REPRESENTATION TYPE OF LEVEL 1 KLR ALGEBRAS $R^{\Lambda_k}(\beta)$ IN TYPE C

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Abstract. We determine the representation type for block algebras of the quiver Hecke algebras $R^{\Lambda_k}(\beta)$ of type $C_\ell^{(1)}$ for all k, generalising results of Ariki–Park for $\Lambda = \Lambda_0$.

1. Introduction

KLR algebras, also called *(affine) quiver Hecke algebras*, were introduced by Khovanov–Lauda [20] and Rouquier [25] to give a categorification of the negative half of quantum groups. KLR algebras have natural finite-dimensional quotients $R^\Lambda(\beta)$ for a fixed dominant integral weight $\Lambda \in \mathsf{P}^+$ and varying $\beta \in \mathsf{Q}^+$ a non-negative integral linear combination of simple roots, called the *cyclotomic KLR algebras*, or *cyclotomic quiver Hecke algebras*. The module categories of cyclotomic KLR algebras $R^\Lambda(\beta)$ for various β, together with the induction and restriction functors between them give a categorification for the irreducible highest weight module $V(\Lambda)$ over a quantum group. Brundan and Kleshchev [9] showed that the block algebras of the cyclotomic Hecke algebras are isomorphic to the cyclotomic KLR algebras of type $A_\ell^{(1)}$, and so cyclotomic KLR algebras are a vast generalization of certain cyclotomic Hecke algebras, which are well understood.

In affine type $A_\ell^{(1)}$, further advances have been made in understanding the structure of cyclotomic KLR algebras. In addition to the Brundan–Kleshchev isomorphism theorem, we also have a categorification theorem due to Ariki [1], Brundan and Kleshchev which implies that in characteristic zero the simple objects in these categories correspond to the dual canonical basis of the integral highest weight module $V_{\mathbb{A}}(\Lambda)$ over the affine type A Kac–Moody Lie algebra, using a Fock space construction. Beyond type A, there is still much to discover about cyclotomic KLR algebras and their block algebras. Classically, the representation type of block algebras of the Iwahori–Hecke algebra of the symmetric group was described by Erdmann and Nakano. Beyond type A, a general representation type classification in the style of Erdmann–Nakano for the block algebras of cyclotomic KLR algebras is a subject of active research. For the special case $\Lambda = \Lambda_0$, this has been investigated in a series of recent papers by Ariki–Iijima–Park for type $A_\ell^{(1)}$ and Ariki–Park for types $A_{2l}^{(2)}, D_{\ell+1}^{(2)}$ and $C_\ell^{(1)}$ [3, 5, 6, 7], resulting in a Lie theoretic classification of representation type for $R^{\Lambda_0}(\beta)$ in the spirit of Erdmann–Nakano. In affine types A (both twisted and untwisted) and $D_{\ell+1}^{(2)}$, the representation types turned out to be governed by the weight, also called defect, introduced by Fayers [15], as a natural generalisation of the classification by Erdmann–Nakano. In affine type C Ariki–Park showed that $R^{\Lambda_0}(\delta)$ is no longer of finite representation type; in fact the algebra $R^{\Lambda_0}(\delta)$ is wild except in type $C_2^{(1)}$, where it is tame [7]. Also, for arbitrary fundamental weight $\Lambda = \Lambda_k$ (i.e. level 1) in untwisted affine ADE type, the set of maximal weights $\max(\Lambda)$ of $V(\Lambda)$ form a single Weyl group orbit [17, Lem. 12.6]. The same is not true for affine type C; as in this case $\max(\Lambda_k)$ consists of several Weyl group orbits, whose maximal representatives were given

2020 Mathematics subject classification: 05E10, 20C08, 81R10 by Ariki and Park in [7] when $k = 0$. Nevertheless, we can show in this paper that the defect governs the representation type in affine type C for arbitrary $\Lambda = \Lambda_k$. As a first step, we enumerate a set of maximal weight representatives

$$\xi_{k,i} := \alpha_{k+1} + 2\alpha_{k+2} + \cdots + (i-1)\alpha_{k+i-1} + i(\alpha_{k+i} + \alpha_{k+i+1} + \cdots + \alpha_{\ell-1}) + \frac{i}{2}\alpha_\ell;$$
$$\xi_{k,-i} := \alpha_{k-1} + 2\alpha_{k-2} + \cdots + (i-1)\alpha_{k-i+1} + i(\alpha_{k-i} + \alpha_{k-i-1} + \cdots + \alpha_1) + \frac{i}{2}\alpha_0$$

for $k \pm i \in I$. Thus, we need only investigate $R^{\Lambda_k}(m\delta - \xi_{k,\pm i})$ for $m \geq i/2$, a fact we will also use to show that the defect is non-negative in level one.

We explain the cases where we need different arguments than [7]. First, we consider $i = 0$ and $m = 1$, that is the representation type of $R^{\Lambda_k}(\delta)$. For the case $\ell = 2$ and $k = 1$, we give an explicit description of the indecomposable projective modules and prove that the algebra has tame representation type. For larger ℓ and k, we are able to apply a recent interesting result of Ariki ([2], see also Lemma 2.16) which reduces the problem to finding two appropriate idempotents. We show that if $\ell \neq 2$ and $0 \leq k \leq \ell$, $R^{\Lambda_k}(\delta)$ is of wild representation type. Then, we will show that $R^{\Lambda_k}(\delta - \xi_{k,\pm 2})$ is of finite representation type, and hence that block algebras of defect one are Brauer tree algebras whose Brauer tree is a straight line with no exceptional vertex. We will also see that depending on k, there can be two inequivalent Morita equivalence classes of blocks of defect one, with distinct number of simple modules $k+1$ or $\ell - k + 1$ respectively. Next, we deal with the representation type of $R^{\Lambda_k}(2\delta - \xi_{k,\pm 4})$ and using Lemma 2.16 again, we arrive at the result that this algebra has wild representation type as well. Finally, we handle the remaining cases by using the same arguments as in [7] with some slight modifications. Our results are summarised in Theorem 6.5.

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2. Background

2.1. **Lie theory notation.** We mostly follow [17] and use standard notation for the root datum.

Let $\ell \in \{2, 3, \dots\}$ and $I = \{0, 1, 2, \dots, \ell\}$.

The affine Cartan matrix of type $C_\ell^{(1)}$ is given by

$$\mathsf{A} = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

We have *simple roots* $\Pi = \{\alpha_i \mid i \in I\}$ and *fundamental weights* $\{\Lambda_i \mid i \in I\}$ in the *weight lattice* P, and simple coroots $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ in the *dual weight lattice* $\mathsf{P}^\vee$. Each α_i gives rise to a $\mathbb{Z}$-linear transformation r_i acting on P by $r_i\Lambda = \Lambda - \langle \alpha_i^\vee, \Lambda \rangle \alpha_i$, for $\Lambda \in \mathsf{P}$. Let $d \in P^\vee$ be the element such that $\alpha_i(d) = \delta_{0,i}$. We denote by W the Weyl group, the group generated by $\{r_i \mid i \in I\}$.

There is a W-invariant symmetric bilinear form $(-,-)$ on P satisfying the following:

(1) $(\Lambda_i, \alpha_j) = d_j \delta_{ij}$; and
(2) $(\alpha_i, \alpha_j) = d_i a_{ij}$ where $d = (2, 1 \dots, 1, 2)$ if $\ell < \infty$.

We denote the set of *dominant integral weights* by

$$\mathsf{P}^+ = \{\Lambda \in \mathsf{P} \mid \langle \alpha_i^\vee, \Lambda \rangle \geq 0 \text{ for all } i \in I\},$$

where $\langle\ ,\ \rangle$ is the natural pairing. We call $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the *root lattice* and $\mathsf{Q}^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ the *positive cone* of the root lattice. For $C_\ell^{(1)}$, we have the *null root* given by

$$\delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

The *defect* of $\beta \in \mathsf{Q}^+$ (relative to Λ) is given by

$$\mathrm{def}_\Lambda(\beta) = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta).$$

When it is clear from context, we will omit Λ from the subscript and write $\mathrm{def}(\beta)$ instead of $\mathrm{def}_\Lambda(\beta)$.

We set $\xi_{k,0} := 0$, and for $k + i \in I$ or $k - i \in I$ we set

$$\begin{aligned} \xi_{k,i} &:= \alpha_{k+1} + 2\alpha_{k+2} + \cdots + (i-1)\alpha_{k+i-1} + i(\alpha_{k+i} + \alpha_{k+i+1} + \cdots + \alpha_{\ell-1}) + \frac{i}{2}\alpha_\ell; \\ \xi_{k,-i} &:= \alpha_{k-1} + 2\alpha_{k-2} + \cdots + (i-1)\alpha_{k-i+1} + i(\alpha_{k-i} + \alpha_{k-i-1} + \cdots + \alpha_1) + \frac{i}{2}\alpha_0 \end{aligned} \tag{1}$$

respectively.

Note that if $i \neq 0$ then

$$\xi_{k,\pm i}(\alpha_j^\vee) = \begin{cases} -1 & \text{if } j = k, \\ 1 & \text{if } j = k \pm i, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and they form a basis for $\sum_{i \in I \setminus \{k\}} \mathbb{Q}\alpha_i$.

