4-MOVE INEQUIVALENT HANDLEBODY-LINKS AND f-TWISTED ALEXANDER MATRICES

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Abstract. A 4-move is a local move of links replacing two parallel arcs with 4 half twists. The notion of 4-moves can be extended to handlebody-links naturally. In this paper, we detect 4-move inequivalent handlebody-links by using Alexander type invariants obtained from an f-twisted Alexander matrix, which is defined by way of a derivative for multiple conjugation quandles. We give a link-homotopically trivial handlebody-link which cannot be reduced to a trivial handlebody-link by 4-moves.

1. Introduction

A k-move is a well-known local move for classical links replacing two parallel arcs with k half twists, which may reduce a link to a trivial link in some cases. A 2-move is identical with a crossing change, which is an unknotting operation. The Montessinos–Nakanishi 3-move conjecture [19] stated that any link can be reduced to a trivial link by 3-moves, but it was refuted in [8]. The Nakanishi 4-move conjecture [19, 26] states that any knot can be reduced to the trivial knot by 4-moves, and it remains as an open problem. As a generalization of this conjecture, it was expected that if two links are link-homotopic, that is, one can be obtained from the other by self-crossing changes, then they can be transformed into each other by 4-moves, but Dabkowski and Przytycki [9] resolved this conjecture in the negative by constructing a three component link-homotopically trivial link which can not be reduced to a trivial link by 4-moves. Behavior of 4-moves for classical links has been studied in, for example, [2, 7, 10, 18, 27, 28, 29, etc.].

The notion of k-moves for classical links can be extended to handlebody-links naturally. A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere, which is a generalization of a classical link to higher genera. A handlebody-link can be also regarded as a quotient structure of a spatial graph. In the same as classical links, a 2-move is an unknotting operation for handlebody-links. There have been some studies of crossing changes of handlebody-links in [1, 17, 23], for example. However, unlike the case of classical links, properties of k-moves for handlebody-links have not been studied well yet.

A quandle [20, 21] is an algebra whose axioms correspond to the Reidemeister moves for links. A quandle yields various invariants for links such as quandle coloring numbers, quandle cocycle invariants [6] and so on. A multiple conjugation quandle (MCQ) [12] is an algebra whose axioms correspond to the Reidemeister moves for handlebody-links.

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As same as a quandle, an MCQ yields various invariants for handlebody-links such as MCQ coloring numbers, MCQ cocycle invariants [5] and so on. The author [24] introduced a pair of maps called an MCQ Alexander pair and showed that any linear extension of an MCQ can be realized by using it. Using an MCQ Alexander pair f, Ishii and the author [16] defined the f-twisted Alexander matrix, which produces some Alexander type invariants of handlebody-links. In this paper, we show that these invariants obtained from a certain MCQ Alexander pair detect 4-move inequivalences of handlebody-links. We give a link-homotopically trivial handlebody-link which can not be reduced to a trivial handlebody-link by 4-moves.

This paper is organized as follows. In Section 2, we introduce k-moves of handlebody-links and some facts briefly. In Section 3, we recall the notions of a multiple conjugation quandle (MCQ) and an MCQ Alexander pair. We see an example of an MCQ Alexander pair used in the main theorem in Section 6. In Section 4, we recall the notions of an MCQ presentation and the fundamental MCQ of a handlebody-link, which is an invariant of handlebody-links. In Section 5, we review the f-twisted Alexander matrix, which provides Alexander type invariants of handlebody-links, with an MCQ Alexander pair f. In Section 6, we introduce some approaches to detect k-move inequivalences of handlebody-links and show that the invariants defined in [16] (described in Section 5) can detect 4-move inequivalences of them. We prove that a certain link-homotopically trivial handlebody-link is not 4-move equivalent to any trivial handlebody-link.

2. Handlebody-links and k-moves

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. A handlebody-knot is *trivial* if its exterior is a handlebody. An n-component handlebody-link is *trivial* if there exist disjoint n 3-balls in S^3 whose each component contains a trivial handlebody-knot. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 sending one to the other.

A k*-move* is a local move on handlebody-links as illustrated in Fig. 1. Two handlebody-links are k*-move equivalent* if they are related by a finite sequence of k-moves and isotopies of S^3. A 2-move is identical with a crossing change, which is an unknotting operation. Two handlebody-links are *link-homotopic* if they are related by a finite sequence of self-crossing changes, which are crossing changes on the same components, and isotopies of S^3. A handlebody-link is *link-homotopically trivial* if it is link-homotopic to a trivial handlebody-link. We know that every genus 2 handlebody-knot up to 6 crossings [15] is 3- and 4-move equivalent to the genus 2 trivial handlebody-knot. Moreover we can see that every non-split irreducible handlebody-link with $n > 1$ components having total genus $n+1$ up to 6 crossings [3] is 3-move equivalent to the genus 2 trivial handlebody-knot.

3. Multiple conjugation quandles and MCQ Alexander pairs

A *quandle* [20, 21] is a non-empty set Q with a non-associative binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

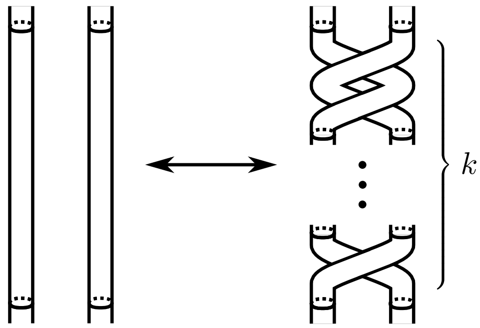

FIGURE 1. A k-move for a handlebody-link.

