THE BOUNDARIES OF BOUNDED TYPE FIXED SIEGEL DISKS OF SOME TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we extend Zakeri's result in [27] on boundaries of bounded type Siegel disks of some entire functions to some transcendental meromorphic functions as follows: We consider a one parameter family of some transcendental meromorphic functions with one pole, two critical points, one finite asymptotic value zero, and bounded type fixed Siegel disks centered at the origin. We show that if two critical values coincide, then the boundary of the Siegel disk is a quasicircle containing exactly one critical point, and the set Ω_1 of all parameters for which two critical values coincide is countably infinite. We also show that there exist uncountable sets Ω_2 and Ω_3 such that the boundary of the Siegel disk is a quasicircle containing exactly one critical point for any parameter in Ω_2 and the boundary of the Siegel disk is a quasicircle containing exactly two critical points for any parameter in Ω_3. Furthermore, we can construct Ω_2 so that for uncountably many parameters in Ω_2, the critical values which are the images of the critical points outside the boundaries of the Siegel disks are in the Siegel disks, on the boundaries of the Siegel disks, and outside the closures of the Siegel disks.

1. Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function. The nth iteration $f^n(z)$ is defined for all points in $\mathbb{C}$ except for the countable set consisting of the preimages of ∞ by $f, f^2, \cdots, f^{n-1}$.

A point $z_0 \in \mathbb{C}$ is called an *irrationally indifferent p-periodic point* if there exists a minimum integer p such that $f^p(z_0) = z_0$ and $\lambda := (f^p)'(z_0) = e^{2\pi i\theta}$ $(\theta \in \mathbb{R} \setminus \mathbb{Q})$. The λ is called the *multiplier* of z_0. In addition, the point z_0 is called a *fixed point* if $p = 1$. The point z_0 is called a *Siegel point* if there exist a maximal f^p-invariant domain $D \subset \widehat{\mathbb{C}}$ and an analytic homeomorphism $\phi : D \to \mathbb{D}$ such that $\phi(f^p(\phi^{-1}(z))) = \lambda z$ and $\phi(z_0) = 0$. Otherwise, z_0 is called a *Cremer point*. In the former case, the domain D is simply connected and we call D the *Siegel disk of period p centered at z_0*. In addition, D is called *fixed* if $p = 1$. If θ satisfies the following condition:

$$\sum_n \frac{\log q_{n+1}}{q_n} < \infty,$$

where p_n/q_n is the nth convergent of θ obtained from the continued fraction expansion, then z_0 is a Siegel point (see [8] and [22] or [21, p.132, Theorem 11.10]). An irrational number is called a *Brjuno number* if it satisfies the condition above. The set $\mathcal{B}$ of all Brjuno numbers is uncountable and dense in $\mathbb{R}$. An irrational number is called *of bounded type* if $\{a_k\}_{k=0}^{\infty}$ is bounded, where $[a_0; a_1, a_2, \ldots, a_k, \ldots]$ is its continued fraction. An irrational number of bounded type is always a Brjuno number. Hence if θ is of bounded type, then z_0 is a Siegel point. In this case, we call z_0 (or the Siegel disk D centered at z_0) *bounded type*.

Points $c \in \mathbb{C}$ and $f(c)$ are called a *critical point* and a *critical value* respectively if $f'(c) = 0$. A point $a \in \widehat{\mathbb{C}}$ is called an *asymptotic value* if there exists a continuous curve $\gamma(t)$ $(0 \leq t < 1)$ with $\lim_{t\to 1} \gamma(t) = \infty$ and $\lim_{t\to 1} f(\gamma(t)) = a$. We call critical values, asymptotic values, and their accumulation points *singular values*.

Let S be the set of all transcendental entire functions of the form

$$P(z) \exp(Q(z)),$$

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where P and Q are polynomials. The set S is a proper subset of the *Speiser class* consisting of all entire functions with finitely many singular values. Functions in S are also called *structurally finite* in the sense of [25]. Zakeri studied boundaries of bounded type fixed Siegel disks centered at the origin for functions in S. His result is as follows:

Theorem ([27]). *Let $f \in S$. If f has a bounded type fixed Siegel disk centered at the origin, then the boundary of the Siegel disk is a quasicircle containing at least one critical point.*[1)]

Let $\tilde{S}$ be the set of all transcendental meromorphic functions of the form

$$R(z)\exp(Q(z)),$$

where $R(z)$ and $Q(z)$ are a rational map which has at least one pole and a polynomial respectively. Functions in $\tilde{S}$ and functions in S share many important properties. For example, they have finitely many critical points, two asymptotic values 0 and ∞, and finitely many zeros.[2)] Thus we can expect the result for functions in $\tilde{S}$ similar to that for functions in S. We ask the following question:

Question 1. *Let $f \in \tilde{S}$. Suppose that f has a bounded type fixed Siegel disk centered at the origin. Is the fixed Siegel disk bounded by a quasicircle containing at least one critical point?*

We consider the easiest case as follows: Henceforth fix any irrational number θ of bounded type. Suppose that $f \in \tilde{S}$, the degrees of R and Q are 1, and f has a bounded type Siegel fixed point at the origin with multiplier $\lambda = e^{2\pi i\theta}$. The function f is conformally conjugate to

$$g_\alpha(z) := e^{2\pi i\theta}\frac{z}{1-\frac{\alpha+1}{\alpha}z}e^{\alpha z}$$

for some $\alpha \in \mathbb{C}\setminus\{0,-1\}$ (see Proposition 3.1). The one parameter family $\{g_\alpha\}_{\alpha\in\mathbb{C}\setminus\{0,-1\}}$ has the following properties:

(1) g_α has two critical points 1 and
$$c_\alpha := \frac{-1}{\alpha+1},$$
two asymptotic values 0 and ∞, and one pole
$$t_\alpha := \frac{\alpha}{\alpha+1}$$
(see Proposition 3.1).

(2) $g_{\alpha'} \neq g_\alpha$ is conformally conjugate to g_α if and only if $\alpha' = 1/(\alpha+1)-1$ (see Proposition 3.2).

Main Theorem. *Let $\triangle_\alpha$ be the bounded type fixed Siegel disk of g_α centered at the origin. Then:*

(i) *If two critical values $g_\alpha(1)$ and $g_\alpha(c_\alpha)$ coincide, then $\triangle_\alpha$ is bounded by a quasicircle containing exactly one critical point. Moreover, the set $\Omega_1 := \{\alpha \mid g_\alpha(1) = g_\alpha(c_\alpha)\}$ is countably infinite.*

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1)Zakeri's original statement includes Shishikura's result which says that all bounded type fixed Siegel disks of polynomials of arbitrary degree are bounded by quasicircles containing critical points. On the other hand, since the space S is not invariant under affine conjugations moving the origin, Theorem does not say anything about bounded type fixed Siegel disks centered at points other than the origin. Zakeri mentioned this technical problem in his paper [27]. In [18], Kisaka and Naba construct some functions in S with bounded type fixed Siegel disks centered at points other than the origin, whose boundaries are quasicircles containing critical points.

2)Meromorphic functions with finitely many critical points and asymptotic values share important dynamical properties (see [4], [5], [6], [11], and [23]).

(ii) *There exists an uncountable set* Ω_2 *such that if* $\alpha \in \Omega_2$, *then* $\triangle_\alpha$ *is bounded by a quasicircle containing exactly one critical point. Moreover, the quasicircle constant can be taken so that it is independent of* $\alpha \in \Omega_2$.

(iii) *There exists an uncountable set* Ω_3 *such that if* $\alpha \in \Omega_3$, *then* $\triangle_\alpha$ *is bounded by a quasicircle containing exactly two critical points. Moreover, the quasicircle constant can be taken so that it is independent of* $\alpha \in \Omega_3$.

(iv) *We can construct* Ω_2 *so that it is connected and it consists of three uncountable sets* $\Omega_{2,1}$, $\Omega_{2,2}$, *and* $\Omega_{2,3}$ *such that:*

  (a) *If* $\alpha \in \Omega_{2,j}$ $(j = 1, 2, 3)$, *then* $v(\alpha) \in \triangle_\alpha$, $v(\alpha) \in \partial\triangle_\alpha$, *and* $v(\alpha) \notin \overline{\triangle_\alpha}$ *respectively, where* $v(\alpha)$ *is the critical value of* g_α *for* $\alpha \in \Omega_2$ *which is the image of the critical point outside the boundary* $\partial\triangle_\alpha$.

  (b)

$$\Omega_{2,2} \subset \partial\Omega_{2,1} \cap \partial\Omega_{2,3}, \quad \Omega_3 \subset \partial\Omega_{2,3}.$$

Remark 1.1. We give two constructions of Ω_2 in Section 5 and Section 7. The second construction of Ω_2 will show the Main Theorem (iv) (see Section 7).

