Orbits of the isotropy group action on quaternionic symmetric spaces

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Abstract

We investigate some geometric properties of orbits of the isotropy group action on quaternionic irreducible symmetric spaces of compact type. We show that such orbits, except for a one-point set, satisfy one of some four properties and classify which orbits satisfy which properties in each such symmetric space. In a symmetric space, a connected component of the fixed point set of a geodesic symmetry, except for a one-point set, is called a polar. A polar is a totally geodesic submanifold and an orbit of the isotropy group action. By the classification, we show that an orbit which is a quaternionic submanifold or the image of a totally complex immersion is a polar, and a polar becomes a quaternionic submanifold or the image of a totally complex immersion.

1 Introduction

We study some geometric properties of orbits of the isotropy group action on quaternionic irreducible symmetric spaces of compact type with respect to the quaternionic structure. In [8], Enoyoshi and Tsukada show that a polar is the image of a totally complex immersion in the associative Grassmann manifold which is a quaternionic symmetric space. In a symmetric space, a polar is a connected component, except for a one-point set, of the fixed point set of

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a geodesic symmetry and it is known that a polar is a totally geodesic submanifold and an orbit of the isotropy group action [7]. In [12], the author studies orbits of the isotropy group action in the associative Grassmann manifold. In the present paper, we study orbits of the isotropy group action in each quaternionic irreducible symmetric space of compact type.

First, recall the definition of a quaternionic Kähler manifold. Let M be a $4n$-dimensional ($n \geq 2$) Riemann manifold and g be the Riemann metric and $\tilde{Q}$ be a 3-dimensional subbundle of $\mathrm{End}\, TM$ satisfying the following conditions:

(1) For any $x \in M$, there is a local frame field $\{I, J, K\}$ defined in a neighborhood U of x such that for any $p \in U$

$$I_p^2 = J_p^2 = K_p^2 = -\mathrm{Id}_{T_pM},$$
$$I_pJ_p = -J_pI_p = K_p, \quad J_pK_p = -K_pJ_p = I_p, \quad K_pI_p = -I_pK_p = J_p.$$

(2) For any $x \in M, I \in \tilde{Q}_x$ and $X, Y \in T_xM$,

$$g\big(I(X), Y\big) + g\big(X, I(Y)\big) = 0.$$

(3) $\tilde{Q}$ is parallel with respect to the Riemann connection of g.

Then, we call $(M, g, \tilde{Q})$ a quaternionic Kähler manifold and $\tilde{Q}$ a quaternionic structure of M. $\bigcup_{x \in M}\{J \in \tilde{Q}_x \ ; \ J^2 = -\mathrm{Id}_{T_xM}\}$ is denoted by Q. Then, Q is an S^2-bundle over M and called the twistor space of M. It is known that Q is a complex manifold and has a holomorphic contact structure [11]. In $\tilde{Q}$, we define an inner product $\langle \ , \ \rangle_{\tilde{Q}}$ as follows:

$$\langle A, B\rangle_{\tilde{Q}} = -\frac{1}{4n}\mathrm{tr}(AB) \qquad (A, B \in \tilde{Q}_x).$$

Then, $Q = \{A \in \tilde{Q} \ ; \ \langle A, A\rangle_{\tilde{Q}} = 1\}$. Also, the Riemann connection of g is metric with respect to $\langle \ , \ \rangle_{\tilde{Q}}$.

Next, we recall some submanifolds of a quaternionic Kähler manifold. Let N be a manifold and $f : N \to M$ be an immersion. We denote by f^*Q the pullback bundle of Q by f. If there is $I \in \Gamma(f^*Q)$ such that $I\big(df(T_xN)\big) \subset df(T_xN)$ for any $x \in N$, we call f an almost complex immersion and I the almost complex structure of f. We set $Q^I = \{J \in f^*Q \ ; \ \langle J, I\rangle_{\tilde{Q}} = 0\} = \{J \in f^*Q \ ; \ IJ = -JI\}$. Then, Q^I is an S^1-bundle over N. If $J\big(df(T_pN)\big) \perp df(T_pN)$ for any $p \in N$ and $J \in Q^I_p$, then we call f a totally complex immersion. It is known that if f is totally complex, then the almost complex structure of f is integrable [16]. Totally complex submanifolds are studied well by several authors ([2],[10],[13],[15]).

In an almost Hermitian manifold, CR submanifolds are defined as an analogy of almost complex submanifolds [3]. Let L be an almost Hermitian manifold. We denote the almost complex structure of L by I. Let U be a submanifold of L. If there is a distribution H on U such that $I(H) \subset H$ and the orthogonal complemental distribution $H^\perp$ of H in TU satisfies $I(H_x^\perp) \subset (T_xU)^\perp$ for any $x \in U$, we call U a CR submanifold of L [3]. U is an almost complex submanifold if $H = TU$ and U is a totally real submanifold if $H^\perp = TU$.

We naturally consider an analogy of an almost complex immersion of a quaternionic Kähler manifold. Let M be a quaternionic Kähler manifold, N be a manifold and $f : N \to M$ be an immersion. If there is a section $I \in \Gamma(f^*Q)$ and a distribution V, W of N such that

$$V + W = TN, \quad df(V) \perp df(W), \quad I(df(V)) \subset df(V), \quad I(df(W)) \subset \big(T(f(N))\big)^\perp,$$

where $\big(T(f(N))\big)^\perp$ is the normal bundle of $f(N)$ in TM, then we call f a CR immersion and I a CR structure of f. We denote the dimension of V by c_I. If $V = TN$, then f is an almost complex immersion. Moreover, if for any $p \in N$ and $J \in (Q_I)_p$ there are subspaces $V_J, W_J \subset T_pN$ such that

$$V_J + W_J = T_pN, \quad df(V_J) \perp df(W_J), \quad J(df(V_J)) \subset df(V_J), \quad J(df(W_J)) \subset \big(T(f(N))\big)^\perp$$

and $\dim V_J$ is independent of the choice of $p \in N$ and $J \in (Q_I)_p$, then we call f a totally CR immersion. We denote $\dim V_J$ by c'_I. A totally complex immersion is a totally CR immersion.

We recall QR submanifolds [4]. Let $N \subset M$ be a submanifold and $(TN)^\perp$ be the normal bundle of N. If there are subbundles $\mu, \nu \subset (TN)^\perp$ such that

$$\mu + \nu = (TN)^\perp, \quad \mu \perp \nu, \quad J(\mu) \subset TN, \quad J(\nu) \subset \nu$$

for any $J \in Q_x$ $(x \in N)$, then we call N a QR submanifold. A typical example of a QR submanifold is a hypersurface. QR submanifolds are studied in [4], [5]. We say that a QR submanifold is a quaternionic submanifold if $\mu = \{0\}$, that is TN is invariant under the quaternionic structure. It is known that a quaternionic submanifold of a quaternionic Kähler manifold is totally geodesic [1]. Moreover, we say that a submanifold N is totally real if $J(X) \in (T_pN)^\perp$ for any $p \in N, X \in T_pN, J \in Q_p$.

We obtain Theorem 1.1 as the main result of the present paper.

Theorem 1.1. Let M be a quaternionic irreducible symmetric space of compact type, Q be the twistor space of M and G be the identity component of the isometry group of M. Fix

$o \in M$ and let $K = \{g \in G \ ; \ g(o) = o\}$. For each $p \in M$, we set $K_p = \{k \in K \ ; \ k(p) = p\}$ and denote the identity component of K_p by $(K_p)_0$. Then, each K-orbit $K(p)$, except for a one-point set, satisfies one of the following properties.

(i) Let $f : K/(K_p)_0 \to K(p) \ ; \ k(K_p)_0 \mapsto k(p)$. Then, f is a K-equivariant totally CR immersion by each K-invariant section I of the induced bundle f^*Q of Q by f. Moreover, all K-invariant sections correspond to each point of the 2-dimensional sphere one-to-one and c_I, c'_I are independent of the choice of I. Also, $K(p)$ is a QR submanifold.

(ii) f is a K-equivariant totally CR immersion by each K-invariant section of f^*Q and K-invariant sections are unique up to the sign.

(iii) For any $x \in K(p)$ and $J \in Q_x$, there are subspaces $V, W \subset T_xK(p)$ such that

$$V + W = T_xK(p), \ V \perp W, \ J(V) \subset V, \ J(W) \subset (T_xK(p))^{\perp}.$$

Moreover, K acts on the restricted bundle of Q to $K(p)$ transitively.

(iv) For any $x \in K(p)$ and $J \in Q_x$, there are no subspaces of $T_xK(p)$ satisfying the property of (iii). K acts on the restricted bundle of Q to $K(p)$ transitively.

In the present paper, we classify which orbits satisfy which properties of Theorem 1.1 in each quaternionic irreducible symmetric space of compact type (Table 2, 3, 4, 5, 6). By this classification, we obtain Theorem 1.2.

Theorem 1.2. If a K-orbit $K(p)$ is a quaternionic submanifold or $f : K/(K_p)_0 \to K(p)$ is a totally complex immersion, then $K(p)$ is a polar. Conversely, a polar is a quaternionic submanifold or the image of a totally complex immersion.

This paper is organized as follows. In Section 2, we observe some results of quaternionic symmetric spaces classified by Wolf [17]. It is known that the rank of a quaternionic irreducible symmetric space is 1,2,3, or 4. Also, we observe some facts of orbits of the isotropy group action on a compact symmetric space. Moreover, we study orbits of the quaternionic projective space $\mathbb{H}P^n$ $(n \geq 2)$. In Section 3, we study orbits of a quaternionic symmetric space M in the case of rank$M = 4$, that is $M = SO(n)/SO(4) \times SO(n-4)$ $(n \geq 8)$, $F_4/((Sp(1) \times Sp(3))/\mathbb{Z}_2)$, $E_6/((Sp(1) \times SU(6))/\mathbb{Z}_2)$, $E_7/((Sp(1) \times Spin(12))/\mathbb{Z}_2)$, $E_8/((Sp(1) \times E_7)/\mathbb{Z}_2)$. In subsection 3.7, we classify which orbits satisfy which properties of Theorem 1.1. In Section 4, we consider the case of rank$M = 2$, that is $M = SU(n)/S(U(2) \times U(n-2))$ $(n \geq 4)$ and $G_2/SO(4)$. We only consider $M = SU(n)/S(U(2) \times U(n-2))$. In the case of

$M = G_2/SO(4)$, we refer to [12]. In Section 5, we consider the case of rank = 3, that is $M = SO(7)/SO(4) \times SO(3)$.

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2 Preliminaries

2.1 Quaternionic symmetric spaces

Let $(M, g, \tilde{Q})$ be a quaternionic Kähler manifold. We call M a quaternionic symmetric space if M is a symmetric space and $\tilde{Q}_p$ is contained in the linear holonomy group $\mathcal{J}_p$ of (M, g) for each $p \in M$. In the present paper, we consider quaternionic irreducible symmetric spaces of compact type. By Wolf [17], all quaternionic irreducible symmetric spaces of compact and noncompact type are constructed from complex simple Lie algebras. We shall review this construction in this section.

Let $\tilde{\mathfrak{g}}$ be a complex simple Lie algebra which is not of type A_1, A_2, B_2. Let τ be a complex conjugation of $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ be the compact real form of $\tilde{\mathfrak{g}}$ corresponding to τ. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ and $\tilde{\mathfrak{h}}$ be the complexification of $\mathfrak{h}$. Then, $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Denote the root system of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$ by Σ. Let $(\ ,\ )$ be an invariant non-degenerate symmetric bilinear form of $\tilde{\mathfrak{g}}$. Set $\mathfrak{h}_0 = i\mathfrak{h}$. For each $\gamma \in \Sigma$, we set $H_\gamma \in \mathfrak{h}_0$ such that $(H_\gamma, H) = \gamma(H)$ for any $H \in \mathfrak{h}_0$. Let $A_\gamma = \frac{2}{(H_\gamma, H_\gamma)} H_\gamma$. For any $\alpha, \beta \in \Sigma$, we set the Cartan integer $a_{\alpha,\beta} = (A_\alpha, H_\beta) = \frac{2(H_\alpha, H_\beta)}{(H_\alpha, H_\alpha)} \in \mathbb{Z}$. Take some linear order on $\mathfrak{h}_0$ and let β be the highest root of Σ and Σ^+ be the set of all positive roots. For $n \in \mathbb{Z}$, we set $\Sigma_n = \{\gamma \in \Sigma\ ;\ a_{\beta,\gamma} = n\}$. Then, $\Sigma_2 = \{\beta\}, \Sigma_{-2} = \{-\beta\}$ and $\Sigma = \Sigma_{-2} \cup \Sigma_{-1} \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$. Let $\theta = \exp(\mathrm{ad}\pi i A_\beta)$. Then, θ is an involutive automorphism of $\mathfrak{g}$. Set $\mathfrak{k} = \{X \in \mathfrak{g}\ ;\ \theta(X) = X\}$ and $\mathfrak{m} = \{X \in \mathfrak{g}\ ;\ \theta(X) = -X\}$. Then, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

Let G be the simply connected compact Lie group whose Lie algebra is $\mathfrak{g}$. Moreover, we denote by the same symbol the induced involutive automorphism of G by θ. Let $K = \{g \in G\ ;\ \theta(g) = g\}$. Since G is simply connected, K is connected. Let $M = G/K$ and $\pi : G \to M$ be the natural projection. Denote $o = \pi(e)$. Then, $T_oM = \mathfrak{m}$. Let $\langle\ ,\ \rangle$ be the G-invariant Riemann metric on M induced by $c(\ ,\ )|_{\mathfrak{m}\times\mathfrak{m}}$, where c is a negative constant. Then, $(M, \langle\ ,\ \rangle)$ is a simply connected irreducible symmetric space of compact type.

For $\gamma \in \Sigma$, let X_γ be a root vector of γ, that is X_γ satisfies $[H, X_\gamma] = \gamma(H) X_\gamma$ for $H \in \tilde{\mathfrak{h}}$.

Let $Z_\gamma = X_\gamma + \tau(X_\gamma)$ and $W_\gamma = i(X_\gamma - \tau(X_\gamma))$ for $\gamma \in \Sigma^+$. Then, $Z_\gamma, W_\gamma \in \mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\gamma\in\Sigma^+} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma).$$

Moreover, by the definition of θ

$$\mathfrak{k} = \mathfrak{h} + (\mathbb{R}Z_\beta + \mathbb{R}W_\beta) + \sum_{\gamma\in\Sigma^+\cap\Sigma_0} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma), \quad \mathfrak{m} = \sum_{\gamma\in\Sigma_1} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma).$$

Let $\mathfrak{s} = \mathbb{R}(iA_\beta) + \mathbb{R}Z_\beta + \mathbb{R}W_\beta$. Then, $\mathfrak{s}$ is a 3-dimensional ideal of $\mathfrak{k}$ and $\mathrm{Ad}(k)(\mathfrak{s}) \subset \mathfrak{s}$ for any $k \in K$ because K is connected. By the restriction of the linear isotropy representation of $\mathfrak{k}$ on $\mathfrak{m}$ to $\mathfrak{s}$, we may consider $\mathfrak{s} \subset \mathrm{End}\mathfrak{m} = \mathrm{End}T_oM$. Then, $G \times_K \mathfrak{s}$ defines a quaternionic structure $\tilde{Q}$ on M, where $G \times_K \mathfrak{s} = (G \times \mathfrak{s})/\sim$ and $(g_1, X_1) \sim (g_2, X_2) \in G \times \mathfrak{s}$ if and only if (g_1, X_1) and (g_2, X_2) satisfy $g_1^{-1}g_2 \in K$ and $X_1 = \mathrm{Ad}(g_1^{-1}g_2)X_2$. Let $S(\mathfrak{s}) = \{X \in \mathfrak{s}\ ;\ ((\mathrm{ad}X)|_{\mathfrak{m}})^2 = -\mathrm{Id}\} = \{a(iA_\beta) + bZ_\beta + cW_\beta\ ;\ a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\}$. Then, $G \times_K S(\mathfrak{s})$ is the twistor space of M since the action of K on $\mathfrak{s}$ is isometric and $\mathrm{Ad}(K)(S(\mathfrak{s})) \subset S(\mathfrak{s})$. Thus, we construct a quaternionic irreducible symmetric space of compact type. Conversely, any quaternionic irreducible symmetric space of compact type is given by this method. All quaternionic irreducible symmetric spaces of compact type are classified as Table 1.

G	K	$\dim M$	rank$M$	G	K	$\dim M$	rank$M$
$Sp(n+1)$	$Sp(1) \times Sp(n)$	$4n$ $(n \geq 2)$	1	G_2	$SO(4)$	8	2
$SU(n+2)$	$S(U(2) \times U(n))$	$4n$ $(n \geq 2)$	2	F_4	$(Sp(1) \times Sp(3))/\mathbb{Z}_2$	28	4
$SO(7)$	$SO(4) \times SO(3)$	12	3	E_6	$(Sp(1) \times SU(6))/\mathbb{Z}_2$	40	4
$SO(n+4)$	$SO(4) \times SO(n)$	$4n$ $(n \geq 4)$	4	E_7	$(Sp(1) \times Spin(12))/\mathbb{Z}_2$	64	4
				E_8	$(Sp(1) \times E_7)/\mathbb{Z}_2$	112	4

Table 1: quaternionic irreducible symmetric spaces of compact type

2.2 Orbits of the isotropy group action

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and R be the restricted root system with respect to $\mathfrak{a}$. For $\omega \in R$, we set $\tilde{\mathfrak{g}}_\omega = \{X \in \tilde{\mathfrak{g}}\ ;\ [A, X] = \omega(A)X\ (A \in \mathfrak{a})\}$. Remark that $\alpha(H) \in i\mathbb{R}$ for any $\alpha \in R$ and $H \in \mathfrak{a}$. Take a linear order on $i\mathfrak{a}$ and the set of all positive roots is denoted by R^+. For each $\omega \in R^+$, we set

$$\mathfrak{k}_\omega = \mathfrak{k} \cap (\tilde{\mathfrak{g}}_\omega + \tilde{\mathfrak{g}}_{-\omega}) = \{S \in \mathfrak{k}\ ;\ (\mathrm{ad}A)^2 S = -\omega(A)^2 S\ (A \in \mathfrak{a})\},$$
$$\mathfrak{m}_\omega = \mathfrak{m} \cap (\tilde{\mathfrak{g}}_\omega + \tilde{\mathfrak{g}}_{-\omega}) = \{T \in \mathfrak{m}\ ;\ (\mathrm{ad}A)^2 T = -\omega(A)^2 T\ (A \in \mathfrak{a})\}.$$

It is obvious that $\mathrm{ad}A(\mathfrak{m}_\omega) \subset \mathfrak{k}_\omega$ and $\mathrm{ad}A(\mathfrak{k}_\omega) \subset \mathfrak{m}_\omega$ for any $A \in \mathfrak{a}$. Let $\mathfrak{k}_0$ be the set of all centralizers of $\mathfrak{a}$ in $\mathfrak{k}$. Then,

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\omega \in R^+} \mathfrak{k}_\omega, \qquad \mathfrak{m} = \mathfrak{a} + \sum_{\omega \in R^+} \mathfrak{m}_\omega.$$

Lemma 2.1. [14] For each $\omega \in R^+$, there is an orthonormal basis $S_1, \cdots, S_{n(\omega)}$ of $\mathfrak{k}_\omega$ and $T_1, \cdots T_{n(\omega)}$ of $\mathfrak{m}_\omega$ such that

$$\begin{aligned}
&[H, S_i] = i\alpha(H)T_i, \quad [H, T_i] = -i\alpha(H)S_i, \\
&\mathrm{Ad}(\exp H)S_i = \cos(i\alpha(H))S_i + \sin(i\alpha(H))T_i, \\
&\mathrm{Ad}(\exp H)T_i = -\sin(i\alpha(H))S_i + \cos(i\alpha(H))T_i
\end{aligned}$$

for any $H \in \mathfrak{a}$ and $1 \leq i \leq n(\omega)$, where $n(\omega)$ is the multiplicity of ω.

For each $H \in \mathfrak{a}$, we denote $\pi(\exp(-H)K\exp H)$ by $\mathcal{O}_H$. Let $K_H = \{k \in K\ ;\ \pi(k\exp H) = \pi(\exp H)\}$ and $\mathfrak{k}_H = \{X \in \mathfrak{k}\ ;\ \mathrm{Ad}(\exp H)X \in \mathfrak{k}\}$. Then, the Lie algebra of K_H is $\mathfrak{k}_H$. Denote the identity component of K_H by $(K_H)_0$. Define a K-action on $\mathcal{O}_H$ such that $K \times \mathcal{O}_H \ni (k, \pi(p)) \mapsto \pi(\exp(-H)k(\exp H)p) \in \mathcal{O}_H$. Then, $\mathcal{O}_H = K/K_H$. For each $H \in \mathfrak{a}$, we set $R_H^+ = \{\alpha \in R^+\ ;\ i\alpha(H) \in \pi\mathbb{Z}\}$. Then, the following direct sum decompositions are true.

$$\mathfrak{k}_H = \mathfrak{k}_0 + \sum_{\omega \in R_H^+} \mathfrak{k}_\omega, \qquad T_o\mathcal{O}_H = \sum_{\omega \in R^+,\ \omega \notin R_H^+} \mathfrak{m}_\omega, \qquad (T_o\mathcal{O}_H)^\perp = \mathfrak{a} + \sum_{\omega \in R_H^+} \mathfrak{m}_\omega,$$

where $(T_o\mathcal{O}_H)^\perp$ is the orthogonal complement of $T_o\mathcal{O}_H$ in $\mathfrak{m} = T_oM$.