- For any $a \in Q$, $a \triangleleft a = a$.
- For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote the iterated map $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. We define the *type* of a quandle Q by

$$\operatorname{type} Q = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ for any } x, y \in Q\},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$, and $\mathbb{Z}_{>0}$ denotes the set of positive integers. Any finite quandle has a finite type.

Let G be a group. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = b^{-1}ab$. Then $(G, \triangleleft)$ is a quandle, called the *conjugation quandle* of G and denoted by $\operatorname{Conj} G$. We define another binary operation $\triangleleft$ on G by $a \triangleleft b = ba^{-1}b$. Then $(G, \triangleleft)$ is a quandle, called the *core quandle* of G and denoted by $\operatorname{Core} G$. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then $(\mathbb{Z}_n, \triangleleft)$ is a quandle, called the *dihedral quandle* of order n and denoted by R_n.

Definition 3.1 ([12]). A *multiple conjugation quandle (MCQ)* X is a disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a non-associative binary operation $\triangleleft : X \times X \to X$ satisfying the following axioms:

- For any $a, b \in G_\lambda$, $a \triangleleft b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x \triangleleft e_\lambda = x$ and $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$, where $a \triangleleft x, b \triangleleft x \in G_\mu$ for some $\mu \in \Lambda$.

In this paper, we often omit parentheses. When doing so, we apply binary operations from left on expressions, except for group operations, which we always apply first. For example, we write $a \triangleleft_1 b \triangleleft_2 cd \triangleleft_3 (e \triangleleft_4 f \triangleleft_5 g)$ for $((a \triangleleft_1 b) \triangleleft_2 (cd)) \triangleleft_3 ((e \triangleleft_4 f) \triangleleft_5 g)$ simply, where each $\triangleleft_i$ is a binary operation, and c and d are elements of the same group.

For two MCQs $X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ and $X_2 = \bigsqcup_{\mu \in M} G_\mu$, an *MCQ homomorphism* $\rho : X_1 \to X_2$ is defined to be a map from X_1 to X_2 satisfying $\rho(x \triangleleft y) = \rho(x) \triangleleft \rho(y)$ for any $x, y \in X_1$ and $\rho(ab) = \rho(a)\rho(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. An MCQ homomorphism from X_1 to X_2 is also called an *MCQ representation* of X_1 to X_2. We denote by $\mathrm{Hom}(X_1, X_2)$ the set of MCQ homomorphisms from X_1 to X_2.

We recall the definition of a G-family of quandles. A G-family of quandles is an algebraic system which yields an MCQ.

Definition 3.2 ([14]). Let G be a group with the identity element e. A *G-family of quandles* is a non-empty set X with a family of binary operations $\triangleleft^g : X \times X \to X$ ($g \in G$) satisfying the following axioms:

- For any $x \in X$ and $g \in G$, $x \triangleleft^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x \triangleleft^e y = x$ and $x \triangleleft^{gh} y = (x \triangleleft^g y) \triangleleft^h y$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x \triangleleft^g y) \triangleleft^h z = (x \triangleleft^h z) \triangleleft^{h^{-1}gh} (y \triangleleft^h z)$.

Let $(Q, \triangleleft)$ be a quandle. Then $(Q, \{\triangleleft^i\}_{i \in \mathbb{Z}_{\mathrm{type}\, Q}})$ is a $\mathbb{Z}_{\mathrm{type}\, Q}$-family of quandles, where we put $\mathbb{Z}_\infty := \mathbb{Z}$. Let $(X, \{\triangleleft^g\}_{g \in G})$ be a G-family of quandles. Then $X \times G = \bigsqcup_{x \in X} (\{x\} \times G)$ is an MCQ with

$$(x, g) \triangleleft (y, h) := (x \triangleleft^h y, h^{-1} g h), \qquad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$ [12]. We call it the *associated MCQ* of $(X, \{\triangleleft^g\}_{g \in G})$.

Then we recall the definition of MCQ Alexander pairs. Throughout this paper, we assume that every ring has the multiplicative identity $1 \neq 0$.

Definition 3.3 ([24]). Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be an MCQ and R a ring. The pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$ is an *MCQ Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a, b \in G_\lambda$,
$$f_1(a, b) + f_2(a, b) = f_1(a, a^{-1} b).$$
- For any $a, b \in G_\lambda$ and $x \in X$,
$$\begin{aligned} f_1(a, x) &= f_1(b, x), \\ f_2(ab, x) &= f_2(a, x) + f_1(b \triangleleft x, a^{-1} \triangleleft x) f_2(b, x). \end{aligned}$$
- For any $x \in X$ and $a, b \in G_\lambda$,
$$\begin{aligned} f_1(x, e_\lambda) &= 1, \\ f_1(x, ab) &= f_1(x \triangleleft a, b) f_1(x, a), \\ f_2(x, ab) &= f_1(x \triangleleft a, b) f_2(x, a). \end{aligned}$$
- For any $x, y, z \in X$,
$$\begin{aligned} f_1(x \triangleleft y, z) f_1(x, y) &= f_1(x \triangleleft z, y \triangleleft z) f_1(x, z), \\ f_1(x \triangleleft y, z) f_2(x, y) &= f_2(x \triangleleft z, y \triangleleft z) f_1(y, z), \\ f_2(x \triangleleft y, z) &= f_1(x \triangleleft z, y \triangleleft z) f_2(x, z) + f_2(x \triangleleft z, y \triangleleft z) f_2(y, z). \end{aligned}$$

An MCQ Alexander pair is related to a linear extension of an MCQ [24, 25]. Several examples of MCQ Alexander pairs are given in [16]. The MCQ Alexander pair in the following example will be used in Section 6

Example 3.4 ([16, Example 3.7]). Let $Q := \operatorname{Core} G$ be the core quandle of a group G. Let $X := Q \times \mathbb{Z}_2$ be the associated MCQ of a $\mathbb{Z}_2$-family of quandles $(Q, \{\triangleleft^i\}_{i\in\mathbb{Z}_2})$. We define maps $f_1, f_2 : X \times X \to R[G]/I$ by

$$f_1((x,a),(y,b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise,} \end{cases}$$

$$f_2((x,a),(y,b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1, \end{cases}$$

where $R[G]$ is the group ring of G over a ring R, and I is a two-sided ideal of $R[G]$. Then the pair (f_1, f_2) is an MCQ Alexander pair.