Keen and Zhang studied the one parameter family

$$\{\tilde{g}_\alpha(z) := (e^{2\pi i\theta} z + \alpha z^2)e^z\}_{\alpha \in \mathbb{C}\setminus\{0\}},$$

where θ is of bounded type (see [17]). Like g_α, $\tilde{g}_\alpha$ has two critical points, two asymptotic values 0 and ∞, and a bounded type fixed Siegel disk $\tilde{\triangle}_\alpha$ centered at the origin. They showed that for every $\alpha \in \mathbb{C} \setminus \{0\}$, $\tilde{\triangle}_\alpha$ is bounded by a quasicircle containing critical points and that for α in some uncountable set, the boundary $\partial\tilde{\triangle}_\alpha$ contains exactly two critical points. However, they did not provide the information on the position of the critical values of $\tilde{g}_\alpha$ as in the Main Theorem (iv). It is natural to expect that Keen and Zhang's proof is applicable to our case and we obtain the result on the Siegel disk $\triangle_\alpha$ of g_α as in [17]. Unfortunately, since g_α has one pole t_α, we cannot use their method as in [17] (and cannot use the method as in [27]). In particular, we have the difficulty of making the number of critical points in the boundaries $\partial\triangle_\alpha$ exactly one or two. Hence in order to show the Main Theorem, we have to modify Keen and Zhang's argument. We use the result of [9] in order to prove the Main Theorem (i). The proofs of the Main Theorem (ii), (iii), and (iv) are inspired by quasiconformal surgery methods of [9], [17], and [26]. We modify some meromorphic functions f_β (defined in Section 5) into g_α with the bounded type fixed Siegel disks $\triangle_\alpha$ bounded by quasicircles containing critical points. The advantage of our surgery technique is that we obtain such g_α for uncountably many parameters α and that we control the number of critical points in the boundaries $\partial\triangle_\alpha$ and the position of critical values as in the Main Theorem (iv). This is done by choosing f_β carefully.

This paper is organized as follows: In Section 2, we introduce basic definitions and facts. We characterize the family $\{g_\alpha\}_{\alpha \in \mathbb{C}\setminus\{0,-1\}}$ in Section 3. In Section 4, Section 5 and Section 6, we prove the Main Theorem (i), (ii), and (iii) respectively. In Section 7, we give another construction of Ω_2 and show the Main Theorem (iv). We devote Section 8 to some concluding remarks.

2. Preliminaries

We introduce preliminary definitions and results.

Definition 2.1 (Quasiregular mappings). Let U be an open subset of $\mathbb{C}$. A continuous mapping $\varphi : U \to \mathbb{C}$ is a K-quasiregular mapping if φ is locally K-quasiconformal except at a discrete set of points in U for some $K \geq 1$. The constant K is called a quasiregular constant.[3)]

Note that quasiconformal mappings or quasiregular mappings between Riemann surfaces are defined by their local coordinates.

Definition 2.2 (Quasicircles). A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a K-quasicircle if there exists a K-quasiconformal mapping $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma = \phi(\mathbb{S}^1)$, where $\mathbb{S}^1 := \{z \mid |z| = 1\}$. This K is called a quasicircle constant of γ. We call γ a quasicircle if it is a K-quasicircle for some $K \geq 1$.

We can tell whether a Jordan curve is a quasicircle or not by the following lemma:

Lemma 2.3 ([1], [12, p.23, Theorem 2.2.5]). *Let $\gamma \subset \widehat{\mathbb{C}}$ be a Jordan curve and let* $\mathrm{Diam}(X)$ *be the Euclidean diameter of a set $X \subset \mathbb{C}$. Then γ is a K-quasicircle for some $K \geq 1$ if and only if there exists a constant $A \geq 1$ such that for every pair of two distinct points $z_1, z_2 \in \gamma \setminus \{\infty\}$,*

$$\min_{j=1,2} \mathrm{Diam}(\gamma_j) \leq A|z_1 - z_2|,$$

where γ_1 and γ_2 are the components of $\gamma \setminus \{z_1, z_2\}$. Moreover, K and A depend only on each other.

We prepare the following lemma:

Lemma 2.4 ([27, p.488, Lemma 2.2]). *Let $\gamma \subset \widehat{\mathbb{C}}$ be a K-quasicircle, let U be a component of $\widehat{\mathbb{C}} \setminus \gamma$, and let $g : \mathbb{D} \to U$ be a conformal mapping. Then g extends to a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$.*

3. Characterization of the family $\{g_\alpha\}_{\alpha \in \mathbb{C} \setminus \{0,-1\}}$

In this section, we characterize the one parameter family $\{g_\alpha\}_{\alpha \in \mathbb{C} \setminus \{0,-1\}}$ defined in the introduction by the following propositions:

Proposition 3.1. *Let $f \in \tilde{S}$ have the following properties:*

(a) *f can be written by*

$$f(z) = \frac{az + b}{cz + d} e^{tz},$$

where $ad - bc$, c, and t are non-zero;

(b) *f has a bounded type Siegel fixed point at the origin with multiplier $\lambda = e^{2\pi i \theta}$.*

Then f is conformally conjugate to

$$g_\alpha(z) = e^{2\pi i \theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z}$$

for some $\alpha \in \mathbb{C} \setminus \{0, -1\}$. Moreover, g_α has two critical points 1 and $c_\alpha = -1/(\alpha + 1)$, two asymptotic values 0 and ∞, and one pole $t_\alpha = \alpha/(\alpha + 1)$.

Proof. Since f has a fixed point at the origin, we have $b = 0$, and hence $ad \neq 0$. In addition, it follows from the assumption (b) that $f'(0) = a/d = e^{2\pi i \theta}$. Set

$$s := -c/d \neq 0.$$

Then we can write

$$f(z) = e^{2\pi i \theta} \frac{z}{1 - sz} e^{tz}.$$

[3)] This is one of the equivalent definitions of quasiregular mappings. See [7] for alternative definitions. For basic properties of quasiconformal mappings, see also [2] or [20].

An easy calculation shows that

$$f'(z) = e^{2\pi i\theta + tz}\frac{-stz^2 + tz + 1}{(1 - sz)^2}.$$

Hence f has two non-zero critical points u and v which are roots of $-stz^2 + tz + 1 = 0$. Let

$$L(z) := uz.$$

It follows that $L^{-1} \circ f \circ L$ has two critical points 1 and v/u. Moreover, we obtain

$$\tilde{f}(z) := L^{-1} \circ f \circ L(z) = e^{2\pi i\theta}\frac{z}{1 - \tilde{s}z}e^{\tilde{t}z},$$

where $\tilde{s} = su \neq 0$ and $\tilde{t} = tu \neq 0$. Since $\tilde{f}'(1) = 0$, we have

$$-\tilde{s}\tilde{t} \cdot 1^2 + \tilde{t} \cdot 1 + 1 = 0,$$

and hence $\tilde{s} = (\tilde{t} + 1)/\tilde{t}$. It follows from this, $\tilde{s} \neq 0$, and $\tilde{t} \neq 0$ that $\tilde{t} \in \mathbb{C} \setminus \{0, -1\}$, and hence $\tilde{f}(z) = g_\alpha(z)$, where $\alpha = \tilde{t}$. By the construction, g_α has two critical points 1 and c_α, and one pole t_α. Since the map $z \mapsto e^{\alpha z}$ has two asymptotic values 0 and ∞, and

$$e^{2\pi i\theta}\frac{z}{1 - \frac{\alpha+1}{\alpha}z} \to -e^{2\pi i\theta}\frac{\alpha}{\alpha + 1} \quad (z \to \infty),$$

g_α has two asymptotic values 0 and ∞. □

Proposition 3.2. *Let α and α' be two distinct points in $\mathbb{C} \setminus \{0, -1\}$. Then g_α and $g_{\alpha'}$ are conformally conjugate if and only if $\alpha' = 1/(\alpha + 1) - 1$.*

Proof. Suppose that $\alpha' = 1/(\alpha + 1) - 1$ and

$$l(z) := -(\alpha + 1)z.$$

An easy calculation shows that $l^{-1} \circ g_{\alpha'} \circ l = g_\alpha$.