Let $F = \{\omega_1. \cdots, \omega_n\}$ be the set of all simple roots of R^+ and η be the highest root. Let $\mathcal{F} = F \cup \{\eta\}$. Set

$$Q = \{H \in \mathfrak{a}\ ;\ 0 < i\lambda(H) < \pi\ (\lambda \in \mathcal{F})\}.$$

Then, each K-orbit intersects $\pi(\exp\overline{Q})$ at only one point. For any subset $\Delta \subset \mathcal{F}$ such that $\Delta \neq \{\eta\}$, we set

$$Q_\Delta = \left\{ H \in \overline{Q}\ ;\ \begin{array}{ll} 0 < i\lambda(H)\ (\lambda \in \Delta \cap F), & i\eta(H) < \pi\ (\eta \in \Delta), \\ 0 = i\mu(H)\ (\mu \in F - \Delta), & i\eta(H) = \pi\ (\eta \notin \Delta). \end{array} \right\}$$

Then, $\overline{Q} = \sqcup_{\Delta \subset \mathcal{F}, \Delta \neq \{\eta\}} Q_\Delta$ and R_H^+ is independent of the choice of $H \in Q_\Delta$ and depends on the choice of Δ.

Let $\pi_\mathfrak{s} : \mathfrak{k} \to \mathfrak{s}$ be the orthogonal projection. Then, $\pi_\mathfrak{s}(\mathfrak{k}_H)$ is a subalgebra of $\mathfrak{s}$ for any $H \in \mathfrak{a}$. Since $\mathfrak{s} \cong \mathfrak{sp}(1)$, $\dim \pi_\mathfrak{s}(\mathfrak{k}_H) = 0, 1, 3$. If $\dim \pi_\mathfrak{s}(\mathfrak{k}_H) = 0$, then $\pi_s(\mathfrak{k}_H)$ is trivial. If

$\dim \pi_{\mathfrak{s}}(\mathfrak{k}_H) = 1$, then $\pi_{\mathfrak{s}}(\mathfrak{k}_H)$ is isomorphic to $\mathfrak{u}(1)$. If $\dim \pi_{\mathfrak{s}}(\mathfrak{k}_H) = 3$, then $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathfrak{s}$. We say $H \in \mathfrak{a}$ is type I if $\dim \pi_{\mathfrak{s}}(\mathfrak{k}_H) = 0$, H is type II if $\dim \pi_{\mathfrak{s}}(\mathfrak{k}_H) = 1$, and H is type III if $\dim \pi_{\mathfrak{s}}(\mathfrak{k}_H) = 3$. Remark that $(K_H)_0$ acts on $\mathfrak{s}$ and $S(\mathfrak{s})$ since K acts on them. Because $(K_H)_0$ is connected and $\mathfrak{s}$ is an ideal of $\mathfrak{k}$, it is ture that $(K_H)_0$ acts on $S(\mathfrak{s})$ trivially if H is type I, acts on as rotations if H is type II, and acts on transitively if H is type III.

We consider the following immersion:

$$f_H : K/(K_H)_0 \to \mathcal{O}_H \ ; \ k(K_H)_0 \mapsto \pi(\exp(-H)k\exp H).$$

Let $f_H^* Q$ be the pull-back bundle of Q by f_H. Set $o' = e(K_H)_0$. Then, $(K_H)_0$ acts on $(f_H^*Q)_{o'}$. If H is type I, then $(K_H)_0$ acts on $(f_H^*Q)_{o'}$ trivially, so for any $A \in S(\mathfrak{s})$ a section $J : K/(K_H)_0 \to f_H^*Q \ ; \ k(K_H)_0 \mapsto dk \circ A \circ dk^{-1}$ $(A \in S(\mathfrak{s}))$ is a K-invariant section of f_H^*Q. Thus, we can construct K-invariant sections of f_H^*Q corresponding to each point of $S(\mathfrak{s}) \cong S^2$. If H is type II, then $(K_H)_0$ acts on $S(\mathfrak{s})$ as rotations, so there is unique $B \in S(\mathfrak{s})$ such that $\pm B$ is fixed by $(K_H)_0$. By the similar way, we can construct the K-invariant section I of f_H^*Q by $\pm B$. In particular, K-invariant sections of f_H^*Q are unique up to sign. Let $Q_I := \{J \in f_H^*Q \ ; \ IJ = -JI\}$. Then, Q_I is given by $S_B(\mathfrak{s}) := \{C \in S(\mathfrak{s}) \ : \ C \perp B\}$. Since $(K_H)_0$ acts on $S_B(\mathfrak{s})$ transitively, K acts on Q_I transitively. Let Q_H be the restricted bundle of Q to $\mathcal{O}_H$. If H is type III, then K_H acts on $S(\mathfrak{s})$ transtively, so K acts on Q_H transitively. Summarizing these arguments, we obtain Proposition 2.2.

Proposition 2.2. Let $H \in \mathfrak{a}$ and $f_H : K/(K_H)_o \to \mathcal{O}_H \ ; \ k(K_H)_0 \mapsto \pi(\exp(-H)k\exp H)$.

(i) If H is type I, then there is a K-invariant section of f_H^*Q and all K-invariant sections correspond to each point of $S(\mathfrak{s}) \cong S^2$ one-to-one.

(ii) If H is type II, then there is a K-invariant section of f_H^*Q and K-invariant sections are unique up to sign. Let I be a K-invariant section of f_H^*Q and $Q_I := \{J \in f_H^*Q \ ; \ IJ = -JI\}$. Then, K acts on Q_I transitively.

(iii) Let Q_H be the restricted bundle of Q to $\mathcal{O}_H$. If H is type III, then K acts on Q_H transitively.

We say that $\mathcal{O}_H$ and $\pi(K\exp H)$ are type I (resp. II,III) if H is type I (resp. II,III). In the present paper, for each quaternionic irreducible symmetric space of compact type, we study that each orbit of the isotropy group action becomes which of type I, type II, and type III and has what properties these K-invariant sections have.

At the end of this section, we consider the quaternionic irreducible symmetric space M of compact type whose rank is 1, that is the quaternionic projective space $\mathbb{H}P^n$ $(n \geq 2)$. In

$\mathbb{H}P^n$, it is known that orbits of the isotropy group action become one of the following: the trivial point, principal orbits, or $\mathbb{H}P^{n-1}$ which is a polar [7]. We see easily that the polar is type III and a quaternionic totally geodesic submanifold. In general, if $\mathcal{O}_H$ is a principal orbit, then $(T_o\mathcal{O}_H)^\perp = \mathfrak{a}$ and $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \{0\}$, so $\mathcal{O}_H$ is type I. Since $\mathrm{rank}\mathbb{H}P^n = 1$, each principal orbit $\mathcal{O}_H$ is a hypersurface of $\mathbb{H}P^n$. Thus, principal orbits are QR submanifolds. For each $X \in \mathfrak{s}$, set subspaces V_X, W_X of $T_o\mathcal{O}_H$ as follows: $W_X = \mathrm{ad}X(\mathfrak{a})$ and V_X is the orthogonal complement of W_X in $T_o\mathcal{O}_H$. Then, V_X, W_X satisfy

$$V_X \perp W_X, \quad V_X + W_X = T_o\mathcal{O}_H, \quad \mathrm{ad}X(V_X) \subset V_X, \quad \mathrm{ad}X(W_X) \subset (T_o\mathcal{O}_H)^\perp.$$

Thus, f_H is a K-equivariant totally CR immersion by each K-invariant section of f_H^*Q. Summarizing these arguments, we obtain Theorem 2.3.

Theorem 2.3. In $\mathbb{H}P^n$ $(n \geq 2)$, an orbit of the isotropy group action is one of the following.:

(i) the trivial point,

(ii) $\mathbb{H}P^{n-1}$ which is a quaternionic totally geodesic submanifold,

(iii) a principal orbit which is a QR-submanifold.

If $\mathcal{O}_H$ is a principal orbit, the immersion f_H is a K-equivariant totally CR immersion by any K-invariant section I of f_H^*Q and all K-invariant sections correspond to each point of S^2 one-to-one. Moreover, c_I, c'_I are independent of the choice of I.

3 The case of $\mathrm{rank}M = 4$

In this section, we consider the case of $\mathrm{rank}M = 4$, that is $G = SO(n)$ $(n \geq 8), F_4, E_6, E_7, E_8$ and $\tilde{\mathfrak{g}} = \mathfrak{so}(n, \mathbb{C}), \mathfrak{f}_4^{\mathbb{C}}, \mathfrak{e}_6^{\mathbb{C}}, \mathfrak{e}_7^{\mathbb{C}}, \mathfrak{e}_8^{\mathbb{C}}$. In subsection 3.1, 3.2, and 3.3, we consider an explicit descripion of the restricted root system and some preparations for this description. In subsection 3.4, we consider $\mathrm{ad}X|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ $(X \in \mathfrak{s})$ for studying the quaternionic structure. In subsection 3.5, we study $H \in \mathfrak{a}$ satisfying $\omega(H) \in i\pi\mathbb{Z}$ for some restricted roots ω and in subsection 3.6, we study orbits of the action of the isotropy group of the isometry group. In subsection 3.7, we summarize properties of each orbit with respect to the quaternionic structure.

3.1 H-orbit

Let $(\ ,\ )$ be the Killing form of $\tilde{\mathfrak{g}}$ and $\{X_\alpha\ ;\ \alpha \in \Sigma\}$ be a Chevalley basis, that is X_α satisfies

(i) $[X_\alpha, X_{-\alpha}] = A_\alpha$,

(ii) $[H, X_\alpha] = \alpha(H)X_\alpha$ $(H \in \tilde{\mathfrak{h}})$,

(iii) For any $\alpha, \gamma \in \Sigma$, $[X_\alpha, X_\gamma] = 0$ if $\alpha + \gamma \notin \Sigma$ and $[X_\alpha, X_\gamma] = N_{\alpha,\gamma}X_{\alpha+\gamma}$ if $\alpha + \gamma \in \Sigma$, where $N_{\alpha,\gamma} = \pm(p + 1)$ and p is the greatest positive number such that $\gamma - p\alpha \in \Sigma$.

Take a linear order in $\mathfrak{h}_0$ and denote the set of all positive roots by Σ^+ and let β be the highest root. For each $n \in \mathbb{Z}$, we set Σ_n as section 1. Set the complex conjugation τ such that

$$\tau(A_\alpha) = -A_\alpha, \quad \tau(X_\alpha) = -X_{-\alpha} \quad (\alpha \in \Sigma^+).$$

Let $Z_\alpha = X_\alpha + \tau(X_\alpha) = X_\alpha - X_{-\alpha}$ and $W_\alpha = i(X_\alpha - \tau(X_\alpha)) = i(X_\alpha + X_{-\alpha})$ for each $\alpha \in \Sigma$. Then, $\mathfrak{g} = \{X \in \tilde{\mathfrak{g}}\ ;\ \tau(X) = X\}$ is a compact real from and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Sigma^+} (\mathbb{R}Z_\alpha + \mathbb{R}W_\alpha).$$

By simple computations, we obtain Lemma 3.1 and Lemma 3.2.

Lemma 3.1. $N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta}$ for $\alpha, \beta \in \Sigma$. Moreover, is $\alpha, \beta, \gamma \in \Sigma$ satisfy $\alpha + \beta + \gamma = 0$ and $|\beta| = |\gamma|, |\alpha| = \sqrt{k}|\beta|$ $(k \in \mathbb{N})$, then it follows that $N_{\alpha,\beta} = \frac{1}{k}N_{\beta,\gamma} = N_{\gamma,\alpha}$.

Lemma 3.2. For any $\alpha, \beta \in \Sigma$ $(\beta \neq -\alpha)$,

$$\begin{aligned}
[Z_\alpha, Z_\beta] &= N_{\alpha,\beta}Z_{\alpha+\beta} - N_{-\alpha,\beta}Z_{-\alpha+\beta},\\
[Z_\alpha, W_\beta] &= N_{\alpha,\beta}W_{\alpha+\beta} - N_{-\alpha,\beta}W_{-\alpha+\beta},\\
[W_\alpha, W_\beta] &= -N_{\alpha,\beta}Z_{\alpha+\beta} - N_{-\alpha,\beta}Z_{-\alpha+\beta},\\
[W_\alpha, Z_\beta] &= N_{\alpha,\beta}W_{\alpha+\beta} + N_{-\alpha,\beta}W_{-\alpha+\beta}.
\end{aligned}$$

Set $\theta = \exp(\mathrm{ad}(\pi i A_\beta))$ and $\mathfrak{k}, \mathfrak{m}$ as section 2. Since the rank of M is 4, there are $\alpha_1, \cdots, \alpha_4 \in \Sigma_1$ such that they are longest roots and $\alpha_i \pm \alpha_j \notin \Sigma$ $(1 \leq i \neq j \leq 4)$ and the subspace $\mathfrak{a} = \sum_{i=1}^4 \mathbb{R}Z_{\alpha_i}$ is a maximal abelian subspace of $\mathfrak{m}$. In $\mathfrak{h}_0$, the reflection with respect to H_γ $(\gamma \in \Sigma)$ is denoted by τ_γ, that is $\tau_\gamma(X) = X - \frac{2(H_\gamma,X)}{(H_\gamma,H_\gamma)}H_\gamma$ $(X \in \mathfrak{h}_0)$. Let H be the subgroup of the Weyl group generated by $\tau_{\alpha_1}, \cdots, \tau_{\alpha_4}$. Since $(H_{\alpha_i}, H_{\alpha_j}) = 0$ $(1 \leq i \neq j \leq 4)$, $\tau_{\alpha_1}, \cdots, \tau_{\alpha_4}$ commute to each other and $H \cong (\mathbb{Z}_2)^4$. We consider the action of H on Σ.

Obviously, the H-orbit through α_i is $\{\alpha_i\}$ for each $1 \le i \le 4$. For an H-orbit Σ' such that $\Sigma' \cap \Sigma_1 \ne \phi$ and $\Sigma' \ne \{\alpha_i\}$ $(1 \le i \le 4)$, set $\mathfrak{m}_{\Sigma'}$ and $\mathfrak{k}_{\Sigma'}$ as follows:

$$\mathfrak{m}_{\Sigma'} = \sum_{\gamma \in (\Sigma_1 \cup \Sigma_{-1}) \cap \Sigma'} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma), \quad \mathfrak{k}_{\Sigma'} = \sum_{\gamma \in (\Sigma_0 \cup \Sigma_2) \cap \Sigma'} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma).$$

Then, $\mathrm{ad}(\mathfrak{a})(\mathfrak{m}_{\Sigma'}) \subset \mathfrak{k}_{\Sigma'}$ and $\mathrm{ad}(\mathfrak{a})(\mathfrak{k}_{\Sigma'}) \subset \mathfrak{m}_{\Sigma'}$. In the following, we study H-orbits intersecting Σ_1.

Denote by Σ_β the H-orbit through β, that is

$$\Sigma_\beta = \left\{ \begin{matrix} \beta, & & & \\ \beta - \alpha_1, & \beta - \alpha_2, & \beta - \alpha_3, & \beta - \alpha_4, \\ \beta - (\alpha_1 + \alpha_2), & \beta - (\alpha_1 + \alpha_3), & \beta - (\alpha_1 + \alpha_4), & \\ \beta - (\alpha_2 + \alpha_3), & \beta - (\alpha_2 + \alpha_4), & \beta - (\alpha_3 + \alpha_4), & \\ \beta - (\alpha_1 + \alpha_2 + \alpha_3), & \beta - (\alpha_1 + \alpha_2 + \alpha_4), & \beta - (\alpha_1 + \alpha_3 + \alpha_4), & \beta - (\alpha_2 + \alpha_3 + \alpha_4), \\ \beta - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) & & & \end{matrix} \right\}.$$

By the definition of $\beta, \alpha_1, \cdots, \alpha_4$, it is obvious that $\beta - (\alpha_1 + \cdots + \alpha_4) = -\beta$ since $a_{\beta, \beta - (\alpha_1 + \cdots + \alpha_4)} = -2$. Thus, any $\gamma \in \Sigma_\beta$ satisfies $-\gamma \in \Sigma_\beta$ and $\Sigma_\beta \cup \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\}$ is a subsystem of Σ which is isomorphic to D_4. Set

$$\Sigma_\beta^+ = \left\{ \begin{matrix} \beta - \alpha_1, & \beta - \alpha_2, & \beta - \alpha_3, & \beta - \alpha_4, \\ \beta, & \beta - (\alpha_1 + \alpha_2), & \beta - (\alpha_1 + \alpha_3), & \beta - (\alpha_1 + \alpha_4) \end{matrix} \right\}.$$

Then, $\Sigma_\beta^+ \cup (-\Sigma_\beta^+) = \Sigma_\beta$, where for any subset $A \subset \Sigma$ we set $-A = \{-\gamma \ ; \ \gamma \in A\}$. We see $\Sigma_\beta^+ \cap \Sigma_1 = \{\beta - \alpha_i \ ; \ 1 \le i \le 4\}, \Sigma_\beta^+ \cap \Sigma_2 = \{\beta\}, \Sigma_\beta^+ \cap \Sigma_0 = \{\beta - (\alpha_1 + \alpha_i) \ ; \ 2 \le i \le 4\}$. Thus,

$$\mathfrak{m}_{\Sigma_\beta} = \sum_{i=1}^{4} (\mathbb{R}Z_{\beta - \alpha_i} + \mathbb{R}W_{\beta - \alpha_i}), \quad \mathfrak{k}_{\Sigma_\beta} = (\mathbb{R}Z_\beta + \mathbb{R}W_\beta) + \sum_{i=2}^{4} \big(\mathbb{R}Z_{\beta - (\alpha_1 + \alpha_i)} + \mathbb{R}W_{\beta - (\alpha_1 + \alpha_i)}\big).$$

Let $\gamma \in \Sigma_1$ be a longest root and $\gamma \notin \Sigma_\beta$. Denote by Σ_γ the H-orbit through γ. Then, we see that $a_{\alpha_i, \gamma} = a_{\alpha_j, \gamma} = 1$ for some $1 \le i < j \le 4$ and $a_{\alpha_k, \gamma} = a_{\alpha_l, \gamma} = 0$ for $1 \le k < l \le 4$ such that $k, l \ne i, j$. Also, $a_{\alpha_k, \beta - \gamma} = a_{\alpha_l, \beta - \gamma} = 1$ and $a_{\alpha_i, \beta - \gamma} = a_{\alpha_j, \beta - \gamma} = 0$. Hence, $\Sigma_\gamma = \{\gamma, \gamma - \alpha_i, \gamma - \alpha_j, \gamma - (\alpha_i + \alpha_j)\}$. Then, $\Sigma_{-(\gamma - (\alpha_i + \alpha_j))} = -\Sigma_\gamma$. Because $\Sigma_\gamma \cap \Sigma_1 = \{\gamma\}, \Sigma_\gamma \cap \Sigma_{-1} = \{\gamma - (\alpha_i + \alpha_j)\}, \Sigma_\gamma \cap \Sigma_0 = \{\gamma - \alpha_i, \gamma - \alpha_j\}$,

$$\mathfrak{m}_{\Sigma_\gamma} = \big(\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma\big) + \big(\mathbb{R}Z_{\gamma - (\alpha_i + \alpha_j)} + \mathbb{R}W_{\gamma - (\alpha_i + \alpha_j)}\big),$$
$$\mathfrak{k}_{\Sigma_\gamma} = \big(\mathbb{R}Z_{\gamma - \alpha_i} + \mathbb{R}W_{\gamma - \alpha_i}\big) + \big(\mathbb{R}Z_{\gamma - \alpha_j} + \mathbb{R}W_{\gamma - \alpha_j}\big).$$

We say that an H-orbit through such $\gamma \in \Sigma_1$ is type $L(i, j)$ or simply type L. Let $\delta \in \Sigma_1$ be a shortest root and denote by Σ_δ the H-orbit through δ. It is easily seen that $a_{\alpha_i, \delta} = a_{\alpha_j, \delta} = 1$

for some $1 \leq i < j \leq 4$ and $a_{\alpha_k,\delta} = a_{\alpha_l,\delta} = 0$ for $1 \leq k < l \leq 4$ such that $k, l \neq i, j$. Moreover, $a_{\alpha_k,\beta-\delta} = a_{\alpha_l,\beta-\delta} = 1$ and $a_{\alpha_i,\beta-\delta} = a_{\alpha_j,\beta-\delta} = 0$. Thus, $\Sigma_\delta = \{\delta, \delta - \alpha_i, \delta - \alpha_j, \delta - (\alpha_i + \alpha_j)\}$. We see that $\delta - (\alpha_i + \alpha_j) = -\delta$ since $a_{\delta,\delta-(\alpha_i+\alpha_j)} = -2$ and Σ_δ is a subsystem of Σ and isomorphic to $A_1 \cup A_1$. Because $\Sigma_\delta \cap \Sigma_1 = \{\delta\}$ and $\Sigma_\delta \cap \Sigma_0 = \{\pm(\delta - \alpha_i)\}$,

$$\mathfrak{m}_{\Sigma_\delta} = \mathbb{R}Z_\delta + \mathbb{R}W_\delta, \qquad \mathfrak{k}_{\Sigma_\delta} = \mathbb{R}Z_{\delta-\alpha_i} + \mathbb{R}W_{\delta-\alpha_i}.$$

We say that an H-orbit through such $\delta \in \Sigma_1$ is type $S(i,j)$ or simply type S.

