DIRECT LIMITS OF GORENSTEIN INJECTIVE MODULES

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Abstract. One of the open problems in Gorenstein homological algebra is: when is the class of Gorenstein injective modules closed under arbitrary direct limits? It is known that if the class of Gorenstein injective modules, $\mathcal{GI}$, is closed under direct limits, then the ring is noetherian. The open problem is whether or not the converse holds. We give equivalent characterizations of $\mathcal{GI}$ being closed under direct limits. More precisely, we show that the following statements are equivalent:
(1) The class of Gorenstein injective left R-modules is closed under direct limits.
(2) The ring R is left noetherian and the character module of every Gorenstein injective left R-module is Gorenstein flat.
(3) The class of Gorenstein injective modules is covering and it is closed under pure quotients.
(4) $\mathcal{GI}$ is closed under pure submodules.

1. introduction

Gorenstein homological algebra is the most developed branch of relative homological algebra. It studies modules using resolutions, just like classical homological algebra does, but it replaces the classical resolutions with the Gorenstein ones. There is a big difference though: while the existence of the classical (injective, projective, flat) resolutions over arbitrary rings is well known, there are still some open questions regarding the existence of the Gorenstein (injective, projective, flat) resolutions. This is the reason why the existence of the Gorenstein (injective, projective, flat) covers and envelopes has been studied intensively in recent years (see for example [5], [6], [11], [12], and [14]).

One question that is still open is: "When (for what rings) is the class of Gorenstein injective left R-modules, $\mathcal{GI}$, a covering class?"

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It is known that, if the class $\mathcal{GI}$ is closed under direct limits, then it is a covering class (this follows from [2, Proposition 3.15], and from [5, Proposition 3]). For example, this is the case when R is an Iwanaga Gorenstein ring (by [7, Lemma 11.1.2]).

So the next question to consider is: when is the class of Gorenstein injective left R-modules closed under direct limits?

In [6] we considered this question over commutative noetherian rings, and we gave a sufficient condition in order for $\mathcal{GI}$ to be closed under direct limits. We proved ([6, Theorem 1]) that, if the character modules of Gorenstein injective modules are Gorenstein flat, then the class of Gorenstein injective modules is closed under direct limits. For example, this is the case when R is a commutative noetherian ring with a dualizing complex ([1, Proposition 5.1]). Consequently, we recover [1, Theorem 6.9]: if R is a commutative ring with a dualizing complex, then a direct limit of Gorenstein injective modules is Gorenstein injective.

We consider here again the question of closure of the class of Gorenstein injective modules under direct limits, but without assuming commutativity of the ring. In this paper we give necessary and sufficient conditions for the class $\mathcal{GI}$ being closed under direct limits. Proposition 1 extends [6, Theorem 1] to the noncommutative case: we prove that if R is a left noetherian ring such that the character modules of Gorenstein injective left R-modules are Gorenstein flat, then the class of Gorenstein injective left R-modules is closed under direct limits.

Then we prove (Proposition 2) that the converse also holds: if the class of Gorenstein injective left R-modules is closed under direct limits, then the ring R is left noetherian and the character module of any left Gorenstein injective module is Gorenstein flat. Propositions 1 and 2 give the answer to the open question we consider: the class of Gorenstein injective left R-modules is closed under direct limits if and only if the ring R is left noetherian, and the character module of any Gorenstein injective left R-module is Gorenstein flat. We also give examples of such rings (in section 3).

Theorem 2 gives more equivalent characterizations of the condition that $\mathcal{GI}$ is closed under direct limits. More precisely, we prove that the following statements are equivalent:
(1) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is closed under direct limits.
(2) The ring R is left noetherian, and the character module of every

Gorenstein injective left R-module is a Gorenstein flat right R-module.
(3) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is covering and closed under pure quotients.
(4) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is closed under pure submodules.

2. PRELIMINARIES

Throughout the paper R will denote an associative ring with identity. Unless otherwise stated, by *module* we mean *left* R-module.

We will denote by Inj the class of all injective R-modules. We recall that an R-module M is Gorenstein injective if there exists an exact and $Hom(Inj, -)$ exact complex of injective modules
$\mathbf{I} = \ldots \to I_1 \to I_0 \to I_{-1} \to \ldots$ such that $M = Ker(I_0 \to I_{-1})$.
We will use the notation $\mathcal{GI}$ for the class of Gorenstein injective modules.