Suppose that there exists a conformal map $\tilde{l} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{l}^{-1} \circ g_{\alpha'} \circ \tilde{l} = g_\alpha$. Since both $g_{\alpha'}$ and g_α have an essential singularity at ∞ and only two asymptotic values 0 and ∞, $\tilde{l}$ fixes 0 and ∞. It follows that $\tilde{l}(z) = kz$ for some $k \neq 0$. Moreover, since $\tilde{l}(1) = k$ is a critical point of $g_{\alpha'}$, we have $k = 1$ or $k = -1/(\alpha' + 1)$. Since $g_{\alpha'} \neq g_\alpha$, we have $k \neq 1$, and hence $k = -1/(\alpha' + 1)$ and $\alpha' \neq -2$. Since $g_{\alpha'}$ has another critical point $\tilde{l}(-1/(\alpha + 1)) = 1/\{(\alpha' + 1)(\alpha + 1)\} = 1$, we obtain $\alpha' = 1/(\alpha + 1) - 1$. □

4. Proof of the Main Theorem (i)

We use the following result of [9] to prove the Main Theorem (i):

Lemma 4.1 ([9, p.2140, Theorem 1.5.])**.** *Let $U \subset \widehat{\mathbb{C}}$ be an open set and let a meromorphic function $f : U \to \widehat{\mathbb{C}}$ have the following properties:*

(a) *The set of all singular values of f is contained in $\{a, b, c\}$ for some $a, b, c \in \widehat{\mathbb{C}}$;*

(b) *$a \in U$ and a is a bounded type Siegel fixed point;*

(c) *$c \in \widehat{\mathbb{C}} \setminus U$ or $f(c) = c$.*

Moreover, let γ' be an injective path which goes from a to b while avoiding $\{a, b, c\}$ in between and let γ be the lift of γ' by f which has an endpoint a. (Note that $f(\gamma) \subset \gamma'$.) Then one and only one of the following three cases occurs:

(1) *γ ends on a non-critical point in U. In addition, $U = \widehat{\mathbb{C}}$ and f is a Möbius transformation.*

(2) *γ ends on a critical point. (We call the critical point the main critical point.) In addition, the Siegel disk $\triangle$ centered at a is bounded by a quasicircle which contains the main critical point and does not contain other critical points.*

(3) *γ leaves every compact subset of U. In addition, $\triangle$ does not compactly contained in U.*

Proof of the Main Theorem (i). By the assumption, g_α has exactly one critical value $g_\alpha(1) = g_\alpha(c_\alpha)$ and two asymptotic value 0 and ∞. Hence we can apply Lemma 4.1 to g_α by putting $U = \mathbb{C}$, $f = g_\alpha$, $a = 0$, $b = g_\alpha(1)$, and $c = \infty$. Since g_α is transcendental, either of the cases (2) and (3) holds. Since $b = g_\alpha(1)$ is not an asymptotic value, the case (3) does not occur. Therefore, the case (2) occurs.

Next, we show the existence of Ω_1. Put $g_\alpha(1) = g_\alpha(c_\alpha)$. Then it follows that

$$F(\alpha) := \frac{1}{(\alpha+1)^2} e^{-\alpha/(\alpha+1)} - e^{\alpha} = 0.$$

$F(\alpha)$ has an essential singularity at $\alpha = -1$ and does not have an asymptotic value 0 at $\alpha = -1$. By Picard's theorem and Iversen's theorem, the set $\Omega_1 := \{\alpha \mid g_\alpha(1) = g_\alpha(c_\alpha)\}$ is countably infinite (see [16] or [10, p.8, Theorem 1.6] for Iversen's theorem). □

Remark 4.2. Two critical points 1 and $c_\alpha = -1/(\alpha+1)$ of g_α coincides only when $\alpha = -2$. By the Main Theorem (i), $\triangle_{-2}$ is bounded by a quasicircle containing the critical point 1 of g_{-2}.

5. Proof of the Main Theorem (ii)

For $\beta \in \mathbb{C} \setminus \{0\}$, we define

$$f_\beta(z) := \begin{cases} \frac{z}{1-(\beta+1)z/\beta} e^{\beta z} & (\beta \in \mathbb{C} \setminus \{0, -1\}) \\ z e^{-z} & (\beta = -1). \end{cases}$$

Note that if $\beta \to -1$, then $f_\beta \to f_{-1}$ locally uniformly. By the argument in Section 3, when $\beta \in \mathbb{C} \setminus \{0, -1\}$, f_β has two critical points 1 and $c_\beta = -1/(\beta+1)$, two asymptotic values 0 and ∞, and one pole $t_\beta = \beta/(\beta+1)$. We have $c_\beta, t_\beta \to \infty$ as $\beta \to -1$. For any $r > 0$, we define

$$B_r := (-1, -1+r].$$

Henceforth we restrict β to B_r (or $\overline{B_r} = B_r \cup \{-1\}$). We prove the Main Theorem (ii) by going through the following three steps:

Step 1. By choosing a small enough $r > 0$ and using f_β, we construct an M-quasiregular mapping $F_\beta : \mathbb{C} \to \widehat{\mathbb{C}}$ for every $\beta \in B_r$ with the following properties:

(1) $F_\beta(0) = 0$, $F_\beta(\mathbb{D}) = (\mathbb{D})$, and $F_\beta|_{\mathbb{S}^1}$ is a critical circle map;

(2) F_β and

$$R_\theta(z) := e^{2\pi i\theta} z$$

are quasiconformally conjugate on $\mathbb{D}$;

(3) F_β depends continuously on $\beta \in B_r$;

(4) The constant M is independent of $\beta \in B_r$.

Step 2. We show that there exists an M_1-quasiconformal mapping $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1, and ∞, and has the following properties:

(1) For some $\alpha \in \mathbb{C} \setminus \{0, -1\}$,

$$G_\beta(z) := \varphi_\beta \circ F_\beta \circ \varphi_\beta^{-1}(z) = e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z} = g_\alpha,$$

where g_α is as in the introduction.

(2) $g_\alpha (= G_\beta)$ has the Siegel disk $\triangle_\alpha$ centered at the origin whose boundary $\partial\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point 1;

(3) The constant M_1 is independent of $\beta \in B_r$.

Step 3. From Step 2, we define the surgery map

$$\mathcal{S} : B_r \to \mathbb{C} \setminus \{0, -1\}, \quad \beta \mapsto \alpha,$$

where $G_\beta = g_\alpha$. We show that the surgery map $\mathcal{S}$ is continuous and $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$. Since the set $\mathcal{S}(B_r)$ is uncountable, and $\partial\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point 1 for any $\alpha \in \mathcal{S}(B_r)$, we obtain the Main Theorem (ii) by taking

$$\Omega_2 := \mathcal{S}(B_r).$$

We prepare the following lemmas for the steps above:

Lemma 5.1. *Let $\beta \in \overline{B_r}$, let*

$$D_\beta := \{z \mid |z| < |f_\beta(1)|\},$$

and let U_β be the connected component of $f_\beta^{-1}(D_\beta)$ which contains the origin. (Note that $f_\beta(0) = 0$.) If $r > 0$ is small enough, then $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and U_β is simply connected. Moreover, U_β has the following properties:

(1) *∂U_β is a piecewise smooth Jordan curve containing exactly one critical point* 1*:*

(2) *$U_\beta \subset \overline{\mathbb{D}}$.*

Proof. Suppose that $\beta \in B_r$. f_β has two critical values $f_\beta(1)$ and $f_\beta(c_\beta)$. We have

$$f_\beta(1) = -\beta e^\beta, \quad f_\beta(c_\beta) = -\frac{\beta}{(1+\beta)^2} e^{-\beta/(1+\beta)}.$$

Since $f_\beta(1) \to e^{-1}$ and $f_\beta(c_\beta) \to \infty$ as $\beta \to -1$, we have $f_\beta(c_\beta) \notin D_\beta$ for $r > 0$ small enough. By [9, p.2155, Lemma 5.3], $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and U_β is simply connected. Obviously, ∂D_β does not contain the asymptotic values 0 and ∞ of f_β. It follows from this that ∂U_β is a Jordan curve (see [9, p.2155, Lemma 5.4]). Since ∂U_β is a preimage of ∂D_β by f_β, ∂U_β is piecewise smooth. By the construction, we have $f_\beta([0,1)) \in \mathbb{R}$, $f'_\beta(z) \neq 0$ for any $z \in [0,1)$, $f_\beta(1) > 0$, and $f_\beta(0) = 0$. It follows that $f'_\beta(z) > 0$ for any $z \in [0,1)$, and hence $[0,1) \subset U_\beta$. This implies that ∂U_β contains the critical point 1. An easy calculation shows that $|f_\beta(z)| > f_\beta(1)$ for any $z \in \mathbb{S}^1 \setminus \{1\}$, and hence $U_\beta \subset \overline{\mathbb{D}}$. By the construction, another critical point c_β is not in ∂U_β for $r > 0$ small enough.

Similarly, we can show the case $\beta = -1$. We omit the details. □

Lemma 5.2. *If $r > 0$ is small enough, then there exists a constant $K \geq 1$ such that ∂U_β is a K-quasicircle for all $\beta \in B_r$.*

Proof. The proof is similar to that of [17, p.142, Lemma 2.4]. We have to pay attention to the existence of the pole t_β of f_β for $\beta \in B_r$ and modify the argument.