Let $\Sigma^L(1), \cdots, \Sigma^L(n)$ be H-orbits of type L such that $\Sigma^L(1), -\Sigma^L(1), \cdots, \Sigma^L(n), -\Sigma^L(n)$ are all H-orbits of type L. Moreover, let $\Sigma^S(1), \cdots, \Sigma^S(m)$ be all H-orbits of type S. Then, the following direct sum decomposition follows:

$$\mathfrak{m} = \mathfrak{a} + \mathbb{R}W_{\alpha_1} + \cdots + \mathbb{R}W_{\alpha_4} + \mathfrak{m}_{\Sigma_\beta} + \sum_{a=1}^{n} \mathfrak{m}_{\Sigma^L(a)} + \sum_{b=1}^{m} \mathfrak{m}_{\Sigma^S(b)}.$$

3.2 Structure coefficient $N_{\alpha,\beta}$

In the Chevalley basis $\{X_\alpha \ ; \ \alpha \in \Sigma\}$, the sign of the structure coefficient $N_{\alpha,\beta}$ depends on an orientation of each X_α. In the following, we fix orientations of some X_α and decide the sign of some structure coefficients. First, we fix an orientation of $X_\beta, X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_3}$ and set $w_i = \exp\frac{\pi}{2}Z_{\alpha_i}$ $(i = 1, 2, 3)$. For each $\gamma = \beta - (\epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \epsilon_3\alpha_3) \in \Sigma_\beta^+$ $(\epsilon_i = 0, 1, \ i = 1, 2, 3)$, we set an orientation of X_γ such that

$$X_{\beta-(\epsilon_1\alpha_1+\epsilon_2\alpha_2+\epsilon_3\alpha_3)} = \mathrm{Ad}(w_1^{\epsilon_1} w_2^{\epsilon_2} w_3^{\epsilon_3}) X_\beta.$$

By the commutativity of w_1, w_2, w_3, these orientations are well-defined. For any $\gamma \in \Sigma$ and $t \in \mathbb{R}$,

$$\begin{aligned} \gamma - \alpha_i \in \Sigma \text{ and } \gamma + \alpha_i \notin \Sigma \ &\Longrightarrow\ \mathrm{Ad}(\exp tZ_{\alpha_i})X_\gamma = \cos t X_\gamma - N_{-\alpha_i,\gamma} \sin t X_{\gamma-\alpha_i}, \\ \gamma - \alpha_i \notin \Sigma \text{ and } \gamma + \alpha_i \in \Sigma \ &\Longrightarrow\ \mathrm{Ad}(\exp tZ_{\alpha_i})X_\gamma = \cos t X_\gamma + N_{\alpha_i,\gamma} \sin t X_{\gamma+\alpha_i}. \end{aligned}$$

Hence, $N_{-\alpha_i,\gamma} = -1$ if $\gamma - \alpha_i \in \Sigma$ and $\gamma + \alpha_i \notin \Sigma$, and $N_{\alpha_i,\gamma} = -1$ if $\gamma - \alpha_i \notin \Sigma$ and $\gamma + \alpha_i \in \Sigma$. Next, we fix an orientation of X_{α_4} such that $\mathrm{Ad}(w_1 \cdots w_4)X_\beta = -X_{-\beta}$.

Lemma 3.3. $N_{-\alpha_4,\beta} = N_{-\alpha_4,\beta-(\alpha_i+\alpha_j)} = -1$ and $N_{-\alpha_4,\beta-\alpha_i} = N_{-\alpha_4,\beta-(\alpha_1+\alpha_2+\alpha_3)} = 1$ for any $1 \leq i \neq j \leq 3$.

Proof. First,

$$\begin{aligned}-N_{-\alpha_4,\beta}X_{\beta-\alpha_4} &= \mathrm{Ad}(w_4)X_\beta = -\mathrm{Ad}(w_1w_2w_3)^{-1}X_{-\beta} = \mathrm{Ad}(w_1w_2w_3)^{-1}\tau(X_\beta)\\ &= \tau(\mathrm{Ad}(w_1w_2w_3)^{-1}X_\beta) = \tau(-X_{\beta-(\alpha_1+\alpha_2+\alpha_3)}) = X_{-\beta+(\alpha_1+\alpha_2+\alpha_3)} = X_{\beta-\alpha_4},\end{aligned}$$

so we obtain $N_{-\alpha_4,\beta} = -1$. Moreover, $N_{-\alpha_4,\beta-(\alpha_1+\alpha_2+\alpha_3)} = 1$ since $N_{-\alpha_4,\beta} = N_{-\beta+\alpha_4,-\alpha_4} = -N_{-\alpha_4,\beta-(\alpha_1+\alpha_2+\alpha_3)}$ by Lemma 3.1. Next, we will show $N_{-\alpha_4,\beta-\alpha_1} = 1$. The other cases are proved by the similar way.

$$\begin{aligned}-N_{-\alpha_4,\beta-\alpha_1}X_{\beta-(\alpha_1+\alpha_4)} &= \mathrm{Ad}(w_1w_4)X_\beta = -\mathrm{Ad}(w_2w_3)^{-1}X_{-\beta} = \mathrm{Ad}(w_2w_3)^{-1}\tau(X_\beta)\\ &= \tau(\mathrm{Ad}(w_2w_3)^{-1}X_\beta) = \tau(X_{\beta-(\alpha_2+\alpha_3)}) = -X_{-\beta+(\alpha_2+\alpha_3)} = -X_{\beta-(\alpha_1+\alpha_4)},\end{aligned}$$

so $N_{-\alpha_4,\beta-\alpha_1} = 1$. Also, $N_{-\alpha_4,\beta-(\alpha_2+\alpha_3)} = -1$ because $N_{-\alpha_4,\beta-\alpha_1} = N_{-\alpha_4,-\beta+(\alpha_2+\alpha_3+\alpha_4)} = N_{\beta-(\alpha_2+\alpha_3),-\alpha_4} = -N_{-\alpha_4,\beta-(\alpha_2+\alpha_3)}$. □

By Lemma 3.1 and Lemma 3.3, we obtain Corollary 3.4 immediately.

Corollary 3.4. For any $1 \le i \ne j \le 3$, $N_{-\alpha_i,\beta-\alpha_4} = N_{\alpha_i,\beta-(\alpha_i+\alpha_4)} = N_{-\alpha_i,\beta-(\alpha_j+\alpha_4)} = N_{\alpha_4,\beta-(\alpha_i+\alpha_4)} = 1$, and $N_{\alpha_4,\beta-\alpha_4} = -1$.

Let Σ^L be an H-orbit of type $L(i,j)$ $(1 \le i < j \le 4)$ and $\Sigma^L \cap \Sigma_1 = \{\gamma\}$. Fix an orientation of X_γ and set an orientation of $X_{\gamma-\alpha_i}, X_{\gamma-\alpha_j}, X_{\gamma-(\alpha_i+\alpha_j)}$ such that

$$X_{\gamma-\alpha_i} = \mathrm{Ad}(w_i)X_\gamma,\ X_{\gamma-\alpha_j} = \mathrm{Ad}(w_j)X_\gamma,\ X_{\gamma-(\alpha_i+\alpha_j)} = \mathrm{Ad}(w_iw_j)X_\gamma.$$

Then, we can prove that for any $\epsilon \in \Sigma^L$ and $k \in \{i,j\}$ it is true that $N_{\alpha_k,\epsilon} = -1$ if $\epsilon+\alpha_k \in \Sigma$ and $N_{-\alpha_k,\epsilon} = -1$ if $\epsilon - \alpha_k \in \Sigma$ by the similar way to the above arguments.

Let Σ^S be an H-orbit of type $S(i,j)$ $(1 \le i < j \le 4)$ and $\Sigma^1 \cap \Sigma^S = \{\delta\}$. Then, $\delta - (\alpha_i + \alpha_j) = -\delta$. Fix an orientation of X_δ and set an orientation of $X_{\delta-\alpha_i}$ such that $X_{\delta-\alpha_i} = \mathrm{Ad}(w_i)X_\delta$. Then, we easily see $N_{-\alpha_i,\delta} = -1$ and $N_{-\alpha_j,\delta-\alpha_i} = N_{\delta,-\alpha_j}$.

3.3 Restricted root system

It is known that the restricted root system of quaternionic irreducible symmetric space of compact type whose rank is 4 is type D_4, B_4 or F_4 [9]. In this subsection, using the Chevalley basis $\{X_\alpha\ ;\ \alpha \in \Sigma\}$ and the structure coefficient $N_{\alpha,\beta}$, we describe the restricted root system explicitly. Let R be the restricted root system of $(\mathfrak{g},\mathfrak{k})$ with respect to $\mathfrak{a}$. Let $A = \sum_{i=1}^4 \lambda_i Z_{\alpha_i} \in \mathfrak{a}$ $(\lambda_i \in \mathbb{R})$. If the linear form ω of $\mathfrak{a}$ satisies $\omega(A) = \sum_{i=1}^4 a_i\lambda_i$ $(a_i \in \mathbb{R})$, then we often denote ω by $\sum_{i=1}^4 a_i\lambda_i$. Conversely, $\sum_{i=1}^4 a_i\lambda_i$ often means the linear form ω of $\mathfrak{a}$ such that $\omega(A) = \sum_{i=1}^4 a_i\lambda_i$. For any linear form ω of $\mathfrak{a}$, we denote the extension of ω as complex linearly to $\mathfrak{a}^{\mathbb{C}}$ by the same symbol. Moreover, for any subset $W \subset \mathfrak{a}^*$, $\{\pm i\omega \in (\mathfrak{a}^{\mathbb{C}})^* \ ; \ \omega \in W\}$ is denoted by $\pm iW$, where for any vector space V the dual space of V is denoted by V^*.

First, we study $\mathrm{ad}(A)|_{\mathfrak{m}_{\Sigma_\beta}} : \mathfrak{m}_{\Sigma_\beta} \to \mathfrak{k}_{\Sigma_\beta}$ and $\mathrm{ad}(A)|_{\mathfrak{k}_{\Sigma_\beta}} : \mathfrak{k}_{\Sigma_\beta} \to \mathfrak{m}_{\Sigma_\beta}$. We set a basis of $\mathfrak{m}_{\Sigma_\beta}$ as follows:

$$
\begin{aligned}
T_{\lambda_1+\lambda_2+\lambda_3+\lambda_4} &:= Z_{\beta-\alpha_1} + Z_{\beta-\alpha_2} + Z_{\beta-\alpha_3} + Z_{\beta-\alpha_4},\\
T_{\lambda_1+\lambda_2-\lambda_3-\lambda_4} &:= Z_{\beta-\alpha_1} + Z_{\beta-\alpha_2} - Z_{\beta-\alpha_3} - Z_{\beta-\alpha_4},\\
T_{\lambda_1-\lambda_2+\lambda_3-\lambda_4} &:= Z_{\beta-\alpha_1} - Z_{\beta-\alpha_2} + Z_{\beta-\alpha_3} - Z_{\beta-\alpha_4},\\
T_{\lambda_1-\lambda_2-\lambda_3+\lambda_4} &:= -Z_{\beta-\alpha_1} + Z_{\beta-\alpha_2} + Z_{\beta-\alpha_3} - Z_{\beta-\alpha_4}.\\
T_{\lambda_1+\lambda_2+\lambda_3-\lambda_4} &:= W_{\beta-\alpha_1} + W_{\beta-\alpha_2} + W_{\beta-\alpha_3} - W_{\beta-\alpha_4},\\
T_{\lambda_1+\lambda_2-\lambda_3+\lambda_4} &:= W_{\beta-\alpha_1} + W_{\beta-\alpha_2} - W_{\beta-\alpha_3} + W_{\beta-\alpha_4},\\
T_{\lambda_1-\lambda_2+\lambda_3+\lambda_4} &:= W_{\beta-\alpha_1} - W_{\beta-\alpha_2} + W_{\beta-\alpha_3} + W_{\beta-\alpha_4},\\
T_{\lambda_1-\lambda_2-\lambda_3-\lambda_4} &:= -W_{\beta-\alpha_1} + W_{\beta-\alpha_2} + W_{\beta-\alpha_3} + W_{\beta-\alpha_4}.
\end{aligned}
$$

and $T_{2\lambda_i} := W_{\alpha_i}$ $(1 \le i \le 4)$. Next, we define a basis of $\mathfrak{k}_{\Sigma_\beta}$ as follows:

$$
\begin{aligned}
S_{\lambda_1+\lambda_2+\lambda_3+\lambda_4} &:= -Z_\beta + Z_{\beta-(\alpha_1+\alpha_2)} + Z_{\beta-(\alpha_1+\alpha_3)} - Z_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1+\lambda_2-\lambda_3-\lambda_4} &:= -Z_\beta + Z_{\beta-(\alpha_1+\alpha_2)} - Z_{\beta-(\alpha_1+\alpha_3)} + Z_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1-\lambda_2+\lambda_3-\lambda_4} &:= -Z_\beta - Z_{\beta-(\alpha_1+\alpha_2)} + Z_{\beta-(\alpha_1+\alpha_3)} + Z_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1-\lambda_2-\lambda_3+\lambda_4} &:= Z_\beta + Z_{\beta+(\alpha_1+\alpha_2)} + Z_{\beta-(\alpha_1+\alpha_3)} + Z_{\beta-(\alpha_1+\alpha_4)}.\\
S_{\lambda_1+\lambda_2+\lambda_3-\lambda_4} &:= -W_\beta + W_{\beta-(\alpha_1+\alpha_2)} + W_{\beta-(\alpha_1+\alpha_3)} + W_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1+\lambda_2-\lambda_3+\lambda_4} &:= -W_\beta + W_{\beta-(\alpha_1+\alpha_2)} - W_{\beta-(\alpha_1+\alpha_3)} - W_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1-\lambda_2+\lambda_3+\lambda_4} &:= -W_\beta - W_{\beta-(\alpha_1+\alpha_2)} + W_{\beta-(\alpha_1+\alpha_3)} - W_{\beta-(\alpha_1+\alpha_4)},\\
S_{\lambda_1-\lambda_2-\lambda_3-\lambda_4} &:= W_\beta + W_{\beta+(\alpha_1+\alpha_2)} + W_{\beta-(\alpha_1+\alpha_3)} - W_{\beta-(\alpha_1+\alpha_4)}.
\end{aligned}
$$

and $S_{2\lambda_i} = iA_{\alpha_i}$ $(1 \le i \le 4)$. Set

$$
R_\beta = \left\{ \begin{array}{llll}
\lambda_1+\lambda_2+\lambda_3+\lambda_4, & \lambda_1+\lambda_2-\lambda_3-\lambda_4, & \lambda_1-\lambda_2+\lambda_3-\lambda_4, & \lambda_1-\lambda_2-\lambda_3+\lambda_4,\\
\lambda_1+\lambda_2+\lambda_3-\lambda_4, & \lambda_1+\lambda_2-\lambda_3+\lambda_4, & \lambda_1-\lambda_2+\lambda_3+\lambda_4, & \lambda_1-\lambda_2-\lambda_3-\lambda_4,\\
2\lambda_1, & 2\lambda_2, & 2\lambda_3, & 2\lambda_4
\end{array} \right\}.
$$

Lemma 3.5. $\mathrm{ad}(A)(T_\omega) = \omega(A)S_\omega$ and $\mathrm{ad}(A)(S_\omega) = -\omega(A)T_\omega$ for any $\omega \in R_\beta$.

Proof. By results of the structure coefficient in subsection 2.2,

$$
\begin{aligned}
\mathrm{ad}A(Z_{\beta-\alpha_1}) &= \lambda_1 N_{\alpha_1,\beta-\alpha_1} Z_\beta - \lambda_2 N_{-\alpha_2,\beta-\alpha_1} Z_{\beta-(\alpha_1+\alpha_2)}\\
&\qquad - \lambda_3 N_{-\alpha_3,\beta-\alpha_1} Z_{\beta-(\alpha_1+\alpha_3)} - \lambda_4 N_{-\alpha_4,\beta-\alpha_1} Z_{\beta-(\alpha_1+\alpha_4)}\\
&= (-\lambda_1) Z_\beta + \lambda_2 Z_{\beta-(\alpha_1+\alpha_2)} + \lambda_3 Z_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_4) Z_{\beta-(\alpha_1+\alpha_4)}.
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}
&\mathrm{ad}A(Z_{\beta-\alpha_2}) = (-\lambda_2)Z_\beta + \lambda_1 Z_{\beta-(\alpha_1+\alpha_2)} + \lambda_4 Z_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_3)Z_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(Z_{\beta-\alpha_3}) = (-\lambda_3)Z_\beta + \lambda_4 Z_{\beta-(\alpha_1+\alpha_2)} + \lambda_1 Z_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_2)Z_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(Z_{\beta-\alpha_4}) = (-\lambda_4)Z_\beta + \lambda_3 Z_{\beta-(\alpha_1+\alpha_2)} + \lambda_2 Z_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_1)Z_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(Z_\beta) = \lambda_1 Z_{\beta-\alpha_1} + \lambda_2\ Z_{\beta-\alpha_2} + \lambda_3\ Z_{\beta-\alpha_3} + \lambda_4 Z_{\beta-\alpha_4},\\
&\mathrm{ad}A(Z_{\beta-(\alpha_1+\alpha_2)}) = -\big(\lambda_2 Z_{\beta-\alpha_1} + \lambda_1 Z_{\beta-\alpha_2} + \lambda_4 Z_{\beta-\alpha_3} + \lambda_3 Z_{\beta-\alpha_4}\big),\\
&\mathrm{ad}A(Z_{\beta-(\alpha_1+\alpha_3)}) = -\big(\lambda_3 Z_{\beta-\alpha_1} + \lambda_4 Z_{\beta-\alpha_2} + \lambda_1 Z_{\beta-\alpha_3} + \lambda_2 Z_{\beta-\alpha_4}\big),\\
&\mathrm{ad}A(Z_{\beta-(\alpha_1+\alpha_3)}) = \lambda_4 Z_{\beta-\alpha_1} + \lambda_3 Z_{\beta-\alpha_2} + \lambda_2 Z_{\beta-\alpha_3} + \lambda_1 Z_{\beta-\alpha_4},\\
&\mathrm{ad}A(W_{\beta-\alpha_1}) = (-\lambda_1)W_\beta + \lambda_2 W_{\beta-(\alpha_1+\alpha_2)} + \lambda_3 W_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_4)W_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(W_{\beta-\alpha_2}) = (-\lambda_2)W_\beta + \lambda_1 W_{\beta-(\alpha_1+\alpha_2)} + (-\lambda_4)W_{\beta-(\alpha_1+\alpha_3)} + \lambda_3 W_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(W_{\beta-\alpha_3}) = (-\lambda_3)W_\beta + (-\lambda_4)W_{\beta-(\alpha_1+\alpha_2)} + \lambda_1 W_{\beta-(\alpha_1+\alpha_3)} + \lambda_2 W_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(W_{\beta-\alpha_4}) = (-\lambda_4)W_\beta + (-\lambda_3)W_{\beta-(\alpha_1+\alpha_2)} + (-\lambda_2)W_{\beta-(\alpha_1+\alpha_3)} + (-\lambda_1)W_{\beta-(\alpha_1+\alpha_4)},\\
&\mathrm{ad}A(W_\beta) = \lambda_1 W_{\beta-\alpha_1} + \lambda_2\ W_{\beta-\alpha_2} + \lambda_3\ W_{\beta-\alpha_3} + \lambda_4 W_{\beta-\alpha_4},\\
&\mathrm{ad}A(W_{\beta-(\alpha_1+\alpha_2)}) = (-\lambda_2)W_{\beta-\alpha_1} + (-\lambda_1)\ W_{\beta-\alpha_2} + \lambda_4\ W_{\beta-\alpha_3} + \lambda_3 W_{\beta-\alpha_4},\\
&\mathrm{ad}A(W_{\beta-(\alpha_1+\alpha_3)}) = (-\lambda_3)W_{\beta-\alpha_1} + \lambda_4\ W_{\beta-\alpha_2} + (-\lambda_1)\ W_{\beta-\alpha_3} + \lambda_2 W_{\beta-\alpha_4},\\
&\mathrm{ad}A(W_{\beta-(\alpha_1+\alpha_3)}) = \lambda_4 W_{\beta-\alpha_1} + (-\lambda_3)\ W_{\beta-\alpha_2} + (-\lambda_2)\ W_{\beta-\alpha_3} + \lambda_1 W_{\beta-\alpha_4}.
\end{aligned}$$

Moreover, $\mathrm{ad}A(W_{\alpha_i}) = 2\lambda_i(iA_{\alpha_i}), \mathrm{ad}A(iA_{\alpha_i}) = -2\lambda_i W_{\alpha_i}$ $(1 \leq i \leq 4)$. By these results, we obtain the statement.

□

Thus, $\pm iR_\beta \subset R$ because $\mathbb{C}(T_\omega \pm iS_\omega) \subset \tilde{\mathfrak{g}}_{\mp i\omega} = \{X \in \tilde{\mathfrak{g}}\ ;\ \mathrm{ad}A(X) = \mp i\omega(A)X\}$ for each $\omega \in R_\beta$. Moreover, we can easily check that $\pm iR_\beta$ is a subsystem of type D_4.

Let Σ^L be an H-orbit of type $L(i,j)$ $(1 \leq i < j \leq 4)$ and $\Sigma^L \cap \Sigma_1 = \{\gamma\}$. Then, $\Sigma^L = \{\gamma, \gamma-\alpha_i, \gamma-\alpha_j, \gamma-(\alpha_i+\alpha_j)\}$. Set a basis of $\mathfrak{m}_{\Sigma^L}$ as follows:

$$\begin{aligned}
T^{\gamma,1}_{\lambda_i+\lambda_j} &:= Z_\gamma - Z_{\gamma-(\alpha_i+\alpha_j)}, & T^{\gamma,1}_{\lambda_i-\lambda_j} &:= Z_\gamma + Z_{\gamma-(\alpha_i+\alpha_j)},\\
T^{\gamma,2}_{\lambda_i+\lambda_j} &:= W_\gamma - W_{\gamma-(\alpha_i+\alpha_j)}, & T^{\gamma,2}_{\lambda_i-\lambda_j} &:= W_\gamma + W_{\gamma-(\alpha_i+\alpha_j)}.
\end{aligned}$$

Moreover, we set a basis of $\mathfrak{k}_{\Sigma^L}$ as follows:

$$\begin{aligned}
S^{\gamma,1}_{\lambda_i+\lambda_j} &:= Z_{\gamma-\alpha_i} + Z_{\gamma-\alpha_j}, & S^{\gamma,1}_{\lambda_i-\lambda_j} &:= Z_{\gamma-\alpha_i} - Z_{\gamma-\alpha_j},\\
S^{\gamma,2}_{\lambda_i+\lambda_j} &:= W_{\gamma-\alpha_i} + W_{\gamma-\alpha_j}, & S^{\gamma,2}_{\lambda_i-\lambda_j} &:= W_{\gamma-\alpha_i} - W_{\gamma-\alpha_j}.
\end{aligned}$$

Set $R_{\Sigma^L} = \{\lambda_i \pm \lambda_j\}$.

Lemma 3.6. $\mathrm{ad}A(T^{\gamma,k}_{\omega}) = \omega(A)S^{\gamma,k}_{\omega}$ and $\mathrm{ad}A(S^{\gamma,k}_{\omega}) = -\omega(A)T^{\gamma,k}_{\omega}$ for any $\omega \in R_{\Sigma^L}$ and $k = 1, 2$.