Lemma 2.1. *For $m \geq i/2$ and so $m\delta - \xi_{k,\pm i} \in \mathsf{Q}^+$, we have that $\mathrm{def}(m\delta - \xi_{k,\pm i}) = 2m - \frac{1}{2}i$ (relative to Λ_k).*

Proof. We have the following calculation:

$$\begin{aligned} \mathrm{def}_{\Lambda_k}(m\delta - \xi_{k,\pm i}) &= (\Lambda_k, m\delta - \xi_{k,\pm i}) - \frac{1}{2}(m\delta - \xi_{k,\pm i}, m\delta - \xi_{k,\pm i}) \\ &= (\Lambda_k, m\delta) - \frac{1}{2}(\xi_{k,\pm i}, \xi_{k,\pm i}) \\ &= 2m - \frac{1}{2}i \end{aligned}$$

as required. □

Let $\mathfrak{g} = \mathfrak{g}(\mathsf{A})$ be the affine Kac–Moody algebra associated with the Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ and let $U_q(\mathfrak{g})$ be its quantum group. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra generated by f_i, e_i ($i \in I$) and q^h ($h \in \mathsf{P}^\vee$) with certain relations (more details can be found in [16, Chap. 3]). Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ and denote by $U_{\mathbb{A}}^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)} := f_i^n/[n]_i!$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, where $q_i = q^{d_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i.$$

For some $\Lambda \in \mathsf{P}^+$, let $V(\Lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight Λ and $V_{\mathbb{A}}(\Lambda)$ the $U_{\mathbb{A}}^-(\mathfrak{g})$-submodule of $V(\Lambda)$ generated by the highest weight vector.

The Fock space representation for $U_q(C_\ell^{(1)})$ was first constructed in [19] by folding the Fock space representation for $U_q(A_{2\ell-1}^{(1)})$ via the Dynkin diagram automorphism (see eq. (3)). Later, the combinatorics of the Fock space and its crystal base were described in terms of tableaux and Young diagrams ([21, 23]). The next section focuses on explaining this combinatorial realisation for $V(\Lambda_k)$.

2.2. **Partitions and tableaux.**

Definition 2.2. For $n \geq 0$, a *partition* of n is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is equal to n. If λ is a partition of n we write $\lambda \vdash n$. We write $\varnothing$ for the unique partition of 0. We will denote the set of partitions of n by $\mathcal{P}_n$.

For any $\lambda \in \mathcal{P}_n$, we define its *Young diagram* $[\lambda]$ to be the set

$$\{(r, c) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \mid c \leq \lambda_r\}.$$

Note that we will depict a Young diagram of a partition using the English convention (i.e. successive rows of the diagram are lower down the page).

We define $f_\ell : \mathbb{Z} \to I$ by $k \mapsto |k|$ if $\ell = \infty$ and, if $\ell \neq \infty$, $f_\ell : \mathbb{Z}/2\ell\mathbb{Z} \to I$ by

$$\begin{aligned} &f_\ell(0 + 2\ell\mathbb{Z}) = 0, \quad f_\ell(\ell + 2\ell\mathbb{Z}) = \ell, \\ &f_\ell(k + 2\ell\mathbb{Z}) = f_\ell(2\ell - k + 2\ell\mathbb{Z}) = k \quad \text{ for } 1 \leq k \leq \ell - 1. \end{aligned} \tag{3}$$

Let p be the natural projection $\mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$. We set $\pi_\ell = f_\ell \circ p : \mathbb{Z} \to I$. If there is no confusion, we will denote $\pi_\ell(k)$ by $\overline{k}$, for $k \in \mathbb{Z}$.

Given a *charge* $\kappa \in \mathbb{Z}$, we define $\Lambda_\kappa \in \mathsf{P}^+$ by $\Lambda_\kappa := \Lambda_{\overline{\kappa}}$. If λ is a partition of n, then to any node $A = (r, c) \in [\lambda]$ we can associate its *residue* defined by

$$\mathrm{res}(A) = \overline{\kappa + c - r}.$$

If $\mathrm{res}(A) = i$, we call A an i-node.

Example 2.3. If $\lambda = (8, 6, 6, 5, 2)$, $\kappa = 2$ and $\ell = 4$, then λ has the following residue pattern.

2	3	4	3	2	1	0	1
1	2	3	4	3	2		
0	1	2	3	4	3		
1	0	1	2	3			
2	1						

We say that a node $A \in [\lambda]$ is *removable* (resp. *addable*) if $[\lambda] \setminus A$ (resp. $[\lambda] \cup A$) is a valid Young diagram for a partition of $n-1$ (resp. $n+1$). We write $\lambda \nearrow A$ (resp. $\lambda \swarrow A$) as shorthand for the partition whose Young diagram is $[\lambda] \setminus A$ (resp. $[\lambda] \cup A$). For an i-node $A \in [\lambda]$, we set

$$\begin{aligned} &N_0(\lambda) = \#\{0\text{-coloured boxes in } \lambda\}, \\ &d_i(\lambda) = \#\{\text{addable } i\text{-nodes of } [\lambda]\} - \#\{\text{removable } i\text{-nodes of } [\lambda]\}, \\ &d_A(\lambda) = d_i \cdot (\#\{\text{addable } i\text{-nodes of } [\lambda] \text{ below } A\} - \#\{\text{removable } i\text{-nodes of } [\lambda] \text{ below } A\}), \\ &d^A(\lambda) = d_i \cdot (\#\{\text{addable } i\text{-nodes of } [\lambda] \text{ above } A\} - \#\{\text{removable } i\text{-nodes of } [\lambda] \text{ above } A\}) \end{aligned}$$

where d_i is given in Subsection 2.1. We define the *Fock space* $\mathcal{F}(\kappa)$ *with charge* κ to be the $\mathbb{Q}(q)$-vector space with basis consisting of partitions of n. For a Young diagram $[\lambda]$,

$\mathcal{F}(\kappa)$ has a $U_q(\mathfrak{g}(\mathsf{A}))$-module structure defined by

$$q^d\lambda = q^{-N_0(\lambda)}\lambda, \qquad e_i\lambda = \sum_A q^{d_A(\lambda)}\lambda \nearrow A,$$
$$q^{\alpha_i^\vee}\lambda = q^{d_i(\lambda)}\lambda, \qquad f_i\lambda = \sum_A q^{-d^A(\lambda)}\lambda \swarrow A,$$

where A runs over all removable i-nodes and all addable i-nodes respectively [23, Theorem 2.3]).

We identify the basis of $\mathcal{F}(\kappa)$ with the set of all Young diagrams. Its crystal structure can be described by considering the usual *i-signature*. For a Young diagram $[\lambda]$, we consider all addable or removable i-nodes $a_1, a_2, \dots, a_m$ of $[\lambda]$ from top to bottom and to each a_j of $[\lambda]$, we assign its signature s_j as $+$ (resp. $-$) if it is addable (resp. removable). We cancel out all possible $(-,+)$ pairs in the i-signature $(s_1, \dots, s_m)$ to obtain the *reduced i-signature*, which is a sequence of $+$'s is followed by $-$'s. We call the removable node corresponding to the leftmost $-$ in the i-signature a *good node* and the addable node corresponding to the rightmost $+$ in the i-signature a *cogood node*. We define $\tilde{f}_i\lambda$ to be the Young diagram obtained from $[\lambda]$ by adding a box at the cogood node. Similarly, we define $\tilde{e}_i\lambda$ to be a Young diagram obtained from $[\lambda]$ by removing the box at the good node. The operators $\tilde{e}_i$ and $\tilde{f}_i$ defined above coincide with Kashiwara's operators cf. [23, Theorem 3.3]. Here our choice of convention is compatible with that of [10, §3.6] which deals with the type A Fock space, and differs from [21, 23] in the choice of convention used when reducing the i-signature.

Then (see [23, Section 2]), the $U_q(\mathfrak{g})$-submodule of $\mathcal{F}(\kappa)$ with $\Lambda_\kappa = \Lambda_k$ generated by the empty partition $\varnothing_k$ is isomorphic to the irreducible integrable highest weight module $V(\Lambda_k)$, and the crystal graph for the $U_q(\mathfrak{g})$-crystal $V(\Lambda_k)$ is the directed coloured graph with vertices the set of partitions that can be obtained from repeated application of the operators $\tilde{f}_i$, $i \in I$ to $\varnothing_k$ and i-coloured edges $\lambda \xrightarrow{i} \mu$ whenever $\mu = \tilde{f}_i\lambda$ (or equivalently, $\tilde{e}_i\mu = \lambda$). We will call a partition *Kleshchev* if it is a vertex in the crystal graph of $V(\Lambda_k) \subset \mathcal{F}(\kappa)$ cf. [14, §6F]. By Theorem C of [14], partitions of n that are Kleshchev form a complete set of labels for the set of simple R_n^Λ-modules.

Definition 2.4. Let $\lambda \in \mathcal{P}_n$. A *$\lambda$-tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \dots, n\}$. We depict $\mathtt{T}$ by filling each node $(r,c) \in [\lambda]$ with $\mathtt{T}(r,c)$. We call a tableau $\mathtt{T}$ *standard* if the entries increase along rows and down the columns. We denote the set of standard tableaux by $\mathsf{Std}(\lambda)$. We will denote by $\mathtt{T}^\lambda$ the unique λ-tableau where nodes are filled by $1, 2, \dots, n$ along the successive rows and call $\mathtt{T}^\lambda$ the *initial tableau*.