Suppose that $r > 0$ is small enough so that the statement of Lemma 5.1 holds. We take two distinct points x and y in ∂U_β so that they divide ∂U_β into two Jordan arcs I and I'. (We mean that $I \cup I' = \partial U_\beta$ and $I \cap I' = \{x, y\}$.) For any piecewise smooth arc segment J, let $|J|$ be the Euclidean length of J. We can assume that $|f_\beta(I)| \le |f_\beta(I')|$ without loss of generality. Let $\mathrm{Diam}(X)$ be as in Lemma 2.3. By Lemma 2.3, we have only to show that there exists a constant $A > 0$ independent of $\beta \in B_r$, x, and y such that

$$Q(\beta, x, y) := \frac{\mathrm{Diam}(I)}{|x-y|} < A. \tag{5.1}$$

Since $f_\beta(I) \subset \partial D_\beta$ and $\partial D_\beta = \{z \mid |z| = f_\beta(1)\}$ is a circle, we have

$$|f_\beta(I)| \le (\pi/2)|f_\beta(x) - f_\beta(y)|. \tag{5.2}$$

Henceforth let L be the closed straight line segment joining x and y. It follows from (5.2) and $|f_\beta(x) - f_\beta(y)| \le |f_\beta(L)|$ that

$$|f_\beta(I)| \le (\pi/2)|f_\beta(L)|. \tag{5.3}$$

By Lemma 5.1, we have $L \subset \overline{\mathbb{D}}$. In addition, recall that f_β has the pole t_β with $t_\beta \to \infty$ as $\beta \to -1$. Thus if $r > 0$ is small enough, then $t_\beta \notin \overline{\mathbb{D}}$, and hence $t_\beta \notin L$. Therefore, there exists a $q \in L$ such that $|f'_\beta(q)| = \max_{z\in L} |f'_\beta(z)| > 0$. It follows that

$$|f_\beta(L)| \le |f'_\beta(q)||L|. \tag{5.4}$$

By the definition of a diameter, there exist points $b_1, b_2 \in I$ such that $|b_1 - b_2| = \mathrm{Diam}(I)$. Moreover, there also exists a $j = 1$ or 2 such that:

$$1 \notin \{z \mid |z - b_j| \le \mathrm{Diam}(I)/5\}.$$

Let $\tilde{I}$ be the connected component of

$$\{z \mid |z - b_j| \le \mathrm{Diam}(I)/10\} \cap I$$

which contains b_j. By definition, it follows that:

$$|\tilde{I}| \ge \mathrm{Diam}(I)/10; \tag{5.5}$$

$$|z - 1| \ge \mathrm{Diam}(I)/10 \quad \text{for any } z \in \tilde{I}. \tag{5.6}$$

Since $\tilde{I}$ does not contain critical points 1 and c_β of f_β, there exists a $p \in \tilde{I}$ such that $|f'_\beta(p)| = \min_{z\in\tilde{I}} |f'_\beta(z)| > 0$. It follows that

$$|f_\beta(\tilde{I})| \ge |f'_\beta(p)||\tilde{I}|. \tag{5.7}$$

From (5.4), (5.5), (5.7), the definition of $Q(\beta, x, y)$, and $\tilde{I} \subset I$, we see that

$$\begin{aligned} \frac{|f'_\beta(q)|}{|f'_\beta(p)|} &\ge \frac{|f_\beta(L)|}{|L|} \cdot \frac{|\tilde{I}|}{|f_\beta(\tilde{I})|} \\ &= \frac{|f_\beta(L)|}{|f_\beta(\tilde{I})|} \cdot \frac{|\tilde{I}|}{\mathrm{Diam}(I)} \cdot \frac{\mathrm{Diam}(I)}{|L|} \\ &\ge \frac{1}{10} \frac{|f_\beta(L)|}{|f_\beta(I)|} \cdot Q(\beta, x, y). \end{aligned} \tag{5.8}$$

It follows from (5.3) that

$$\frac{|f_\beta(I)|}{|f_\beta(L)|} \le \frac{\pi}{2}. \tag{5.9}$$

The inequalities (5.8) and (5.9) yield

$$Q(\beta, x, y) \le 5\pi \frac{|f'_\beta(q)|}{|f'_\beta(p)|}. \tag{5.10}$$

An easy calculation shows that

$$f'_\beta(z) = -\beta^2 \frac{(z-1)(z+1/(\beta+1))}{(\beta+1)(z-\beta/(\beta+1))^2} e^{\beta z}.$$

Thus we have

$$\frac{|f'_\beta(q)|}{|f'_\beta(p)|} = \frac{|p-\beta/(\beta+1)|^2}{|q-\beta/(\beta+1)|^2} \cdot \frac{|q-1|}{|p-1|} \cdot \frac{|q+1/(\beta+1)|}{|p+1/(\beta+1)|} \cdot |e^{\beta(q-p)}|. \tag{5.11}$$

Since $L \subset \overline{\mathbb{D}}$ and $\tilde{I} \subset I \subset \overline{\mathbb{D}}$, we have $|p| \le 1$ and $|q| \le 1$. Thus we obtain for every $\beta \in B_r$,

$$|e^{\beta(q-p)}| < e^{2(1+r)}. \tag{5.12}$$

Moreover, when $r > 0$ is small enough, it follows that for every $\beta \in B_r$,

$$\frac{|p-\beta/(\beta+1)|^2}{|q-\beta/(\beta+1)|^2} < 2; \tag{5.13}$$

$$\frac{|q+1/(\beta+1)|}{|p+1/(\beta+1)|} < 2. \tag{5.14}$$

(This is because the left-hand sides of (5.13) and (5.14) converge to 1 as $\beta \to -1$.) From the triangle inequality, $q \in L$, and the definition of a diameter, we see that

$$\begin{aligned} |q-1| &\le |q-p| + |p-1| \\ &\le |q-x| + |x-p| + |p-1| \\ &\le |x-y| + |x-p| + |p-1| \\ &\le 2\mathrm{Diam}(I) + |p-1|. \end{aligned} \tag{5.15}$$

The inequalities (5.6) and (5.15) show that

$$\begin{aligned} \frac{|q-1|}{|p-1|} &\le \frac{2\mathrm{Diam}(I) + |p-1|}{|p-1|} \\ &= \frac{2\mathrm{Diam}(I)}{|p-1|} + 1 \\ &\le \frac{2\mathrm{Diam}(I)}{\mathrm{Diam}(I)/10} + 1 \\ &= 21. \end{aligned} \tag{5.16}$$

It follows from (5.10)–(5.16) that if $r > 0$ is small enough, then for any $\beta \in B_r$ and any pair of x and y in U_β,

$$Q(\beta, x, y) < 420\pi e^4 =: A,$$

as required. □

Henceforth we suppose that $r > 0$ is small enough so that the statements of Lemma 5.1 and Lemma 5.2 hold.

Lemma 5.3. *Let $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ be a sequence with $\beta_n \to \beta_\infty \in \overline{B_r}$ as $n \to \infty$. Then $\partial U_{\beta_n} \to \partial U_{\beta_\infty}$ as $n \to \infty$ with respect to the Hausdorff metric.*

Proof. Suppose that there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ and a $\delta > 0$ such that the Hausdorff metric between $\partial U_{\beta'_n}$ and $\partial U_{\beta_\infty}$ is greater than δ for any $n \ge 1$. By the Riemann mapping theorem and Carathéodory's theorem, we can take a homeomorphism $\tilde{\omega}_{\beta_n} : \overline{\mathbb{D}} \to \overline{U}_{\beta_n}$ which is conformal in $\mathbb{D}$, and fixes 0 and 1. By Lemma 2.4, we can extend $\tilde{\omega}_{\beta_n}$ into a K^2-quasiconformal mapping ω_{β_n} of $\widehat{\mathbb{C}}$ fixing 0 and 1, where K is as in Lemma 5.2. From the construction, every limit function of $\{\omega_{\beta_n}|_{\widehat{\mathbb{C}}\setminus\{0,1\}}\}_{n\in\mathbb{N}}$ cannot be the constant 0 or 1. Therefore, there exists a subsequence $\{\beta''_n\}_{n\in\mathbb{N}} \subset \{\beta'_n\}_{n\in\mathbb{N}}$ such that $\omega_{\beta''_n} \to \omega$ locally uniformly on $\mathbb{C}$, where ω is a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 0 and 1. Let

$$\gamma := \omega(\mathbb{S}^1) \subset \mathbb{C}.$$

By the construction, γ is a K^2-quasicircle with $\partial U_{\beta''_n} \to \gamma$ (as $n \to \infty$) with respect to the Hausdorff metric. By Lemma 5.1, we have $U_{\beta''_n} \subset \overline{\mathbb{D}}$ for any $n \geq 1$, and hence $\gamma \subset \overline{\mathbb{D}}$. In addition, from the fact that $f_{\beta''_n} \to f_{\beta_\infty}$ uniformly on $\overline{\mathbb{D}}$ and the definition of D_β, it follows that

$$f_{\beta''_n}(\partial U_{\beta''_n}) \to f_{\beta_\infty}(\gamma), \quad \partial D_{\beta''_n} \to \partial D_{\beta_\infty}$$

with respect to the Hausdorff metric. Since $\partial D_{\beta''_n} = f_{\beta''_n}(\partial U_{\beta''_n})$, we obtain $f_{\beta_\infty}(\gamma) = \partial D_{\beta_\infty}$. By Hurwitz's theorem, f_{β_∞} is univalent on the bounded component of $\mathbb{C} \setminus \gamma$, and hence $\gamma = \partial U_{\beta_\infty}$. It follows that $\partial U_{\beta''_n} \to \partial U_{\beta_\infty}$ with respect to the Hausdorff metric. This contradicts the fact that $\{\beta''_n\}_{n\in\mathbb{N}} \subset \{\beta'_n\}_{n\in\mathbb{N}}$. □