4. The fundamental MCQ of a handlebody-link

In this section, we recall the notions of presentations of MCQs and the fundamental MCQ of a handlebody-link briefly. For details see [13].

For a set of pairwise disjoint sets $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$, the free MCQ $F_{\mathsf{MCQ}}(S_\Lambda)$ over S_Λ is a free object in the category of MCQs. It is known that every MCQ has a *presentation* $\langle S_\Lambda \mid R\rangle$, which is also denoted $\langle S_\lambda\ (\lambda \in \Lambda) \mid R\rangle$ for $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$. We call S_Λ the *generating set* of $\langle S_\Lambda \mid R\rangle$ and an element of R a *relator* of $\langle S_\Lambda \mid R\rangle$. A relator (a, b) is also written as $a = b$. For $x \in \bigcup S_\Lambda$, we use the same symbol x for the element of $\langle S_\Lambda \mid R\rangle$ represented by x. A presentation $\langle S_\Lambda \mid R\rangle$ is called a *finite presentation* if $\bigcup S_\Lambda$ and R are finite. For a finitely presented MCQ, we often write

$$\begin{aligned} &\langle x_{1,1}, \dots, x_{1,n_1}; \dots; x_{l,1}, \dots, x_{l,n_l} \mid r_1, \dots, r_m\rangle \\ &:= \langle \{x_{1,1}, \dots, x_{1,n_1}\}, \dots, \{x_{l,1}, \dots, x_{l,n_l}\} \mid \{r_1, \dots, r_m\}\rangle. \end{aligned}$$

A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves depicted in Fig. 2 [11]. Let D be a diagram of a handlebody-link. A *Y-orientation* of D is a collection of orientations of all edges of D without sources and sinks with respect to the orientation as shown in Fig. 3, where an edge of D is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of D is also regarded as an edge of D. It is known that every diagram has a Y-orientation. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on a diagram. A vertex of a Y-oriented diagram can be allocated a sign; the vertex is said to have a sign $+1$ or -1 as shown in Fig. 3.

Let H be a handlebody-link represented by a Y-oriented diagram D. We denote by $C(D)$, $V(D)$ and $\mathcal{A}(D)$ the sets of crossings, vertices and arcs of D, respectively. For each $c \in C(D)$, we denote by v_c the over-arc of c, and we denote by u_c and w_c the under-arcs of c such that the normal orientation of v_c points from u_c to w_c as illustrated in the left of Fig. 4. For each $\tau \in V(D)$, if τ has a sign $+1$ (resp. -1), we

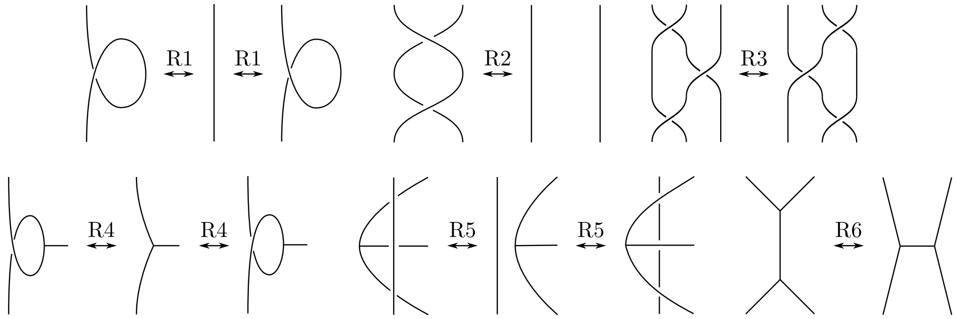

FIGURE 2. The Reidemeister moves of handlebody-links.

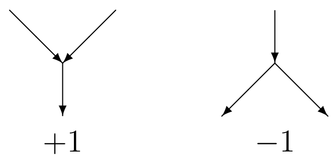

FIGURE 3. Y-orientations and signs of verteces.

denote by w_τ the arc whose initial (resp. terminal) vertex is τ, and we denote by u_τ and v_τ the arcs incident to τ such that the normal orientation of w_τ points from u_τ to v_τ as illustrated in the center and right of Fig. 4. We denote by $\mathcal{A}^\sqcup(D)$ the quotient set of $\mathcal{A}(D)$ by the equivalence relation generated by $\bigcup_{\tau\in V(D)}\{u_\tau, v_\tau, w_\tau\}^2$, that is, two arcs $x, x' \in \mathcal{A}(D)$ are equivalent if there exist arcs $x_1, x_2, \ldots, x_n \in \mathcal{A}(D)$ such that $x = x_1, x' = x_n$, and that x_i and x_{i+1} have a common vertex of D for each i. For example, for the Y-oriented diagram D of a handlebody-knot depicted in Fig. 5, we have $\mathcal{A}^\sqcup(D) = \{\{x_1, x_2, x_3\}, \{x_4, \ldots, x_{10}\}, \{x_{11}\}, \ldots, \{x_{14}\}\}$. Then we define

$$\mathrm{MCQ}(D) := \langle \mathcal{A}^\sqcup(D) \,|\, r_c, r_\tau \ (c \in C(D), \tau \in V(D)) \rangle ,$$

where r_c and r_τ denote the relators $(u_c \triangleleft v_c, w_c)$ and $(u_\tau v_\tau, w_\tau)$, respectively. The isomorphism class of $\mathrm{MCQ}(D)$ does not depend on the choice of a diagram D of H and its Y-orientation [13]. We then define $\mathrm{MCQ}(H) := \mathrm{MCQ}(D)$ and call it the *fundamental MCQ* of H. This presentation is called the *Wirtinger presentation of* $\mathrm{MCQ}(H)$ *with respect to* D.