Proof. By the simialr way to the proof of Lemma 3.5, we obtain the followings and the statement is true.

$$\begin{aligned}
&\mathrm{ad}A(Z_{\gamma}) = \lambda_i Z_{\gamma-\alpha_i} + \lambda_j Z_{\gamma-\alpha_j}, && \mathrm{ad}A(Z_{\gamma-(\alpha_i+\alpha_j)}) = -\lambda_j Z_{\gamma-\alpha_i} - \lambda_i Z_{\gamma-\alpha_j},\\
&\mathrm{ad}A(Z_{\gamma-\alpha_i}) = -\lambda_i Z_{\gamma} + \lambda_j Z_{\gamma-(\alpha_i+\alpha_j)}, && \mathrm{ad}A(Z_{\gamma-\alpha_j}) = -\lambda_j Z_{\gamma} + \lambda_i Z_{\gamma-(\alpha_i+\alpha_j)},\\
&\mathrm{ad}A(W_{\gamma}) = \lambda_i W_{\gamma-\alpha_i} + \lambda_j W_{\gamma-\alpha_j}, && \mathrm{ad}A(W_{\gamma-(\alpha_i+\alpha_j)}) = -\lambda_j W_{\gamma-\alpha_i} - \lambda_i W_{\gamma-\alpha_j},\\
&\mathrm{ad}A(W_{\gamma-\alpha_i}) = -\lambda_i W_{\gamma} + \lambda_j W_{\gamma-(\alpha_i+\alpha_j)}, && \mathrm{ad}A(W_{\gamma-\alpha_j}) = -\lambda_j W_{\gamma} + \lambda_i W_{\gamma-(\alpha_i+\alpha_j)}.
\end{aligned}$$

□

Let Σ^S be an H-orbit of type $S(i,j)$ $(1 \le i < j \le 4)$ and $\Sigma^S \cap \Sigma_1 = \{\delta\}$. Then, $\Sigma^S = \{\delta, \delta - \alpha_i\}$. Set $c_\delta := N_{-\alpha_j,\delta}$. Then, $c_\delta = \pm 1$. Set a basis of $\mathfrak{m}_{\Sigma^S}$ as follows:

$$T^{\delta}_{\lambda_i + c_\delta \lambda_j} := Z_\delta, \quad T^{\delta}_{\lambda_i - c_\delta \lambda_j} := W_\delta.$$

Moreover, we set a basis of $\mathfrak{k}_{\Sigma^S}$ as follows:

$$S^{\delta}_{\lambda_i + c_\delta \lambda_j} := Z_{\delta-\alpha_i}, \quad S^{\delta}_{\lambda_i - c_\delta \lambda_j} := W_{\delta-\alpha_i}.$$

Set $R_{\Sigma^S} = \{\lambda_i \pm c_\delta \lambda_j\} = \{\lambda_i \pm \lambda_j\}$. By the similar way to Lemma 3.5, we obtain Lemma 3.7.

Lemma 3.7. $\mathrm{ad}A(T^{\delta}_{\omega}) = \omega(A)S^{\delta}_{\omega}$ and $\mathrm{ad}A(S^{\delta}_{\omega}) = -\omega(A)T^{\delta}_{\omega}$ for any $\omega \in R_{\Sigma^S}$.

For an H-orbit Σ^L of type L, if $\Sigma^L \cap \Sigma_1 = \{\gamma\}$, then we denote $\mathfrak{m}_{\Sigma^L}, \mathfrak{k}_{\Sigma^L}, R_{\Sigma^L}$ by $\mathfrak{m}_\gamma, \mathfrak{k}_\gamma, R_\gamma$. Similarly, for an H-orbit Σ^S of type S, if $\Sigma^S \cap \Sigma_1 = \{\delta\}$, then we denote $\mathfrak{m}_{\Sigma^S}, \mathfrak{k}_{\Sigma^S}, R_{\Sigma^S}$ by $\mathfrak{m}_\delta, \mathfrak{k}_\delta, R_\delta$. Let $\Sigma^L(1), \cdots, \Sigma^L(n)$ be H-orbits of type L such that $\Sigma^L(1), -\Sigma^L(1), \cdots, \Sigma^L(n)$, $-\Sigma^L(n)$ are all H-orbits of type L. Moreover, let $\Sigma^S(1), \cdots, \Sigma^S(m)$ be all H-orbits of type S. Let $\Sigma^L(p) \cap \Sigma_1 = \{\gamma_p\}$ $(1 \le p \le n)$ and $\Sigma^S(q) \cap \Sigma_1 = \{\delta_q\}$ $(1 \le q \le m)$. For $\omega \in iR$, we set $\mathfrak{m}_\omega = \{T \in \mathfrak{m} \ ; \ (\mathrm{ad}A)^2 T = -\omega(A)^2 T \ (A \in \mathfrak{a})\}$. We denote $\mathfrak{a} + \sum_{\omega \in R_\beta} \mathfrak{m}_\omega$ by $\mathfrak{m}_\beta$. Then, the following direct sum decomposition is true.

$$\mathfrak{m} = \mathfrak{m}_\beta + \sum_{p=1}^{n} \mathfrak{m}_{\gamma_p} + \sum_{q=1}^{m} \mathfrak{m}_{\delta_q}$$

Moreover, the restricted root system R with respect to $\mathfrak{a}$ is given by

$$R = \pm i\Big(R_\beta \cup \bigcup_{p=1}^{n} R_{\gamma_p} \cup \bigcup_{q=1}^{m} R_{\delta_q}\Big).$$

3.4 The representation of $\mathfrak{s}$ on $\mathfrak{m}$

In this subsection, we study $\mathrm{ad}X|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ for each $X \in \mathfrak{s}$. Since $iA_\beta, Z_\beta, W_\beta$ is a basis of $\mathfrak{s}$, we consider $\mathrm{ad}(iA_\beta), \mathrm{ad}(Z_\beta), \mathrm{ad}(W_\beta)$. Remark that $(\mathrm{ad}(iA_\beta)|_{\mathfrak{m}})^2 = (\mathrm{ad}(Z_\beta)|_{\mathfrak{m}})^2 = (\mathrm{ad}(W_\beta)|_{\mathfrak{m}})^2 = -\mathrm{id}_{\mathfrak{m}}$.

We easily see $\mathrm{ad}X(\mathfrak{m}_\beta) \subset \mathfrak{m}_\beta$ for any $X \in \mathfrak{s}$ since $\mathfrak{m}_\beta = \sum_{i=1}^4 (\mathbb{R}Z_{\alpha_i} + \mathbb{R}W_{\alpha_i}) + \sum_{i=1}^4 (\mathbb{R}Z_{\beta-\alpha_i} + \mathbb{R}W_{\beta-\alpha_i})$. Denote each element of R_β as follows:

$$\omega_i^1 = 2\lambda_i \ (1 \le i \le 4),$$
$$\omega_1^2 = \lambda_1+\lambda_2+\lambda_3+\lambda_4, \quad \omega_2^2 = \lambda_1+\lambda_2-\lambda_3-\lambda_4, \quad \omega_3^2 = \lambda_1-\lambda_2+\lambda_3-\lambda_4, \quad \omega_4^2 = \lambda_1-\lambda_2-\lambda_3+\lambda_4,$$
$$\omega_1^3 = \lambda_1+\lambda_2+\lambda_3-\lambda_4, \quad \omega_2^3 = \lambda_1+\lambda_2-\lambda_3+\lambda_4, \quad \omega_3^3 = \lambda_1-\lambda_2+\lambda_3+\lambda_4, \quad \omega_4^3 = \lambda_1-\lambda_2-\lambda_3-\lambda_4.$$

Set $R_\beta^k = \{\omega_i^k \ ; \ 1 \le i \le 4\}$ and $\mathfrak{m}_\beta^k = \sum_{\omega \in R_\beta^k} \mathbb{R}T_\omega$ for each $1 \le k \le 3$. Then, $\mathfrak{m}_\beta = \mathfrak{a} + \mathfrak{m}_\beta^1 + \mathfrak{m}_\beta^2 + \mathfrak{m}_\beta^3$. By direct computations and using $N_{-\beta,\beta-\alpha_i} = N_{\beta,-\alpha_i}$ $(1 \le i \le 4)$, we obtain Lemma 3.8, Lemma 3.9, Lemma 3.10.

Lemma 3.8. $\mathrm{ad}(iA_\beta)\mathfrak{a} \subset \mathfrak{m}_\beta^1$ and $\mathrm{ad}(iA_\beta)\mathfrak{m}_\beta^2 \subset \mathfrak{m}_\beta^3$. Moreover, the representation matrices of $\mathrm{ad}(iA_\beta)|_{\mathfrak{a}}$ with respect to Z_{α_i} $(1 \le i \le 4)$ and $T_{\omega_i^1}$ $(1 \le i \le 4)$ and of $\mathrm{ad}(iA_\beta)|_{\mathfrak{m}_\beta^2}$ with respect to $T_{\omega_i^2}$ $(1 \le i \le 4)$ and $T_{\omega_i^3}$ $(1 \le i \le 4)$ are

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

where empty components are 0.

Lemma 3.9. $\mathrm{ad}(Z_\beta)\mathfrak{a} \subset \mathfrak{m}_\beta^2$ and $\mathrm{ad}(Z_\beta)\mathfrak{m}_\beta^1 \subset \mathfrak{m}_\beta^3$. Moreover, the representation matrices of $\mathrm{ad}(Z_\beta)|_{\mathfrak{a}}$ with respect to Z_{α_i} $(1 \le i \le 4)$ and $T_{\omega_i^2}$ $(1 \le i \le 4)$ and of $\mathrm{ad}(Z_\beta)|_{\mathfrak{m}_\gamma^1}$ with respect to $T_{\omega_i^1}$ $(1 \le i \le 4)$ and $T_{\omega_i^3}$ $(1 \le i \le 4)$ are

$$-\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma 3.10. $\mathrm{ad}(W_\beta)\mathfrak{a} \subset \mathfrak{m}_\beta^3$ and $\mathrm{ad}(W_\beta)\mathfrak{m}_\beta^1 \subset \mathfrak{m}_\beta^2$. Moreover, the representation matrice of $\mathrm{ad}(W_\beta)|_{\mathfrak{a}}$ with respect to Z_{α_i} $(1 \le i \le 4)$ and $T_{\omega_i^3}$ $(1 \le i \le 4)$ and of $\mathrm{ad}(W_\beta)|_{\mathfrak{m}_\beta^1}$ with respect to $T_{\omega_i^1}$ $(1 \le i \le 4)$ and $T_{\omega_i^2}$ $(1 \le i \le 4)$ are

$$-\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad -\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Let Σ^L be an H-orbit of type $L(i,j)$ $(1 \leq i < j \leq 4)$ and $\Sigma^L \cap \Sigma_1 = \{\gamma\}$. Then, we see that the H-orbit through $\beta - \gamma$ is type $L(k,l)$ $(1 \leq k \neq l \leq 4,\ k,l \neq i,j)$ and $\mathrm{ad}X(\mathfrak{m}_\gamma + \mathfrak{m}_{\beta-\gamma}) \subset \mathfrak{m}_\gamma + \mathfrak{m}_{\beta-\gamma}$ for any $X \in \mathfrak{s}$. For each $\omega \in iR$, we denote $(\mathfrak{m}_\gamma)_\omega = \mathfrak{m}_\gamma \cap \mathfrak{m}_\omega$. Then, $\mathfrak{m}_\gamma = (\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j} + (\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j}$ and $(\mathfrak{m}_\gamma)_{\lambda_i\pm\lambda_j} = \mathbb{R}T^{\gamma,1}_{\lambda_i\pm\lambda_j} + \mathbb{R}T^{\gamma,2}_{\lambda_i\pm\lambda_j}$. By direct computations, we obtain Lemma 3.11 immediately.

Lemma 3.11. $\mathrm{ad}(iA_\beta)(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j} \subset (\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j}$ The representation matrix of $\mathrm{ad}(iA_\beta)|_{(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j}}$ with respect to $T^{\gamma,a}_{\lambda_i+\lambda_j}$ $(a=1,2)$ and $T^{\gamma,a}_{\lambda_i-\lambda_j}$ $(a=1,2)$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 3.12. $N_{-\beta,\gamma} = -N_{\beta,\gamma-(\alpha_i+\alpha_j)}$

Proof. Since $\mathrm{Ad}(w_1\cdots w_4)X_\gamma = X_{\gamma-(\alpha_i+\alpha_j)}$ and $\mathrm{Ad}(w_1\cdots w_4)X_{-\beta} = -X_\beta$, we obtain

$$\begin{aligned}
\mathrm{Ad}(w_1\cdots w_4)[X_{-\beta}, X_\gamma] &= N_{-\beta,\gamma}\mathrm{Ad}(w_1\cdots w_4)X_{-\beta+\gamma} = -N_{-\beta,\gamma}\tau(\mathrm{Ad}(w_1\cdots w_4)X_{\beta-\gamma}) \\
&= -N_{-\beta,\gamma}\tau(X_{\beta-\gamma-(\alpha_k+\alpha_l)}) = N_{-\beta,\gamma}X_{-\beta+\gamma+(\alpha_k+\alpha_l)} \\
\mathrm{Ad}(w_1\cdots w_4)[X_{-\beta}, X_\gamma] &= [\mathrm{Ad}(w_1\cdots w_4)X_{-\beta}, \mathrm{Ad}(w_1\cdots w_4)X_\gamma] = -[X_\beta, X_{\gamma-(\alpha_i+\alpha_l)}] \\
&= -N_{\beta,\gamma-(\alpha_i+\alpha_l)}X_{\beta+\gamma-(\alpha_i+\alpha_j)} = -N_{\beta,\gamma-(\alpha_i+\alpha_j)}X_{-\beta+\gamma+(\alpha_k+\alpha_l)}.
\end{aligned}$$

Thus, $N_{-\beta,\gamma} = -N_{\beta,\gamma-(\alpha_i+\alpha_j)}$.

□

Remark $N_{-\beta,\gamma} = \pm 1$. By direct computations and Lamme 3.12, we obtain Lemma 3.13, Lemma 3.14.

Lemma 3.13. $\mathrm{ad}(Z_\beta)(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j} \subset (\mathfrak{m}_{\beta-\gamma})_{\lambda_k+\lambda_l}$ and $\mathrm{ad}(Z_\beta)(\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j} \subset (\mathfrak{m}_{\beta-\gamma})_{\lambda_k-\lambda_l}$. Moreover, the representation matrix of $\mathrm{ad}(Z_\beta)|_{(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j}}$ with respect to $T^{\gamma,a}_{\lambda_i+\lambda_j}$ $(a=1,2)$ and $T^{\gamma,a}_{\lambda_k+\lambda_l}$ $(a=1,2)$ and of $\mathrm{ad}(Z_\beta)|_{(\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j}}$ with respect to $T^{\gamma,a}_{\lambda_i-\lambda_j}$ $(a=1,2)$ and $T^{\gamma,a}_{\lambda_k-\lambda_l}$ $(a=1,2)$ is

$$\begin{pmatrix} N_{-\beta,\gamma} & 0 \\ 0 & -N_{-\beta,\gamma} \end{pmatrix}.$$

Lemma 3.14. $\mathrm{ad}(W_\beta)(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j} \subset (\mathfrak{m}_\gamma)_{\lambda_k-\lambda_l}$ and $\mathrm{ad}(W_\beta)(\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j} \subset (\mathfrak{m}_\gamma)_{\lambda_k+\lambda_l}$. Moreover, the representation matrix of $\mathrm{ad}(W_\beta)|_{(\mathfrak{m}_\gamma)_{\lambda_i+\lambda_j}}$ with respect to $T^{\gamma,a}_{\lambda_i+\lambda_j}$ $(a=1,2)$ and $T^{\gamma,a}_{\lambda_k-\lambda_l}$ $(a=1,2)$ and of $\mathrm{ad}(iA_\beta)|_{(\mathfrak{m}_\gamma)_{\lambda_i-\lambda_j}}$ with respect to $T^{\gamma,a}_{\lambda_i-\lambda_j}$ $(a=1,2)$ and $T^{\gamma,a}_{\lambda_k+\lambda_l}$ $(a=1,2)$ is

$$\begin{pmatrix} 0 & N_{-\beta,\gamma} \\ N_{-\beta,\gamma} & 0 \end{pmatrix}.$$

Let Σ^S be an H-orbit of type $S(i,j)$ $(1 \leq i < j \leq 4)$ and $\Sigma^S \cap \Sigma_1 = \{\delta\}$. Then, we see that the H-orbit through $\beta - \delta$ is type $S(k,l)$ $(1 \leq k \neq l \leq 4,\ k,l, \neq i,j,\ k < l)$ and $\mathrm{ad}X(\mathfrak{m}_\delta + \mathfrak{m}_{\beta-\delta}) \subset \mathfrak{m}_\delta + \mathfrak{m}_{\beta-\delta}$ for any $X \in \mathfrak{s}$. Let $c_\delta = N_{-\alpha_j,\delta}$ and $c_{\beta-\delta} = N_{-\alpha_l,\beta-\delta}$. Then, c_δ and $c_{\beta-\delta}$ are ± 1. For each $\omega \in iR$, we set $(\mathfrak{m}_\delta)_\omega = \mathfrak{m}_\delta \cap \mathfrak{m}_\omega$. Then, $(\mathfrak{m}_\delta)_{\lambda_i \pm c_\delta \lambda_j} = \mathbb{R}T^\delta_{\lambda_i \pm c_\delta \lambda_j}$ and $\mathfrak{m}_\delta = (\mathfrak{m}_\delta)_{\lambda_i+\lambda_j} + (\mathfrak{m}_\delta)_{\lambda_i-\lambda_j}$. Since $T^\delta_{\lambda_i + c_\delta \lambda_j} = Z_\delta$ and $T^\delta_{\lambda_i - c_\delta \lambda_j} = W_\delta$, we obtain Lemma 3.15 by direct computations.

Lemma 3.15. It is true that $\mathrm{ad}(iA_\beta)(\mathfrak{m}_\delta)_{\lambda_i \pm \lambda_j} \subset (\mathfrak{m}_\delta)_{\lambda_i \mp \lambda_j}$, $\mathrm{ad}(Z_\beta)(\mathfrak{m}_\delta)_{\lambda_i \pm c_\delta \lambda_j} \subset (\mathfrak{m}_\delta)_{\lambda_k \pm c_{\beta-\delta}\lambda_l}$ and $\mathrm{ad}(W_\beta)(\mathfrak{m}_\delta)_{\lambda_i \pm c_\delta \lambda_j} \subset (\mathfrak{m}_\delta)_{\lambda_k \mp c_{\beta-\delta}\lambda_l}$.

Summarizing the above arguments we obtain Proposition 3.16.

Proposition 3.16. Let $1 \leq i < j \leq 4$ and $1 \leq k < l \leq 4$ such that $k, l \neq i, j$. Then,

$$\begin{aligned}
&\mathrm{ad}(iA_\beta)(\mathfrak{m}_{\lambda_i+\lambda_j}) \subset \mathfrak{m}_{\lambda_i-\lambda_j},\\
&\mathrm{ad}(Z_\beta)(\mathfrak{m}_{\lambda_i+\lambda_j} + \mathfrak{m}_{\lambda_i-\lambda_j}) \subset \mathfrak{m}_{\lambda_k+\lambda_l} + \mathfrak{m}_{\lambda_k-\lambda_l}\\
&\mathrm{ad}(W_\beta)(\mathfrak{m}_{\lambda_i+\lambda_j} + \mathfrak{m}_{\lambda_i-\lambda_j}) \subset \mathfrak{m}_{\lambda_k+\lambda_l} + \mathfrak{m}_{\lambda_k-\lambda_l}.
\end{aligned}$$

3.5 Root system $D_4 \subset B_4 \subset F_4$

For $H \in \mathfrak{a}$ and any subset $\Delta \subset \mathfrak{a}^*$, set $\Delta_H = \{\omega \in \Delta\ ;\ \omega(H) \in \pi\mathbb{Z}\}$. We easily check that the following are true.

$$\begin{cases} -\omega_1^1 + \omega_1^2 + \omega_4^3 = 0 \\ -\omega_1^1 + \omega_2^2 + \omega_3^3 = 0 \\ -\omega_1^1 + \omega_3^2 + \omega_2^3 = 0 \\ -\omega_1^1 + \omega_4^2 + \omega_1^3 = 0 \end{cases}, \quad \begin{cases} -\omega_2^1 + \omega_1^2 - \omega_3^3 = 0 \\ -\omega_2^1 + \omega_2^2 - \omega_4^3 = 0 \\ \omega_2^1 + \omega_3^2 - \omega_1^3 = 0 \\ \omega_2^1 + \omega_4^2 - \omega_2^3 = 0 \end{cases},$$
$$\begin{cases} -\omega_3^1 + \omega_1^2 - \omega_2^3 = 0 \\ \omega_3^1 + \omega_2^2 - \omega_1^3 = 0 \\ -\omega_3^1 + \omega_3^2 - \omega_4^3 = 0 \\ \omega_3^1 + \omega_4^2 - \omega_3^3 = 0 \end{cases}, \quad \begin{cases} -\omega_4^1 + \omega_1^2 - \omega_1^3 = 0 \\ \omega_4^1 + \omega_2^2 - \omega_2^3 = 0 \\ \omega_4^1 + \omega_3^2 - \omega_3^3 = 0 \\ -\omega_4^1 + \omega_4^2 - \omega_4^3 = 0 \end{cases}. \qquad \cdots\cdots \quad (*)$$

Lemma 3.17. Let $H \in \mathfrak{a}$. If $\#(R^1_\beta)_H = 1$, then $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 0$ or $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 1$.

Proof. If $\#(R^2_\beta)_H \geq 2$, then we obtain $\#(R^1_\beta)_H \geq 2$ by $(*)$, but this contradicts to the assumption. (For example, we assume $\omega_1^1 \in (R^1_\beta)_H$ and $\omega_1^2, \omega_2^2 \in (R^2_\beta)_H$. By $(*)$, $-\omega_1^1 - \omega_1^2 - \omega_4^3 = 0, -\omega_2^1 + \omega_2^2 - \omega_4^3 = 0$ and we obtain $\omega_2^1 \in (R^1_\beta)_H$.) Thus, $\#(R^2_\beta)_H, \#(R^3_\beta)_H = 0, 1$, and $\#(R^2_\beta)_H = 1$ if and only if $\#(R^3_\beta)_H = 1$, and $\#(R^2_\beta)_H = 0$ if and only if $\#(R^3_\beta)_H = 0$. Thus, the statement follows. $\square$

Lemma 3.18. Let $H \in \mathfrak{a}$. If $\#(R^1_\beta)_H = 2$, then $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 0$ or $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 2$.

