1,\gamma}(f)(x,t).$$

□

Proof of Theorem 1.2 By Lemma 4.2 and Lemma 2.9 we get

$$\|T_j^*(f)\|_{L^q(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $p'_\alpha \le p < \frac{n+4}{\gamma}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n+4}$, where $\gamma = 4(\beta - \alpha) - j$.

By duality, we get

$$\|T_j(f)\|_{L^q(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $\frac{n+4}{n+4-\gamma} < q \le p_\alpha$, and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n+4}$. These conditions are equivalent to

$$1 < p \le \frac{1}{\frac{1}{p_\alpha} + \frac{\gamma}{n+4}}, \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n+4}.$$

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