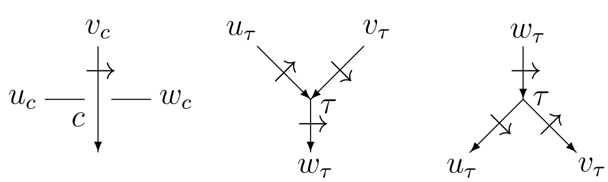

FIGURE 4. Notations of arcs.

Let D be a Y-oriented diagram of a handlebody-link H and let X be an MCQ. An *X-coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau)C(v_\tau) = C(w_\tau)$$

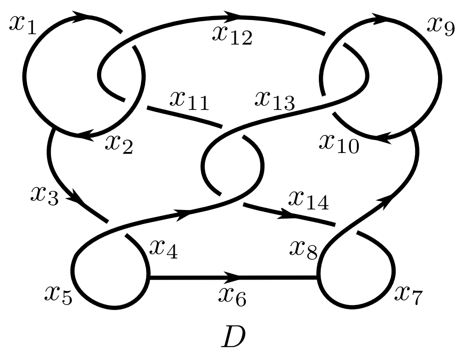

FIGURE 5. A Y-oriented diagram of a handlebody-knot.

for each $c \in C(D)$ and $\tau \in V(D)$. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. An X-coloring of D can be regarded as an MCQ representation of $\mathrm{MCQ}(D)$ to X, that is, we can then identify $\mathrm{Col}_X(D)$ with $\mathrm{Hom}(\mathrm{MCQ}(D), X)$. Hence its cardinality is an invariant for the handlebody-link, called the *MCQ coloring number*.

Let D be a Y-oriented diagram of a handlebody-link H and D' a Y-oriented diagram of H obtained by changing the Y-orientation of D. We then obtain the MCQ isomorphism $f_{(D,D')} : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D')$ sending x into $x^{\varepsilon(x)}$ for any $x \in \mathcal{A}(D)$, where $\varepsilon(x) = 1$ if the Y-orientations of D and D' coincide on x; otherwise $\varepsilon(x) = -1$ (see [13]). Moreover, let D'' a Y-oriented diagram of H obtained by applying one of Reidemeister moves preserving the Y-orientation to D once. We then obtain a unique MCQ isomorphism $f_{(D,D'')} : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D'')$ sending x into x for any $x \in \mathcal{A}(D \cap D'')$, where $\mathcal{A}(D \cap D'')$ denotes the set of arcs in the outside of the disk where the move is applied. Let H and H' be handlebody-links represented by Y-oriented diagrams D and D', respectively. Let $\rho : \mathrm{MCQ}(D) \to X$ and $\rho' : \mathrm{MCQ}(D') \to X$ be MCQ representations. Then (H, ρ) and (H', ρ') are *equivalent*, denoted by $(H, \rho) \cong (H', \rho')$, if there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves preserving the Y-orientation and Y-orientation changes such that $\rho' = \rho \circ f^{-1}_{(D_1,D_2)} \circ \cdots \circ f^{-1}_{(D_{n-1},D_n)}$. Clearly, if two handlebody-links H and H' represented by Y-oriented diagrams D and D' respectively are equivalent, there is a bijection $\Phi : \mathrm{Hom}(\mathrm{MCQ}(D), X) \to \mathrm{Hom}(\mathrm{MCQ}(D'), X)$ such that $(H, \rho) \cong (H', \Phi(\rho))$ for any MCQ representation $\rho : \mathrm{MCQ}(D) \to X$.

5. f-TWISTED ALEXANDER MATRICES FOR HANDLEBODY-LINKS

In this section, we recall f-twisted Alexander matrices for handlebody-links with an MCQ Alexander pair f. See [16] for more details.

Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a finite set of pairwise disjoint finite sets and $x_1, \ldots, x_n$ the elements of $\bigcup S_\Lambda$. Let $X = \langle S_\Lambda \,|\, \{r_1, \ldots, r_m\} \rangle$ be a finitely presented MCQ. Let $F_{\mathsf{MCQ}}(S_\Lambda)$ be the free MCQ on S_Λ and $\mathrm{pr} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ the canonical projection. We often omit "pr" to represent $\mathrm{pr}(x)$ as x. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. We denote by G_μ a direct summand of $F_{\mathsf{MCQ}}(S_\Lambda)$, that is, $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{\mu \in \overline{\Lambda}} G_\mu$ for some index set $\overline{\Lambda}$. For $j \in \{1, \ldots, n\}$, the f-*derivative* *with respect to* x_j [16] is a map $\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$ satisfying