Proof. $\#(R^2_\beta)_H \neq 1$ by $(*)$. (For example, we assume $\omega^1_1, \omega^1_2 \in (R^1_\beta)_H$ and $\omega^2_1 \in (R^2_\beta)_H$. By $(*)$, $-\omega^1_1 - \omega^2_1 - \omega^3_4 = 0, -\omega^1_2 + \omega^2_2 - \omega^3_4 = 0$ and $\omega^2_2 \in (R^2_\beta)_H$.) Moreover, $\#(R^2_\beta)_H \leq 2$ by $(*)$. (For example, we assume $\omega^1_1, \omega^1_2 \in (R^1_\beta)_H$ and $\omega^2_1, \omega^2_2, \omega^2_3 \in (R^2_\beta)_H$. By $(*)$, $-\omega^1_1 - \omega^2_1 - \omega^3_4 = 0$, $-\omega^1_3 + \omega^2_3 - \omega^3_4 = 0$ and $\omega^1_3 \in (R^3_\beta)_H$. This contradicts to the assumption.) Thus, $\#(R^2_\beta)_H = 0, 2$. We see $\#(R^2_\beta)_H = 0$ if and only if $\#(R^3_\beta)_H = 0$. Also, $\#(R^2_\beta)_H = 2$ if and only if $\#(R^3_\beta)_H = 2$. In particular, if $\#(R^1_\beta)_H = \#(R^2_\beta)_H = \#(R^3_\beta)_H = 2$, then $((R^1_\beta)_H, (R^2_\beta)_H, (R^3_\beta)_H))$ is one of the following:

$$\begin{aligned}
&\Big(\{\omega^1_1,\omega^1_2\},\{\omega^2_1,\omega^2_2\},\{\omega^3_3,\omega^3_4\}\Big), \quad \Big(\{\omega^1_1,\omega^1_2\},\{\omega^2_3,\omega^2_4\},\{\omega^3_1,\omega^3_2\}\Big), \quad \Big(\{\omega^1_3,\omega^1_4\},\{\omega^2_1,\omega^2_2\},\{\omega^3_1,\omega^3_2\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_3\},\{\omega^2_1,\omega^2_3\},\{\omega^3_2,\omega^3_4\}\Big), \quad \Big(\{\omega^1_1,\omega^1_3\},\{\omega^2_2,\omega^2_4\},\{\omega^3_1,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_4\},\{\omega^2_1,\omega^2_3\},\{\omega^3_1,\omega^3_3\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_4\},\{\omega^2_2,\omega^2_3\},\{\omega^3_2,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_3\},\{\omega^2_1,\omega^2_4\},\{\omega^3_2,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_3\},\{\omega^2_2,\omega^2_3\},\{\omega^3_1,\omega^3_4\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_4\},\{\omega^2_1,\omega^2_4\},\{\omega^3_1,\omega^3_4\}\Big), \quad \Big(\{\omega^1_2,\omega^1_4\},\{\omega^2_2,\omega^2_4\},\{\omega^3_2,\omega^3_4\}\Big), \quad \Big(\{\omega^1_3,\omega^1_4\},\{\omega^2_3,\omega^2_4\},\{\omega^3_3,\omega^3_4\}\Big).
\end{aligned}$$

□

Lemma 3.19. Let $H \in \mathfrak{a}$. If $\#(R^1_\beta)_H = 3$, then $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 0$.

Proof. By $(*)$, the statement follows. (For example, we assume $\omega^1_1, \omega^1_2, \omega^1_3 \in (R^1_\beta)_H$ and $\omega^2_1 \in (R^2_\beta)_H$. Then, by $(*)$, $-\omega^1_1 - \omega^2_1 - \omega^3_4 = 0$, $-\omega^1_2 + \omega^2_2 - \omega^3_4 = 0$, $-\omega^1_3 + \omega^2_3 - \omega^3_4 = 0$ and $\omega^2_2, \omega^2_3 \in (R^2_\beta)_H$. Moreover, we see $(R^3_\beta)_H = R^3_\beta$. Hence, $(R^2_\beta)_H = R^2_\beta$ and $(R^1_\beta)_H = R^1_\beta$. This contradicts to the assumption.)

□

By similar arguments to the proof of Lemma 3.19, we obtain Lemma 3.20.

Lemma 3.20. Let $H \in \mathfrak{a}$. If $\#(R^1_\beta)_H = 4$, then $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 0$ or $\#(R^2_\beta)_H = \#(R^3_\beta)_H = 4$.

Summarizing the above arguments, we obtain Proposition 3.21 by the homogeneity of D_4.

Proposition 3.21. For each $H \in \mathfrak{a}$, $\big(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H\big)$ is one of the following:

$$\begin{aligned}
&(0,0,0), (1,1,1), (2,2,2), (4,4,4),\\
&(1,0,0), (0,1,0), (0,0,1), (2,0,0), (0,2,0), (0,0,2),\\
&(3,0,0), (0,3,0), (0,0,3), (4,0,0), (0,4,0), (0,0,4).
\end{aligned}$$

If $H \in \mathfrak{a}$ satisfies $\big(\#(R_\beta^1)_H, \#(R_\beta^2)_H, \#(R_\beta^3)_H\big) = (0,0,0)$, then we say H is type I. If $H \in \mathfrak{a}$ satisfies $\big(\#(R_\beta^1)_H, \#(R_\beta^2)_H, \#(R_\beta^3)_H\big) = (n,0,0),(0,n,0),(0,0,n)$ $(n = 1,2,3,4)$, then we say H is type II. If $H \in \mathfrak{a}$ satisfies $\big(\#(R_\beta^1)_H, \#(R_\beta^2)_H, \#(R_\beta^3)_H\big) = (n,n,n)$ $(n = 1,2,4)$, then we say H is type III. Let $\pi_{\mathfrak{s}} : \mathfrak{k} \to \mathfrak{s}$ be the orthogonal projection. Set $\mathfrak{k}_\beta^a = [\mathfrak{a}, \mathfrak{m}_\beta^a]$ $(a = 1,2,3)$. Then, $\pi_{\mathfrak{s}}(\mathfrak{k}_\beta^1) = \mathbb{R}(iA_\beta), \pi_{\mathfrak{s}}(\mathfrak{k}_\beta^2) = \mathbb{R}Z_\beta, \pi_{\mathfrak{s}}(\mathfrak{k}_\beta^3) = \mathbb{R}W_\beta$. Moreover, since $\mathfrak{k}_H = \mathfrak{k}_0 + \sum_{\omega \in R_H^+} \mathfrak{k}_\omega$ for each $H \in \mathfrak{a}$ and $\pi_{\mathfrak{s}}(X) = \{0\}$ for any $X \in \mathfrak{k}$ which is orthogonal to $[\mathfrak{a}, \mathfrak{m}_\beta]$, we see that $H \in \mathfrak{a}$ is type a $(a = \text{I,II,III})$ if and only if the K-orbit through $\pi(\exp H)$ is type a.

3.6 Orbits of the isotropy group action

We consider properties of each K-orbit with respect to the quaternionic structure. Let $\mathfrak{m}'_\beta = \mathfrak{m}_\beta^1 + \mathfrak{m}_\beta^2 + \mathfrak{m}_\beta^3$. Then, $\mathfrak{m}_\beta = \mathfrak{a} + \mathfrak{m}'_\beta$. Set $R_0 = iR \cap \{\lambda_i \pm \lambda_j \ ; \ 1 \le i < j \le 4\}$ and $\mathfrak{m}_0 = \sum_{\omega \in R_0} \mathfrak{m}_\omega$. Moreover, for each $1 \le i < j \le 4$, set $R_{ij} = \{\lambda_i \pm \lambda_j, \lambda_k \pm \lambda_l\}$, where $1 \le k < l \le 4,\ k,l \ne i,j$. Then, $R_0 \subset R_{12} \cup R_{13} \cup R_{14}$. We set $\mathfrak{m}_{ij} = \sum_{\omega \in R_{ij}} \mathfrak{m}_\omega$. Then, $\mathfrak{m}_0 \subset \mathfrak{m}_{12} + \mathfrak{m}_{13} + \mathfrak{m}_{14}$.

Let $H \in \mathfrak{a}$ be type I. Recall the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$. Then, $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \{0\}$ and each $X \in S(\mathfrak{s})$ defines the K-invariant section J of f_H^*Q. We study each K-invariant section J of f_H^*Q.

Lemma 3.22. For each $1 \le i < j \le 4$, $\#(R_{ij})_H \le 1$.

Proof. We see $a+b, a-b \in \pm R_\beta$ for any $a, b \in R_{ij}$ $(a \ne b)$. Since $(R_\beta)_H = \phi$, the statement follows. □

Lemma 3.23. For any $X \in \mathfrak{s}$, there are subspaces V_o, W_o of $T_o\mathcal{O}_H \cap \mathfrak{m}_0$ such that

$$V_0 \perp W_0, \quad V_0 + W_0 = T_o\mathcal{O}_H \cap \mathfrak{m}_0, \quad \mathrm{ad}X(V_0) \subset V_0, \quad \mathrm{ad}X(W_0) \subset (T_o\mathcal{O}_H)^\perp$$

Proof. By Proposition 3.16, $\mathrm{ad}X(\mathfrak{m}_{1i}) \subset \mathfrak{m}_{1i}$ $(i = 2,3,4)$. By Lemma 3.22, for each $1 \le i \le 3$, there is some $\omega_i \in (R_{1i})_H$ such that

$$T_o\mathcal{O}_H \cap \mathfrak{m}_{1i} = \sum_{\omega \in R_{1i}, \omega \ne \omega_i} \mathfrak{m}_\xi, \qquad (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{1i} = \mathfrak{m}_\omega$$

or $T_o\mathcal{O}_H \cap \mathfrak{m}_{1i} = \mathfrak{m}_{1i}, (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{1i} = \{0\}$. In any case, $\mathrm{ad}X((T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_0) \subset T_o\mathcal{O}_H \cap \mathfrak{m}_0$. Set $W_o = \mathrm{ad}X((T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_0)$ and let V_0 be the orthogonal complement of W_0 in $T_o\mathcal{O}_H \cap \mathfrak{m}_0$. Then, V_0, W_0 satisfy the statement. □

Since H is type I, we see $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \mathfrak{m}'_\beta, (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \mathfrak{a}$ and $\mathrm{ad}X(\mathfrak{a}) \subset \mathfrak{m}'_\beta = T_o\mathcal{O}_H \cap \mathfrak{m}_\beta$ for any $X \in \mathfrak{s}$ by Lemma 3.8, Lemma 3.9, Lemma 3.10. Set $W_\beta = \mathrm{ad}X(\mathfrak{a})$ and let V_β be the orthogonal complement of W_β in $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta$. Then, V_β, W_β satisfy

$$V_\beta \perp W_\beta, \quad V_\beta + W_\beta = \mathfrak{m}_\beta, \quad \mathrm{ad}(X)(V_\beta) \subset V_\beta, \quad \mathrm{ad}(X)(W_\beta) \subset (T_o\mathcal{O}_H)^\perp$$

Summarizing these arguments and subsection 1.3, we obtain Proposition 3.24.

Proposition 3.24. Let $H \in \mathfrak{a}$ be type I and $f_H : K/(K_H)_0 \to \mathcal{O}_H$ be the immersion. Then, $\mathcal{O}_H$ is type I and f_H is a K-equivariant totally CR immersion by any K-invariant section of f_H^*Q. Moreover, for any K-invariant section I, $c_I = c'_I$ and c_I is independent of the choice of I. Also, $\mathcal{O}_H$ is a QR submanifold.

Let $H \in \mathfrak{s}$ be type II. We can assume $(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H) = (a, 0, 0)$ $(a = 1, 2, 3, 4)$. Then, $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathbb{R}(iA_\beta)$ and $\mathrm{ad}(iA_\beta)$ defines the K-invariant section of f_H^*Q. Set $\mathfrak{s}' = \mathbb{R}Z_\beta + \mathbb{R}W_\beta$. $(K_H)_0$ acts on $\mathfrak{s}'$ as $U(1)$-action.

Lemma 3.25. Let $X \in \mathbb{R}(iA_\beta) \cup \mathfrak{s}'$. There are subspaces V_0, W_0 of $T_o\mathcal{O}_H \cap \mathfrak{m}_0$ such that

$$V_0 \perp W_0, \quad V_0 + W_0 = T_o\mathcal{O}_H \cap \mathfrak{m}_0, \quad \mathrm{ad}X(V_0) \subset V_0, \quad \mathrm{ad}X(W_0) \subset (T_o\mathcal{O}_H)^\perp.$$

Proof. For each $1 \le i < j \le 4$, since H is type II, $(T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{ij}$ is one of the following:

$$\{0\}, \quad \mathfrak{m}_{\lambda_i-\lambda_j}, \quad \mathfrak{m}_{\lambda_i+\lambda_j}, \quad \mathfrak{m}_{\lambda_i-\lambda_j} + \mathfrak{m}_{\lambda_i+\lambda_j}, \quad \mathfrak{m}_{\lambda_k-\lambda_l}, \quad \mathfrak{m}_{\lambda_k+\lambda_l}, \quad \mathfrak{m}_{\lambda_k-\lambda_l} + \mathfrak{m}_{\lambda_k+\lambda_l}, \quad \mathfrak{m}_{ij},$$

where $1 \le k < l \le 4,\ k, l \ne i, j$. Set $W_0 = \mathrm{ad}X\big(\sum_{j=2}^4((T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{1j})\big) \cap T_o\mathcal{O}_H$ and let V_0 be the orthogonal complement of W_0 in $\sum_{j=2}^4(T_o\mathcal{O}_H \cap \mathfrak{m}_{1j})$. Then, V_0, W_0 satisfy the statement. □

We remark

$$T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \mathfrak{m}^2_\beta + \mathfrak{m}^3_\beta + \sum_{\omega \in R^1_\beta - (R^1_\beta)_H} \mathfrak{m}_\omega, \quad (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \mathfrak{a} + \sum_{\omega \in (R^1_\beta)_H} \mathfrak{m}_\omega.$$

Let $V_A = \mathfrak{m}^2_\beta + \mathfrak{m}^3_\beta$ and $W_A = \sum_{\omega \in R^1_\beta - (R^1_\beta)_H} \mathfrak{m}_\omega$. Since $\mathrm{ad}(iA_\beta)(\mathfrak{a}) \subset \mathfrak{m}^1_\beta$, V_A and W_A satisfy

$$V_A \perp W_A, \quad V_A + W_A = T_o\mathcal{O}_H \cap \mathfrak{m}_\beta, \quad \mathrm{ad}(iA_\beta)(V_A) \subset V_A, \quad \mathrm{ad}(iA_\beta)(W_A) \subset (T_o\mathcal{O}_H)^\perp.$$

Let $X \in \mathfrak{s}'$. Then, $\mathrm{ad}X(\mathfrak{a} + \mathfrak{m}^1_\beta) \subset \mathfrak{m}^2_\beta + \mathfrak{m}^3_\beta$. Set $W_X = \mathrm{ad}X((T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta)$ and let V_X be the orthogonal complement of W_X in $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta$. Then, V_X, W_X satisfy

$$V_X \perp W_X, \quad V_X + W_X = T_o\mathcal{O}_H \cap \mathfrak{m}_0, \quad \mathrm{ad}X(V_X) \subset V_X, \quad \mathrm{ad}X(W_X) \subset (T_o\mathcal{O}_H)^\perp.$$

Summarizing these arguments we obatin Proposition 3.26.

Proposition 3.26. Let $H \in \mathfrak{a}$ be type II and $f_H : K/(K_H)_0 \to \mathcal{O}_H$ be the immersion. Then, $\mathcal{O}_H$ is type II and f_H is a K-equivariant totally CR immersion by the K-invariant section I of $f_H^* Q$.

Let $H \in \mathfrak{a}$ be type III. Then, since $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathfrak{s}$ and $(K_H)_0$ acts on $\mathfrak{s}$ as $SO(3)$-action. Thus, we only consider $\mathrm{ad}(iA_\beta)$. Let $(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H) = (4,4,4)$. Then, $\mathfrak{m}_\beta \subset (T_o\mathcal{O}_H)^\perp$. Moreover, for each $1 \leq i < j \leq 4$, $(R_{ij})_H = \phi$ or $(R_{ij})_H = R_{ij}$ since $a \pm b \in \pm R_\beta$ for any $a, b \in R_{ij}$ $(a \neq b)$. Hence, $T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \{0\}$ or $T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \mathfrak{m}_{ij}$. Since $\mathrm{ad}(iA_\beta)\mathfrak{m}_{ij} \subset \mathfrak{m}_{ij}$, we obtain Proposition 3.27 immediately.

Proposition 3.27. Let $H \in \mathfrak{a}$ be type III and $(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H) = (4,4,4)$. Then, $\mathcal{O}_H$ is a one-point set or a quaternionic submanifold.

Next, let $H \in \mathfrak{a}$ satisfy $(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H) = (2,2,2)$. Then, by the proof of Lemma 3.18, $\big((R^1_\beta)_H, (R^2_\beta)_H, (R^3_\beta)_H\big)$ is one of the following:

$$\begin{aligned}
&\Big(\{\omega^1_1,\omega^1_2\},\{\omega^2_1,\omega^2_2\},\{\omega^3_3,\omega^3_4\}\Big), \quad \Big(\{\omega^1_1,\omega^1_2\},\{\omega^2_3,\omega^2_4\},\{\omega^3_1,\omega^3_2\}\Big), \quad \Big(\{\omega^1_3,\omega^1_4\},\{\omega^2_1,\omega^2_2\},\{\omega^3_1,\omega^3_2\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_3\},\{\omega^2_1,\omega^2_3\},\{\omega^3_2,\omega^3_4\}\Big), \quad \Big(\{\omega^1_1,\omega^1_3\},\{\omega^2_2,\omega^2_4\},\{\omega^3_1,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_4\},\{\omega^2_1,\omega^2_3\},\{\omega^3_1,\omega^3_3\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_4\},\{\omega^2_2,\omega^2_3\},\{\omega^3_2,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_3\},\{\omega^2_1,\omega^2_4\},\{\omega^3_2,\omega^3_3\}\Big), \quad \Big(\{\omega^1_2,\omega^1_3\},\{\omega^2_2,\omega^2_3\},\{\omega^3_1,\omega^3_4\}\Big),\\
&\Big(\{\omega^1_1,\omega^1_4\},\{\omega^2_1,\omega^2_4\},\{\omega^3_1,\omega^3_4\}\Big), \quad \Big(\{\omega^1_2,\omega^1_4\},\{\omega^2_2,\omega^2_4\},\{\omega^3_2,\omega^3_4\}\Big), \quad \Big(\{\omega^1_3,\omega^1_4\},\{\omega^2_3,\omega^2_4\},\{\omega^3_3,\omega^3_4\}\Big).
\end{aligned}$$

Let $(R^a_\beta)_H = \{\eta^a_1, \eta^a_2\}$ $(a = 1,2,3)$ and $R^a_\beta - (R^a_\beta)_H = \{\eta^a_3, \eta^a_4\}$. Then,

$$T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \sum_{a=1}^{3}(\mathfrak{m}_{\eta^a_1} + \mathfrak{m}_{\eta^a_2}), \qquad (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \sum_{a=1}^{3}(\mathfrak{m}_{\eta^a_3} + \mathfrak{m}_{\eta^a_4}).$$

By Lemma 3.8, $\mathrm{ad}(iA_\beta)(\mathfrak{m}_{\eta^1_1} + \mathfrak{m}_{\eta^1_2}) \subset \mathfrak{a} \subset (T_o\mathcal{O}_H)^\perp$. Moreover, we see that unique $v_2 \in \{T_{\eta^2_1} \pm T_{\eta^2_2}\}$ satisfies $\mathrm{ad}(iA_\beta)v_2 \in \mathfrak{m}_{\eta^3_1} + \mathfrak{m}_{\eta^3_2} \subset T_o\mathcal{O}_H$ and the other $w_2 \in \{T_{\eta^2_1} \pm T_{\eta^2_2}\}$ satisfies $\mathrm{ad}(iA_\beta)w_2 \in \mathfrak{m}_{\eta^3_3} + \mathfrak{m}_{\eta^3_4} \subset (T_o\mathcal{O}_H)^\perp$. Similarly, unique $v_3 \in \{T_{\eta^3_1} \pm T_{\eta^3_2}\}$ satisfies $\mathrm{ad}(iA_\beta)v_3 \in \mathfrak{m}_{\eta^2_1} + \mathfrak{m}_{\eta^2_2} \subset T_o\mathcal{O}_H$ and the other $w_3 \in \{T_{\eta^3_1} \pm T_{\eta^3_2}\}$ satisfies $\mathrm{ad}(iA_\beta)w_3 \in \mathfrak{m}_{\eta^2_3} + \mathfrak{m}_{\eta^2_4} \subset (T_o\mathcal{O}_H)^\perp$. In particular, $\mathrm{ad}(iA_\beta)(w_2) \in \mathbb{R}w_3$. Thus, $V_\beta = \mathbb{R}v_2 + \mathbb{R}v_3$ and $W_\beta = \mathfrak{m}_{\eta^1_1} + \mathfrak{m}_{\eta^1_2} + \mathbb{R}w_2 + \mathbb{R}w_3$ satisfy

$$V_\beta \perp W_\beta, \quad V_\beta + W_\beta = T_o\mathcal{O}_H \cap \mathfrak{m}_\beta, \quad \mathrm{ad}(iA_\beta)(V_\beta) \subset V_\beta, \quad \mathrm{ad}(iA_\beta)(W_\beta) \subset (T_o\mathcal{O}_H)^\perp.$$

Because $\big((R^1_\beta)_H, (R^2_\beta)_H, (R^3_\beta)_H\big)$ is one of the above, we obtain Lemma 3.28.

