THE BETTI NUMBERS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPES E_7 AND E_8

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Abstract. We compute the rational Betti numbers of the real toric varieties associated to Weyl chambers of types E_7 and E_8, completing the computations for all types of root systems.

1. Introduction

It is known that a root system of type R generates a non-singular complete fan Σ_R by its Weyl chambers and co-weight lattice [10], and that Σ_R corresponds to a smooth compact (complex) toric variety X_R by the fundamental theorem for toric geometry. In particular, the real locus of X_R is called *the real toric variety associated to the Weyl chambers*, denoted by $X^{\mathbb{R}}_R$.

It is natural to ask for the topological invariants of $X^{\mathbb{R}}_R$. By [6], the $\mathbb{Z}_2$-Betti numbers of $X^{\mathbb{R}}_R$ can be completely computed from the face numbers of Σ_R. In general, however, computing the rational Betti numbers of a real toric variety is much more difficult. In 2012, Henderson [8] computed the rational Betti numbers of $X^{\mathbb{R}}_{A_n}$. The computation of other classical and exceptional types has been carried out using the formulae for rational Betti numbers developed in [13] or [5]. At the time of writing this paper, results have been established for $X^{\mathbb{R}}_R$ of all types except E_7 and E_8.

For the classical types $R = A_n, B_n, C_n$, and D_n, the kth Betti numbers β_k of $X^{\mathbb{R}}_R$ are known to be as follows (see [8], [4], [3]):

$$\begin{aligned}
\beta_k(X^{\mathbb{R}}_{A_n};\mathbb{Q}) &= \binom{n+1}{2k} a_{2k},\\
\beta_k(X^{\mathbb{R}}_{B_n};\mathbb{Q}) &= \binom{n}{2k} b_{2k} + \binom{n}{2k-1} b_{2k-1},\\
\beta_k(X^{\mathbb{R}}_{C_n};\mathbb{Q}) &= \binom{n}{2k-2}\left(2^n - 2^{2k-2}\right) a_{2k-2} + \binom{n}{2k}(2b_{2k} - 2^{2k}a_{2k}), \text{ and}\\
\beta_k(X^{\mathbb{R}}_{D_n};\mathbb{Q}) &= \binom{n}{2k-4}\left(2^{2k-4} + (n-2k+2)2^{n-1}\right) a_{2k-4} + \binom{n}{2k}(2b_{2k} - 2^{2k}a_{2k}),
\end{aligned}$$

where a_r is the rth Euler zigzag number (A000111 in [11]) and b_r is the rth generalized Euler number (A001586 in [11]).

For the exceptional types $R = G_2, F_4$, and E_6, the Betti numbers of $X^{\mathbb{R}}_R$ are as in Table 1 (see [2, Proposition 3.3]).

The purpose of this paper is to compute the Betti numbers for the remaining exceptional types E_7 and E_8. The reason these cases have remained unsolved to date is that, as

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$\beta_k(X^{\mathbb{R}}_R)$	$R = G_2$	$R = F_4$	$R = E_6$
$k = 0$	1	1	1
$k = 1$	9	57	36
$k = 2$	0	264	1,323
$k = 3$	0	0	4,392

TABLE 1. Nonzero Betti numbers of $X^{\mathbb{R}}_{G_2}$, $X^{\mathbb{R}}_{F_4}$, and $X^{\mathbb{R}}_{E_6}$

remarked in [2], the corresponding fans are too large to be dealt with. We provide a technical method to decompose all facets of the Coxeter complex; using this method, we obtain explicit subcomplexes K_S that play an important role in our main computation. Furthermore, we obtain a smaller simplicial complex by removing vertices in K_S without changing its homology groups so that the Betti numbers can be computed.

Theorem 1.1. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_7}$ and $X^{\mathbb{R}}_{E_8}$ are as follows.*

$$\beta_k(X^{\mathbb{R}}_{E_7};\mathbb{Q}) = \begin{cases} 1, & \textit{if } k=0 \\ 63, & \textit{if } k=1 \\ 8{,}127, & \textit{if } k=2 \\ 131{,}041, & \textit{if } k=3 \\ 122{,}976, & \textit{if } k=4 \\ 0, & \textit{otherwise.} \end{cases}$$

$$\beta_k(X^{\mathbb{R}}_{E_8};\mathbb{Q}) = \begin{cases} 1, & \textit{if } k=0 \\ 120, & \textit{if } k=1 \\ 103{,}815, & \textit{if } k=2 \\ 6{,}925{,}200, & \textit{if } k=3 \\ 23{,}932{,}800, & \textit{if } k=4 \\ 0, & \textit{otherwise.} \end{cases}$$

2. Real toric varieties associated to the Weyl chambers

We recall some known facts about the real toric varieties associated to the Weyl chambers, following the notation in [2] unless otherwise specified.