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(x \triangleleft y) = f_1(x,y)\frac{\partial_f}{\partial x_j}(x) + f_2(x,y)\frac{\partial_f}{\partial x_j}(y),\\
&\frac{\partial_f}{\partial x_j}(ab) = \frac{\partial_f}{\partial x_j}(a) + f_1(a,a^{-1})\frac{\partial_f}{\partial x_j}(b),\\
&\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}
\end{aligned}$$

for any $x, y \in F_{\mathsf{MCQ}}(S_\Lambda)$, $a, b \in G_\mu$ and $i \in \{1, \dots, n\}$, where δ_{ij} denotes the Kronecker delta. By using the second condition, the equations $\frac{\partial_f}{\partial x_j}(e_\mu) = 0$ and $\frac{\partial_f}{\partial x_j}(a^{-1}) = -f_1(a,a)\frac{\partial_f}{\partial x_j}(a)$ hold. For a relator $r_i = (r_i', r_i'')$, we define

$$\frac{\partial_f}{\partial x_j}(r_i) := \frac{\partial_f}{\partial x_j}(r_i') - \frac{\partial_f}{\partial x_j}(r_i'').$$

Let R be a ring. We denote by $M(m,n;R)$ the set of $m \times n$ matrices over R. Two matrices A_1 and A_2 over R are *equivalent*, denoted by $A_1 \sim A_2$, if they are related by a finite sequence of the following transformations:

- $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n) \leftrightarrow (\boldsymbol{a}_1, \dots, \boldsymbol{a}_i + \boldsymbol{a}_j r, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n)$ $(r \in R)$,
- $\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \leftrightarrow \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + r\boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$ $(r \in R)$,
- $A \leftrightarrow \begin{pmatrix} A \\ \boldsymbol{0} \end{pmatrix}$,
- $A \leftrightarrow \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}$.

Let R be a commutative ring, and let $A \in M(m,n;R)$. A *k-minor* of A is the determinant of a $k \times k$ submatrix of A. For any $d \in \mathbb{Z}_{\geq 0}$, the *$d$-th elementary ideal* $E_d(A)$ of A is the ideal of R generated by all $(n-d)$-minors of A if $n - m \leq d < n$, and

$$E_d(A) := \begin{cases} 0 & \text{if } d < n - m, \\ R & \text{if } n \leq d. \end{cases}$$

If $A \sim B$, then it follows $E_d(A) = E_d(B)$.

Let $X = \langle \boldsymbol{x} \mid \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a finitely presented MCQ and $\rho : X \to Y$ an MCQ representation. For an MCQ Alexander pair $f = (f_1, f_2)$ of maps $f_1, f_2 : Y \times Y \to R$, we set $f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$, which is also an MCQ Alexander pair. Then the *f-twisted Alexander matrix* of (X, ρ) (with respect to the presentation $\langle \boldsymbol{x} \mid \boldsymbol{r} \rangle$) [16] is defined by

$$A(X, \rho; f) = \begin{pmatrix} \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_n}(r_m) \end{pmatrix} \in M(m,n;R).$$

Let H be a handlebody-link represented by a Y-oriented diagram D. Let $\rho : \mathrm{MCQ}(D) \to X$ be an MCQ representation. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. Then we define the *$f$-twisted Alexander matrix of (H, ρ) (with respect to D)* by

$$A(H, \rho; f) := A(\mathrm{MCQ}(D), \rho; f).$$

We also define

$$E_d(H, \rho; f) := E_d(A(\mathrm{MCQ}(D), \rho; f))$$

if R is a commutative ring. These are invariants of the pair of the handlebody-link H and the MCQ representation ρ, that is, if $(H, \rho) \cong (H', \rho')$, then we have $A(H, \rho; f) \sim A(H', \rho'; f)$ and $E_d(H, \rho; f) = E_d(H', \rho'; f)$ [16].

These invariants take the following values for trivial handlebody-links (see [16, Proposition 6.5]). Let O_g be a trivial handlebody-link having total genus g. Let D_g be a Y-oriented diagram of O_g. For any MCQ representation $\rho : \mathrm{MCQ}(D_g) \to X$ and MCQ Alexander pair $f = (f_1, f_2)$ of maps $f_1, f_2 : X \times X \to R$, we have

$$A(O_g, \rho; f) \sim \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \in M(1, g; R).$$

Especially, we have

$$E_d(O_g, \rho; f) = \begin{cases} 0 & \text{if } d < g, \\ R & \text{if } g \leq d \end{cases}$$

if R is a commutative ring.

6. Detecting k-move inequivalent handlebody-links

In this section, we provide some methods to distinguish k-move equivalence classes of handlebody-links. In particular, we show that the invariants introduced in [16], (described in Section 5), detect 4-move inequivalent handlebody-links.

It is well-known that $2k$-moves for two component classical links do not change the linking numbers modulo k for any $k \in \mathbb{Z}_{>0}$. In the following, we consider a similar property for handlebody-links. Let H be a two component handlebody-link, and let H_1, H_2 be its components and m, n be genera of them, respectively. Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be bases of the first homology groups of H_1 and H_2, respectively. We can regard e_i and f_j as closed oriented circles embedded in S^3. Then the invariant factors, also called elementary divisors, $d_1, \ldots, d_l$ of $(lk(e_i, f_j)) \in M(m, n; \mathbb{Z})$ is an invariant of H up to multiplication by ± 1, where $lk(e_i, f_j)$ denotes the linking number of e_i and f_j. In [22], the linking number of H is defined by

$$lk(H) = \begin{cases} \{|d_1|, \ldots, |d_l|\} & \text{if } 0 < l, \\ \{0\} & \text{othewise} \end{cases}$$

as a multiset. Clearly, link-homotopic two component handlebody-links have the same linking number. We can also regard $(lk(e_i, f_j))$ as an $m \times n$ matrix over $\mathbb{Z}_k$ for $k \in \mathbb{Z}_{>0}$. It is known that any matrix over a principal ideal ring has unique invariant factors up to multiplication by a unit (see [4, Theorem 15.24]). Since $\mathbb{Z}_k$ is a principal ideal ring, the matrix $(lk(e_i, f_j)) \in M(m, n; \mathbb{Z}_k)$ has unique invariant factors $d_1, \ldots, d_l$ up to multiplication by a unit of $\mathbb{Z}_k$. We then have the following proposition.