Next, we introduce the following version of the Herman-Świątek theorem:

Lemma 5.4 ([9, p.2147, Theorem 3.8], [14], [15], and [24]). *Let $\mathcal{F}$ be a family of holomorphic maps defined in a neighborhood of $\mathbb{S}^1$ with the following properties:*

- (a) *There exists an open annulus A containing $\mathbb{S}^1$ such that every $f \in \mathcal{F}$ is defined in A;*
- (b) *$f(\mathbb{S}^1) = \mathbb{S}^1$ and $f|_{\mathbb{S}^1}$ is a critical circle map;*
- (c) *There exists an $R > 0$ such that for every $f \in \mathcal{F}$, the rotation number of $f|_{\mathbb{S}^1}$ has all its entries of the continued fraction less than or equal to R;*
- (d) *$\mathcal{F}$ is precompact on A for the Euclidean metric.*

Then there exists a $k > 1$ such that for every $f \in \mathcal{F}$, $f|_{\mathbb{S}^1}$ is k-quasisymmetrically conjugate to rotation.[4)]

Proof of the Main Theorem (ii). Our proof is divided into the three steps which we mentioned at the beginning of this section. Recall that we restricted β to B_r (or $\overline{B_r}$) and $r > 0$ is small enough for the statements of Lemma 5.1 and Lemma 5.2 to hold.

Step 1: By the Riemann mapping theorem and Carathéodory's theorem, for $\beta \in \overline{B_r}$, we can take a homeomorphism $\rho_\beta : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus U_\beta$ which is conformal in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and satisfies $\rho_\beta(\infty) = \infty$ and $\rho_\beta(1) = 1$. By Lemma 2.4, we can extend ρ_β into a K^2-quasiconformal mapping $\hat{\rho}_\beta$ of $\widehat{\mathbb{C}}$ fixing 1 and ∞, where K is as in Lemma 5.2.

For any sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ with $\beta_n \to \beta_\infty \in \overline{B_r}$ as $n \to \infty$, it follows from the construction that every limit function of $\{\hat{\rho}_{\beta_n}|_{\widehat{\mathbb{C}}\setminus\{1,\infty\}}\}_{n\in\mathbb{N}}$ cannot be the constant 1 or ∞. Thus there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ such that $\hat{\rho}_{\beta'_n} \to \sigma$ locally uniformly on $\mathbb{C}$, where σ is a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 1 and ∞. It follows from Lemma 5.3 that $\sigma|_{\widehat{\mathbb{C}}\setminus\mathbb{D}} = \rho_{\beta_\infty}$, and hence $\hat{\rho}_{\beta'_n}|_{\widehat{\mathbb{C}}\setminus\mathbb{D}} = \rho_{\beta'_n} \to \rho_{\beta_\infty}$ locally uniformly on $\mathbb{C}\setminus\mathbb{D}$. This implies that the set of all limit functions of $\{\rho_{\beta_n}\}_{n\in\mathbb{N}}$ contains only ρ_{β_∞}, and hence $\rho_{\beta_n} \to \rho_{\beta_\infty}$ locally uniformly on $\mathbb{C} \setminus \mathbb{D}$. Therefore, ρ_β depends continuously on $\beta \in \overline{B_r}$. The map $f_\beta \circ \rho_\beta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial D_\beta$ is a homeomorphism, where D_β is as in Lemma 5.1. From the standard theory about the rotation number, there exists a unique

4) For the definition of quasisymmetric mappings from $\mathbb{S}^1$ to itself, see [13] or [9, p.2144, Definition 3.2]. The rotation is the map $z \mapsto e^{2\pi i\theta}z$, where θ is the rotation number of $f|_{\mathbb{S}^1}$.

$\theta_\beta \in [0,1)$ such that for

$$L_\beta(z) := \frac{e^{2\pi i\theta_\beta} z}{|f_\beta(1)|},$$

the rotation number of $L_\beta \circ f_\beta \circ \rho_\beta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is the θ which was fixed at the beginning (see [7, p.103, Theorem 3.20]). By the construction, L_β depends continuously on $\beta \in \overline{B_r}$. For $\beta \in \overline{B_r}$, we define

$$\tilde{F}_\beta(z) := L_\beta \circ f_\beta \circ \rho_\beta(z) \quad (z \in \mathbb{C} \setminus \mathbb{D}).$$

The Schwarz reflection principle shows that if $r > 0$ is small enough, then there exists an $l > 1$ such that for any β, $\tilde{F}_\beta$ is extended to a holomorphic map $\hat{F}_\beta$ in $\{z \mid |z| > 1/l\}$. Henceforward, we fix a small enough $r > 0$ so that such extension goes well and the statements of Lemma 5.1 and Lemma 5.2 hold. Set

$$A_l := \{z \mid 1/l < |z| < l\}.$$

By the construction, $\hat{F}_\beta|_{A_l}$ depends continuously on $\beta \in \overline{B_r}$, and hence the family $\{\hat{F}_\beta|_{A_l}\}_{\beta \in B_r}$ satisfies the assumption of Lemma 5.4. By Lemma 5.4, there exists a k-quasisymmetric mapping $s_\beta : \mathbb{S}^1 \to \mathbb{S}^1$ for $\beta \in B_r$ such that

$$s_\beta \circ \hat{F}_\beta|_{\mathbb{S}^1} \circ s_\beta^{-1} = R_\theta, \quad s_\beta(1) = 1,$$

where $k > 1$ is independent of β and $R_\theta(z) = e^{2\pi i\theta} z$. By the theory of Ahlfors-Beurling, we can extend s_β as a homeomorphism $\hat{s}_\beta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is an M-quasiconformal mapping in $\mathbb{D}$ with $s_\beta(0) = 0$, where M depends only on k, and hence M is independent of β (see [9, p.2148, Lemma 3.10]). Since $\hat{F}_\beta|_{\mathbb{S}^1} = \tilde{F}_\beta|_{\mathbb{S}^1}$ depends continuously on β, one can show that s_β depends continuously on $\beta \in B_r$. Then it follows from the way of its extension that $\hat{s}_\beta$ also depends continuously on β. For $\beta \in B_r$, we define F_β as follows:

$$F_\beta(z) := \begin{cases} \tilde{F}_\beta(z) & (z \in \mathbb{C} \setminus \mathbb{D}) \\ \hat{s}_\beta^{-1} \circ R_\theta \circ \hat{s}_\beta(z) & (z \in \mathbb{D}). \end{cases}$$

Since $\tilde{F}_\beta|_{\mathbb{S}^1} = \hat{F}_\beta|_{\mathbb{S}^1} = s_\beta^{-1} \circ R_\theta \circ s_\beta$, F_β is continuous. In addition, $F_\beta : \mathbb{C} \to \widehat{\mathbb{C}}$ is locally M-quasiconformal except the two preimages of the two critical points of f_β by ρ_β. Thus F_β is an M-quasiregular mapping. By the construction, F_β satisfies the following properties:

(1) $F_\beta(0) = 0$, $F_\beta(\mathbb{D}) = (\mathbb{D})$, and $F_\beta|_{\mathbb{S}^1}$ is a critical circle map;

(2) F_β and R_θ are quasiconformally conjugate on $\mathbb{D}$;

(3) F_β depends continuously on $\beta \in B_r$;

(4) The constant M is independent of $\beta \in B_r$.

Thus, we achieve the goal of Step 1.

Step 2: We construct an F_β-invariant almost complex structure on $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_β satisfying $||\mu_\beta||_\infty < k'$ for some $k' < 1$ independent of β as follows: Let $\mu_{\hat{s}_\beta}$ be the Beltrami coefficient of $\hat{s}_\beta$ in $\mathbb{D}$. If $z \in F_\beta^{-n}(\mathbb{D})$ for some integer $n \geq 0$, then we define $\mu_\beta(z)$ as the pullback of $\mu_{\hat{s}_\beta}(F_\beta^n(z))$ by F_β^n. Otherwise, set $\mu_\beta(z) := 0$. Since the almost complex structure on $\mathbb{D}$ with Beltrami coefficient $\mu_{\hat{s}_\beta}$ is F_β-invariant, the almost complex structure on $\widehat{\mathbb{C}}$ with coefficient μ_β is well-defined and F_β-invariant. We have $||\mu_\beta||_\infty < k'$ for some $k' < 1$, since F_β is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $F_\beta(\overline{\mathbb{D}}) = (\overline{\mathbb{D}})$. Moreover, we can take $k' < 1$ independent of β, since the quasiregular constant M of F_β is independent of β. By the integrability theorem (see [7, p.40, Theorem 1.28]), there exists a quasiconformal mapping $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which solves the Beltrami equation with coefficient μ_β and fixes 0, 1, and ∞. Therefore,

$$G_\beta := \varphi_\beta \circ F_\beta \circ \varphi_\beta^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is meromorphic. By the construction, G_β has the only one zero 0 and the only one pole. Thus there exist an entire function $h(z)$ and non-zero constants b and p such that

$$G_\beta(z) = b\frac{z}{z-p}e^{h(z)}.$$

We can show that $h(z)$ is a polynomial of degree 1 as follows: When $|z|$ is large enough, we have

$$(*) \qquad \phi_1 \circ G_\beta(z) = f_\beta \circ \phi_2(z),$$

where

$$\phi_1 := L_\beta^{-1} \circ \varphi_\beta^{-1}, \qquad \phi_2 := \rho_\beta \circ \varphi_\beta^{-1}.$$