For each λ-tableau $\mathtt{T}$, we have the associated residue sequence

$$\begin{aligned}\mathbf{i}^{\mathtt{T}} &= (i_1, i_2, \dots, i_n)\\ &= (\mathrm{res}(\mathtt{T}^{-1}(1)),\ \mathrm{res}(\mathtt{T}^{-1}(2)), \dots, \mathrm{res}(\mathtt{T}^{-1}(n))).\end{aligned}$$

We will write $\mathbf{i}^\lambda$ for $\mathbf{i}^{\mathtt{T}^\lambda}$.

Example 2.5. If $\lambda = (4,3,3,2)$, $\kappa = 1$ and $\ell = 3$, then

$$\mathtt{T}^\lambda = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6&7\\ \cline{1-3} 8&9&10\\ \cline{1-3} 11&12\\ \cline{1-2}\end{array}$$

and we have that $\mathbf{i}^\lambda = (123201210121)$.

Let $\mathtt{T}$ be a λ-tableau and choose some $0 \leq m \leq n$. We denote by $\mathtt{T}_{\downarrow m}$ the set of nodes of $[\lambda]$ whose entries are less than or equal to m. If $\mathtt{T} \in \mathsf{Std}(\lambda)$, then $\mathtt{T}_{\downarrow m}$ is a standard tableau for some partition, which we call $\mathrm{Shp}(\mathtt{T}_{\downarrow m})$.

For any $\lambda \in \mathcal{P}_n$ and $\mathtt{T} \in \mathsf{Std}(\lambda)$ we define the *degree* $\deg \mathtt{T}$ of $\mathtt{T}$ as follows. If $n = 0$ then $\mathtt{T}$ is the unique $\varnothing$-tableau and we set $\deg \mathtt{T} := 0$. Otherwise, let $A = \mathtt{T}^{-1}(n) \in [\lambda]$ and suppose A is an i-node. We set inductively

$$\deg \mathtt{T} := \deg \mathtt{T}_{\downarrow n-1} + d_A(\lambda).$$

Example 2.6. If $\lambda = (3, 3, 1, 1)$, $\kappa = 1$ and $\ell = 2$, then λ has the following residue pattern

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 0 & 1 & 2 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

and if $\mathtt{T}$ is the tableau

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 8 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

then

$$\deg \mathtt{T} = 0 + 0 + 0 + (1 + 1) + 1 + 0 + 0 - 2 = 1.$$

Definition 2.7. We define the *content* of the partition λ to be

$$\mathrm{cont}(\lambda) = \sum_{A \in [\lambda]} \alpha_{\mathrm{res}(A)} \in \mathsf{Q}^+.$$

The *defect* of a partition λ is

$$\mathrm{def}(\lambda) := \mathrm{def}(\mathrm{cont}(\lambda)).$$

Recall from [17, § 12.6] that a weight μ of $V(\Lambda)$ is *maximal* if $\mu + \delta$ is not a weight of $V(\Lambda)$. Let $\max(\Lambda)$ be the set of all maximal weights of $V(\Lambda)$. The following is a generalisation of Ariki and Park's result for $\Lambda = \Lambda_0$ to $\Lambda = \Lambda_k$:

Proposition 2.8. *For the weight system of the $\mathfrak{g}(\mathsf{A})$-module $V(\Lambda_k)$ in type $C_\ell^{(1)}$, we have*

(1) $\max(\Lambda_k) \cap \mathsf{P}^+ = \{\Lambda_k + \xi_{k,\pm i} - \frac{i}{2}\delta \mid k \pm i \in I,\ i$ *is an even nonnegative integer*$\}$, *and*

(2) μ *is a weight of* $V(\Lambda_k)$ *if and only if* $\mu = w\eta - m\delta$ *for some* $w \in \mathsf{W}$, $\eta \in \max(\Lambda_k) \cap \mathsf{P}^+$ *and* $m \in \mathbb{Z}_{\geq 0}$.

Proof. The details here are similar to [7, Proof of Prop. 5.1]: Let $\mu \in \max(\Lambda_0) \cap \mathsf{P}^+$. Since $\{\xi_{k,\pm i} \mid k \pm i \in I\}$ satisfies eq. (2) and $\{\xi_{k,\pm i} \mid i \neq 0$ and $k \pm i \in I\}$ forms a basis of $\sum_{i \in I \setminus \{k\}} \mathbb{Q}\alpha_i$, by the same computations $\mu = \Lambda_k + \xi_{k,\pm i} + t\delta$ for some $i \neq 0$ such that $k \pm i \in I$ or $\mu = \Lambda_k + t\delta$. In the latter case, μ maximal implies that $\mu = \Lambda_k = \Lambda_k + \xi_{k,0}$. In the former case, $\Lambda_k - \mu \in \mathsf{Q}^+$ and (1) together imply that i is even.

Then, to show that $t = -\frac{i}{2}$, we need to consider the cases $\mu = \Lambda_k + \xi_{k,-i} + t\delta$ and $\mu = \Lambda_k + \xi_{k,i} + t\delta$ separately. For the case $\mu = \Lambda_k + \xi_{k,-i} + t\delta$, using the partition

$$\lambda(-i) = (\underbrace{\ell - k + i/2, \ell - k + i/2, \ldots, \ell - k + i/2}_{i})$$

in the Fock space $\mathcal{F}(\kappa)$ with $\Lambda_\kappa = \Lambda_k$ and considering its residue pattern we have the corresponding weight

$$\Lambda_k - \left(\frac{i}{2}\alpha_\ell + i(\alpha_{\ell-1} + \cdots + \alpha_k) + (i-1)\alpha_{k-1} + (i-2)\alpha_{k-2} \cdots + \alpha_{k-i+1}\right)$$
$$= \Lambda_k + \xi_{k,-i} - \frac{i}{2}\delta,$$

and so by Theorem 2.13 in the next section that $\dim R^{\Lambda_k}(\frac{i}{2}\delta - \xi_{k,-i}) \neq 0$ and so $\Lambda_k + \xi_{k,-i} - \frac{i}{2}\delta$ is a weight of $V(\Lambda_k)$. Furthermore, $\Lambda_k + \xi_{k,-i} - \frac{i}{2}\delta$ is maximal since $(-\xi_{k,-i} + \frac{i}{2}\delta) - \delta \notin \mathsf{Q}^+$.

For the case $\mu = \Lambda_k + \xi_{k,+i} + t\delta$, we use the partition

$$\lambda(+i) = (\underbrace{i, i, \ldots, i}_{k+i/2})$$

in the Fock space $\mathcal{F}(\kappa)$ with $\Lambda_\kappa = \Lambda_k$ instead, which has corresponding weight

$$\Lambda_k - \left(\frac{i}{2}\alpha_0 + i(\alpha_1 + \cdots + \alpha_k) + (i-1)\alpha_{k+1} + (i-2)\alpha_{k+2} \cdots + \alpha_{k+i-1}\right)$$
$$= \Lambda_k + \xi_{k,+i} - \frac{i}{2}\delta,$$

and so $\Lambda_k + \xi_{k,+i} - \frac{i}{2}\delta$ is a weight of $V(\Lambda_k)$; it is maximal since $(-\xi_{k,+i} + \frac{i}{2}\delta) - \delta \notin \mathsf{Q}^+$.

The second part is also argued similarly: $\max(\Lambda_k)$ is W-invariant by [17, Prop. 10.1] and furthermore $\max(\Lambda_k) = \mathsf{W}(\max(\Lambda_k) \cap \mathsf{P}^+)$ by [17, Cor. 10.1]. Thus, given any weight μ of $V(\Lambda_k)$, by [17, (12.6.1)] there exists a unique $\zeta \in \max(\Lambda_k)$ and a unique $m \in \mathbb{Z}_{\geq 0}$ such that $\mu = \zeta - m\delta$. □

2.3. **Quiver Hecke algebras.** Let $\mathbb{F}$ be an algebraically closed field and $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ the Cartan datum from Subsection 2.1. We set polynomials $\mathcal{Q}_{i,j}(u,v) \in \mathbb{F}[u,v]$, for $i, j \in I$, of the form

$$\mathcal{Q}_{i,j}(u,v) = \begin{cases} \sum_{p(\alpha_i|\alpha_i)+q(\alpha_j|\alpha_j)+2(\alpha_i|\alpha_j)=0} t_{i,j;p,q}u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;p,q} \in \mathbb{F}$ are such that $t_{i,j;-a_{ij},0} \neq 0$ and $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$. The symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \ldots, n-1 \rangle$ acts on I^n by place permutations.