Proposition 6.1. *Let H be a two component handlebody-link and let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be bases of the first homology groups of the components of H, respectively. Then for any $k \in \mathbb{Z}_{>0}$, the invariant factors $d_1, \ldots, d_l$ of $(lk(e_i, f_j)) \in M(m, n; \mathbb{Z}_k)$ is invariant up to multiplication by a unit of $\mathbb{Z}_k$ under $2k$-moves for H.*

Proof. Since $lk(e_i, f_j) \in \mathbb{Z}_k$ is invariant under $2k$-moves for H, then $(lk(e_i, f_j)) \in M(m, n; \mathbb{Z}_k)$ is also invariant under that. Furthermore, a replacement of a basis of the first homology group of a component of H causes multiplying an invertible matrix on $\mathbb{Z}_k$ to $(lk(e_i, f_j))$. This operation does not change the invariant factors of $(lk(e_i, f_j))$ up to multiplication by a unit of $\mathbb{Z}_k$. □

In Proposition 6.1, the invariant factors $d_1, \ldots, d_l$ of $(lk(e_i, f_j)) \in M(m, n; \mathbb{Z}_k)$ can be identified with $lk(H)$ regarded as a multiset over $\mathbb{Z}_k$.

For example, let H be the two component handlebody-link depicted in Fig. 6. Then we have $lk(H) = \{1\}$. On the other hand, for any two component trivial handlebody-link H_0, we have $lk(H_0) = \{0\}$. Hence H is not 4-move equivalent to a trivial handlebody-link by Proposition 6.1.

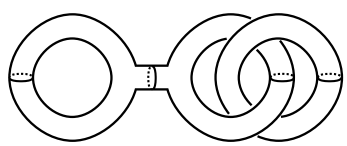

FIGURE 6. A two component handlebody-link H.

Let R_{4k} be the dihedral quandle for $k \in \mathbb{Z}_{>0}$ and $X := R_{4k} \times \mathbb{Z}_2$ the associated MCQ of the $\mathbb{Z}_2$-family of quandles $(R_{4k}, \{\triangleleft^i\}_{i \in \mathbb{Z}_2})$. Let H_1 and H_2 be handlebody-links which are deformed into each other by a $4k$-move. Let D_1 and D_2 be Y-oriented diagrams of H_1 and H_2, respectively. We may assume that D_1 and D_2 are identical except in the disk where the $4k$-move is applied. For any X-coloring ρ_1 of D_1, we obtain the unique X-coloring ρ_2 of D_2 which coincides with ρ_1 except in the disk where the $4k$-move is applied as depicted in Fig. 7. Then the map from $\mathrm{Col}_X(D_1)$ to $\mathrm{Col}_X(D_2)$ sending ρ_1 into ρ_2 is bijective. Therefore, $\#\mathrm{Col}_X(D_1)$ is invariant under $4k$-moves for H_1.

Theorem 6.2. *Let R_4 be the dihedral quandle and $X := R_4 \times \mathbb{Z}_2$ the associated MCQ of the $\mathbb{Z}_2$-family of quandles $(R_4, \{\triangleleft^i\}_{i \in \mathbb{Z}_2})$, where we regard R_4 as the core quandle* $\mathrm{Core}\langle t \mid t^4 \rangle$. *Let $f = (f_1, f_2)$ be the MCQ Alexander pair of maps $f_1, f_2 : X \times X \to \mathbb{Z}_4[t^{\pm 1}]/(t^2 + 1)$ or $f_1, f_2 : X \times X \to \mathbb{Z}_2[t^{\pm 1}]/(t^3 + t^2 + t + 1)$ introduced in Example 3.4,*

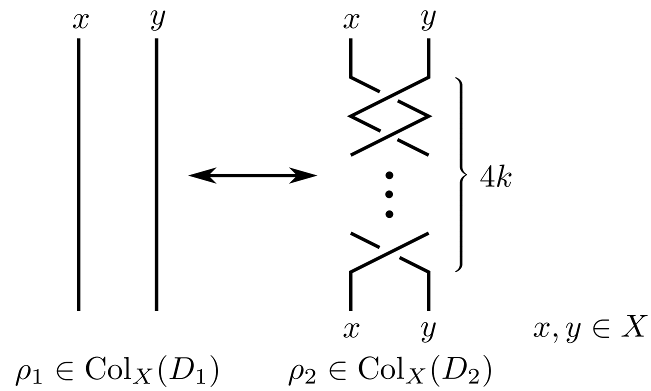

FIGURE 7. The X-colorings $\rho_1 \in \mathrm{Col}_X(D_1)$ and $\rho_2 \in \mathrm{Col}_X(D_2)$.

that is,

$$f_1((x,a),(y,b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise,} \end{cases}$$

$$f_2((x,a),(y,b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1. \end{cases}$$

Then for any handlebody-link H, the multiset

$$\{E_d(H,\rho;f) \mid \rho \in \mathrm{Hom}(\mathrm{MCQ}(H),X)\}$$

is an invariant under 4-moves for H for each $d \in \mathbb{Z}_{\geq 0}$.

Proof. First, we remark that the two MCQ Alexander pairs in the statement are given by settings $R = \mathbb{Z}_4, I = (t^2+1)$ and $R = \mathbb{Z}_2, I = (t^3+t^2+t+1)$ in Example 3.4, respectively.

Let H_1 and H_2 be handlebody-links which are deformed into each other by a 4-move. Let D_1 and D_2 be Y-oriented diagrams of H_1 and H_2, respectively. We may assume that D_1 and D_2 are identical except in the disk where the 4-move is applied as depicted in Fig. 8.