We recall the definitions for Gorenstein injective precovers and covers.

Definition 1. *A homomorphism $\phi : G \to M$ is a Gorenstein injective precover of M if G is Gorenstein injective, and if for any Gorenstein injective module G', and any $\phi' \in Hom(G', M)$ there exists $u \in Hom(G', G)$ such that $\phi' = \phi u$.*
A Gorenstein injective precover ϕ is said to be a cover if any $u \in End_R(G)$ such that $\phi u = \phi$ is an automorphism of G.

We note that a Gorenstein injective precover (a Gorenstein injective cover, respectively) does not have to be surjective.
The class of Gorenstein injective modules is said to be *covering* if every R-module M has a Gorenstein injective cover.

It is known that, if the class $\mathcal{GI}$ is closed under direct limits, then it is a covering class (this follows from [2, Proposition 3.15], and from [5, Proposition 3])

Given a class of R-modules $\mathcal{F}$, we will denote as usual by ${}^{\perp}\mathcal{F}$ the class of all R-modules N such that $Ext^1(N, F) = 0$ for every $F \in \mathcal{F}$. The right orthogonal class of $\mathcal{F}$, $\mathcal{F}^{\perp}$, is defined in a similar manner: $\mathcal{F}^{\perp} = \{X, Ext^1(F, X) = 0, \forall F \in \mathcal{F}\}$.

We recall that a pair $(\mathcal{L}, \mathcal{C})$ is a *cotorsion pair* if $\mathcal{L}^{\perp} = \mathcal{C}$ and $^{\perp}\mathcal{C} = \mathcal{L}$. A cotorsion pair $(\mathcal{L}, \mathcal{C})$ is *complete* if for every $_RM$ there exists exact sequences $0 \to C \to L \to M \to 0$ and $0 \to M \to C' \to L' \to 0$ with C, C' in $\mathcal{C}$ and L, L' in $\mathcal{L}$.

Definition 2. *A cotorsion pair $(\mathcal{L}, \mathcal{C})$ is called hereditary if one of the following equivalent statements hold:*

(1) *$\mathcal{L}$ is resolving, that is, $\mathcal{L}$ is closed under taking kernels of epimorphisms.*
(2) *$\mathcal{C}$ is coresolving, that is, $\mathcal{C}$ is closed under taking cokernels of monomorphisms.*
(3) *$Ext^i(F, C) = 0$ for any $F \in \mathcal{F}$ and $C \in \mathcal{C}$ and $i \geq 1$.*

It is known ([14, Theorem 4.6]) that $(^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair over any ring R.

Since we use Gorenstein flat modules as well, we recall their definition.

Definition 3. *A right R-module G is Gorenstein flat if there is an exact complex of flat right R-modules $\boldsymbol{F} = \ldots \to F_1 \to F_0 \to F_{-1} \to \ldots$ such that $\boldsymbol{F} \otimes I$ is exact for every injective left R-module I, and such that $G = Ker(F_0 \to F_{-1})$.*

We will use $\mathcal{GF}$ to denote the class of Gorenstein flat right R-modules. We denote by $\mathcal{F}lat$ the class of flat right R-modules. We recall that the modules in the right orthogonal class of $\mathcal{F}lat$ are called cotorsion modules. We will use $\mathcal{C}ot$ to denote this class of modules.

We will also use duality pairs. They were introduced by Holm and Jorgensen in [10]. We recall the definition.

Definition 4. *([10]) A* duality pair *over R is a pair $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is a class of R-modules and $\mathcal{C}$ is a class of right R-modules, satisfying the following conditions:*

(1) *$M \in \mathcal{M}$ if and only if $M^+ \in \mathcal{C}$ (where M^+ is the character module of M, $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$).*
(2) *$\mathcal{C}$ is closed under direct summands and finite direct sums.*

A duality pair $(\mathcal{M}, \mathcal{C})$ is called (co)product closed if the class $\mathcal{M}$ is closed under (co)products in the category of all R-modules.