Obviously, ϕ_1 and ϕ_2 are quasiconformal mappings. Since ϕ_1 and ϕ_2^{-1} are Hölder continuous at ∞, there exist positive constants $K' > 1$, C_1, and C_2 such that

$$|\phi_1(z)| \geq C_1|z|^{1/K'}, \quad |\phi_2(z)| \leq C_2|z|^{K'} \quad \text{for } |z| \text{ large enough.}$$

From this and $|f_\beta(z)| \leq e^{|z|^2}$ ($|z| \to \infty$), there exist positive constants A and N such that

$$\max_{|z|=R} e^{h(z)} \leq e^{AR^N} \quad \text{for } R > 0 \text{ large enough.}$$

Thus $h(z)$ is a polynomial. In addition, the relation $(*)$ implies that both of f_β and G_β have only one positive (or negative) sector in a punctured neighborhood of ∞ in the sense of [27, p.495]. Therefore, we deduce that $h(z)$ is a polynomial of degree 1.

By the construction, we have $G'_\beta(0) = e^{2\pi i\theta}$ and $G'_\beta(1) = 0$. Hence as in the proof of Proposition 3.1, we obtain for some $\alpha \in \mathbb{C} \setminus \{0, -1\}$,

$$G_\beta(z) = g_\alpha(z) = e^{2\pi i\theta}\frac{z}{1 - \frac{\alpha+1}{\alpha}z}e^{\alpha z}.$$

It follows from the construction that $g_\alpha (= G_\beta)$ has the Siegel disk $\triangle_\alpha = \varphi_\beta(\mathbb{D})$ centered at the origin. Since $||\mu_\beta||_\infty < k'$ for $k' < 1$ independent of β, there exists a constant $M_1 \geq 1$ independent of β such that φ_β is M_1-quasiconformal. Thus the boundary $\partial\triangle_\alpha = \varphi_\beta(\mathbb{S}^1)$ is an M_1-quasicircle containing exactly one critical point 1 of g_α. Therefore, the argument above completes Step 2.

Step 3: From Step 2, we can define the surgery map

$$\mathcal{S} : B_r \to \mathbb{C} \setminus \{0, -1\}, \quad \beta \mapsto \alpha,$$

where $G_\beta = g_\alpha$. In order to show that $\mathcal{S}$ is continuous, we claim the following assertion, whose proof is similar to the argument in [17, p.157, Section 5] or [26, p.218, Section 11]:

Assertion. *Let $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ be any sequence with $\beta_n \to \beta_\infty \in B_r$ as $n \to \infty$. Then there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ such that $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$.*

Proof of the assertion. By Step 2, there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ and an M_1-quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi_{\beta'_n} \to \varphi$ locally uniformly on $\mathbb{C}$ (as $n \to \infty$). We define

$$\varsigma := \varphi \circ F_{\beta_\infty} \circ \varphi^{-1}, \quad \varsigma_n := \varphi_{\beta'_n} \circ F_{\beta'_n} \circ \varphi_{\beta'_n}^{-1}, \quad \varsigma_\infty := \varphi_{\beta_\infty} \circ F_{\beta_\infty} \circ \varphi_{\beta_\infty}^{-1}$$

If $\varsigma = \varsigma_\infty$, then $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$. The proof is completed in the case. Henceforth we suppose that $\varsigma \neq \varsigma_\infty$.

We can show that if $\varsigma \neq \varsigma_\infty$, then $\mu_{\beta'_n} \to \mu_{\beta_\infty}$ with respect to the spherical measure as follows: For a measurable set $E \subset \widehat{\mathbb{C}}$, let Area$(E)$ be the Lebesgue area of E in the spherical metric. In addition, we define

$$Q_n^\varepsilon := \{z \in \mathbb{C} \mid |\mu_{\beta'_n}(z) - \mu_{\beta_\infty}(z)| > \varepsilon\},$$

for $\varepsilon > 0$ and $n \geq 1$. It suffices to show that for any $\varepsilon > 0$ and any $C > 0$, if n is large enough, then Area$(Q_n^\varepsilon) < C$. By the definitions of $\mu_{\beta'_n}$ and μ_{β_∞}, we obtain

$$Q_n^\varepsilon \subset \bigcup_{k \geq 0} F_{\beta'_n}^{-k}(\mathbb{D}) \cup \bigcup_{k \geq 0} F_{\beta_\infty}^{-k}(\mathbb{D}). \tag{5.17}$$

Obviously, ς and ς_∞ are quasiconformally conjugate. It follows from $\varsigma \neq \varsigma_\infty$, $\varsigma_n \to \varsigma$ locally uniformly, and the argument similar to that in [26, p.201] or [17, p.157, p.158] that for n large enough, there exist quasiconformal mappings $\xi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(i) ξ_n fixes 0, 1, and ∞;

(ii) ξ_n satisfies
$$\xi_n \circ \varsigma = \varsigma_n \circ \xi_n;$$

(iii) The complex dilatations χ_n of ξ_n are uniformly bounded, and
$$||\chi_n||_\infty \to 0 \quad (n \to \infty).$$

Hence we have

$$\tau_n \circ F_{\beta_\infty} = F_{\beta'_n} \circ \tau_n,$$

where $\tau_n := \varphi_{\beta'_n}^{-1} \circ \xi_n \circ \varphi$. It follows from the construction that for every $n \geq 1$,

$$\tau_n(\mathbb{D}) = \mathbb{D}, \quad \tau_n(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \quad \tau_n(0) = 0, \quad \tau_n(\infty) = \infty,$$

and the complex dilatations of quasiconformal mappings τ_n are uniformly bounded. Thus from this, the fact that the area of the Riemann sphere is finite, and the area distortion theorem (see [3, p.37, Theorem 1.1]), we deduce that for any $\delta > 0$, there exists an integer $N \geq 1$ such that:

$$\text{Area}\left(\bigcup_{k \geq 0} F_{\beta_\infty}^{-k}(\mathbb{D}) \setminus \bigcup_{0 \leq k \leq N} F_{\beta_\infty}^{-k}(\mathbb{D})\right) < \delta, \tag{5.18}$$

and for n large enough,

$$\text{Area}\left(\bigcup_{k \geq 0} F_{\beta'_n}^{-k}(\mathbb{D}) \setminus \bigcup_{0 \leq k \leq N} F_{\beta'_n}^{-k}(\mathbb{D})\right) < \delta. \tag{5.19}$$

Note that every connected component of $F_{\beta'_n}^{-k}(\mathbb{D})$ is the image of some connected component of $F_{\beta_\infty}^{-k}(\mathbb{D})$ by τ_n. It follows from the properties (i) and (iii) of ξ_n that $\xi_n \to \text{Id}_{\widehat{\mathbb{C}}}$ locally uniformly, and hence $\tau_n \to \text{Id}_{\widehat{\mathbb{C}}}$ locally uniformly. We have for n large enough,

$$\text{Area}\left(\bigcup_{0 \leq k \leq N} F_{\beta'_n}^{-k}(\mathbb{D}) \setminus \bigcup_{0 \leq k \leq N} F_{\beta_\infty}^{-k}(\mathbb{D})\right) < \delta. \tag{5.20}$$

From the construction, $\hat{s}_{\beta'_n} \circ F_{\beta'_n}^N \to \hat{s}_{\beta_\infty} \circ F_{\beta_\infty}^N$ locally uniformly on $\bigcup_{0 \leq k \leq N} F_{\beta_\infty}^{-k}(\mathbb{D})$ as $n \to \infty$. In addition, when $z \in \bigcup_{0 \leq k \leq N} F_{\beta_\infty}^{-k}(\mathbb{D})$ and n is large enough, the complex dilatation of $\hat{s}_{\beta'_n} \circ F_{\beta'_n}^N$

at z and that of $\hat{s}_{\beta_\infty} \circ F^N_{\beta_\infty}$ at z are $\mu_{\beta'_n}(z)$ and $\mu_{\beta_\infty}(z)$ respectively. It follows from this and the construction that for n large enough,

$$\text{Area}\left(Q^\varepsilon_n \cap \bigcup_{0\le k\le N} F^{-k}_{\beta_\infty}(\mathbb{D}) \right) < \delta. \tag{5.21}$$

From (5.17)–(5.21), we obtain

$$\text{Area}(Q^\varepsilon_n) < 4\delta.$$

Since $\delta > 0$ is arbitrary, we can take $4\delta = C$. This implies that $\mu_{\beta'_n} \to \mu_{\beta_\infty}$ with respect to the spherical measure.