Let Φ_R be an irreducible root system of type R in a finite dimensional Euclidean space and W_R its Weyl group. The connected components of the complement of the reflection hyperplanes are called the *Weyl chambers*. We fix a particular Weyl chamber, called the *fundamental Weyl chamber* Ω, and *the fundamental co-weights* $\omega_1, \ldots, \omega_n$ *form the set of its rays*. Then, $\mathbb{Z}(\{\omega_1, \ldots, \omega_n\})$ has a lattice structure, called the co-weight lattice. Consider the set of Weyl chambers as a nonsingular complete fan Σ_R with the co-weight lattice. From the set $V = \{v_1, \ldots, v_m\}$ of rays spanning Σ_R we obtain the simplicial complex K_R, called *the Coxeter complex* of type R on V, whose faces in K_R are obtained via the corresponding faces in Σ_R (see [1] for more details). The directions of rays on the co-weight lattice give a linear map $\lambda_R \colon V \to \mathbb{Z}^n$. In addition, the composition map $\Lambda_R \colon V \overset{\lambda_R}{\to} \mathbb{Z}^n \overset{\text{mod } 2}{\longrightarrow} \mathbb{Z}_2^n$ can be expressed as an $n \times m$ (mod 2) matrix, called a (mod 2) *characteristic matrix*. Let S be an element of the row space $Row(\Lambda_R)$ of Λ_R, the vector space spanned by the row vectors of Λ_R. Since each column of Λ_R corresponds to a vertex $v \in V$, S can be regarded as a subset of V. Let us consider the induced subcomplex K_S

of K_R with respect to S. It is known that the reduced Betti numbers of K_S related to the Betti numbers of $X_R^{\mathbb{R}}$.

Theorem 2.1. [2] *For any root system Φ_R of type R, let W_R be the Weyl group of Φ_R. Then, there is a W_R-module isomorphism*

$$H_*(X_R^{\mathbb{R}}) \cong \bigoplus_{S \in Row(\Lambda_R)} \widetilde{H}_{*-1}(K_S),$$

where K_S is the induced subcomplex of K_R with respect to S.

The definition of the W_R-action on $Row(\Lambda_R)$ is explained in Lemma 3.1 in [2], and implies that

$$K_S \cong K_{gS} \text{ for } S \in Row(\Lambda_R) \text{ and } g \in W_R. \tag{2.1}$$

Combining Theorem 2.1 with (2.1), we need only investigate representatives K_S of the W_R-orbits in $Row(\Lambda_R)$.

Proposition 2.2. [2] *For type E_7, there are 127 nonzero elements in $Row(\Lambda_{E_7})$. In addition, there are exactly three orbits (whose representatives are denoted by S_1, S_2, and S_3), and the numbers of elements for each orbit are $63, 63$, and 1, respectively.*

For type E_8, there are 255 nonzero elements in $Row(\Lambda_{E_8})$. There are only two orbits (whose representatives are denoted by S_4 and S_5), and the numbers of elements for each orbit are 120 and 135, respectively.

Thus, for our purpose, it is enough to compute the (reduced) Betti numbers of K_{S_i} for $1 \leq i \leq 5$. For practical reasons such as memory constraints and high time complexity, it is not easy to obtain K_S directly by computer programs. The remainder of this section is devoted to introducing an effective way to obtain K_S.

For a fixed fundamental co-weight ω, let H_ω be the isotropy subgroup of ω in W_R, and let K_ω be the subcomplex of K_R such that the set of facets of K_ω is $\{h \cdot \Omega \mid h \in H_\omega\}$, where Ω is the fundamental Weyl chamber.