Definition 2.9. The *cyclotomic quiver Hecke algebra* R_n^Λ associated with polynomials $(\mathcal{Q}_{i,j}(u,v))_{i,j\in I}$ and $\Lambda_k \in \mathsf{P}$ is the $\mathbb{Z}$-graded $\mathbb{F}$-algebra generated by three sets of generators

$$\{e(\nu) \mid \nu = (\nu_1, \ldots, \nu_n) \in I^n\},\ \{x_r \mid 1 \leq r \leq n\},\ \{\psi_j \mid 1 \leq j \leq n-1\}$$

subject to the following list of relations:

$$\begin{aligned}
e(\nu)e(\nu') &= \delta_{\nu,\nu'}e(\nu); \\
\sum_{\nu \in I^n} e(\nu) &= 1; \\
x_r e(\nu) &= e(\nu)x_r; \\
\psi_r e(\nu) &= e(s_r\nu)\psi_r; \\
x_r x_s &= x_s x_r; \\
\psi_r x_s &= x_s \psi_r && \text{if } s \neq r, r+1; \\
\psi_r \psi_s &= \psi_s \psi_r && \text{if } |r-s| > 1;
\end{aligned}$$

$$\begin{aligned}
x_r\psi_r e(\nu) &= (\psi_r x_{r+1} - \delta_{\nu_r,\nu_{r+1}})e(\nu);\\
x_{r+1}\psi_r e(\nu) &= (\psi_r x_r + \delta_{\nu_r,\nu_{r+1}})e(\nu);\\
\psi_r^2 e(\nu) &= Q_{\nu_r,\nu_{r+1}}(x_r, x_{r+1})e(\nu);\\
(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)e(\nu) &= \begin{cases} \frac{Q_{\nu_r,\nu_{r+1}}(x_r,x_{r+1})-Q_{\nu_r,\nu_{r+1}}(x_{r+2},x_{r+1})}{x_r-x_{r+2}}e(\nu) & \text{if } \nu_r = \nu_{r+2},\\ 0 & \text{otherwise;}\end{cases}
\end{aligned}$$

for all admissible r, s, ν, ν', and $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda\rangle} e(\nu) = 0$ for $\nu \in I^n$.

The algebra R_n^Λ is $\mathbb{Z}$-graded by setting

$$\deg(e(\nu)) = 0, \qquad \deg(x_r e(\nu)) = (\alpha_{\nu_r}, \alpha_{\nu_r}), \qquad \deg(\psi_s e(\nu)) = -(\alpha_{\nu_s}, \alpha_{\nu_{s+1}})$$

for all admissible r, s and ν.

For some $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Then $e(\beta) := \sum_{\nu\in I^\beta} e(\nu)$ is a central idempotent. We define $R^\Lambda(\beta) := R_n^\Lambda e(\beta)$, which is also an $\mathbb{F}$-algebra. It is clear that $R^\Lambda(\beta)$ may be defined by the same set of generators and relations if we replace I^n with I^β in both. The cyclotomic quiver Hecke algebra R_n^Λ can be decomposed into a direct sum of $\mathbb{F}$-algebras:

$$R_n^\Lambda = \bigoplus_{\substack{\beta\in\mathsf{Q}^+\\ \mathrm{ht}(\beta)=n}} R^\Lambda(\beta),$$

(see [8, § 2.1]) and we refer to $R^\Lambda(\beta)$ as a *block algebra* of R_n^Λ. In general, it is not known that $R^\Lambda(\beta)$ is indecomposable, but as we show later this is the case in level one when β has defect zero and one (Proposition 2.15 and Corollary 4.3 below). The block algebras for $\Lambda = \Lambda_0$ were referred to as *finite quiver Hecke algebras* in [7]. If we drop the relation $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda\rangle} e(\nu) = 0$ for $\nu \in I^\beta$, we obtain the *quiver Hecke algebra* $R(\beta)$.

We define the *defect* of a nonzero block algebra $R^\Lambda(\beta)$ to be the defect of $\beta \in \mathsf{Q}^+$.

Lemma 2.10. *The defect of a nonzero block algebra $R^{\Lambda_k}(\beta)$ is non-negative.*

Proof. By the categorification theorem, $R^\Lambda(\beta) \neq 0$ precisely when $\Lambda - \beta$ is a weight of $V(\Lambda)$. By Proposition 2.8, when $\Lambda = \Lambda_k$ this means that $\Lambda - \beta = w\Lambda + w\xi_{k,\pm i} - m\delta$ for $k \pm i \in I$, $m \geq i/2$ and $w \in \mathsf{W}$ and so $\beta = \Lambda - w\Lambda - w\xi_{k,\pm i} + m\delta$. By direct computation for $\beta \in \mathsf{Q}^+$, $\Lambda - w^{-1}\Lambda + w^{-1}\beta$ and β have the same defect and so the defect of $R^{\Lambda_k}(\beta)$ is $\mathrm{def}(\beta) = \mathrm{def}(m\delta - \xi_{k,\pm i}) = 2m - \frac{i}{2} \geq \frac{i}{2} \geq 0$ by Lemma 2.1. □

This is true in higher levels as well, details will appear in an upcoming paper [11].

In the rest of this section, we recall some important results which will be used in our proofs. Recall that by direct computation, $R^\Lambda(\beta)$ and $R^\Lambda(\Lambda - w\Lambda + w\beta)$ for $w \in \mathsf{W}$ have the same defect. In fact, a much stronger statement holds:

Proposition 2.11 (cf. [5, Cor. 4.8])**.** *For $w \in \mathsf{W}$, $R^\Lambda(\beta)$ and $R^\Lambda(\Lambda - w\Lambda + w\beta)$ are derived equivalent; furthermore, they have the same number of simple modules and the same representation type.*

We denote the direct sum of the split Grothendieck groups of the categories $R^\Lambda(\beta)\text{-}\mathrm{proj}$ of finitely generated projective graded $R^\Lambda(\beta)$-modules by

$$K_0(R^\Lambda) = \bigoplus_{\beta\in\mathsf{Q}^+} K_0(R^\Lambda(\beta)\text{-}\mathrm{proj}).$$

Note that $K_0(R^\Lambda)$ has a free $\mathbb{A}$-module structure induced from the $\mathbb{Z}$-grading on $R^\Lambda(\beta)$, i.e. $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$. Let $e(\nu, i)$ be the idempotent corresponding to the concatenation of ν and (i), and set $e(\beta, i) = \sum_{\nu\in I^\beta} e(\nu, i)$ for $\beta \in \mathsf{Q}^+$. Then we define the induction functor $F_i : R^\Lambda(\beta)\text{-mod} \to R^\Lambda(\beta+\alpha_i)\text{-mod}$ and the restriction functor $E_i : R^\Lambda(\beta+\alpha_i)\text{-mod} \to R^\Lambda(\beta)\text{-mod}$ by

$$F_i(M) = R^\Lambda(\beta+\alpha_i)e(\beta, i) \otimes_{R^\Lambda(\beta)} M, \qquad E_i(N) = e(\beta, i)N,$$

for an $R^\Lambda(\beta)$-module M and an $R^\Lambda(\beta+\alpha_i)$-module N.

Theorem 2.12 ([18, Thm. 5.2])**.** *Let $l_i = \langle \alpha_i^\vee, \Lambda - \beta\rangle$, for $i \in I$. Then one of the following isomorphisms of endofunctors on $R^\Lambda(\beta)$-mod holds.*

(1) If $l_i \geq 0$, then

$$q_i^{-2}F_iE_i \oplus \bigoplus_{k=0}^{l_i-1} q_i^{2k}\mathrm{id} \xrightarrow{\sim} E_iF_i.$$

(2) If $l_i \leq 0$, then

$$q_i^{-2}F_iE_i \xrightarrow{\sim} E_iF_i \oplus \bigoplus_{k=0}^{-l_i-1} q_i^{-2k-2}\mathrm{id}.$$

For $\nu \in I^n$, let

$$K_q(\lambda, \nu) := \sum_{\substack{\mathtt{T}\in\mathsf{Std}(\lambda)\\ \mathrm{res}(\mathtt{T})=\nu}} q^{\deg(\mathtt{T})}, \qquad K_q(\lambda) := \sum_{\mathtt{T}\in\mathsf{Std}(\lambda)} q^{\deg(\mathtt{T})}.$$

Theorem 2.13 ([8, Thm.2.5])**.** *For $\nu, \nu' \in I^\beta$, we have*

$$\dim_q e(\nu)R^\Lambda(\beta)e(\nu') = \sum_{\substack{\lambda\vdash n\\ \mathrm{wt}(\lambda)=\Lambda-\beta}} K_q(\lambda, \nu)K_q(\lambda, \nu'),$$

$$\dim_q R^\Lambda(\beta) = \sum_{\substack{\lambda\vdash n\\ \mathrm{wt}(\lambda)=\Lambda-\beta}} K_q(\lambda)^2,$$

$$\dim_q R_n^\Lambda = \sum_{\lambda\vdash n} K_q(\lambda)^2,$$

where $\dim_q M := \sum_{k\in\mathbb{Z}} \dim_{\mathbb{F}}(M_k)q^k$ for a free graded $\mathbb{F}$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$.

The statement below is an immediate consequence of the dimension formula:

Corollary 2.14 (cf. [7, Cor. 2.7])**.** *(1) Let $\nu \in I^n$. Then, $e(\nu) \neq 0$ in $R^{\Lambda_k}(n)$ if and only if ν may be obtained from a standard tableau $\mathtt{T}$ as $\nu = \mathrm{res}(\mathtt{T})$.*

(2) For a natural number n, we have $\dim R_n^{\Lambda_k} = n!$.