Theorem 1. [10] *Let $(\mathcal{M}, \mathcal{C})$ be a duality pair. Then the following hold:*

1. *$\mathcal{M}$ is closed under pure submodules, pure quotients, and pure extensions.*
2. *If $(\mathcal{M}, \mathcal{C})$ is coproduct-closed then M is covering.*

3. RESULTS

Lemma 1. *Let R be a left noetherian ring. Then $K^+ \in \mathcal{GF}^\perp$ for any $K \in ^\perp \mathcal{GI}$*

Proof. Let $K \in ^\perp \mathcal{GI}$. For any Gorenstein flat right R-module F, we have that $Ext^1(F, K^+) \simeq Ext^1(K, F^+) = 0$ (since, by [9, Theorem 3.6], F^+ is a Gorenstein injective module). So $K^+ \in \mathcal{GF}^\perp$. □

The following result is basically [12, Theorem 4]. It was stated there over two sided noetherian rings. But the proof does not use R being a right noetherian ring. For completness, we include the proof.

Lemma 2. *(*[12, Theorem 4]*) Let R be a left noetherian ring such that the character module of every Gorenstein injective left R-module is a Gorenstein flat right R-module. Then $(\mathcal{GI}, \mathcal{GF})$ is a duality pair.*

Proof. We show first that K is Gorenstein injective if and only if K^+ is Gorenstein flat.
One implication is a hypothesis that we made on the ring.
For the other implication: assume that K^+ is Gorenstein flat. Since $(^\perp\mathcal{GI}, \mathcal{GI})$ is a complete cotorsion pair ([14, Theorem 4.6]), there is an exact sequence $0 \to K \to G \to L \to 0$ with G Gorenstein injective, and with $L \in ^\perp \mathcal{GI}$. This gives an exact sequence $0 \to L^+ \to G^+ \to K^+ \to 0$ with K^+ Gorenstein flat by hypothesis, and with $L^+ \in \mathcal{GF}^\perp$ (by Lemma 1). Thus $Ext^1(K^+, L^+) = 0$. Therefore $G^+ \simeq K^+ \oplus L^+$. Since G^+ is Gorenstein flat, it follows that $L^+ \in \mathcal{GF}$. So $L^+ \in \mathcal{GF} \bigcap \mathcal{GF}^\perp = \mathcal{F}lat \bigcap \mathcal{C}ot$ ([8, Proposition 3.1]). Thus L^+ is a flat right R-module. Since the ring R is left noetherian, it follows that L is injective (by [7, Theorem 3.2.16]).
The short exact sequence $0 \to K \to G \to L \to 0$ with G Gorenstein injective and with L injective, gives that $G.i.d.K \leq 1$. By [1, Lemma 2.18], there is a short exact sequence $0 \to B \to H \to K \to 0$ with B Gorenstein injective, and with $i.d.H = G.i.d.K < \infty$. This gives a short exact sequence $0 \to K^+ \to H^+ \to B^+ \to 0$ with both K^+ and B^+ Gorenstein flat. Thus H^+ is also Gorenstein flat. Since H

has finite injective dimension, we have that H^+ has finite flat dimension. But then H^+ is Gorenstein flat of finite flat dimension. By [7, Corollary 10.3.4], H^+ is flat. Therefore H is injective (by [7, Theorem 3.2.16]).
The short exact sequence $0 \to B \to H \to K \to 0$ with B Gorenstein injective and with H injective gives that K is Gorenstein injective.
So K is Gorenstein injective if and only if K^+ is Gorenstein flat.

We can prove now that $(\mathcal{GI}, \mathcal{GF})$ is a duality pair:
By the above, K is Gorenstein injective if and only if K^+ is Gorenstein flat. Since $\mathcal{GF}$ is closed under direct summands ([9, Theorem 3.7]) and direct sums, $(\mathcal{GI}, \mathcal{GF})$ is a duality pair. □

Proposition 1. *Let R be a left noetherian ring such that the character module of every Gorenstein injective left R-module is a Gorenstein flat right R-module. Then the class of Gorenstein injective left R-modules, $\mathcal{GI}$, is closed under direct limits.*