Lemma 2.3. *The set of facets of K_R is decomposed as the union of the sets of all facets of $K^g = g \cdot K_\omega$ for all $g \in W_R/H_\omega$.*

Proof. For each facet $\sigma \in K_R$, there uniquely exists $g_\sigma \in W_R$ such that $g_\sigma \cdot \Omega = \sigma$ by Propositions 8.23 and 8.27 in [7]. Thus, there is exactly one $g_\sigma \cdot H_\omega \in W_R/H_\omega$ such that σ is a facet of K^{g_σ} as desired. □

Obviously, the set of facets of K_S is then obtainable as the union of the sets of all facets of K_S^g for all $g \in W_R/H_\omega$.

In this paper, we fix the fundamental co-weight ω to correspond to α_1 for type E_7, and to correspond to α_8 for type E_8 in Figure 1.

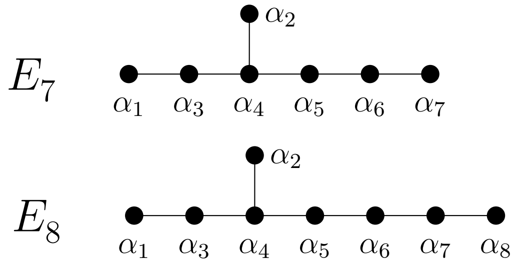

FIGURE 1. The Dynkin diagrams for types E_7 and E_8

However, since K^g still has many facets, it is not easy to obtain K^g_S from K^g directly; see Table 2.

	$R = E_7$	$R = E_8$
# vertices of K_R	17,642	881,760
# facets of K_R	2,903,040	696,729,600
$\lvert W_R/H_\omega \rvert$	126	240
# facets of K^g	23,040	2,903,040

TABLE 2. Statistics for K_R when $R = E_7$ and E_8

Hence, we establish a lemma to improve the time complexity. Denote by V^g_S the set of vertices in K^g_S.

Lemma 2.4. *Let $g, h \in W_R/H_\omega$. If $g \cdot V^h_S = V^{gh}_S$, then $g \cdot K^h_S = K^{gh}_S$.*

Proof. For $g \in W_R/H_\omega$, we naturally consider g a simplicial isomorphism from K^h to K^{gh}. If $g \cdot V^h_S = V^{gh}_S$, then the restriction of g to K^h_S is well-defined. Thus, g is also regarded as a simplicial isomorphism between K^h_S and K^{gh}_S. □

By the above lemma, when $g \cdot V^h_S = V^{gh}_S$, K^{gh}_S is obtainable without any computation. Since checking the hypothesis of the lemma is much easier than forming K^g_S from K^g, a good deal of time can be saved. Using this method, one can obtain K_S within a reasonable time with standard computer hardware.

3. SIMPLICIAL COMPLEXES FOR TYPES E_7 AND E_8

Since each K_S for the types E_7 or E_8 is too large for direct computation, it is impossible to compute their Betti numbers directly using existing methods. In this section, we introduce the specific smaller simplicial complex $\widehat{K}_S$ whose homology group is isomorphic as a group to that of K_S.

Let K be a simplicial complex. The *link* $Lk_K(v)$ of v in K is a set of all faces $\sigma \in K$ such that $v \notin \sigma$ and $\{v\} \cup \sigma \in K$, while the (closed) *star* $St_K(v)$ of v in K is a set of all faces $\sigma \in K$ such that $\{v\} \cup \sigma \in K$. For a vertex v of K_S satisfying $Lk_K(v) \neq \emptyset$, we consider the following Mayer-Vietoris sequence:

$$\cdots \to \widetilde{H}_k(Lk_K(v)) \to \widetilde{H}_k(K - v) \oplus \widetilde{H}_k(St_K(v)) \to \widetilde{H}_k(K) \to \widetilde{H}_{k-1}(Lk_K(v)) \to \cdots,$$

where $K - v = \{\sigma \backslash \{v\} \mid \sigma \in K\}$ and k is a positive integer. We note that $\widetilde{H}_k(St_K(v)) = 0$ for $k \geq 0$ since $St_K(v)$ is a topological cone. Therefore, for $k \geq 0$, if $\widetilde{H}_k(Lk_K(v))$ is trivial, then $\widetilde{H}_k(K - v) \cong \widetilde{H}_k(K)$ as groups. In this case, we call v a *removable vertex* of K.

Let us consider the canonical action of the Weyl group W_R on the vertex set V_R of K_R. It is known that there are exactly n vertex orbits $V_1, \ldots, V_n$ of K_R, where n is the number of simple roots of W_R.