Let ρ_1 be an X-coloring of D_1, and let ρ_2 be the X-coloring of D_2 which coincides with ρ_1 except in the disk where the 4-move is applied as depicted in Fig. 7. Then it is sufficient to show $A(H_1,\rho_1;f) \sim A(H_2,\rho_2;f)$. Let $\mathrm{MCQ}(D_1)$ and $\mathrm{MCQ}(D_2)$ be the Wirtinger presentations of $\mathrm{MCQ}(H_1)$ and $\mathrm{MCQ}(H_2)$ with respect to D_1 and D_2, respectively. We then have

$$\mathrm{MCQ}(D_1) = \langle x_1,\dots,x_n | \boldsymbol{r_1} \rangle,$$
$$\mathrm{MCQ}(D_2) = \left\langle x_1,\dots,x_{n+4} \,\middle|\, \begin{array}{c} \boldsymbol{r_2},\ x_1 \triangleleft x_2^{\varepsilon} = x_{n+1},\ x_2 \triangleleft x_{n+1} = x_{n+2}, \\ x_{n+1} \triangleleft x_{n+2}^{\varepsilon} = x_{n+3},\ x_{n+2} \triangleleft x_{n+3} = x_{n+4} \end{array} \right\rangle,$$

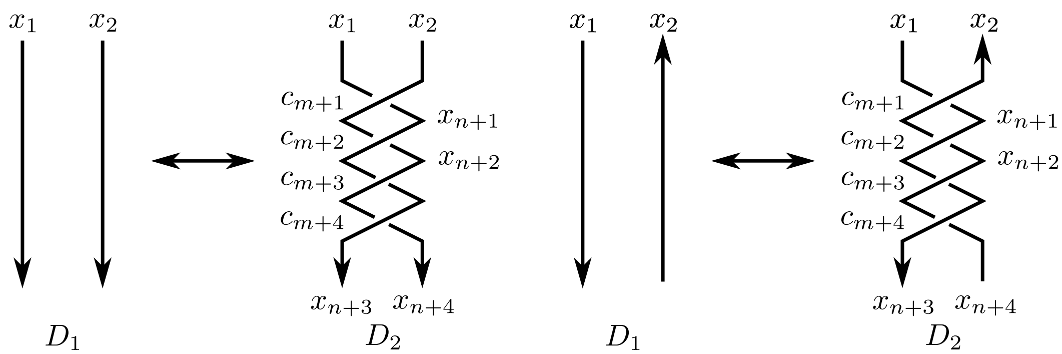

FIGURE 8. Y-oriented diagrams D_1 and D_2.

which can be transformed into

$$\left\langle x_1, \ldots, x_{n+4} \;\middle|\; \begin{array}{l} \boldsymbol{r_2},\ x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1 \triangleleft x_2^{\varepsilon} = x_{n+3}, \\ x_2 \triangleleft x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1 \triangleleft x_2^{\varepsilon} = x_{n+4} \end{array} \right\rangle$$

by using certain transformations of presentations of MCQs equipped with MCQ representations, so-called "Tietze transformations" [13, 16], which do not change equivalence classes of f-twisted Alexander matrices, for some $\varepsilon \in \{1, -1\}$ and some relations $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ satisfying $\boldsymbol{r_2}|_{x_{n+3}=x_1, x_{n+4}=x_2} = \boldsymbol{r_1}$. In the following, we show that

$$\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1 \triangleleft x_2^{\varepsilon}) = \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1), \tag{1}$$

$$\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2 \triangleleft x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1 \triangleleft x_2^{\varepsilon}) = \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2) \tag{2}$$

for each $j \in \{1, \ldots, n+4\}$, $\varepsilon \in \{1, -1\}$ and an MCQ representation $\rho : \mathrm{MCQ}(D_2) \to X$. We write $f_i \circ (\rho \times \rho)$ as f_i^{ρ} for each $i = 1, 2$. We then have

$$\begin{aligned}
&\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1 \triangleleft x_2^{\varepsilon}) \\
&= f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1) + f_2^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon}) \\
&= \cdots \\
&= f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon}, x_1) f_1^{\rho}(x_1, x_2^{\varepsilon}) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1) \\
&\quad + f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon}, x_1) f_2^{\rho}(x_1, x_2^{\varepsilon}) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon}) \\
&\quad + f_1^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) f_2^{\rho}(x_1 \triangleleft x_2^{\varepsilon}, x_1) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1) \\
&\quad + f_2^{\rho}(x_1 \triangleleft x_2^{\varepsilon} \triangleleft x_1, x_2^{\varepsilon}) \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon}).
\end{aligned} \tag{3}$$

When $\rho(x_1)=(t^p,1)$ and $\rho(x_2)=(t^q,1)$ for some integers p and q, we have

$$\begin{aligned}(3) &= (-t^{-p+q}-t^{-2p+2q}-t^{-3p+3q})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)\\ &\quad + (1+t^{-p+q}+t^{-2p+2q}+t^{-3p+3q})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon})\\ &= \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)\end{aligned}$$

since $1+t^{-p+q}+t^{-2p+2q}+t^{-3p+3q}=0$ in $\mathbb{Z}_4[t^{\pm1}]/(t^2+1)$ and in $\mathbb{Z}_2[t^{\pm1}]/(t^3+t^2+t+1)$, and otherwise we can easily see that $(3)=\dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)$. Hence we obtain the equality (1). Next we have

$$\begin{aligned}&\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1\triangleleft x_2^{\varepsilon})\\ &= f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1)+f_2^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon})\\ &= \cdots\\ &= f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}, x_1)f_1^{\rho}(x_2\triangleleft x_1, x_2^{\varepsilon})f_1^{\rho}(x_2, x_1)\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2)\\ &\quad + f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}, x_1)f_1^{\rho}(x_2\triangleleft x_1, x_2^{\varepsilon})f_2^{\rho}(x_2, x_1)\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)\\ &\quad + f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}, x_1)f_2^{\rho}(x_2\triangleleft x_1, x_2^{\varepsilon})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon})\\ &\quad + f_1^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})f_2^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}, x_1)\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)\\ &\quad + f_2^{\rho}(x_2\triangleleft x_1\triangleleft x_2^{\varepsilon}\triangleleft x_1, x_2^{\varepsilon})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon}). \qquad (4)\end{aligned}$$