Proof. By Lemma 2, $(\mathcal{GI}, \mathcal{GF})$ is a duality pair. By [10, Theorem 2.10], $\mathcal{GI}$ is closed under pure submodules and pure quotients.
Since $\mathcal{GI}$ is closed under direct products, and since the direct sum of modules is a pure submodule of the direct product of the modules, it follows that $\mathcal{GI}$ is closed under arbitrary direct sums.
Since $\mathcal{GI}$ is closed under direct sums and under pure quotients, and since a direct limits of modules is a pure quotient of the direct sum of the modules, it follows that $\mathcal{GI}$ is closed under direct limits. □

Proposition 2. *If the class of Gorenstein injective left R-modules is closed under direct limits, then the ring R is left noetherian and the character module of every Gorenstein injective left R-module is a Gorenstein flat right R-module.*

Proof. Since $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair with $\mathcal{GI}$ closed under direct limits, it follows (by [15, Theorem 3.5]), that $\mathcal{GI}$ is a definable class. So, in particular, $\mathcal{GI}$ is closed under pure submodules. Since $\mathcal{GI}$ is closed under direct products, and a direct sum is a pure submodule of a direct product, $\mathcal{GI}$ is also closed under direct sums. By [2, Proposition 3.15], the ring R is left noetherian.
Also, since $\mathcal{GI}$ is a definable class, we have (by [8, Lemma 1.1]), that a module M is Gorenstein injective if and only if the module M^{++} is Gorenstein injective.

So consider a Gorenstein injective module M. By [8, Lemma 1.1], the module $M^{++} = (M^+)^+$ is Gorenstein injective. Since the ring R is noetherian (therefore coherent), by [9, Theorem 3.6], M^+ is a Gorenstein flat module. □

The following result is immediate from the fact that $\mathcal{GI}$ is the right half of a hereditary cotorsion pair, so it is closed under cokernels of monomorphisms.

Lemma 3. *If the class of Gorenstein injective modules, $\mathcal{GI}$ is closed under pure submodules then it is also closed under pure quotients.*

Proof. Let $0 \to M' \to M \to M" \to 0$ be a pure exact sequence with M a Gorenstein injective module. By hypothesis, M' is Gorenstein injective. Since the class $\mathcal{GI}$ is closed under cokernels of monomorphisms ([14]) it follows that $M"$ is also Gorenstein injective. So $\mathcal{GI}$ is also closed under pure quotients in this case. □

Theorem 2. *The following statements are equivalent:*
(1) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is closed under direct limits.
(2) The ring R is left noetherian, and the character module of every Gorenstein injective left R-module is a Gorenstein flat right R-module.
(3) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is covering and closed under pure quotients.
(4) The class of Gorenstein injective left R-modules, $\mathcal{GI}$, is closed under pure submodules.

Proof. $1 \Rightarrow 2$ is Proposition 2.
$2 \Rightarrow 1$ is Proposition 1.
$1 \Rightarrow 4$. By [15, Theorem 3.5], $\mathcal{GI}$ being closed under direct limits implies that $\mathcal{GI}$ is a definable class, so it is closed under pure submodules (by definition).
$4 \Rightarrow 1$. Since $\mathcal{GI}$ is closed under pure submodules, it follows that $\mathcal{GI}$ is also closed under pure quotients (by Lemma 3). Since $\mathcal{GI}$ is closed under direct products, and since the direct sum of modules is a pure submodule of the direct product of the modules, it follows that $\mathcal{GI}$ is closed under arbitrary direct sums.
Since $\mathcal{GI}$ is closed under direct sums and under pure quotients, and since a direct limits of modules is a pure quotient of the direct sum of modules, it follows that $\mathcal{GI}$ is closed under direct limits.
$1 \Rightarrow 3$ By [5, Proposition 3], and [1, Proposition 3.15], if $\mathcal{GI}$ is closed under direct limits then $\mathcal{GI}$ is a covering class.

And by [15, Theorem 3.5], the class $\mathcal{GI}$ is definable, therefore it is closed under pure submodules. By Lemma 3, $\mathcal{GI}$ is also closed under pure quotients.