Lemma 3.28. Let $H \in \mathfrak{a}$ satisfy $(\#(R^1_\beta)_H, \#(R^2_\beta)_H, \#(R^3_\beta)_H) = (2,2,2)$. Then, for each $1 \leq i < j \leq 4$, $\#(R^+_{ij})_H \neq 2$.

Thus, by some $\omega \in R_{ij}$, we obtain

$$\begin{array}{llll} & T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \{0\} & \text{and} & (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{ij} = \mathfrak{m}_{ij}, \\ \text{or} & T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \mathfrak{m}_{ij} & \text{and} & (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{ij} = \{0\}, \\ \text{or} & T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \mathfrak{m}_\omega & \text{and} & (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{ij} = \sum_{\eta \in R_{ij}^+, \eta \neq \omega} \mathfrak{m}_\eta, \\ \text{or} & T_o\mathcal{O}_H \cap \mathfrak{m}_{ij} = \sum_{\eta \in R_{ij}^+, \eta \neq \omega} \mathfrak{m}_\eta & \text{and} & (T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_{ij} = \mathfrak{m}_\omega. \end{array}$$

In any case of the above, there are subspaces V_{ij}, W_{ij} of $T_o\mathcal{O}_H \cap \mathfrak{m}_{ij}$ such that

$$V_{ij} \perp W_{ij}, \quad V_{ij} + W_{ij} = T_o\mathcal{O}_H \cap \mathfrak{m}_{ij}, \quad \mathrm{ad}(iA_\beta)(V_{ij}) \subset V_{ij}, \quad \mathrm{ad}(iA_\beta)(W_{ij}) \subset (T_o\mathcal{O}_H)^\perp.$$

Summarizing these arguments we obtain Proposition 3.29.

Proposition 3.29. Let $H \in \mathfrak{a}$ be type III and $(\#(R_\beta^1)_H, \#(R_\beta^2)_H, \#(R_\beta^3)_H) = (2, 2, 2)$. Then, for any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are subspaces V, W of $T_p\mathcal{O}_H$ such that

$$V \perp W, \quad V + W = T_p\mathcal{O}_H, \quad J(V) \subset V, \quad J(W) \subset (T_p\mathcal{O}_H)^\perp.$$

Let $H \in \mathfrak{a}$ be type III and $(\#(R_\beta^1)_H, \#(R_\beta^2)_H, \#(R_\beta^3)_H) = (1, 1, 1)$. Then, $((R_\beta^1)_H, (R_\beta^2)_H, (R_\beta^3)_H)$ is one of the following:

$$\begin{array}{llll} (\{\omega_1^1\}, \{\omega_1^2\}, \{\omega_4^3\}), & (\{\omega_2^1\}, \{\omega_1^2\}, \{\omega_3^3\}), & (\{\omega_3^1\}, \{\omega_1^2\}, \{\omega_2^3\}), & (\{\omega_4^1\}, \{\omega_1^2\}, \{\omega_1^3\}), \\ (\{\omega_1^1\}, \{\omega_2^2\}, \{\omega_3^3\}), & (\{\omega_2^1\}, \{\omega_2^2\}, \{\omega_4^3\}), & (\{\omega_3^1\}, \{\omega_2^2\}, \{\omega_1^3\}), & (\{\omega_4^1\}, \{\omega_2^2\}, \{\omega_2^3\}), \\ (\{\omega_1^1\}, \{\omega_3^2\}, \{\omega_2^3\}), & (\{\omega_2^1\}, \{\omega_3^2\}, \{\omega_1^3\}), & (\{\omega_3^1\}, \{\omega_3^2\}, \{\omega_4^3\}), & (\{\omega_4^1\}, \{\omega_3^2\}, \{\omega_3^3\}), \\ (\{\omega_1^1\}, \{\omega_4^2\}, \{\omega_1^3\}), & (\{\omega_2^1\}, \{\omega_4^2\}, \{\omega_2^3\}), & (\{\omega_3^1\}, \{\omega_4^2\}, \{\omega_3^3\}), & (\{\omega_4^1\}, \{\omega_4^2\}, \{\omega_4^3\}). \end{array}$$

By Lemma 3.8, we see that there are no subspaces V, W of $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta$ such that

$$V \perp W, \quad V + W = T_p\mathcal{O}_H \cap \mathfrak{m}_\beta, \quad \mathrm{ad}(iA_\beta)(V) \subset V, \quad \mathrm{ad}(iA_\beta)(W) \subset (T_p\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta.$$

Summarizing results in this subsection, we obtain Theorem 3.30.

Theorem 3.30. Let $H \in \mathfrak{a}$.

(i) If $\mathcal{O}_H$ is type I, then the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section I of f_H^*Q and such K-invariant sections correspond to each point of the 2-dimensional sphere one-to-one. Moreover, $c_I = c_I'$ and c_I is independent of the choice of I. Also, $\mathcal{O}_H$ is a QR submanifold.

(ii) If $\mathcal{O}_H$ is type II, then the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by the K-invariant section of f_H^*Q. Such K-invariant sections are unique up to the sign.

(iii) If $\mathcal{O}_H$ is type III, then $\mathcal{O}_H$ satisfies one of the following:

(iii-1) $\mathcal{O}_H$ is a one-point set or a quaternionic submanifold.

(iii-2) For any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V$ and $J(W) \subset (T_p\mathcal{O}_H)^\perp$.

(iii-3) For any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are no subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V$ and $J(W) \subset (T_p\mathcal{O}_H)^\perp$.

3.7 Classification

In this subsection, we decide what each K-orbit become one of (i), (ii), (iii-1), (iii-2), (iii-3) in Theorem 3.30. Since rank $M = 4$, G is one of $G = SO(n)$ $(n \geq 8), F_4, E_6, E_7, E_8$. In this subsection, we shall follow the notations of irreducible root systems in [6], that is

$$B_n = \{\pm e_p \ ; \ 1 \leq p \leq n\} \cup \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq n\},$$

$$D_n = \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq n\},$$

$$F_4 = \{\pm e_p \ ; \ 1 \leq p \leq 4\} \cup \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq 4\} \cup \left\{ \frac{1}{2}\sum_{p=1}^{4} a_p e_p \ ; \ a_p = \pm 1 \right\},$$

$$E_6 = \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq 5\} \cup \left\{ \frac{1}{2}\sum_{p=1}^{8} a_p e_p \ ; \ a_p = \pm 1, \prod_{p=1}^{8} a_p = 1, a_6 = a_7 = a_8 \right\},$$

$$E_7 = \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq 6\} \cup \{\pm(e_7 + e_8)\} \cup \left\{ \frac{1}{2}\sum_{p=1}^{8} a_p e_p \ ; \ a_p = \pm 1, \prod_{p=1}^{8} a_p = 1, a_7 = a_8 \right\},$$

$$E_8 = \{\pm e_p \pm e_q \ ; \ 1 \leq p < q \leq 8\} \cup \left\{ \frac{1}{2}\sum_{p=1}^{8} a_p e_p \ ; \ a_p = \pm 1, \prod_{p=1}^{8} a_p = 1 \right\}.$$

Take some linear order in each type such that the highest root is $\beta = e_1 + e_2$. Let $\alpha_1 = e_1 + e_3, \alpha_2 = e_1 - e_3, \alpha_3 = e_2 + e_4, \alpha_4 = e_2 - e_4$. Then, $\alpha_i \in \Sigma_1$ and $\alpha_i \pm \alpha_j \notin \Sigma$ $(1 \leq i \neq j \leq 4)$.

In the case of $G = SO(8)$, Σ is type D_4. Then, we see $\Sigma = \Sigma_\beta$ and $R = \pm i R_\beta$. Thus, R is type D_4.

In the case of $G = SO(2n)$ $(n \geq 5)$, then Σ is type D_n. Then, $\Sigma_1 - (\Sigma_\beta \cap \Sigma_1) = \{e_1 \pm e_m, e_2 \pm e_m \ ; \ 5 \leq m \leq n\}$. Thus, $R_{\Sigma'}$ is $\{\lambda_1 \pm \lambda_2\}$ or $\{\lambda_3 \pm \lambda_4\}$ for each H-orbit Σ'. Hence, $R = \pm i(R_\beta \cup R_{12})$ and R is type B_4.

In the case of $G = SO(2n+1)$ $(n \geq 4)$, then Σ is type B_n. Then, $\Sigma_1 - (\Sigma_\beta \cap \Sigma_1) = \{e_1 \pm e_m, e_2 \pm e_m \ ; \ 5 \leq m \leq n\} \cup \{e_1, e_2\}$. Thus, $R_{\Sigma'}$ is $\{\lambda_1 \pm \lambda_2\}$ or $\{\lambda_3 \pm \lambda_4\}$ for each H-orbit Σ'. Thus, $R = \pm i(R_\beta \cup R_{12})$ and R is type B_4.

In the case of $G = F_4$, then Σ is type F_4. Then,

$$\Sigma_1 - (\Sigma_\beta \cap \Sigma_1) = \{e_1 \pm e_m, e_2 \pm e_m \ ; \ 5 \leq m \leq n\} \cup \{e_1, e_2\} \\ \cup \left\{ \frac{1}{2}(\alpha_1 + \alpha_3), \frac{1}{2}(\alpha_1 + \alpha_4), \frac{1}{2}(\alpha_2 + \alpha_3), \frac{1}{2}(\alpha_2 + \alpha_4) \right\}.$$

We see that for any $1 \leq i < j \leq 4$ there is some H-orbit Σ' such that $R_{\Sigma'} = \{\lambda_i \pm \lambda_j\}$. Thus, $R = \pm i(R_\beta \cup \bigcup_{2 \leq i \leq 4} R_{1i})$ and R is type F_4.

In the case of $G = E_n$ $(n = 6, 7, 8)$, then

$$\begin{aligned} &\Sigma_1 - (\Sigma_\beta \cap \Sigma_1) \\ &= \{e_1 \pm e_m, e_2 \pm e_m \ ; \ 5 \leq m \leq n\} \\ &\quad \cup \left(\left\{ \frac{1}{2}(\alpha_1 + \alpha_3) + \delta, \ \frac{1}{2}(\alpha_1 + \alpha_4) + \delta, \ \frac{1}{2}(\alpha_2 + \alpha_3) + \delta, \ \frac{1}{2}(\alpha_2 + \alpha_4) + \delta \ ; \ \delta \in \sum_{k=5}^{8} \mathbb{R} e_m \right\} \cap \Sigma \right) \end{aligned}$$

and $R = \pm i(R_\beta \cup \bigcup_{2 \leq i \leq 4} R_{1i})$. Hence, R is type F_4.

If $R = \pm i R_\beta$, we take some linear order such that $\omega_1 = 2i\lambda_2, \omega_2 = i(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \omega_3 = 2i\lambda_3, \omega_4 = i\lambda_4$ are simple roots. Then, the highest root η is $2i\lambda_4$.

If $R = \pm i(R_\beta \cup R_{12})$, we take some linear order such that $\omega_1 = 2i\lambda_1, \omega_2 = i(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4), \omega_3 = 2i\lambda_3, \omega_4 = i(-\lambda_3 - \lambda_4)$ are simple roots. Then, the highest root η is $2i\lambda_2$.

If $R = \pm i(R_\beta \cup \bigcup_{2 \leq i \leq 4} R_{1i})$, we take some linear order such that $\omega_1 = i(-\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4), \omega_2 = 2i\lambda_3, \omega_3 = i(\lambda_2 - \lambda_3), \omega_4 = i(\lambda_1 - \lambda_2)$ are simple roots. Then, the highest root η is $2i\lambda_4$.

Recall arguments of subsection 2.2. Each K-orbit intersects $\pi(\exp\overline{Q})$ at only one point and $\overline{Q} = \sqcup_{\Delta \subset \mathcal{F}, \Delta \cap F \neq \phi} Q_\Delta$. Moreover, for $H \in Q_\Delta$, it is true that R_H^+ is independent of the choice of H and only depend on Δ. In Table 2,3,4, we summarize that each K-orbit through $\pi(\exp H)$ $(H \in Q_\Delta)$ becomes one of (i),(ii),(iii-1),(iii-2),(iii-3) in Theorem 3.30 in each G. In the list, Δ implies a subset of $\mathcal{F}$. For example $(1, 2, 3)$ implies $\{\omega_1, \omega_2, \omega_3\}$ and $(2, \eta)$ implies $\{\omega_2, \eta\}$. The "type" implies the type of the K-orbit through $\pi(\exp H)$ $(H \in Q_\Delta)$, that is (i),(ii),(iii-1),(iii-2),(iii-3). The "dim" implies the dimension of K-orbit through $\pi(\exp H)$ $(H \in Q_\Delta)$. If $H \in Q_\Delta$ is type (i), then "c" implies c_I of the CR immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ and a K-invariant section I of $f_H^* Q$. If $H \in Q_\Delta$ is type (ii), then "c" and "c'" implies c_I and c_I' of the totally CR immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ and the CR structure I of f_H. If $H \in Q_\Delta$ is type (iii-2), then "c" implies the dimension of V in Theorem 3.30. If the K-orbit becomes a principal orbit, a polar, a pole, a quaternionic submanifold or the image of a totally complex immersion, then we specify this in "remark", where a pole is a polar which is a one-point set [7]. In Table 2 of the case of $G = F_4, E_6, E_7, E_8$, set $m \in \mathbb{Z}$ as $m = 1$ if $G = F_4$, $m = 2$ if $G = E_6$, $m = 4$ if $G = E_7$, $m = 8$ if $G = E_8$. In Table 3 of the case of $G = SO(n)$ $(n \geq 8)$, set $m = n - 3$ if n is odd and $m = n - 4$ if n is even.

Δ	type	dim	c	c'	remark
$(1,2)$	(i)	$9m+12$	$6m+8$		
$(1,2,3,\eta)$	(i)	$11m+12$	$10m+8$		
$(1,2,4,\eta)$	(i)	$11m+12$	$10m+8$		
$(1,2,3,4,\eta)$	(i)	$12m+12$	$12m+8$		principal orbit
(1)	(ii)	$6m+8$	$6m+8$	0	polar, f_H is totally complex
$(1,\eta)$	(ii)	$6m+9$	$6m+8$	2	
$(1,2)$	(ii)	$9m+11$	$6m+8$	$6m+6$	
$(2,\eta)$	(ii)	$9m+11$	$6m+8$	$6m+6$	
$(1,4)$	(ii)	$10m+9$	$10m+8$	$8m+2$	
$(1,4,\eta)$	(ii)	$10m+10$	$10m+8$	$8m+4$	
$(1,2)$	(ii)	$11m+10$	$10m+8$	$10m+4$	
$(1,2,3)$	(ii)	$11m+11$	$10m+8$	$10m+6$	
$(1,2,4)$	(ii)	$11m+11$	$10m+8$	$10m+6$	
$(1,3,\eta)$	(ii)	$11m+11$	$10m+8$	$10m+6$	
$(2,3,\eta)$	(ii)	$11m+11$	$10m+8$	$10m+6$	
$(2,4,\eta)$	(ii)	$11m+11$	$10m+8$	$10m+6$	
$(1,3,4)$	(ii)	$12m+10$	$12m+8$	$12m+4$	
$(1,2,3,4)$	(ii)	$12m+11$	$12m+8$	$12m+6$	
$(1,3,4,\eta)$	(ii)	$12m+11$	$12m+8$	$12m+6$	
$(2,3,4,\eta)$	(ii)	$12m+11$	$12m+8$	$12m+6$	
(4)	(iii-1)	$8m$			polar, quaternionic
(3)	(iii-2)	$11m+6$	$10m+2$		
$(3,4)$	(iii-2)	$12m+6$	$12m+2$		
$(4,\eta)$	(iii-2)	$12m+6$	$8m+2$		
(2)	(iii-3)	$9m+9$			
$(2,3)$	(iii-3)	$11m+9$			
$(2,4)$	(iii-3)	$11m+9$			
$(3,\eta)$	(iii-3)	$11m+9$			
$(2,3,4)$	(iii-3)	$12m+9$			
$(3,4,\eta)$	(iii-3)	$12m+9$			

Table 2: K-orbits in the case of $G = F_4, E_6, E_7, E_8$

Δ	type	dim	c	c'	remark
$(1,2,3,\eta)$	(i)	$3m+12$	$2m+8$		
$(1,2,3,4,\eta)$	(i)	$4m+12$	$4m+8$		principal orbit
(2)	(ii)	$2m+8$	$2m+8$	0	polar, f_H is totally complex
$(2,\eta)$	(ii)	$2m+9$	$2m+8$	2	
$(1,2)$	(ii)	$2m+9$	$2m+8$	2	
$(1,2,\eta)$	(ii)	$2m+10$	$2m+8$	4	
$(2,3)$	(ii)	$3m+10$	$2m+8$	$2m+4$	
$(1,2,3)$	(ii)	$3m+11$	$2m+8$	$2m+6$	
$(1,3,\eta)$	(ii)	$3m+11$	$2m+8$	$2m+6$	
$(2,3,\eta)$	(ii)	$3m+11$	$2m+8$	$2m+6$	
$(2,4)$	(ii)	$4m+9$	$4m+8$	$4m+2$	
$(1,2,4)$	(ii)	$4m+10$	$4m+8$	$4m+4$	
$(2,3,4)$	(ii)	$4m+10$	$4m+8$	$4m+4$	
$(2,4,\eta)$	(ii)	$4m+10$	$4m+8$	$4m+4$	
$(1,2,3,4)$	(ii)	$4m+11$	$4m+8$	$4m+6$	
$(1,2,4,\eta)$	(ii)	$4m+11$	$4m+8$	$4m+6$	
$(1,3,4,\eta)$	(ii)	$4m+11$	$4m+8$	$4m+6$	
$(2,3,4,\eta)$	(ii)	$4m+11$	$4m+8$	$4m+6$	
(1)	(iii-1)	0			pole
(4)	(iii-1)	$4m$	$4m$		polar, quaternionic
$(1,\eta)$	(iii-2)	$m+6$	2		
(3)	(iii-2)	$3m+6$	$2m+2$		
$(1,4)$	(iii-2)	$4m+6$	$4m+2$		
$(3,4)$	(iii-2)	$4m+6$	$4m+2$		
$(4,\eta)$	(iii-2)	$4m+6$	$4m+2$		
$(1,3)$	(iii-3)	$3m+9$			
$(3,\eta)$	(iii-3)	$3m+9$			
$(1,3,4)$	(iii-3)	$4m+9$			
$(1,4,\eta)$	(iii-3)	$4m+9$			
$(3,4,\eta)$	(iii-3)	$4m+9$			

Table 3: K-orbits in the case of $G=SO(n)$ $(n\neq 8)$.

Δ	type	dim	c	c'	remark	Δ	type	dim	c	c'	remark
$(1,2,3,4,\eta)$	(i)	12	8		principal orbit	(1)	(iii-1)	0			pole
$(1,2,3,\eta)$	(ii)	11	8	6		(3)	(iii-1)	0			pole
$(1,2,4,\eta)$	(ii)	11	8	6		(4)	(iii-1)	0			pole
$(1,3,4,\eta)$	(ii)	11	8	6		$(1,3)$	(iii-2)	6	2		
$(2,3,4,\eta)$	(ii)	11	8	6		$(1,4)$	(iii-2)	6	2		
$(1,2,3,4)$	(ii)	11	8	6		$(3,4,\eta)$	(iii-2)	9			
$(1,2,4)$	(ii)	10	8	4		$(1,\eta)$	(iii-2)	6	2		
$(1,2,3)$	(ii)	10	8	4		$(3,\eta)$	(iii-2)	6	2		
$(2,3,4)$	(ii)	10	8	4		$(4,\eta)$	(iii-2)	6	2		
$(1,2,\eta)$	(ii)	10	8	4		(1,3,4)	(iii-3)	9			
$(2,3,\eta)$	(ii)	10	8	4		$(1,3,\eta)$	(iii-3)	9			
$(2,4,\eta)$	(ii)	10	8	4		$(1,4,\eta)$	(iii-3)	9			
$(1,2)$	(ii)	9	8	2		$(3,4,\eta)$	(iii-3)	9			
$(2,3)$	(ii)	9	8	2							
$(2,4)$	(ii)	9	8	2							
$(2,\eta)$	(ii)	9	8	2							
(2)	(ii)	8	8	0	polar, f_H is totally complex						

Table 4: K-orbits in the case of $G = SO(8)$.

4 The case of $\mathrm{rank}M = 2$

In this section, we consider the case of $\mathrm{rank}M = 2$, that is M is a complex Grassmann manifold $SU(n)/S(U(2) \times U(n-2))$ $(n \geq 4)$ or the associative Grassmann manifold $G_2/SO(4)$. In the present paper, we only consider the complex Grassmann manifold. We cite [12] about the associative Grassmann manifold.