Proposition 2.15. *Block algebras of defect 0 are simple.*

Proof. By Lemma 2.1 and Propositions 2.8 and 2.11, a block algebra B of defect 0 is derived equivalent to $R_0^{\Lambda_k}$ which is simple. By [5, Proposition 4.2], these algebras are self-injective and so by Rickard's theorem [24, Theorem 2.1], the derived equivalence between them is in fact a stable equivalence. Since $R_0^{\Lambda_k}$ is a simple algebra, all of its modules are projective and its stable module category consists only of zero objects, hence the same is true for B.

Suppose that the unique indecomposable projective B-module is not simple. Then, the identity automorphism of the simple B-module does not factor through any projective B-module, so that it is a nonzero homomorphism in the stable module category of B, which is a contradiction. Hence, the unique simple B-module is projective-injective and therefore B is a simple algebra as well. $\square$

The next lemma will play a crucial role in our proofs.

Lemma 2.16 ([2, Lem.1.3]). *Suppose that $e = e_1 + e_2$ with $e_1^2 = e_1 \neq 0$, $e_2^2 = e_2 \neq 0$, $e_1 e_2 = e_2 e_1 = 0$ and*

$$\dim_q e_i R^{\Lambda_k}(\beta) e_j - \delta_{ij} - c_{ij} q^2 \in q^3 \mathbb{Z}_{\geq 0}[q],$$

for $i, j = 1, 2$. Then the quiver of $eR^{\Lambda_k}(\beta)e$ has two vertices 1 and 2, c_{ii} loops on the vertex i, for $i = 1, 2$, and there are at least c_{12} arrows and c_{21} reverse arrows between 1 and 2.

Lemma 2.17 ([2, Lem. 3.1]). *Let $\sigma : I \simeq I$ be a Dynkin automorphism, namely a bijective map that satisfies $a_{\sigma(i)\sigma(j)} = a_{ij}$, for $i, j \in I$. For $\beta = \sum_{i \in I} b_i \alpha_i \in \mathsf{Q}^+$ and $\Lambda = \sum_{i \in I} c_i \Lambda_i \in \mathsf{P}^+$, we define*

$$\sigma\beta = \sum_{i \in I} b_i \alpha_{\sigma(i)}, \quad \sigma\Lambda = \sum_{i \in I} c_i \Lambda_{\sigma(i)}.$$

Then, $R^{\Lambda}(\beta)$ defined with $Q_{ij}(u, v)$ is isomorphic to $R^{\sigma\Lambda}(\sigma\beta)$ defined with $Q'_{ij}(u, v)$ such that

$$Q'_{\sigma(i)\sigma(j)}(u, v) = Q_{ij}(u, v).$$

Corollary 2.18. *We have an isomorphism of algebras*

$$R^{\Lambda_k}(\beta)_{Q_{ij}} \cong R^{\Lambda_{\ell-k}}(\sigma\beta)_{Q'_{ij}} \cong R^{\Lambda_{\ell-k}}(\sigma\beta)_{Q_{ij}}$$

Proof. The first isomorphism follows by setting $\sigma(i) = \ell - i$ in Lemma 2.17; the second isomorphism holds by the arguments given in the discussion in [3, Lem. 2.2]. $\square$

Hence in order to determine the representation type of $R^{\Lambda_k}(\beta)$, it is enough to only consider $0 \leq k \leq \ell/2$ by application of the above isomorphism.

3. Representations of $R^{\Lambda_k}(\delta)$

In [7] in order to investigate the representation type of $R^{\Lambda_0}(\delta)$, they built on the work of [22] and constructed the irreducible $R^{\Lambda_0}(\delta)$-modules. However for $R^{\Lambda_k}(\delta)$, we only need to consider the projective modules and their radical series for $k = 1$ and all other cases can be easily proved using Lemma 2.16. Note that $R^{\Lambda_k}(\delta)$ has defect 2.

Lemma 3.1. *For $\ell = 2$, the algebra $R^{\Lambda_k}(\delta)$ has tame representation type.*

Proof. If $k = 0$ this is proved in [7, Thm. 3.7]. Suppose that $k \neq 0$. Then we may assume that $k = 1$ by Corollary 2.18 and we will prove the assertion by explicit construction of the indecomposable projective modules. We have that $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$ and $e(\nu) \neq 0 \in R^{\Lambda_1}(\delta)$ for the following four sequences:

$$\begin{aligned} e_1 &= e(1210) \\ e_2 &= e(1201) \\ e_2' &= e(1021) \\ e_3 &= e(1012). \end{aligned}$$

Using Theorem 2.13, we can easily calculate the q-dimensions of $e(\nu)R^{\Lambda_1}(\delta)R^{\Lambda_1}$:

$e(\nu)\backslash e(\nu')$	1210	1201	1021	1012
1210	$1+q^4$	q^2	q^2	$\cdot$
1201	q^2	$1+q^2+q^4$	$1+q^2+q^4$	q^2
1021	q^2	$1+q^2+q^4$	$1+q^2+q^4$	q^2
1012	$\cdot$	q^2	q^2	$1+q^4$

Let $A = R^{\Lambda_1}(\delta)$. Looking at the table above, we see that A is non-negatively graded, hence its radical consists of linear combinations of elements of positive degree and $A/\mathrm{Rad}(A)$ is semisimple. The basis of $A/\mathrm{Rad}(A)$ contains the degree zero elements:

$$A/\mathrm{Rad}(A) = \mathrm{span}\{e_1, e_2, e_2\psi_2 e_2', e_2'\psi_2 e_2, e_2', e_3\}.$$

Let

$$\begin{aligned}
\mathcal{P}_1 &:= R^{\Lambda_1}(\delta)e_1\\
\mathcal{P}_2 &:= R^{\Lambda_1}(\delta)e_2\\
\mathcal{P}_2' &:= R^{\Lambda_1}(\delta)e_2'\\
\mathcal{P}_3 &:= R^{\Lambda_1}(\delta)e_3
\end{aligned}$$

and define the simple modules $\mathcal{S}_i := \mathcal{P}_i/\mathrm{Rad}(\mathcal{P}_i)$. (Note that $\mathcal{S}_1$ and $\mathcal{S}_3$ are both one dimensional, while $\mathcal{S}_2$ and $\mathcal{S}_2'$ have dimension two.)

As for all $i = 1, 2, 3, 4$ we have that

$$\dim_q e_i A e_i = 1 + \text{higher order terms},$$

the algebras $\mathrm{End}_A(\mathcal{P}_i)$ are local, and we must have that $\mathcal{P}_i$ are indecomposable projective A-modules with simple heads.

Now, we will show that $\mathcal{P}_2 \cong \mathcal{P}_2'$ as left A-modules. Let $f : \mathcal{P}_2 \to \mathcal{P}_2'$ be given by $xe_2 \mapsto xe_2\alpha$ where $\alpha = e_2\psi_2 e_2'$. Because $f(e_2) \neq 0$ and it has degree 0, we conclude that $f(e_2) \notin \mathrm{Rad}(\mathcal{P}_2')$. Thus, it generates $\mathcal{S}_2'$ and f is surjective. Finally, as $\dim \mathcal{P}_2 = \dim \mathcal{P}_2'$, we see that f is indeed an isomorphism of A-modules.

Let $e = e_1 + e_2 + e_3$. By looking at the q-dimensions of $e(\nu)Ae(\nu')$, we see that $\mathcal{P}_i \not\cong \mathcal{P}_j$ for $1 \le i \neq j \le 3$, hence eAe is the basic algebra of A and all simple eAe-modules are one dimensional. We have the following radical series for eAe (as the corresponding module categories of eAe and A are Morita equivalent, abusing notation, we write $\mathcal{P}_i$ and $\mathcal{S}_i$ instead of $e\mathcal{P}_i$ and $e\mathcal{S}_i$):

$$\mathcal{P}_1 \simeq \begin{array}{c}\mathcal{S}_1\\ \mathcal{S}_2\\ \mathcal{S}_1\end{array} \qquad \mathcal{P}_2 \simeq \begin{array}{c}\mathcal{S}_2\\ \mathcal{S}_1\oplus\mathcal{S}_2\oplus\mathcal{S}_3\\ \mathcal{S}_2\end{array} \qquad \mathcal{P}_3 \simeq \begin{array}{c}\mathcal{S}_3\\ \mathcal{S}_2\\ \mathcal{S}_3.\end{array}$$

Thus $eAe \cong \mathbb{F}Q/\mathcal{I}$ where the ideal $\mathcal{I}$ is given by the relations $\alpha_1\alpha_2 = 0 = \beta_2\beta_1, \alpha_1\gamma = 0 = \gamma\beta_1, \gamma\alpha_2 = 0 = \beta_2\gamma, \alpha_2\beta_2 = \gamma^2 = \beta_1\alpha_1$ and the quiver Q has the following form:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] \arrow[loop above, "\gamma"] & 3 \arrow[l, shift left, "\beta_2"] \end{tikzcd}$$

Hence $R^{\Lambda_1}(\delta)$ is of tame representation type by [4, Thm. 1(5)]. $\square$

Lemma 3.2. *For $\ell \geq 3$ and $0 \le k \le \ell$, the algebra $R^{\Lambda_k}(\delta)$ has wild representation type.*

Proof. If $k = 0$, the result follows by [7, Thm. 3.7].