Theorem 3.1. *For a subcomplex L of K_R, the simplicial complex obtained by the algorithm below has the same homology group as L.*

Algorithm

1: $K \leftarrow L$
2: **for** $i = 1, \dots, n$ **do**
3: $\quad W \leftarrow \emptyset$
4: $\quad$ **for** each $v \in V_i$ **do**
5: $\qquad$ **if** v is removable in K **then**
6: $\qquad\quad W \leftarrow W \cup \{v\}$
7: $\qquad$ **end if**
8: $\quad$ **end for**
9: $\quad K \leftarrow K - W := \{\sigma \setminus W \mid \sigma \in K\}$
10: **end for**
11: Return K

Proof. By Proposition 8.29 in [7], for each facet $\mathcal{C}$ of K_R, every vertex orbit of K_R contains exactly one vertex of $\mathcal{C}$. That is, for any $v, w \in V_i$, v and w are not adjacent. Then, for any subcomplex K of K_R and $v, w \in V_i$, v is not contained in $Lk_K(w)$.

Note that, for removable vertices v and w of K, w is still removable in $K - v$ if w is not in the link of v in K, whereas there is no guarantee that w is removable in $K - v$ in general. Thus, we can remove all removable vertices of K in V_i from K at once without changing their homology groups. We do this procedure inductively for every vertex orbit to obtain K, and obviously, that $H_*(K) \cong H_*(L)$ as groups. □

If line 5 of the algorithm above is replaced with '**if** $Lk_K(v)$ forms a cone **then**', simplicial complex K returned in line 11 is unique up to isomorphism, regardless of any changes in the order of vertex orbits [9]. However, Theorem 3.1 is enough to compute the Betti numbers of K_{S_i} for $1 \leq i \leq 5$.

In this paper, we fix the order by size of orbit, with $|V_i| < |V_{i+1}|$. Let $\widehat{K}_S$ be the complex resulting from K_S as obtained by the algorithm in Theorem 3.1. Then, the sizes of $\widehat{K}_S$ obtained as in Table 3 are dramatically smaller than the sizes of K_S.

E_7	$S = S_1$	$S = S_2$	$S = S_3$
K_S	9,176	8,672	4,664
$\widehat{K}_S$	408	928	4,664

E_8	$S = S_4$	$S = S_5$
K_S	432,944	451,200
$\widehat{K}_S$	9,328	15,488

TABLE 3. Numbers of vertices of K_S and $\widehat{K}_S$

The following proposition establishes some properties of K_S and $\widehat{K}_S$.

Proposition 3.2.

(1) *K_{S_1} and K_{S_4} have two connected components; the other K_S are connected.*
(2) *For $S = S_1, S_4$, two components of K_S are isomorphic.*
(3) *All $\widehat{K}_S$ are pure simplicial complexes.*
(4) *Each component of $\widehat{K}_{S_1}$ is isomorphic to some induced subcomplex of K_{D_6}.*
(5) *Each component of $\widehat{K}_{S_4}$ is isomorphic to $\widehat{K}_{S_3}$.*

The above proposition was checked by a computer program. The Python codes used for validation are available at `https://github.com/Seonghyeon-Yu/E7-and-E8`. Note that to verify the correctness of these codes, we computed the Betti numbers for the types already known in Table 1 using the codes.

In conclusion, by Proposition 3.2, we only need to compute the Betti numbers of K_S for $S = S_2, S_3$, and S_5, since the Betti numbers of K_S of K_{D_6} are already computed in [3] for all $S \in Row(\Lambda_{D_6})$.

Remark 3.3.

(1) Each isomorphism in Proposition 3.2 (2) can be represented as one of simple roots; see Figure 1. For the type E_7, the simple root α_3 represents the isomorphism between the components of $\widehat{K}_{S_1}$; for the type E_8, the simple root α_2 represents the isomorphism between the components of $\widehat{K}_{S_4}$.

(2) Denote by $\bar{K}_S$ a connected component of $\widehat{K}_S$. The f-vectors $f(\bar{K}_S)$ of $\bar{K}_S$ as follows:

$$f(\bar{K}_{S_1}) = (204, 1312, 1920) \qquad f(\bar{K}_{S_4}) = (4664, 36288, 60480)$$
$$f(\bar{K}_{S_2}) = (928, 6848, 15360, 11520) \qquad f(\bar{K}_{S_5}) = (15488, 193536, 645120)$$
$$f(\bar{K}_{S_3}) = (4664, 36288, 60480)$$

As seen, the f-vectors of $\bar{K}_{S_3}$ and $\bar{K}_{S_4}$ are the same because of Proposition 3.2 (5). From the f-vectors, we can compute the Euler characteristic of K_S.