Let E_{ij} be the $n \times n$ matrix whose (i,j)-component is 1 and the others are 0. Let $\tilde{\mathfrak{g}} = \mathfrak{sl}(n,\mathbb{C}) = \{X \in M(n,\mathbb{C})\ ;\ \mathrm{tr}X = 0\}$ and $\tilde{\mathfrak{h}} = \{H = \sum_{i=1}^{n} z_i E_{ii}\ ;\ z_i \in \mathbb{C}, \mathrm{tr}H = 0\}$. Set a complex conjugation τ such that $\tau(X) = -{}^t\overline{X}$. Then, $\mathfrak{g} = \{X \in \mathfrak{sl}(n,\mathbb{C})\ ;\ \tau(X) = X\} = \mathfrak{su}(n)$ and $\mathfrak{h} = \tilde{\mathfrak{h}} \cap \mathfrak{g} = \{H = \sum_{j=1}^{n}(ix_j)E_{jj} \in \tilde{\mathfrak{h}}\ ;\ x_j \in \mathbb{R}\}$. Let $G = SU(n)$. Define a linear form ϵ_i $(1 \leq i \leq n)$ of $\tilde{\mathfrak{h}}$ such that $\epsilon_i(\sum_{j=1}^{n} z_j E_{jj}) = z_i$. Then, $\Sigma = \{\pm(\epsilon_i - \epsilon_j)\ ;\ 1 \leq i < j \leq n\}$. Set an invariant nondegenerate symmetric bilinear form $(\ ,\ )$ such that $(X,Y) = \mathrm{tr}(XY)$ $(X,Y \in \mathfrak{sl}(n,\mathbb{C}))$. Then, $H_{\epsilon_i-\epsilon_j} = E_{ii} - E_{jj}$ and $A_{\epsilon_i-\epsilon_j} = H_{\epsilon_i-\epsilon_j}$ for each $1 \leq i \neq j \leq n$. Take some linear order on $i\mathfrak{h}$ such that $\beta = \epsilon_1 - \epsilon_2$ is the highest root. Let Σ^+ be the set of all positive roots. We see $\Sigma_1 = \{\epsilon_1 - \epsilon_k, -\epsilon_2 + \epsilon_k\ ;\ 3 \leq k \leq n\}, \Sigma_0 = \{\epsilon_i - \epsilon_j\ ;\ 3 \leq i < j \leq n\}$. Set a root vector $X_{\epsilon_i-\epsilon_j} = E_{ij}$ for each $1 \leq i \neq j \leq n$. Let $Z_{\epsilon_i-\epsilon_j} := X_{\epsilon_i-\epsilon_j} + \tau(X_{\epsilon_j-\epsilon_i}) = E_{ij} - E_{ji}$ and $W_{\epsilon_i-\epsilon_j} = i(X_{\epsilon_i-\epsilon_j} - \tau(X_{\epsilon_j-\epsilon_i})) = i(E_{ij} + E_{ji})$ for

$1 \leq i \neq j \leq n$. Let $\theta = \exp(\mathrm{ad}(\pi i A_\beta))$. Then, θ is an involutive automorphism of $\mathfrak{g}$ and

$$\mathfrak{m} = \{X \in \mathfrak{g} \ ; \ \theta(X) = -X\} = \sum_{i=3}^{n} (\mathbb{R}Z_{\epsilon_1-\epsilon_i} + \mathbb{R}W_{\epsilon_1-\epsilon_i} + \mathbb{R}Z_{\epsilon_i-\epsilon_2} + \mathbb{R}W_{\epsilon_i-\epsilon_2}),$$
$$\mathfrak{k} = \{X \in \mathfrak{g} \ ; \ \theta(X) = X\} = \mathfrak{h} + \mathbb{R}Z_\beta + \mathbb{R}W_\beta + \sum_{3\leq i<j\leq n} (\mathbb{R}Z_{\epsilon_i-\epsilon_j} + \mathbb{R}W_{\epsilon_i-\epsilon_j}).$$

In particular, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(2)\times\mathfrak{u}(n-2))$. Denote by the same symbol the involution of G induced by θ. Then, $K = \{g \in G \ ; \ \theta(g) = g\} = S(U(2) \times U(n-2))$.

Set $\alpha_1 = \epsilon_1 - \epsilon_3, \alpha_2 = -\epsilon_2 + \epsilon_4 \in \Sigma_1$. Then, $\mathfrak{a} = \mathbb{R}Z_{\alpha_1} + \mathbb{R}Z_{\alpha_2}$ is a maximal abelian subspace of $\mathfrak{m}$. Let $A = \lambda_1 Z_{\alpha_1} + \lambda_2 Z_{\alpha_2}$ $(\lambda_1, \lambda_2 \in \mathbb{R})$. We easily check that the followings are true.

$$\begin{aligned}
&\mathrm{ad}A(Z_{\beta-\alpha_1} \pm Z_{\beta-\alpha_2}) = (\lambda_1 \mp \lambda_2)(Z_\beta \mp Z_{\beta-(\alpha_1+\alpha_2)}),\\
&\mathrm{ad}A(Z_\beta \mp Z_{\beta-(\alpha_1+\alpha_2)}) = -(\lambda_1 \mp \lambda_2)(Z_{\beta-\alpha_1} \pm Z_{\beta-\alpha_2}),\\
&\mathrm{ad}A(W_{\beta-\alpha_1} \pm W_{\beta-\alpha_2}) = (\lambda_1 \mp \lambda_2)(W_\beta \mp W_{\beta-(\alpha_1+\alpha_2)}),\\
&\mathrm{ad}A(W_\beta \mp W_{\beta-(\alpha_1+\alpha_2)}) = -(\lambda_1 \mp \lambda_2)(W_{\beta-\alpha_1} \pm W_{\beta-\alpha_2}),\\
&\mathrm{ad}A(W_{\alpha_1}) = 2\lambda_1(iA_{\alpha_1}), \quad \mathrm{ad}A(iA_{\alpha_i}) = (-2\lambda_1)W_{\alpha_1},\\
&\mathrm{ad}A(W_{\alpha_2}) = 2\lambda_2(iA_{\alpha_2}), \quad \mathrm{ad}A(iA_{\alpha_2}) = (-2\lambda_2)W_{\alpha_2}.
\end{aligned}$$

Moreover, for each $5 \leq k \leq n$,

$$\begin{aligned}
&\mathrm{ad}A(Z_{\epsilon_1-\epsilon_k}) = \lambda_1(-Z_{\epsilon_3-\epsilon_k}), && \mathrm{ad}A(-Z_{\epsilon_3-\epsilon_k}) = (-\lambda_1)Z_{\epsilon_1-\epsilon_k},\\
&\mathrm{ad}A(W_{\epsilon_1-\epsilon_k}) = \lambda_1(-W_{\epsilon_3-\epsilon_k}), && \mathrm{ad}A(-W_{\epsilon_3-\epsilon_k}) = (-\lambda_1)W_{\epsilon_1-\epsilon_k},\\
&\mathrm{ad}A(Z_{-\epsilon_2+\epsilon_k}) = \lambda_2(Z_{-\epsilon_4+\epsilon_k}), && \mathrm{ad}A(Z_{-\epsilon_4+\epsilon_k}) = (-\lambda_2)Z_{-\epsilon_2+\epsilon_k},\\
&\mathrm{ad}A(W_{-\epsilon_2+\epsilon_k}) = \lambda_2(W_{-\epsilon_4+\epsilon_k}), && \mathrm{ad}A(W_{-\epsilon_4+\epsilon_k}) = (-\lambda_2)W_{-\epsilon_2+\epsilon_k}.
\end{aligned}$$

Set elements of $\mathfrak{m}$ as follows:

$$\begin{aligned}
&T^1_{\lambda_1-\lambda_2} = Z_{\beta-\alpha_1} + Z_{\beta-\alpha_2}, \quad T^2_{\lambda_1-\lambda_2} = W_{\beta-\alpha_1} + W_{\beta-\alpha_2},\\
&T^1_{\lambda_1+\lambda_2} = Z_{\beta-\alpha_1} - Z_{\beta-\alpha_2}, \quad T^2_{\lambda_1+\lambda_2} = W_{\beta-\alpha_1} - W_{\beta-\alpha_2}, \quad T_{2\lambda_i} = W_{\alpha_i} \ (i = 1, 2)
\end{aligned}$$

and $T^{k,1}_{\lambda_1} = Z_{\epsilon_1-\epsilon_k}, T^{k,2}_{\lambda_1} = W_{\epsilon_1-\epsilon_k}, T^{k,1}_{\lambda_2} = Z_{-\epsilon_2+\epsilon_k}, T^{k,2}_{\lambda_2} = W_{-\epsilon_2+\epsilon_k}$ for each $5 \leq k \leq n$. Set elements of $\mathfrak{k}$ as follows:

$$\begin{aligned}
&S^1_{\lambda_1-\lambda_2} = Z_\beta - Z_{\beta-(\alpha_1+\alpha_2)}, \quad S^2_{\lambda_1-\lambda_2} = W_\beta - W_{\beta-(\alpha_1+\alpha_2)},\\
&S^1_{\lambda_1+\lambda_2} = Z_\beta + Z_{\beta-(\alpha_1+\alpha_2)}, \quad S^2_{\lambda_1+\lambda_2} = W_\beta + W_{\beta-(\alpha_1+\alpha_2)}, \quad S_{2\lambda_j} = iA_{\alpha_j} \ (j = 1, 2)
\end{aligned}$$

and $S^{k,1}_{\lambda_1} = -Z_{\epsilon_3-\epsilon_k}, S^{k,2}_{\lambda_1} = -W_{\epsilon_3-\epsilon_k}, S^{k,1}_{\lambda_2} = Z_{-\epsilon_4+\epsilon_k}, S^{k,2}_{\lambda_2} = W_{-\epsilon_4+\epsilon_k}$ for each $5 \leq k \leq n$. Let

$R_\beta = \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}$ and $R_0 = \{\lambda_1, \lambda_2\}$. Then, for any $A \in \mathfrak{a}, \omega \in \{\lambda_1 \pm \lambda_2\}$ and $\eta \in R_0$,

$$\begin{aligned}
&\mathrm{ad}A(T^i_\omega) = \omega(A)S^i_\omega, && \mathrm{ad}A(S^i_\omega) = -\omega(A)T^i_\omega, \\
&\mathrm{ad}A(T^{k,i}_\eta) = \eta(A)S^{k,i}_\eta, && \mathrm{ad}A(S^{k,i}_\eta) = -\eta(A)T^{k,i}_\eta \quad (i = 1, 2 \text{ and } 5 \le k \le n), \\
&\mathrm{ad}A(T_{2\lambda_j}) = 2\lambda_j(A)S_{2\lambda_j}, && \mathrm{ad}A(S_{2\lambda_j}) = -2\lambda_j T_{2\lambda_j} \quad (j = 1, 2).
\end{aligned}$$

Thus, the restricted root system R is given by $\pm i(R_\beta \cup R_0)$. For each $\omega \in R_\beta \cup R_0$, we set $\mathfrak{m}_\omega = \{X \in \mathfrak{m} \ ; \ (\mathrm{ad}A)^2(X) = -\omega(A)^2 X \ (A \in \mathfrak{a})\}$. Let $\mathfrak{m}_\beta = \mathfrak{m}_{\lambda_1-\lambda_2} + \mathfrak{m}_{\lambda_1+\lambda_2} + \mathfrak{m}_{2\lambda_1} + \mathfrak{m}_{2\lambda_2}$ and $\mathfrak{m}_k = \mathbb{R}T^{k,1}_{\lambda_1} + \mathbb{R}T^{k,2}_{\lambda_1} + \mathbb{R}T^{k,1}_{\lambda_2} + \mathbb{R}T^{k,2}_{\lambda_2}$ for each $3 \le k \le n$. Then, $\mathfrak{m} = \mathfrak{a} + \mathfrak{m}_\beta + \sum_{k=3}^n \mathfrak{m}_k$. By direct computations, we see $\mathrm{ad}X(\mathfrak{a} + \mathfrak{m}_\beta) \subset \mathfrak{a} + \mathfrak{m}_\beta$ and $\mathrm{ad}X(\mathfrak{m}_k) \subset \mathfrak{m}_k$ for any $X \in \mathfrak{s}$ and $5 \le k \le n$. Moreover, we obtain Lemma 4.1 and Lemma 4.2.

Lemma 4.1. Set subspaces $\mathfrak{m}_-$ and $\mathfrak{m}_+$ of $\mathfrak{a} + \mathfrak{m}_\beta$ as follows:

$$\mathfrak{m}_- = \mathfrak{m}_{\lambda_1-\lambda_2} + \mathbb{R}(Z_{\alpha_1} - Z_{\alpha_2}) + \mathbb{R}(T_{2\lambda_1} - T_{2\lambda_2}), \ \ \mathfrak{m}_+ = \mathfrak{m}_{\lambda_1+\lambda_2} + \mathbb{R}(Z_{\alpha_1} + Z_{\alpha_2}) + \mathbb{R}(T_{2\lambda_1} + T_{2\lambda_2}).$$

Then, $\mathrm{ad}\mathfrak{s}(\mathfrak{m}_-) \subset (\mathfrak{m}_-)$ and $\mathrm{ad}\mathfrak{s}(\mathfrak{m}_+) \subset (\mathfrak{m}_+)$. The representation matrices of $\mathrm{ad}(iA_\beta)|_{\mathfrak{m}_-}$, $\mathrm{ad}Z_\beta|_{\mathfrak{m}_-}$, $\mathrm{ad}W_\beta|_{\mathfrak{m}_-}$ with respect to $T^1_{\lambda_1-\lambda_2}, T^2_{\lambda_1-\lambda_2}, Z_{\alpha_1} - Z_{\alpha_2}, T_{2\lambda_1} - T_{2\lambda_2}$ are

$$\left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right),$$

where empty components are 0. Also, the representation matrices of $\mathrm{ad}(iA_\beta)|_{\mathfrak{m}_+}$, $\mathrm{ad}Z_\beta|_{\mathfrak{m}_+}$, $\mathrm{ad}W_\beta|_{\mathfrak{m}_+}$ with respect to $T^1_{\lambda_1+\lambda_2}, T^2_{\lambda_1+\lambda_2}, Z_{\alpha_1} + Z_{\alpha_2}, T_{2\lambda_1} + T_{2\lambda_2}$ are

$$\left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right),$$

Lemma 4.2. For each $5 \le k \le n$, $\mathrm{ad}\mathfrak{s}(\mathfrak{m}_k) \subset \mathfrak{m}_k$. Moreover, the representation matrices of $\mathrm{ad}(iA_\beta)|_{\mathfrak{m}_k}$, $\mathrm{ad}Z_\beta|_{\mathfrak{m}_k}$, $\mathrm{ad}W_\beta|_{\mathfrak{m}_k}$ with respect to $T^{k,1}_{\lambda_1}, T^{k,2}_{\lambda_1}, T^{k,1}_{\lambda_2}, T^{k,2}_{\lambda_2}$ are

$$\left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & -1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right).$$

where empty components are 0.

For $H \in \mathfrak{a}$, we set $(R_\beta)_H = \{\omega \in R_\beta \ ; \ \omega(H) \in \pi\mathbb{Z}\}$. If $H \in \mathfrak{a}$ satisfies $(R_\beta)_H = \phi$ for each $\omega \in R_\beta$, we say that H is type I. We easily see that if H is type I, then $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \{0\}$ and $\mathcal{O}_H$ is type I. If H satisfies $(R_\beta)_H \subset \{2\lambda_1, 2\lambda_2\}$, then we say that H is type II. We easily see that if H is type II, then $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathbb{R}(iA_\beta)$ and $\mathcal{O}_H$ is type II. If H is not type I and type II, then we say that H is type III. We see that if H is type III, then $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathfrak{s}$ and $\mathcal{O}_H$ is type III.

Let H be type I. Then, $\lambda_i(H) \notin \pi\mathbb{Z}$ $(i = 1, 2)$ and

$$\begin{aligned}
T_o\mathcal{O}_H \cap \mathfrak{m}_- &= \mathfrak{m}_{\lambda_1-\lambda_2} + \mathbb{R}(T_{2\lambda_1} - T_{2\lambda_2}),\\
T_o\mathcal{O}_H \cap \mathfrak{m}_+ &= \mathfrak{m}_{\lambda_1+\lambda_2} + \mathbb{R}(T_{2\lambda_1} + T_{2\lambda_2}),\\
T_o\mathcal{O}_H \cap \mathfrak{m}_k &= \mathfrak{m}_k \ (5 \leq k \leq n).
\end{aligned}$$

For any $X \in \mathfrak{s}$, let $W_X = \mathrm{ad}X(\mathfrak{a})$ and V_X be the orthogonal complement of W_X in $T_o\mathcal{O}_H$. Then V_X, W_X satisfy

$$V_X \perp W_X, \ V_X + W_X = T_o\mathcal{O}_H, \ \mathrm{ad}X(V_X) \subset V_X, \ \mathrm{ad}X(W_X) \subset (T_o\mathcal{O})^\perp.$$

Thus, the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section of f_H^*Q. Moreover, $\mathcal{O}_H$ is a QR submanifold. Thus, we obtain Lemma 4.3.

Lemma 4.3. Let $H \in \mathfrak{a}$ be type I. Then, $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section I of f_H^*Q and K-invariant sections correspond to each point of the 2-dimensional sphere. Moreover, $c_I = c'_I$ and c_I is independent of the choice of I. Also, $\mathcal{O}_H$ is a QR submanifold.

Let H be type II. Then, $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathbb{R}(iA_\beta)$ and $\mathrm{ad}(iA_\beta)$ defines the K-invariant section of f_H^*Q. If $(R_\beta)_H = \{2\lambda_1\}$, then $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \mathfrak{m}_{\lambda_1-\lambda_2} + \mathfrak{m}_{\lambda_1+\lambda_2} + \mathfrak{m}_{2\lambda_2}$ and $(T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \mathfrak{a} + \mathfrak{m}_{2\lambda_1}$. If $(R_\beta)_H = \{2\lambda_2\}$, then $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \mathfrak{m}_{\lambda_1-\lambda_2} + \mathfrak{m}_{\lambda_1+\lambda_2} + \mathfrak{m}_{2\lambda_1}$ and $(T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \mathfrak{a} + \mathfrak{m}_{2\lambda_2}$. If $(R_\beta)_H = \{2\lambda_1, 2\lambda_2\}$, then $T_o\mathcal{O}_H \cap \mathfrak{m}_\beta = \mathfrak{m}_{\lambda_1-\lambda_2} + \mathfrak{m}_{\lambda_1+\lambda_2}$ and $(T_o\mathcal{O}_H)^\perp \cap \mathfrak{m}_\beta = \mathfrak{a} + \mathfrak{m}_{2\lambda_1} + \mathfrak{m}_{2\lambda_2}$. By Lemma 4.1, $\mathrm{ad}(iA_\beta)(\mathfrak{a} + \mathfrak{m}_{2\lambda_1} + \mathfrak{m}_{2\lambda_2}) \subset \mathfrak{a} + \mathfrak{m}_{2\lambda_1} + \mathfrak{m}_{2\lambda_2}$. Moreover, $\mathrm{ad}(iA_\beta)(\mathfrak{m}_{\lambda_j}) \subset \mathfrak{m}_{\lambda_j}$ $(j = 1, 2)$ by Lemma 4.2. Thus, there are subspaces V_A and W_A of $T_o\mathcal{O}_H$ such that

$$V_A \perp W_A, \ V_A + W_A = T_o\mathcal{O}_H, \ \mathrm{ad}(iA_\beta)(V_A) \subset V_A, \ \mathrm{ad}(iA_\beta)(W_A) \subset (T_o\mathcal{O})^\perp.$$

Also, for any $X \in \mathbb{R}Z_\beta + \mathbb{R}W_\beta$ since $\mathrm{ad}X(\mathfrak{m}_{\lambda_1-\lambda_2} + \mathfrak{m}_{\lambda_1+\lambda_2}) \subset \mathfrak{a} + \mathfrak{m}_{2\lambda_1} + \mathfrak{m}_{2\lambda_2}$ by Lemma 4.1 and $\mathrm{ad}X(\mathfrak{m}_{\lambda_1}) \subset \mathfrak{m}_{\lambda_2}$ by Lemma 4.2, we see that there are subspaces V_X, W_X of $T_o\mathcal{O}_H$ such that

$$V_X \perp W_X,\ V_X + W_X = T_o\mathcal{O}_H,\ \mathrm{ad}X(V_X) \subset V_X,\ \mathrm{ad}X(W_X) \subset (T_o\mathcal{O})^\perp.$$

Thus, we obtain Lemma 4.4.

Lemma 4.4. Let $H \in \mathfrak{a}$ be type II. Then, $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by the K-invariant section of f_H^*Q and such K-invariant sections are unique up to sign.

Let H be type III. Since $(K_H)_0$ acts on $\mathfrak{s}$ as $SO(3)$, we only consider $\mathrm{ad}(iA_\beta)$. Then, $(R_\beta)_H = \{\lambda_1 - \lambda_2\}, \{\lambda_1 + \lambda_2\}$ or R_β. In the case of $(R_\beta)_H = \{\lambda_1 - \lambda_2\}$, then $\lambda_i(H) \notin \pi\mathbb{Z}$ $(i = 1, 2)$. Thus, $T_o\mathcal{O}_H = \mathfrak{m}_{\lambda_1+\lambda_2} + \sum_{a=1}^{2}(\mathfrak{m}_{2\lambda_a} + \mathfrak{m}_{\lambda_a})$ and $(T_o\mathcal{O}_H)^\perp = \mathfrak{a} + \mathfrak{m}_{\lambda_1-\lambda_2}$. Let $W_A = \mathrm{ad}(iA_\beta)\mathfrak{a}$ and V_A be the orthogonal complement of W_A in $T_o\mathcal{O}_H$. Then, V_A and W_A satisfy

$$V_A \perp W_A,\ V_A + W_A = T_o\mathcal{O}_H,\ \mathrm{ad}(iA_\beta)V_A \subset V_A,\ \mathrm{ad}(iA_\beta)W_A \subset (T_o\mathcal{O}_H)^\perp.$$

In the case of $(R_\beta)_H = \{\lambda_1 + \lambda_2\}$, we can prove that there are such subspaces by similar way. In the case of $(R_\beta)_H = R_\beta$, we see $T_o\mathcal{O}_H = \{0\}$ or $\mathfrak{m}_{\lambda_1} + \mathfrak{m}_{\lambda_2}$. In the former case, $\mathcal{O}_H$ is a one-point set. In the latter case, $\mathcal{O}_H$ is a quaternionic submanifold. Summarizing these arguments, we obtain Lemma 4.5.

Lemma 4.5. Let $H \in \mathfrak{a}$ be type III. Then, $\mathcal{O}_H$ is type III. If $(R_\beta)_H = \{\lambda_1 - \lambda_2\}$ or $\{\lambda_1 + \lambda_2\}$, then for any $p \in \mathcal{O}_H$ and $J \in Q_p$ there are subspaces V and W such that

$$V \perp W,\ V + W = T_o\mathcal{O}_H,\ J(V) \subset V,\ J(W) \subset (T_o\mathcal{O}_H)^\perp.$$

If $(R_\beta)_H = R_\beta$, then $\mathcal{O}_H$ is a one-point set or a quaternionic submanifold.

Summarizing Lemma 4.3, Lemma 4.4, Lemma 4.5, we obtain Theorem 4.6.

Theorem 4.6. Let $H \in \mathfrak{a}$.