Let $A = eR^{\Lambda_k}(\delta)e$ where $e = e_1 + e_2$. For $\ell = 3$, we set:

$$\begin{aligned} e_1 &= e(123201) \\ e_2 &= e(101232). \end{aligned}$$

Otherwise, we choose

$$\begin{aligned} e_1 &= e(k, k+1, \ldots, \ell, \ldots, k+1, k-1, k, k-2, k-2, \ldots, 0, \ldots, k-1) \\ e_2 &= e(k, k+1, \ldots, \ell, \ldots, k+3, k+2, k-1, k-2, \ldots, 0, \ldots, k+1). \end{aligned}$$

Using Theorem 2.13, we compute the graded dimensions for any $1 \leq k \leq \ell/2$:

$$\begin{aligned} \dim_q e_i R^{\Lambda_k}(\delta) e_i &= 1 + c_{i,\ell} q^2 + q^4; \\ \dim_q e_i R^{\Lambda_k}(\delta) e_j &= \dim_q e_j R^{\Lambda_k}(\beta) e_i = q^2, \end{aligned}$$

where

$$c_{i,\ell} = \begin{cases} 1 & \text{if} \quad \ell = 3, i = 1 \\ 2 & \text{otherwise.} \end{cases}$$

As $c_{i,\ell} > 0$, we have that $eR^{\Lambda_k}(\delta)e_1$ and $eR^{\Lambda_k}(\delta)e_2$ are pairwise non-isomorphic, indecomposable projective A-modules and in particular that $R^{\Lambda_k}(\delta)$ is of wild representation type for $\ell \geq 3$ by Lemma 2.16 and [12, I.10.8(iv)]. □

4. Representations of $R^{\Lambda_k}(\delta - \xi_{k,\pm 2})$

In this section we see that blocks of defect one have finite representation type, and moreover that they are equivalent to a Brauer tree algebra. We also demonstrate that depending on k, there are two distinct possibilities for the number of simple modules in block algebras of defect one. Note that every block algebra of defect one is derived equivalent to $R^{\Lambda_k}(\delta - \xi_{k,\pm i})$ by Lemma 2.1 and Propositions 2.8 and 2.11, where

$$\begin{aligned} \delta - \xi_{k,2} &= \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_k + \alpha_{k+1}, \text{ and} \\ \delta - \xi_{k,-2} &= \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{\ell-1} + \alpha_\ell \end{aligned}$$

for $k \pm 2 \in I$.

Proposition 4.1. *$R^{\Lambda_k}(\delta - \xi_{k,\pm 2})$ is of finite representation type.*

Proof. First consider the case for $\delta - \xi_{k,2}$. The partitions belonging to the block algebra $R^{\Lambda_k}(\delta - \xi_{k,2})$ are $\lambda(i) = (2^{k-i+1}, 1^{2i})$ for $0 \leq i \leq k+1$.

The partition $\lambda(0) = (2^{k+1})$ is not Kleshchev since it has only one removable 1-node, which is not a good node as it is immediately followed by an addable 1-node (when reading from top to bottom), and hence does not show up in the reduced 1-signature, and so (2^{k+1}) cannot correspond to a vertex in the crystal graph. The remaining partitions are all Kleshchev since each of them has a good (removable) node, and we may traverse against the directed edges on the crystal graph by continuing to remove good nodes until the empty partition $\varnothing_k$ is reached. In more detail, starting from each partition we can move up the crystal graph as follows: the first column of $\lambda(i)$ has residue $(k, k-1, \ldots, 1, 0, 1, \ldots i)$ and the second column has residue $(k+1, k, \ldots, i+1)$ and so $\lambda(i)$ has a good i-node. The partition obtained by removing that good node has itself a good $(i-1)$-node, and the partition obtained from by removing that good node has a good $(i-2)$-node, and so on until the partition (2^{k-i+1}) is reached. Here, we have a unique removable node with residue $i+1$, and since $i > 0$ that node is good and can be removed to move up the crystal graph. Continuing as before, the remainder of the second column, and subsequently the rest of the first column, can be removed by a sequence of good nodes until the empty partition $\varnothing_k$ is reached. This is by no means the only possible path back to $\varnothing_k$, but to show that $\lambda(i)$ is Kleshchev for $1 \le i \le k+1$ it suffices to exhibit one such path.

Let e_i be the idempotent corresponding to the residue sequence for the initial tableau for each Kleshchev partition $\lambda(i)$ for $0 \le i \le k$; in particular $e_i = e(\nu(i))$ where

$$\nu(0) = (k, k-1, \dots, 1, 0, 1, \dots, k, k+1)$$

and for $1 \le i \le k$, the residue sequence $\nu(i)$ is

$$\nu(i) = (k, k+1, k-1, k, \dots, k-i+1, k-i+2, k-i, k-i-1, \dots, 1, 0, 1, \dots, k-i+1)$$

From Theorem 2.13 above, we have the following graded dimensions:

$$\dim_q e_i R^{\Lambda_k}(\delta - \xi_{k,2}) e_j = \begin{cases} 1+q^2 & \text{if } j = i, \\ q & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\mathrm{Rad} R^{\Lambda_k}(\delta - \xi_{k,2})$ is spanned by homogeneous elements of positive degree, and $P_i = R^{\Lambda_k}(\delta - \xi_{k,2}) e_i$ are indecomposable and pairwise non-isomorphic projective modules and hence we can conclude that the radical series of P_i is given by

$$P_0 = \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix}, \quad P_i = \begin{matrix} S_i \\ S_{i-1} \oplus S_{i+1} \\ S_i \end{matrix} \quad (1 \le i \le k), \quad P_k = \begin{matrix} S_k \\ S_{k-1} \\ S_k \end{matrix}$$

where $S_i = \mathrm{Top}(P_i)$, and so $R^{\Lambda_k}(\delta - \xi_{k,2})$ is Morita equivalent to a Brauer tree algebra whose Brauer tree is a straight line (cf. [2, Prop. 5.1]). Moreover, the idempotents $e_1, e_2, \dots, e_k$ give a complete list of the pairwise non-isomorphic primitive idempotents of $R^{\Lambda_k}(\delta - \xi_{k,2})$.

For the case $\delta - \xi_{k,-2}$, we can apply the same argument where the partitions belonging to $R^{\Lambda_k}(\delta - \xi_{k,-2})$ are $\mu(i) = (\ell - k + i + 1, \ell - k - i + 1)$ for $0 \le i \le \ell - k + 1$, all Kleshchev except when $i = \ell - k + 1$ (by a similar reasoning to the previous case) and so in this case the idempotents correspond to the residue sequence

$$\upsilon(i) = (k, k+1, \dots, \ell - 1, \ell, \ell - 1, \dots, \ell - i, k-1, k, \dots, \ell - i - 1),$$

for $0 \le i \le \ell - k$ instead. □

Corollary 4.2. *When $k \ne \ell/2$ and $2 \le k \le \ell - 2$, there are two inequivalent Morita equivalence classes of blocks of defect one with distinct number of simple modules $k+1$ or $\ell - k + 1$ respectively.*

Proof. From the proof of Proposition 4.1, we see that $R^{\Lambda_k}(\delta - \xi_{k,-2})$ has $k+1$ simple modules and $R^{\Lambda_k}(\delta - \xi_{k,+2})$ has $\ell - k + 1$ simple modules. By Lemma 2.1 and Propositions 2.8 and 2.11 all blocks of defect one have the same number of simple modules as either one of these two maximal weight cases. □

In [8], the authors construct for cyclotomic KLR algebras of type $C_\ell^{(1)}$ the Specht modules S^λ where λ is a (multi)partition. The following corollary describes the graded decomposition matrices for blocks of defect one in level one, $\Lambda = \Lambda_k$.

Corollary 4.3. *Let $m \in \{k+1, \ell - k + 1\}$ be the number of simple modules for a block algebra of defect one. Then the graded decomposition multiplicity is*

$$[S^{\lambda(i)} : D^{\lambda(j)}]_q = \begin{cases} 1 & \text{if } i = j, \\ q & \text{if } i = j+1 \text{ for } 1 \le i \le m+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda(i)$ is as in the proof of Proposition 4.1 above when $m = k+1$, and $\lambda(i) = \mu(i)$ when $m = \ell - k + 1$.

Proof. From the proof of Proposition 4.1, the ungraded decomposition multiplicities are $[S^{\lambda(i)} : D^{\lambda(j)}] = 1$ when $i = j$ or $i = j+1$ or $1 \leq i \leq m+1$. By Corollary 6E.18 of [14] when the defect d is 1, the bottom-most nonzero entry of each column of the graded decomposition matrix is $[S^{\lambda(i)} : D^{\lambda(i-1)}]_q = q^d = q$ for $1 \leq i \leq m+1$. □

We note here that $R^{\Lambda_k}(\delta - \xi_{k,\pm 2})$ is an indecomposable algebra.