4. Computation of the Betti numbers

In this section, we shall use a computer program *SageMath 9.3* [12], to compute the Betti numbers of the given simplicial complexes. From Proposition 3.2, we already know the Betti numbers of $\widehat{K}_{S_1}$. For S_2 and S_3, we can compute the Betti numbers of $\widehat{K}_S$ within a reasonable time; see Table 4.

$\widetilde{\beta}_k(K_S)$	$S = S_1$	$S = S_2$	$S = S_3$
$k = 0$	1	0	0
$k = 1$	0	129	0
$k = 2$	1,622	0	28,855
$k = 3$	0	1,952	0
# orbit	63	63	1

Table 4. Nonzero reduced Betti numbers of K_S for S in $\mathrm{Row}(\Lambda_{E_7})$

From Table 4, we can immediately conclude the following theorem.

Theorem 4.1. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_7}$ are as follows:*

$$\beta_k(X^{\mathbb{R}}_{E_7}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8{,}127, & \text{if } k = 2 \\ 131{,}041, & \text{if } k = 3 \\ 122{,}976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 3.2 and the above result, we now have the Betti numbers of $\widehat{K}_{S_4}$. For any vertex v of $\widehat{K}_{S_5}$, we can check $\widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) = 0$ by the program.

Hence, we have the Mayer-Vietoris sequence

$$0 = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) \to \widetilde{H}_1(\widehat{K}_{S_5} - v) \oplus \widetilde{H}_1(St_{\widehat{K}_{S_5}}(v)) \to \widetilde{H}_1(\widehat{K}_{S_5}) \to \widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = 0.$$

Since $\widetilde{H}_1(St_{\widehat{K}_{S_5}}(v))$ is trivial, $\widetilde{H}_1(\widehat{K}_{S_5} - v)$ is isomorphic to $\widetilde{H}_1(\widehat{K}_{S_5})$. For the largest vertex orbit V of $\widehat{K}_{S_5}$, by the same proof argument as for Theorem 3.1, $\widetilde{H}_1(\widehat{K}_{S_5} - V)$ is isomorphic to $\widetilde{H}_1(\widehat{K}_{S_5})$. Note that the size of $\widehat{K}_{S_5} - V$ is much smaller than $\widehat{K}_{S_5}$. Thus, $\widetilde{\beta}_1(K_{S_5})$ can be computed within a reasonable time from $\widehat{K}_{S_5} - V$ instead of $\widehat{K}_{S_5}$. However, there is no vertex of $\widehat{K}_{S_5}$ such that $\widetilde{H}_2(Lk_{\widehat{K}_{S_5}}(v)) = 0$. Thus, for $k = 2, 3$ we must compute $\widetilde{\beta}_k(\widehat{K}_{S_5})$ directly, which takes a few days of run time. See Table 5 for the results.

$\widetilde{\beta}_k(K_S)$	$S = S_4$	$S = S_5$
$k = 0$	1	0
$k = 1$	0	769
$k = 2$	57,710	0
$k = 3$	0	177,280
# orbit	120	135

Table 5. Nonzero reduced Betti numbers of K_S for S in $\mathrm{Row}(\Lambda_{E_8})$

Table 5 implies the following theorem.

Theorem 4.2. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_8}$ are as follows:*

$$\beta_k(X^{\mathbb{R}}_{E_8}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103{,}815, & \text{if } k = 2 \\ 6{,}925{,}200, & \text{if } k = 3 \\ 23{,}932{,}800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristic number $\chi(X)$ of a topological space X is equal to the alternating sum of the Betti numbers $\beta_k(X)$ of X. We can use this fact as a confidence check for our results.

Remark 4.3. The $\mathbb{Z}_2$-cohomology ring of a real toric variety is completely determined by its fan [6], and then, it can be obtained that $\chi(X^{\mathbb{R}}_{E_7}) = 0$ and $\chi(X^{\mathbb{R}}_{E_8}) = 17{,}111{,}296$. Obviously, the alternating sums of the Betti numbers based on our results match $\chi(X^{\mathbb{R}}_{E_7})$ and $\chi(X^{\mathbb{R}}_{E_8})$, respectively.

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