(i) If $\mathcal{O}_H$ is type I, $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section I of f_H^*Q and K-invariant sections correspond to each point of the 2-dimensional sphere one-to-one. Moreover, $c_I = c_I'$ and c_I is independent of the choice of I. Also, $\mathcal{O}_H$ is a QR submanifold.

(ii) If $\mathcal{O}_H$ is type II, then the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by a K-invariant section of $f_H^* Q$. Such K-invariant sections are unique up to the sign.

(iii) If $\mathcal{O}_H$ is type III, $\mathcal{O}_H$ satisfies one of the following:

(iii-1) $\mathcal{O}_H$ is a one-point set or a quaternionic submanifold.

(iii-2) For any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, .V + W = T_p\mathcal{O}_H, J(V) \subset V$ and $J(W) \subset (T_p\mathcal{O}_H)^\perp$.

We summarize that each K-orbit becomes one of (i),(ii),(iii-1),(iii-2) as Section 3. Let $\omega_1 = i(\lambda_1 - \lambda_2), \omega_2 = i\lambda_2$. Then, ω_1, ω_2 are simple roots with respect to some linear order of $i\mathfrak{a}$ and the highest root η is $2i\lambda_1$. Let $\mathcal{F} = \{\omega_1, \omega_2, \eta\}$. As the table in Section 3, we make Table 5 in the following.

Δ	type	dim	c	c'	remark
$(1, 2, \eta)$	(i)	$4n-10$	$4n-12$		principal orbit
(1)	(ii)	$2n-4$	$2n-4$	0	polar, f_H is totally complex
$(1, \eta)$	(ii)	$2n-3$	$2n-4$	2	
$(1, 2)$	(ii)	$4n-3$	$4n-4$	$4n-6$	
(2)	(iii-1)	$4n-16$	$4n-16$		pole ($n=4$), polar and quaternionic ($n>4$)
$(2, \eta)$	(iii-2)	$4n-12$	$4n-14$		

Table 5: K-orbits in $G = SU(n)$ $(n \geq 4)$

5 The case of $\mathrm{rank}M = 3$

In this section, we consider the case of $\mathrm{rank}M = 3$, that is M is the oriented real Grassmann manifold as the set of all oriented 3-dimensional subspaces of $\mathbb{R}^7$. In this case, $\tilde{\mathfrak{g}} = \mathfrak{so}(7, \mathbb{C}) = \{X \in M(7, \mathbb{C}) \ ; \ {}^tX = -X\}$. Let $\tau : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \ ; \ X \mapsto \bar{X}$ and $\mathfrak{g} = \{X \in \tilde{\mathfrak{g}} \ ; \ \tau(X) = X\} = \mathfrak{so}(7)$. Set $F_{ij} = E_{ij} - E_{ji}$ for each $1 \leq i \neq j \leq n$. Let $\tilde{\mathfrak{h}} = \{H = z_1F_{12} + z_2F_{34} + z_3F_{56} \ ; \ z_i \in \mathbb{C}\}$. Then, $\mathfrak{h} = \tilde{\mathfrak{h}} \cap \mathfrak{g} = \{x_1F_{12} + x_2F_{34} + x_3F_{56} \ ; \ x_i \in \mathbb{R}\}$ and $\mathfrak{h}$ is a maximal abelian subspace of $\mathfrak{g}$. Let ϵ_j be the linear form of $\tilde{\mathfrak{h}}$ such that $\epsilon_j(z_1F_{12} + z_2F_{34} + z_3F_{56}) = iz_j$ $(1 \leq j \leq 3)$. The root system of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$ is given by $\Sigma = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_k \ ; \ 1 \leq i < j \leq 3, 1 \leq k \leq 3\}$. Set an invariant nondegenerate symmetric bilinear form $(\ , \ )$ of $\tilde{\mathfrak{g}}$ such that $(X, Y) = \mathrm{tr}(XY)$ for $X, Y \in \tilde{\mathfrak{g}}$. For each $\gamma \in \Sigma$, we set the element H_γ of the real part $\mathfrak{h}_0 = i\mathfrak{h}$ by $(H_\gamma, H) = \gamma(H)$, that is $H_{\epsilon_i - \epsilon_j} =$

$-\frac{i}{2}(F_{2i-1,2i}-F_{2j-1,2j}), H_{\epsilon_i+\epsilon_j} = -\frac{i}{2}(F_{2i-1,2i}+F_{2j-1,2j}), H_{\epsilon_i} = -\frac{i}{2}F_{2i-1,2i}$ for $1 \le i \ne j \le 3$. Let $A_\gamma = \frac{2}{(H_\gamma,H_\gamma)}H_\gamma$, that is $A_{\epsilon_i-\epsilon_j} = -i(F_{2i-1,2i} - F_{2j-1,2j}), A_{\epsilon_i+\epsilon_j} = -i(F_{2i-1,2i} + F_{2j-1,2j}), A_{\epsilon_i} = -2iF_{2i-1,2i}$ for $1 \le i \ne j \le 3$. Take some linear order such that the highest root β is $\epsilon_1 + \epsilon_2$ and the set of all positive roots Σ^+ is $\{\epsilon_i \pm \epsilon_j, \epsilon_k \ ; \ 1 \le i < j \le 3, 1 \le k \le 3\}$. Then, $\Sigma_1 = \{\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_2 \pm \epsilon_3\}$ and $\Sigma_0 = \{\pm(\epsilon_1 - \epsilon_2), \pm\epsilon_3\}$. For $1 \le i < j \le 3$ and $1 \le k \le 3$, we set root vectors

$$\begin{aligned} X_{\epsilon_i-\epsilon_j} &= -(F_{2i-1,2j-1} + F_{2i,2j}) + i(F_{2i-1,2j} - F_{2i,2j-1}), \\ X_{\epsilon_i+\epsilon_j} &= (F_{2i-1,2j-1} - F_{2i,2j}) + i(F_{2i-1,2j} + F_{2i,2j-1}), \\ X_{\epsilon_k} &= F_{2k-1,7} + iF_{2k,7}. \end{aligned}$$

For each $\gamma \in \Sigma^+$, set $Z_\gamma = \frac{1}{2}(X_\gamma + \tau(X_\gamma))$ and $W_\gamma = \frac{i}{2}(X_\gamma - \tau(X_\gamma))$, that is for $1 \le i < j \le 3$,

$$\begin{aligned} Z_{\epsilon_i-\epsilon_j} &= -F_{2i-1,2j-1} - F_{2i,2j}, & Z_{\epsilon_i+\epsilon_j} &= F_{2i-1,2j-1} - F_{2i,2j}, \\ W_{\epsilon_i-\epsilon_j} &= -F_{2i-1,2j} + F_{2i,2j-1}, & W_{\epsilon_i+\epsilon_j} &= -F_{2i-1,2j} - F_{2i,2j-1}, \\ Z_{\epsilon_k} &= F_{2k-1,7}, & W_{\epsilon_k} &= -F_{2k,7}. \end{aligned}$$

Let $\theta = \exp(\pi i A_\beta)$. Then,

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \ ; \ \theta(X) = X\} = \mathfrak{h} + \mathbb{R}Z_\beta + \mathbb{R}W_\beta + \mathbb{R}Z_{\epsilon_1-\epsilon_2} + \mathbb{R}W_{\epsilon_1-\epsilon_2} + \mathbb{R}Z_{\epsilon_3} + \mathbb{R}W_{\epsilon_3}, \\ \mathfrak{m} &= \{X \in \mathfrak{g} \ ; \ \theta(X) = -X\} = \sum_{\gamma \in \Sigma_1} (\mathbb{R}Z_\gamma + \mathbb{R}W_\gamma). \end{aligned}$$

Let $G = SO(7)$ and denote by the same symbol the involution of G induced by θ. Let K be the identity component of $\{g \in G \ ; \ \theta(g) = g\}$, that is $K = SO(4) \times SO(3)$.

Let $U_i = F_{i,4+i}$ $(1 \le i \le 3)$ and $\mathfrak{a} = \{A = \sum_{i=1}^3 \lambda_i U_i \ ; \ \lambda_i \in \mathbb{R}\}$. Then, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$. We set elements of $\mathfrak{m}$ as follows:

$$T_{\lambda_1} = \tfrac{1}{2}(W_{\epsilon_2-\epsilon_3} - W_{\epsilon_2+\epsilon_3}) = F_{45}, \quad T_{\lambda_2} = -\tfrac{1}{2}(Z_{\epsilon_2-\epsilon_3} + Z_{\epsilon_2+\epsilon_3}) = F_{46}, \quad T_{\lambda_3} = -W_{\epsilon_2} = F_{47},$$

$$\begin{aligned} T_{\lambda_1+\lambda_2} &= W_{\epsilon_1-\epsilon_3} = F_{25} - F_{16}, & T_{\lambda_1-\lambda_2} &= -W_{\epsilon_1+\epsilon_3} = F_{25} + F_{16}, \\ T_{\lambda_1+\lambda_3} &= \tfrac{1}{2}(-Z_{\epsilon_2-\epsilon_3} + Z_{\epsilon_2+\epsilon_3}) - Z_{\epsilon_1} = F_{35} - F_{17}, & T_{\lambda_1-\lambda_3} &= \tfrac{1}{2}(-Z_{\epsilon_2-\epsilon_3} + Z_{\epsilon_2+\epsilon_3}) + Z_{\epsilon_1} = F_{35} + F_{17}, \\ T_{\lambda_2+\lambda_3} &= \tfrac{1}{2}(-W_{\epsilon_2-\epsilon_3} - W_{\epsilon_2+\epsilon_3}) + W_{\epsilon_1} = F_{36} - F_{27}, & T_{\lambda_2-\lambda_3} &= -\tfrac{1}{2}(W_{\epsilon_2-\epsilon_3} + W_{\epsilon_2+\epsilon_3}) - W_{\epsilon_1} = F_{36} + F_{27}, \end{aligned}$$

These vectors give a basis of the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{m}$. Moreover, we set a basis of $\mathfrak{k}$ as follows:

$$S_{\lambda_1} = \tfrac{1}{2}(W_{\epsilon_1-\epsilon_2} + W_{\epsilon_1+\epsilon_2}) = -F_{14}, \quad S_{\lambda_2} = \tfrac{1}{2}(Z_{\epsilon_1-\epsilon_2} + Z_{\epsilon_1+\epsilon_2}) = -F_{24}, \quad S_{\lambda_3} = -2iH_{\epsilon_2} = -F_{34},$$

$$\begin{aligned} S_{\lambda_1+\lambda_2} &= -2iH_{\epsilon_1-\epsilon_3} = -F_{12} + F_{56}, & S_{\lambda_1-\lambda_2} &= 2iH_{\epsilon_1+\epsilon_3} = F_{12} + F_{56}, \\ S_{\lambda_1+\lambda_3} &= \tfrac{1}{2}(Z_{\epsilon_1-\epsilon_2} - Z_{\epsilon_1+\epsilon_2}) + Z_{\epsilon_3} = -F_{13} + F_{57}, & S_{\lambda_1-\lambda_3} &= -\tfrac{1}{2}(Z_{\epsilon_1-\epsilon_2} - Z_{\epsilon_1+\epsilon_2}) + Z_{\epsilon_3} = F_{13} + F_{57}, \\ S_{\lambda_2+\lambda_3} &= -\tfrac{1}{2}(W_{\epsilon_1-\epsilon_2} - W_{\epsilon_1+\epsilon_2}) - W_{\epsilon_3} = -F_{23} + F_{67}, & S_{\lambda_2-\lambda_3} &= \tfrac{1}{2}(W_{\epsilon_1-\epsilon_2} - W_{\epsilon_1+\epsilon_2}) - W_{\epsilon_3} = F_{23} + F_{67}. \end{aligned}$$

We use the notations used in the previous two sections. Let $R_\beta = \{\lambda_i, \lambda_i \pm \lambda_j \; ; \; 1 \leq i < j \leq 3\}$. Then, for any $\omega \in R_\beta$ and $A \in \mathfrak{a}$,

$$\mathrm{ad}A(T_\omega) = \omega(A)S_\omega, \quad \mathrm{ad}A(S_\omega) = -\omega(A)T_\omega$$

and the restricted root system of $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{a}$ is given by $\pm iR_\beta$. We set $P_j^i \in \mathfrak{m}$ $(1 \leq i \leq 3, 1 \leq j \leq 4)$ as follows:

$$\begin{aligned}
&P_1^1 = \tfrac{1}{2}(-Z_{\epsilon_1-\epsilon_3} + Z_{\epsilon_1+\epsilon_3}) = F_{15}, && P_2^1 = \tfrac{1}{2}(T_{\lambda_1+\lambda_2} - T_{\lambda_1-\lambda_2}) = F_{25}, \\
&P_3^1 = \tfrac{1}{2}(T_{\lambda_1+\lambda_3} + T_{\lambda_1-\lambda_3}) = F_{35}, && P_4^1 = T_{\lambda_1} = F_{45}, \\
&P_1^2 = -\tfrac{1}{2}(T_{\lambda_1+\lambda_2} + T_{\lambda_1-\lambda_2}) = F_{16}, && P_2^2 = -\tfrac{1}{2}(Z_{\epsilon_1-\epsilon_3} + Z_{\epsilon_1+\epsilon_3}) = F_{26}, \\
&P_3^2 = \tfrac{1}{2}(T_{\lambda_2+\lambda_3} + T_{\lambda_2-\lambda_3}) = F_{36}, && P_4^2 = T_{\lambda_2} = P_{46}, \\
&P_1^3 = -\tfrac{1}{2}(T_{\lambda_1+\lambda_3} - T_{\lambda_1-\lambda_3}) = F_{17}, && P_2^3 = -\tfrac{1}{2}(T_{\lambda_2+\lambda_3} - T_{\lambda_2-\lambda_3}) = F_{27}, \\
&P_3^3 = Z_{\epsilon_2} = F_{37}, && P_4^3 = T_{\lambda_3} = F_{47}.
\end{aligned}$$

Remark that $P_i^i \in \mathfrak{a}$ $(1 \leq i \leq 3)$. Let $\mathfrak{m}^i = \sum_{j=1}^4 \mathbb{R}P_j^i$ $(i = 1, 2, 3)$. We obtain Lemma 5.1

Lemma 5.1. For any $X \in \mathfrak{s}$, $\mathrm{ad}X(\mathfrak{m}^i) \subset \mathfrak{m}^i$ $(i = 1, 2, 3)$. Moreover, for each $i = 1, 2, 3$, the representation matrices of $\mathrm{ad}(iA_\beta)|_{\mathfrak{m}^i}$ and $\mathrm{ad}Z_\beta|_{\mathfrak{m}^i}$ and $\mathrm{ad}W_\beta|_{\mathfrak{m}^i}$ with respect to $P_1^i, \cdots, P_4^i$ are

$$\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right),$$

where empty components are 0.

Set subsets $R^1, R^2, R^3 \subset R_\beta$ as follows: $R^1 = \{\lambda_1, \lambda_2 \pm \lambda_3\}, R^2 = \{\lambda_2, \lambda_1 \pm \lambda_3\}, R^3 = \{\lambda_3, \lambda_1 \pm \lambda_2\}$. If $H \in \mathfrak{a}$ satisfies $(R_\beta)_H = \phi$, we say that H is type I. If $H \in \mathfrak{a}$ satisfies $(R_\beta)_H \neq \phi$ and $(R_\beta)_H \subset R^i$ for some $1 \leq i \leq 3$, we say that H is type II. In the other cases, we say that H is type III. Then, we see that $\mathcal{O}_H$ and H have the same type because $\pi_{\mathfrak{s}}(\mathbb{R}S_\omega) = \mathbb{R}W_\beta$ if and only if $\omega \in R^1$ and $\pi_{\mathfrak{s}}(\mathbb{R}S_\omega) = \mathbb{R}Z_\beta$ if and only if $\omega \in R^2$ and $\pi_{\mathfrak{s}}(\mathbb{R}S_\omega) = \mathbb{R}(iA_\beta)$ if and only if $\omega \in R^3$.

If H is type I, then $T_o\mathcal{O} = \sum_{\omega \in R_\beta} \mathfrak{m}_\omega$ and $(T_o\mathcal{O}_H)^\perp = \mathfrak{a}$. By Lemma 5.1, we see that for each $X \in \mathfrak{s}$ there are subspaces V_X, W_X of $T_o\mathcal{O}_H$ such that $V_X \perp W_X, V_X + W_X = T_o\mathcal{O}_H, \mathrm{ad}X(V_X) \subset V_X, \mathrm{ad}X(W_X) \subset (T_o\mathcal{O}_H)^\perp$. Thus, the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section of f_H^*Q and such K-invariant section corresponds to each point of the 2-dimensional sphere. In particular, $\mathcal{O}_H$ is a QR submanifold.

Let H be type II. We may assume $(R_\beta)_H \subset R^1$. Then, $\pi_{\mathfrak{s}}(\mathfrak{k}_H) = \mathbb{R}W_\beta$. We see that there are subspaces V, W of $T_o\mathcal{O}_H$ such that $V \perp W, V + W = T_o\mathcal{O}_H, \mathrm{ad}W_\beta(V) \subset V, \mathrm{ad}W_\beta(W) \subset (T_o\mathcal{O}_H)^\perp$. Moreover, we see that for each $X \in \mathbb{R}(iA_\beta) + \mathbb{R}Z_\beta$ there are subspace V_X, W_X of $T_o\mathcal{O}_H$ such that $V_X \perp W_X, V_X + W_X = T_o\mathcal{O}_H, \mathrm{ad}X(V_X) \subset V_X, \mathrm{ad}X(W_X) \subset (T_o\mathcal{O}_H)^\perp$. Thus, f_H is a totally CR immersion by the K-invariant section of f_H^*Q. In particular, $(R_\beta)_H = R^1$ if and only if f_H is a totally complex immersion.

Let H be type III. By the definition, we see $\#((R_\beta)_H \cap \{\lambda_1, \lambda_2, \lambda_3\}) = 0, 2, 3$. If $\#((R_\beta)_H \cap \{\lambda_1, \lambda_2, \lambda_3\}) = 3$, then obviously $(R_\beta)_H = R_\beta$. Then, $T_o\mathcal{O}_H = \{0\}$ and $\mathcal{O}_H$ is a one-point set. If $\#((R_\beta)_H \cap \{\lambda_1, \lambda_2, \lambda_3\}) = 2$, then $(R_\beta)_H$ is one of $\{\lambda_1, \lambda_2, \lambda_1 \pm \lambda_2\}, \{\lambda_2, \lambda_3, \lambda_2 \pm \lambda_3\}, \{\lambda_3, \lambda_1, \lambda_1 \pm \lambda_3\}$. In this case, for any $p \in \mathcal{O}_H$ and $J \in Q_p$ there are subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V, J(W) \subset (T_p\mathcal{O}_H)^\perp$. If $\#((R_\beta)_H \cap \{\lambda_1, \lambda_2, \lambda_3\}) = 0$, we see that $(R_\beta)_H$ is one of $\{\lambda_i \pm \lambda_j \ ; \ 1 \leq i < j \leq 3\}, \{\lambda_1 - \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}, \{\lambda_1 + \lambda_2, \lambda_1 - \lambda_3, \lambda_2 + \lambda_3\}, \{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 - \lambda_3\}, \{\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \lambda_2 - \lambda_3\}$. If $(R_\beta)_H = \{\lambda_i \pm \lambda_j \ ; \ 1 \leq i < j \leq 3\}$, then $\mathcal{O}_H$ is a totally real submanifold. In the other cases, then for any $p \in \mathcal{O}_H$ and $J \in Q_p$ there are no subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V, J(W) \subset (T_p\mathcal{O}_H)^\perp$.

Summarizing these arguments, we obtain Theorem 5.2.

Theorem 5.2. Let $H \in \mathfrak{a}$.

(i) If $\mathcal{O}_H$ is type I, then $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by each K-invariant section I of f_H^*Q and K-invariant sections correspond to each point of the 2-dimensional sphere one-to-one. Moreover, $c_I = c'_I$ and c_I is independent of the choice of I. Also, $\mathcal{O}_H$ is a QR submanifold.

(ii) If $\mathcal{O}_H$ is type II, then the immersion $f_H : K/(K_H)_0 \to \mathcal{O}_H$ is a K-equivariant totally CR immersion by the K-invariant section of f_H^*Q. Such K-invariant sections are unique up to the sign.

(iii) If $\mathcal{O}_H$ is type III, then $\mathcal{O}_H$ satisfies one of the following:

(iii-1) $\mathcal{O}_H$ is a one-point set or a totally real submanifold.

(iii-2) For any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V, J(W) \subset (T_p\mathcal{O}_H)^\perp$.

(iii-3) For any $p \in \mathcal{O}_H$ and $J \in Q_p$, there are no subspaces V, W of $T_p\mathcal{O}_H$ such that $V \perp W, V + W = T_p\mathcal{O}_H, J(V) \subset V, J(W) \subset (T_p\mathcal{O}_H)^\perp$.

We summarize what type (i), (ii), (iii-1), (iii-2), (iii-3) each K-orbit becomes as Section 3. Let $\omega_1 = i(\lambda_1 - \lambda_2), \omega_2 = i(\lambda_2 - \lambda_3), \omega_3 = i\lambda_3$. Then, $\omega_1, \omega_2, \omega_3$ are simple roots with respect to some linear order of $i\mathfrak{a}$ and the highest root η is $2i\lambda_1$. Let $\mathcal{F} = \{\omega_1, \omega_2, \omega_3, \eta\}$. As the table in Section 3, we make Table 6.

Δ	type	dim	c	c'	remark
$(1,2,3,\eta)$	(i)	9	6		principal orbit
(2)	(ii)	6	6	0	polar, f_H is totally complex
$(1,2)$	(ii)	7	6	2	
$(2,3)$	(ii)	7	6	2	
$(2,\eta)$	(ii)	7	6	2	
$(1,2,3)$	(ii)	8	6	4	
$(1,2,\eta)$	(ii)	8	6	4	
$(1,3,\eta)$	(ii)	8	6	4	
$(2,3,\eta)$	(ii)	8	6	4	
(1)	(iii-1)	0			pole
(3)	(iii-1)	3	0		totally real
$(1,\eta)$	(iii-2)	5	2		
$(3,\eta)$	(iii-3)	6			
$(1,3)$	(iii-3)	6			

Table 6: K-orbits in $G = SO(7)$

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