From the argument above and [19, p.29, Theorem 4.6], we have $\varphi_{\beta'_n} \to \varphi_{\beta_\infty}$ locally uniformly. It follows that $\varsigma = \varsigma_\infty$. On the other hand, we assumed that $\varsigma \neq \varsigma_\infty$. This is a contradiction, and hence we obtain $\varsigma = \varsigma_\infty$ and $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$. This completes the proof of the assertion. ■

The assertion implies that if $\beta_n \to \beta_\infty \in B_r$, then the set $\{\mathcal{S}(\beta_n)\}_{n\in\mathbb{N}}$ is bounded and has only one accumulation point $\mathcal{S}(\beta_\infty)$. It follows that $\mathcal{S}(\beta_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$, and hence $\mathcal{S}$ is continuous.

Finally, we show that $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$. Recall that $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an M_1-quasiconformal mapping fixing 0, 1, and ∞, where M_1 is independent of β, and ρ_β can be extended to a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 1 and ∞, where K is as in Lemma 5.2. Thus

$$\{\psi_\beta := \varphi_\beta \circ \rho_\beta^{-1}\}_{\beta\in B_r}$$

is uniformly Hölder continuous at ∞ in the sense of [20, p.70] (see [20, p.70, Theorem 4.3]). In addition, since $g_{\mathcal{S}(\beta)}$ has a critical point $c_{\mathcal{S}(\beta)} = -1/(\mathcal{S}(\beta)+1) = \psi_\beta(-1/(\beta+1))$, it follows from $-1/(\beta+1) \to \infty$ as $\beta \to -1$ that

$$\frac{-1}{\mathcal{S}(\beta)+1} = \psi_\beta\left(\frac{-1}{\beta+1}\right) \to \infty$$

as $\beta \to -1$. This shows that $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$, and hence $\mathcal{S}(B_r)$ is uncountable. Moreover, by the construction, $\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point 1 when $\alpha \in \mathcal{S}(B_r)$ (see Step 2). Thus we can take

$$\Omega_2 := \mathcal{S}(B_r).$$

Therefore, we have the desired result of the Main Theorem (ii). □

Remark 5.5. From this construction of Ω_2 and Proposition 3.2, there exists an uncountable set $\tilde{\Omega}_2$ such that if $\alpha \in \tilde{\Omega}_2$, then $\partial\triangle_\alpha$ is a quasicircle containing exactly one critical point c_α. Moreover, it follows from the construction that $g_\alpha(c_\alpha) \notin \overline{\triangle_\alpha}$ for $\alpha \in \Omega_2$.

6. Proof of the Main Theorem (iii)

In this section, we show the Main Theorem (iii) by the quasiconformal surgery in Section 5. Let f_β be as in Section 5. For $0 < r < \pi$, we restrict β to the set

$$C_r := \{z = -1 + e^{i\theta} \mid \theta \in [\pi - r, \pi) \cup (\pi, \pi + r]\}.$$

Lemma 6.1. *Let $\beta \in C_r$, let*

$$D_\beta := \{z \mid |z| < |f_\beta(1)|\},$$

and let U_β be the connected component of $f_\beta^{-1}(D_\beta)$ which contains the origin. (*Note that $f_\beta(0) = 0$.*) *If $r > 0$ is small enough, then $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and U_β is simply connected. Moreover, U_β has the following properties:*

(1) *∂U_β is a piecewise smooth Jordan curve containing exactly two critical points* 1 *and* $c_\beta = e^{i(\pi-\theta)}$*;*

(2) *There exists a large enough* $R > 1$ *independent of* β *such that*

$$\overline{U_\beta} \subset E,$$

where $E := \{z \mid |z| \leq R\} \setminus \{z \mid |z - t_\beta| < 1/R\}$*.*

Proof. We show that ∂U_β contains 1 and c_β as follows: An easy calculation shows that for $0 < x < 1$,

$$|f_\beta(x)|^2 = |f_\beta(xe^{i(\pi-\theta)})|^2 = \frac{2x^2(1-\cos\theta)}{(x-(1-\cos\theta))^2+\sin^2\theta} e^{-2x+2x\cos\theta} =: M(x).$$

In addition, we have

$$M'(x) = \frac{4x(1-\cos\theta)^2(1-x)(x^2+(2\cos\theta-1)x+2)}{((x-(1-\cos\theta))^2+\sin^2\theta)^2} e^{-2x+2x\cos\theta}.$$

Therefore, we have $M'(x) > 0$ for any $0 < x < 1$. Thus U_β contains

$$\{x \mid 0 < x < 1\} \cup \{xe^{i(\pi-\theta)} \mid 0 < x < 1\},$$

and hence ∂U_β contains 1 and c_β. From the argument in the proof of Lemma 5.1, it follows that $f|_{U_\beta} : U_\beta \to D_\beta$ is univalent and ∂U_β is a piecewise smooth Jordan curve. This shows (1).

Since

$$\frac{f_\beta}{e^{\beta z}} = \frac{z}{(1-(\beta+1)z/\beta)} \to \frac{-\beta}{\beta+1} \qquad (z \to \infty)$$

and $\partial D_\beta = \{z \mid |z| = |f_\beta(1)|\}$ is bounded away from 0 and ∞, which are asymptotic values of $z \mapsto e^{\beta z}$, there exists some compact set E' such that $\overline{U_\beta} \subset E'$ for any $\beta \in C_r$. Moreover, since the pole $t_\beta \to 2$ and $f_\beta \to f_{-2}$ uniformly in a neighborhood of $t_{-2} = 2$ with respect to the spherical metric as $\beta \to -1 + e^{i\pi} = -2$, we can choose a large enough $R > 0$ such that the property (2) holds if $r > 0$ is small enough. □

Lemma 6.2. *If* $r > 0$ *is small enough, then there exists a constant* $K \geq 1$ *independent of* β *such that* ∂U_β *is a* K*-quasicircle for any* $\beta \in C_r$*.*

Proof. The proof is similar to that of Lemma 5.2. However, we have to modify the treatment of the pole t_β and pay attention to the two critical points 1 and c_β in ∂U_β. Suppose that $r > 0$ is small enough so that the statement of Lemma 6.1 holds. We need to show that there exists a constant $A > 0$ independent of $\beta \in C_r$ and two distinct points x and y in ∂U_β such that

$$Q(\beta, x, y) := \frac{\text{Diam}(I)}{|L|} < A, \tag{6.1}$$

where I and I' are two Jordan arcs with $\partial U_\beta = I \cup I'$, $I \cap I' = \{x, y\}$, and $|f_\beta(I)| \leq |f_\beta(I')|$, and L is the closed straight line segment joining x and y. Let d be the distance from t_β to the straight line segment L. Suppose that $d < 1/(2R)$. Then the property (2) in Lemma 6.1 assures that $|L| > 1/R$. Since $\text{Diam}(I) \leq 2R$, we have

$$Q(\beta, x, y) = \frac{\text{Diam}(I)}{|L|} < \frac{2R}{1/R} = 2R^2. \tag{6.2}$$

Henceforth, we consider the case $d \geq 1/(2R)$. There exist two points b_1 and b_2 in I such that $|b_1 - b_2| = \text{Diam}(I)$. In addition, there exists a connected component $\hat{I}$ of

$$\{z \mid 3\text{Diam}(I)/10 \leq |z - b_1| \leq 2\text{Diam}(I)/5\} \cap I$$

with $|\hat{I}| \geq \mathrm{Diam}(I)/10$. If

$$\{1, c_\beta\} \cap \{z \mid |z - b_j| \leq \mathrm{Diam}(I)/5\} \neq \emptyset$$

for $j = 1, 2$, then we define $\tilde{I} := \hat{I}$. Otherwise, there exists a $j = 1$ or 2 such that:

$$\{1, c_\beta\} \cap \{z \mid |z - b_j| \leq \mathrm{Diam}(I)/5\} = \emptyset.$$

In this case, let $\tilde{I}$ be the connected component of

$$\{z \mid |z - b_j| \leq \mathrm{Diam}(I)/10\} \cap I$$

which contains b_j. By definition, we have

$$|\tilde{I}| \geq \mathrm{Diam}(I)/10; \tag{6.3}$$

$$|z - c_\beta| \geq \mathrm{Diam}(I)/10,\ |z - 1| \geq \mathrm{Diam}(I)/10 \quad \text{for any } z \in \tilde{I}. \tag{6.4}$$