Theorem 4.4. *Let B be a block algebra of defect 1. Then B is of finite representation type and moreover B is a Brauer tree algebra whose Brauer tree is a straight line with no exceptional vertex.*

Proof. By Lemma 2.1 and Propositions 2.8, 2.11 and 4.1, B is of finite type and is derived equivalent to $R^{\Lambda_k}(\delta - \xi_{k,\pm 2})$, which is a Brauer line algebra. Moreover, derived equivalence preserves Hochschild homology so in particular, the zeroth Hochschild homology i.e. the center is preserved and hence B is indecomposable as well. Furthermore, B is (graded) cellular by [14, Thm. A] and symmetric by [14, §4E]. Hence, applying Ohmatsu's theorem and by the argument in [2, §8.2], we conclude that B is a Brauer tree algebra whose tree is a straight line with no exceptional vertex. □

Remark. Thus we see that the algebra $R^{\Lambda_k}(\beta)$ is indecomposable if it has defect 0 or 1. We expect this to be true for any $R^{\Lambda}(\beta)$ in any defect as well.

5. Representations of $R^{\Lambda_k}(2\delta - \xi_{k,\pm 4})$

Note that every block algebra of defect 2 is derived equivalent to $R^{\Lambda_k}(\delta)$ or $R^{\Lambda_k}(2\delta - \xi_{k,\pm 4})$. Observe that

$$\beta := 2\delta - \xi_{k,4} = 2\alpha_0 + 4\sum_{1}^{k} \alpha_j + 3\alpha_{k+1} + 2\alpha_{k+2} + \alpha_{k+3}, \text{ and}$$

$$\gamma := 2\delta - \xi_{k,-4} = \alpha_{k-3} + 2\alpha_{k-2} + 3\alpha_{k-1} + 4\sum_{k}^{\ell-1} \alpha_j + 2\alpha_\ell.$$

Lemma 5.1. *For $k \pm 4 \in I$, the algebra $R^{\Lambda_k}(2\delta - \xi_{k,\pm 4})$ has wild representation type.*

Proof. If $k = 0$, this is proved in [7, Thm. 4.2]. Suppose $k \neq 0$ and $k \leq \ell/2$. Then, $k+4 \in I$ implies that $\ell > 4$ and $k-4 \in I$ implies that $\ell \geq 2k \geq 8$ and so we may assume that $\ell > 4$. Just as in the proof of Lemma 3.2, we set $e = e_1 + e_2$. First, we consider the algebra $A = eR^{\Lambda_k}(\beta)e$ where we choose

$$\begin{aligned} e_1 =&(k, k-1, \dots, 0, \dots, k+3, k+1, k+2, k, \dots, 0, \dots, k+1) \\ e_2 =&(k, k+1, k+2, k+3, k-1, k-2, \dots, 0, \dots, k+2, \\ &k, k+1, k-1, k-2, \dots, 0, \dots, k). \end{aligned}$$

For shorthand notation, let $\underline{i} := (i, i-1, i-2, i-3)$. Let $B = eR^{\Lambda_k}(\gamma)e$ and set

$$\begin{aligned} e_1 =&(\underline{k}, \underline{k+1}, \dots, \underline{\ell}, \ell-1, \ell-2, \ell, \ell-1) \\ e_2 =&(\underline{k}, \underline{k+1}, \dots, \underline{k+d-2}, \ell-1, \ell, \ell-1, \ell-2, \ell-3, \\ &\ell-4, \ell-2, \ell-3, \ell-1, \ell, \ell-1, \ell-2). \end{aligned}$$

Then using Theorem 2.13, we can compute the graded dimensions for $\kappa = \beta, \gamma$ and any $k > 0$:

$$\begin{aligned} \dim_q e_i R^{\Lambda_k}(\kappa) e_i &= 1 + 2q^2 + q^4; \\ \dim_q e_i R^{\Lambda_k}(\kappa) e_j &= \dim_q e_j R^{\Lambda_k}(\beta) e_i = q^2. \end{aligned}$$

Thus $R^{\Lambda_k}(\kappa)$ is wild by the same reasoning as in Lemma 3.2. □

6. Representations of $R^{\Lambda_k}(\beta)$

In this final section, we prove two important lemmas that will enable us to generalise our results for all $R^{\Lambda_k}(\beta)$. After stating the main theorem, we also rewrite it in terms of defect and compare it with the original statement of Erdmann–Nakano.

Lemma 6.1 ([13, Prop. 2.3], [6, Rem. 5.10])**.** *Let A and B be finite dimensional $\mathbb{F}$-algebras and suppose that there exists a constant $C > 0$ and functors*

$$F: \ A\text{-mod} \to B\text{-mod}, \quad G: \ B\text{-mod} \to A\text{-mod}$$

such that, for any A-module M,

(1) M is a direct summand of $GF(M)$ as an A-module,
(2) $\dim F(M) \le C \dim M$.

Then, if A is wild, so is B.

The next lemma is the analogous statement of [8, Lem. 5.3] for Λ_k.

Lemma 6.2. *(1) If $R^{\Lambda_k}(\beta - \alpha_j)$ is wild and $\langle h_j, \Lambda_k - \beta + \alpha_j\rangle \ge 1$, then $R^{\Lambda_k}(\beta)$ is wild.*
(2) Suppose that $R^{\Lambda_k}(n\delta - \xi_{k,\pm i})$ is wild. Then
(a) $R^{\Lambda_k}((n+1)\delta - \xi_{k,\pm i})$ is wild,
(b) if $k \pm (i+2) \in I$, then $R^{\Lambda_k}((n+1)\delta - \xi_{k,\pm(i+2)})$ is wild.

Proof. (1) Considering the functors

$$F_j : R^{\Lambda_k}(\beta - \alpha_j)\text{-mod} \to R^{\Lambda_k}(\beta)\text{-mod}, \quad E_j : R^{\Lambda_k}(\beta)\text{-mod} \to R^{\Lambda_k}(\beta - \alpha_j)\text{-mod},$$

the statement follows from Lemma 6.1 and Theorem 2.12.

(2) First, we will consider $R^{\Lambda_k}((n+1)\delta - \xi_{k,i})$. Notice that

$$\begin{aligned}\Lambda_k + \xi_{k,i+2} - (n+1)\delta &= \Lambda_k + \xi_{k,i} + \alpha_{k+i+1} + 2\sum_{j=k+i+2}^{\ell-1} \alpha_j + \alpha_\ell - (n+1)\delta \\ &= \Lambda_k + \xi_{k,i} - \alpha_0 - 2\sum_{j=1}^{k+i} \alpha_j - \alpha_{k+i+1} - n\delta.\end{aligned}$$

For $0 \le k+i \le \ell - 1$ and $n \in \mathbb{Z}_{\ge 0}$, we compute

$$\begin{aligned}\Lambda_k + \xi_{k,i+2} - (n+1)\delta + \alpha_{k+i+1} &= r_{k+i} r_{k+i-1} \dots r_1 r_0 r_1 \dots r_{k+i}(\Lambda_k + \xi_{k,i} - n\delta), \\ \Lambda_k + \xi_{k,i} - (n+1)\delta + \alpha_\ell &= r_{\ell-1} r_{\ell-2} \dots r_1 r_0 r_1 \dots r_{k+i}(\Lambda_k + \xi_{k,i} - n\delta).\end{aligned}$$

Hence $R^{\Lambda_k}((n+1)\delta - \xi_{k,i+2} - \alpha_{k+i+1})$ and $R^{\Lambda_k}((n+1)\delta - \xi_{k,i} - \alpha_\ell)$ are wild by Proposition 2.11 and the assumption that $R^{\Lambda_k}(n\delta - \xi_{k,\pm i})$ is. Moreover, we also have that

$$\begin{aligned}&\langle h_{k+i+1}, \Lambda_k + \xi_{k,i+2} - (n+1)\delta + \alpha_{k+i+1}\rangle = 2, \\ &\langle h_\ell, \Lambda_k + \xi_{k,i} - (n+1)\delta + \alpha_\ell\rangle = 2,\end{aligned}$$

and now we apply (1) to arrive at the desired conclusion.

Similarly, for $k + i = \ell$ we need to consider

$$\Lambda_k + \xi_{k,i} - (n+1)\delta + \alpha_0 = r_1 r_2 \cdots r_\ell(\Lambda_k + \xi_{k,i} - n\delta).$$

and we also have to check

$$\langle h_0, \Lambda_k + \xi_{k,i} - (n+1)\delta + \alpha_0\rangle = 2.$$

Using Proposition 2.11 and (1) again, (2) follows for $R^{\Lambda_k}(n\delta - \xi_{k,i})$.