$3 \Rightarrow 1$. Since $\mathcal{GI}$ is covering and closed under direct summands, it is also closed under arbitrary direct sums. Since $\mathcal{GI}$ is closed under pure quotients and a direct limit of modules is a pure quotient of the direct sum of the modules, it follows that $\mathcal{GI}$ is closed under direct limits. □

3.1. **Examples of noetherian rings such that the character module of any Gorenstein injective left R-module is a Gorenstein flat right R-module.**

- **Example 1.** By [10, Lemma 2.5(b)], any commutative noetherian ring R with a dualizing complex has the desired property: the character modules of Gorenstein injective modules are Gorenstein flat.

- **Example 2.** Any Iwanaga-Gorenstein ring also has the desired property: if R is an Iwanaga-Gorenstein ring, then $\mathcal{GI}$ is closed under direct limits ([7, Lemma 11.1.2]). Then the result follows form [8, Lemma 1.1].

- **Example 3.** By [11, Theorem 3], if R is a two sided noetherian ring such that ${}_RR$ is a left n-perfect ring, and there exists a dualizing module ${}_RV_R$ for the pair (R, R), then the character modules of Gorenstein injective left R-modules are Gorenstein flat right R-modules.

- **Example 4.** By [13], if R is a left artinian ring such that the injective envelope of every simple left R-module is finitely generated, then the character module of Gorenstein injective left R-modules are Gorenstein flat.

- **Example 5.** The following result (Theorem 3 below) shows that any left noetherian ring R having finite self injective dimension as a right R-module satisfies Theorem 2.

We recall that a module M is called *strongly Gorenstein injective* if there is an exact and $Hom(Inj, -)$ exact complex $\ldots \to E \to E \to E \to \ldots$ with E injective, and with $M = Ker(E \to E)$.

Theorem 3. *Let R be a left noetherian ring such that id. $R_R \leq n$ for some positive integer n. Then the character modules of Gorenstein injective left R-modules are Gorenstein flat.*

Proof. Let M be a strongly Gorenstein injective left R-module, and let I be any injective left R-module. By [16, Proposition 1], the flat dimension of I is less than or equal to n. Then the right R-module I^+ has injective dimension $\leq n$. It follows that $Ext^i(M^+, I^+) = 0$ for any $i \geq n+1$. So we have that $Ext^i(I, M^{++}) \simeq Ext^i(M^+, I^+) = 0$ for all $i \geq n+1$, for any injective ${}_RI$. But M is strongly Gorenstein injective, so there exists an exact sequence $0 \to M \to E \to M \to 0$ with E injective. This gives an exact sequence $0 \to M^{++} \to E^{++} \to M^{++} \to 0$, with E^{++} an injective left R-module. It follows that $Ext^k(-, M^{++}) \simeq Ext^1(-, M^{++})$ for any $k \geq 1$. By the above, we have that $Ext^k(I, M^{++}) = 0$ for any injective ${}_RI$, for any $k \geq 1$. Then the exact sequence $\ldots \to E^{++} \to E^{++} \to M^{++} \to 0$ is also $Hom(Inj,)$ exact. So M^{++} has an exact left injective resolution. Since $Ext^i(I, M^{++}) = 0$ for all $i \geq 1$, for any injective ${}_RI$, and M^{++} has an exact left injective resolution it follows that M^{++} is Gorenstein injective ([7, Proposition 10.1.3]).
Let G be a Gorenstein injective left R module. By [3], there exists a strongly Gorenstein injective left R-module M such that $M \simeq G \oplus H$. It follows that G^{++} is isomorphic to a direct summand of the Gorenstein injective module M^{++}, and therefore it is Gorenstein injective. Since the ring R is right coherent and $G^{++} = (G^+)^+$ is Gorenstein , it follows that G^+ is Gorenstein flat, for any Gorenstein injective ${}_RG$. □

Some open questions:

1. Does every (left) noetherian ring satisfy the condition that character modules of Gorenstein injectives are Gorenstein flat? Or is this class of rings strictly contained in the class of noetherian rings?

2. Does the assumption that $\mathcal{GI}$ is covering imply that $\mathcal{GI}$ is closed under pure quotients? In other words, is it true that the class of Gorenstein injective modules is closed under direct limits if and only if it is a covering class? This would be consistent with the Enochs' conjecture: "Every covering class is closed under direct limits".

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