When $\rho(x_1)=(t^p,1)$ and $\rho(x_2)=(t^q,1)$ for some integers p and q, we have

$$\begin{aligned}(4) &= -t^{-p+q}(1+t^{-p+q}+t^{-2p+2q}+t^{-3p+3q})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_1)\\ &\quad + t^{-2p+2q}\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2)+(2+t^{-p+q}+t^{-3p+3q})\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2^{\varepsilon})\\ &= \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2),\end{aligned}$$

and otherwise we can easily see that $(4) = \dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(x_2)$. Hence we obtain the equality (2). Putting $\boldsymbol{r_2} = \{r_1, \ldots, r_k\}$, by the equalities (1) and (2), we have

$$\begin{aligned} A(H_2,\rho_2;f) &\sim \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & B & \boldsymbol{a_{n+3}} & \boldsymbol{a_{n+4}} \\ 1 & 0 & \mathbf{0} & -1 & 0 \\ 0 & 1 & \mathbf{0} & 0 & -1 \end{pmatrix} \\ &\sim \begin{pmatrix} \boldsymbol{a_1}+\boldsymbol{a_{n+3}} & \boldsymbol{a_2}+\boldsymbol{a_{n+4}} & B & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & -1 & 0 \\ 0 & 0 & \mathbf{0} & 0 & -1 \end{pmatrix} \\ &\sim \begin{pmatrix} \boldsymbol{a_1}+\boldsymbol{a_{n+3}} & \boldsymbol{a_2}+\boldsymbol{a_{n+4}} & B \end{pmatrix} \\ &= A(H_1,\rho_1;f), \end{aligned}$$

where

$$\boldsymbol{a_i} = \begin{pmatrix} \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_i}(r_1) \\ \vdots \\ \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_i}(r_k) \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_3}(r_1) & \cdots & \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_{n+2}}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_3}(r_k) & \cdots & \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_{n+2}}(r_k) \end{pmatrix}.$$

□

Example 6.3. Let H be the link-homotopically trivial three component handlebody-link represented by the Y-oriented diagram D depicted in Fig. 9. Let X and $f = (f_1, f_2)$ be the MCQ and the MCQ Alexander pair of maps $f_1, f_2 : X \times X \to \mathbb{Z}_4[t^{\pm 1}]/(t^2+1)$ that are the same as Theorem 6.2, respectively. Let $\rho : \mathrm{MCQ}(H) \to X$ be the MCQ representation depicted in Fig. 9. Then the Wirtinger presentation of $\mathrm{MCQ}(H)$ with respect to D is given by

$$\left\langle \begin{matrix} x_1, x_2, x_3; x_4; \\ x_5, x_6, x_7; x_8; x_9 \end{matrix} \;\middle|\; \begin{matrix} x_6 \triangleleft x_1 = x_7,\ x_1 \triangleleft x_7 = x_2,\ x_8 \triangleleft x_3 = x_8,\ x_4 \triangleleft x_8 = x_3, \\ x_9 \triangleleft x_4 = x_9,\ x_5 \triangleleft x_9 = x_4,\ x_3x_1 = x_2,\ x_7x_5 = x_6 \end{matrix} \right\rangle.$$

Hence we have

$$\begin{aligned} A(H,\rho;f) &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1-t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -t^{-1} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 2+2t & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and $E_3(H,\rho;f) = (2+2t)$. On the other hand, let H_0 be the three component trivial handlebody-link consisting of one genus 2 component and two genus 1 components. As seen in Section 5, for any MCQ representation $\rho_0 : \mathrm{MCQ}(H_0) \to X$, we have

$$A(H_0,\rho_0;f) \sim \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

and $E_3(H_0, \rho_0; f) = 0$. Consequently, H is not 4-move equivalent to the trivial handlebody-link by Theorem 6.2.

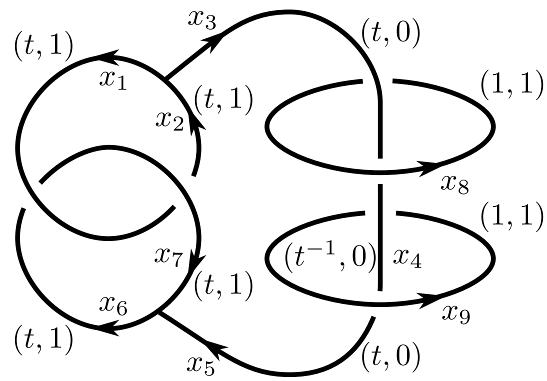

FIGURE 9. A Y-oriented diagram D of the three component handlebody-link H.

Remark 6.4. In Example 6.3, since the handlebody-link H is link-homotopically trivial, the linking number of any two components of H is $\{0\}$ as well as H_0. Furthermore, H and H_0 have the same X-coloring numbers; $\#\mathrm{Hom}(\mathrm{MCQ}(H), X) = \#\mathrm{Hom}(\mathrm{MCQ}(H_0), X) = 1024$.

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