Next, we look at $R^{\Lambda_k}((n+1)\delta - \xi_{k,-i})$. Here we note that

$$\begin{aligned}\Lambda_k + \xi_{k,-(i+2)} - (n+1)\delta &= \Lambda_k + \xi_{k,-i} + \alpha_0 + 2\sum_{j=1}^{k-(i+2)} \alpha_j + \alpha_{k-(i+1)} - (n+1)\delta \\ &= \Lambda_k + \xi_{k,-i} - \alpha_{k-(i+1)} - 2\sum_{j=k-(i+2)}^{\ell-1} \alpha_j - \alpha_\ell - n\delta.\end{aligned}$$

The proof is essentially the same as for $k+(i+2)$, but in this case the Weyl group generators r_i will act in reverse order. For $1 \le k-i \le \ell$ and $n \in \mathbb{Z}_{\ge 0}$, we compute

$$\begin{gathered}\Lambda_k + \xi_{k,-(i+2)} - (n+1)\delta + \alpha_{k-(i-1)} = r_{k-i}r_{k-(i-1)}\dots r_{\ell-1}r_\ell r_{\ell-1}\cdots r_{k-i}(\Lambda_k + \xi_{k,-i} - n\delta), \\ \Lambda_k + \xi_{k,-i} - (n+1)\delta + \alpha_0 = r_1 r_2 \dots r_{\ell-1}r_\ell r_{\ell-1}\dots r_{k-i}(\Lambda_k + \xi_{k,-i} - n\delta).\end{gathered}$$

Moreover, we also have that

$$\begin{aligned}&\langle h_{k-(i+1)}, \Lambda_k + \xi_{k,-(i+2)} - (n+1)\delta + \alpha_{k-(i-1)}\rangle = 2, \\ &\langle h_0, \Lambda_k + \xi_{k,-i} - (n+1)\delta + \alpha_0\rangle = 2.\end{aligned}$$

Finally, for $k-i=0$ we need to consider

$$\Lambda_k + \xi_{k,-k} - (n+1)\delta + \alpha_\ell = r_{\ell-1}r_{\ell-2}\cdots r_1 r_0(\Lambda_k + \xi_{k,-k} - n\delta).$$

and we also have to check

$$\langle h_\ell, \Lambda_k + \xi_{k,-i} - (n+1)\delta + \alpha_\ell\rangle = 2.$$

Thus for $R^{\Lambda_k}(n\delta - \xi_{k,-i})$ (2) follows by the same reasoning as for $k+(i+2)$. □

Theorem 6.3. *The algebra $R^{\Lambda_k}(2\delta)$ is wild.*

Proof. If $\ell \ge 3$, the statement follows from Lemma 3.2 by applying Lemma 6.2 with $i=0$. Now assume $\ell = 2$. Let $e = e_1 + e_2$ and consider $A = eR^{\Lambda_1}(2\delta)e$ where

$$e_1 = e(10121012) \qquad \text{and} \qquad e_2 = e(12012101).$$

Then we have the following graded dimensions:

$$\begin{aligned}&\dim_q e_1 R^{\Lambda_1}(2\delta)e_1 = 1 + 2q^2 + 2q^4 + 2q^6 + q^8 \\ &\dim_q e_2 R^{\Lambda_1}(2\delta)e_2 = 1 + 2q^2 + 3q^4 + 2q^6 + q^8 \\ &\dim_q e_1 R^{\Lambda_1}(2\delta)e_2 = \dim_q e_2 R^{\Lambda_1}(2\delta)e_1 = q^2 + 2q^4 + q^6.\end{aligned}$$

Then $R^{\Lambda_1}(2\delta)$ is wild by the same reasoning as in Lemma 3.2. □

Theorem 6.4. *The algebra $R^{\Lambda_k}(2\delta - \xi_{k,\pm 2})$ is wild.*

Proof. If $k=0$, this is proved in [7, Lem. 5.4]. Suppose $k \ne 0$ and $k \le \ell/2$. Then $k+2 \le \ell$ implies that $\ell \ge 3$ and $k-2 \in I$ implies that $\ell \ge 2k \ge 4$ and so we may assume that $\ell \ge 3$. Notice that for $1 \le k \le \ell-1$ we have that

$$2\delta - \xi_{k,2} = \delta + \alpha_0 + 2\sum_{j=1}^{k} \alpha_j + \alpha_{k+1}$$

$$2\delta - \xi_{k,-2} = \delta + \alpha_{k-1} + 2\sum_{j=k}^{\ell-1} \alpha_j + \alpha_\ell.$$

By Lemma 3.2, we already know that $R^{\Lambda_k}(\delta)$ is wild and we also see that

$$\langle h_k, \Lambda_k - \delta\rangle, \langle h_{k+1}, \Lambda_k - \delta - \alpha_k\rangle, \dots, \langle h_{\ell-1}, \Lambda_k - \delta - \alpha_k - \cdots - \alpha_{\ell-2}\rangle,$$
$$\langle h_\ell, \Lambda_k - \delta - \alpha_k - \cdots - \alpha_{\ell-1}\rangle, \langle h_{k-1}, \Lambda_k - \delta - \alpha_k - \cdots - \alpha_\ell\rangle$$

are all positive, so we have that $R^{\Lambda_k}(\delta + \alpha_{k-1} + \alpha_k + \cdots + \alpha_\ell)$ is also wild by Lemma 6.2. Finally, direct computation shows that

$$\begin{aligned}\Lambda_k - (\delta + \alpha_{k-1} + 2\sum_{j=k}^{\ell-1}\alpha_j + \alpha_\ell) &= \Lambda_k - \delta - \alpha_{k-1} - 2\alpha_k - 2\alpha_{k+1} - \cdots - 2\alpha_{\ell-1} - \alpha_\ell \\ &= r_{\ell-1}r_{\ell-2}\dots r_k(\Lambda_k - \delta - \alpha_{k-1} - \alpha_k - \cdots - \alpha_\ell).\end{aligned}$$

Hence $R^{\Lambda_k}(2\delta - \xi_{k,-2})$ is wild.

Similarly, for $2\delta - \xi_{k,2}$ it is easy to see that

$$\langle h_k, \Lambda_k - \delta\rangle, \langle h_{k-1}, \Lambda_k - \delta - \alpha_k\rangle, \dots, \langle h_1, \Lambda_k - \delta - \alpha_2 - \cdots - \alpha_k\rangle,$$
$$\langle h_0, \Lambda_k - \delta - \alpha_1 - \cdots - \alpha_k\rangle, \langle h_{k+1}, \Lambda_k - \delta - \alpha_0 - \cdots - \alpha_k\rangle$$

are all positive, thus the algebra $R^{\Lambda_k}(\delta + \alpha_0 + \alpha_1 + \cdots + \alpha_k)$ is wild. Using direct computation again, we have that

$$\begin{aligned}&(\Lambda_k - \delta - \alpha_0 - \alpha_1 - 2\alpha_2 - \cdots - 2\alpha_k - \alpha_{k+1}) \\ &\qquad = r_2r_3\dots r_k(\Lambda_k - \delta - \alpha_0 - \alpha_1 - \cdots - \alpha_{k+1})\end{aligned}$$

and that

$$\langle h_1, \Lambda_k - \delta - \alpha_0 - \alpha_1 + 2\sum_{j=2}^{k}\alpha_j + \alpha_{k+1}\rangle = 2$$

thus $R^{\Lambda_k}(2\delta - \xi_{k,2})$ is wild by Lemma 6.2 and we have proved the statement. □

Theorem 6.5. *Let $0 \le k+i \le \ell$ or $0 \le k-i \le \ell$ for some even $i \in I$. For $\beta \in \mathsf{W}(\Lambda_k + \xi_{k,\pm i})$ and $m \ge i/2$, the block algebra $R^{\Lambda_k}(\Lambda_k - \beta + m\delta)$ of type $C_\ell^{(1)}$ is*

(1) a simple algebra if $i = m = 0$;
(2) of finite representation type if $m = 1$ and $i = 2$;
(3) of tame representation type if $i = 0$, $m = 1$ and $\ell = 2$; and
(4) of wild representation type otherwise.

Proof. If $k = 0$ or ℓ, the result follows by [7]. If $1 \le k \le \ell - 1$, (1) comes from Proposition 2.15, (2) comes from Theorem 4.4, (3) is proved in Lemma 3.1 and (4) follows by Lemma 3.2 and applying Lemma 6.2 to Lemma 5.1 and Theorems 6.3 and 6.4. □

Remark. We note here that we can phrase things in terms of defect. Let d be the defect of $R^{\Lambda_k}(\Lambda_k - \beta + m\delta)$ for $m \ge i/2$, $\beta \in \mathsf{W}(\Lambda_k + \xi_{k,\pm i})$ with even $i \in I$ such that $0 \le k+i \le \ell$ or $0 \le k - i \le \ell$. Then the block algebra $R^{\Lambda_k}(\Lambda_k - \beta + m\delta)$ of type $C_\ell^{(1)}$ is

(1) a simple algebra if $d = 0$;
(2) of finite representation type if $d = 1$;
(3) of tame representation type if $\ell = 2$ and $d = 2$; and
(4) of wild representation type otherwise.

Remark. We also summarise the results of Ariki–Ijima–Park, Ariki–Park [3, 5, 7, 6] and ours on the representation type of block algebras in level one in terms of defect. Let d denote the defect of the respective block algebra.

- If $\Lambda = \Lambda_0$ in type $A_{2\ell}^{(2)}$, $d = 0$ implies simple, $d = 1$ implies finite, wild otherwise (tame representation type does not occur here).

- If $\Lambda = \Lambda_0$ in type $D^{(2)}_{\ell+1}$, $d = 0$ implies simple, $d = 1$ implies finite, $d = 2$ implies tame, wild otherwise.
- If $\Lambda = \Lambda_k$ in type $A^{(1)}_\ell$, $d = 0$ implies simple, $d = 1$ implies finite, $d = 2$ and $\ell = 1$ implies tame, wild otherwise. (This follows from the fact that in type $A^{(1)}_\ell$, $R^{\Lambda_0}(\beta) \cong R^{\Lambda_k}(\beta)$ for any $0 \leq k \leq \ell$.)
- If $\Lambda = \Lambda_k$ in type $C^{(1)}_\ell$, $d = 0$ implies simple, $d = 1$ implies finite, $d = 2$ and $\ell = 2$ implies tame, wild otherwise.

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