As in the proof of Lemma 5.2, we can show that there exist points $q \in L$ and $p \in \tilde{I}$ such that

$$Q(\beta, x, y) \leq 5\pi \frac{|f'_\beta(q)|}{|f'_\beta(p)|}. \tag{6.5}$$

From the argument similar to the proof of Lemma 5.2, the property (2) in Lemma 6.1, (6.3), (6.4), and the assumption $d \geq 1/(2R)$, there exists a constant $A' > 0$ independent of β, x, and y such that

$$\frac{|f'_\beta(q)|}{|f'_\beta(p)|} < A'. \tag{6.6}$$

From (6.2), (6.5), and (6.6), we can take $A := \max\{5\pi A', 2R^2\}$. □

Remark 6.3. In Lemma 6.2, we suppose that $r > 0$ is small enough. However, by using the compactness of $C_r \cup \{-2\}$ and modifying the proofs in Lemma 6.1 and Lemma 6.2, one can remove the assumption. Let

$$\tilde{C} := \{z = -1 + e^{i\theta} \mid \theta \in (0, \pi) \cup (\pi, 2\pi)\},$$

and let D_β and U_β be as in Lemma 6.1 for $\beta \in \tilde{C}$. It follows from the proof of Lemma 6.1 that $|f_\beta(1)| = |f_\beta(c_\beta)|$ and ∂U_β contains exactly two critical points of f_β for any $\beta \in \tilde{C}$. However, since f_β is not defined for $\beta = 0 = -1 + e^{2\pi i}$, and $|f_\beta(1)| = |f_\beta(c_\beta)| \to 0$ and the pole $t_\beta \to 0$ as $\beta \to 0$, we do not know whether there exists a constant $K \geq 1$ independent of β such that ∂U_β is a K-quasicircle for any $\beta \in \tilde{C}$ or not.

Proof of the Main Theorem (iii). From Lemma 6.1 and Lemma 6.2, we can apply the quasiconformal surgery technique in Section 5 to f_β for $\beta \in C_r$. Hence there exists a continuous mapping

$$\mathcal{S} : C_r \to \mathbb{C} \setminus \{0, -1\}$$

such that $\partial \triangle_{\mathcal{S}(\beta)}$ is a quasicircle containing exactly two critical points. Note that the construction assures that we can choose the quasicircle constant of $\partial \triangle_{\mathcal{S}(\beta)}$ independent of $\beta \in C_r$. Moreover, there exist quasiconformal mappings ψ_β for $\beta \in C_r$ of $\widehat{\mathbb{C}}$ fixing 1 and ∞ such that:

(1) $\{\psi_\beta\}_{\beta \in C_r}$ is uniformly Hölder continuous at 1 in the sense of [20, p.70];

(2) $g_{\mathcal{S}(\beta)}$ has two critical points 1 and

$$c_{\mathcal{S}(\beta)} = \frac{-1}{\mathcal{S}(\beta) + 1} = \psi_\beta\left(\frac{-1}{\beta + 1}\right).$$

Since $-1/(\beta+1) \to 1$ as $\beta \to -2$, we have

$$c_{\mathcal{S}(\beta)} = \frac{-1}{\mathcal{S}(\beta)+1} \to 1 \quad (\beta \to -2),$$

and hence

$$\mathcal{S}(\beta) \to -2 \quad (\beta \to -2).$$

Thus $\mathcal{S}(C_r)$ is uncountable. We can take

$$\Omega_3 := \mathcal{S}(C_r). \quad \square$$

7. Proof of the Main Theorem (iv)

In this section, we give another construction of Ω_2 and prove the Main Theorem (iv). We extend the surgery map $\mathcal{S} : C_r \to \mathbb{C} \setminus \{0, -1\}$ in the proof of the Main Theorem (iii) into the map $\mathcal{S} : Q_r \to \mathbb{C} \setminus \{0, -1\}$, where $Q_r \supset C_r$ is defined as follows: For any $0 < r < \pi$, let

$$I_r := \{z = k(r) + iy \mid -l(r) \le y \le l(r)\}$$

where $k(r)$ and $l(r)$ are the real part and the imaginary part of $-1 + e^{i(\pi - r)} \in C_r$ respectively, and let Q_r be the bounded closed domain whose boundary is

$$\{-2\} \cup C_r \cup I_r.$$

Note that $k(r) \to -2$ and $l(r) \to 0$ as $r \to 0$. Let D_β and U_β be as in Lemma 6.1 for $\beta \in Q_r$. As in the proofs of Lemma 5.1, Lemma 6.1, and Lemma 6.2, we can easily show the following lemma:

Lemma 7.1. *Let*

$$\hat{B}_r := (-2, k(r)].$$

If $r > 0$ is small enough and $\beta \in \hat{B}_r$, then $|f_\beta(1)| > |f_\beta(c_\beta)|$ and $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and ∂U_β is a piecewise smooth Jordan curve which contains exactly one critical point 1 of f_β. Moreover, there exists a constant $K \ge 1$ independent of β such that ∂U_β is a K-quasicircle for any $\beta \in \hat{B}_r$.

Henceforth let $\tilde{Q}_r$ be the interior of Q_r.

Lemma 7.2. *If $r > 0$ is small enough and $\beta \in \tilde{Q}_r$, then $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and ∂U_β is a piecewise smooth Jordan curve which contains exactly one critical point 1 of f_β. Moreover, there exists a constant $K \ge 1$ independent of β such that ∂U_β is a K-quasicircle for any $\beta \in \tilde{Q}_r$.*

Proof. First of all, we show that ∂U_β contains the critical point 1 of f_β as follows: Let

$$M(x) := |f_\beta(x)|^2,$$

where $\beta = -2 + X + iY \in \tilde{Q}_r$ for $0 < X \le k(r) + 2 < 2$ and $-l(r) < Y < l(r)$, and $0 < x < 1$. An easy calculation shows that

$$M'(x) = L(x) \cdot P(x),$$

where

$$L(x) := \frac{2x(2-X)((2-X)^2+Y^2)(1-x)e^{2(-2+X)x}}{((-2+X+(1-X)x)^2+Y^2(1-x)^2)^2},$$

$$P(x) := ((1-X)^2+Y^2)x^2 - (Y^2+(X-1)(X-3))x + 2 - X + Y^2/(2-X).$$

Obviously, we obtain $L(x) > 0$. Since $0 < X < 2$, we have

$$P(1) = X + Y^2/(2-X) > 0.$$

It follows from this that if $r > 0$ is small enough, then $P(x) > 0$ for $0 < x < 1$. This implies that U_β contains $(0, 1)$, and hence ∂U_β contains the critical point 1 of f_β.

Next, we show that $f_\beta : U_\beta \to D_\beta$ is univalent and $c_\beta \notin \partial U_\beta$ as follows: Let $\beta = -2+X+iY \in \tilde{Q}_r$ for $0 < X \le k(r)+2 < 2$ and $-l(r) < Y < l(r)$. Note that

$$|f_\beta(1)/\beta| = |e^\beta|, \quad |f_\beta(c_\beta)/\beta| = |e^{-\beta/(\beta+1)}/(\beta+1)^2|.$$

Let

$$H(X,Y) := |e^\beta| - |e^{-\beta/(\beta+1)}/(\beta+1)^2| = e^{-2+X} - \frac{1}{(-1+X)^2+Y^2} e^{F(X,Y)},$$

where

$$F(X,Y) := \frac{(2-X)(-1+X)-Y^2}{(-1+X)^2+Y^2}.$$

One can check that

$$\frac{\partial H(X,Y)}{\partial Y} = \frac{2Y}{((-1+X)^2+Y^2)^3} e^{F(X,Y)} \cdot G(X,Y),$$

where

$$G(X,Y) := \left(X - \frac{1}{2}\right)^2 + Y^2 - \frac{1}{4}.$$

For any fixed $0 < X \le k(r)+2$, define $Y(X) > 0$ and $T(X) > 0$ by

$$G(X, \pm Y(X)) = 0, \quad -2+X \pm iT(X) \in C_r.$$

We have $Y(X) < T(X)$, and hence

$$\begin{aligned}
\frac{\partial H(X,Y)}{\partial Y} &< 0 \quad (-T(X) < Y < -Y(X),\ 0 < Y < Y(X)),\\
\frac{\partial H(X,Y)}{\partial Y} &= 0 \quad (Y = 0,\ \pm Y(X)),\\
\frac{\partial H(X,Y)}{\partial Y} &> 0 \quad (-Y(X) < Y < 0,\ Y(X) < Y < T(X)).
\end{aligned}$$

Since $H(X,0) > 0$ from Lemma 7.1 and $H(X, \pm T(X)) = 0$, there exists a constant $W(X) \in (0, Y(X))$ such that:

$$H(X,Y) > 0 \quad (-W(X) < Y < 0,\ 0 < Y < W(X)), \tag{7.1}$$

$$H(X,Y) = 0 \quad (Y = \pm W(X)), \tag{7.2}$$

$$H(X,Y) < 0 \quad (-T(X) < Y < -W(X),\ W(X) < Y < T(X)). \tag{7.3}$$

By Lemma 7.1, we have $c_\beta \notin \overline{U_\beta}$ and $|f_\beta(1)| > |f_\beta(c_\beta)|$ for any $\beta \in \hat{B}_r$. Obviously, the mappings

$$\beta \mapsto c_\beta, \quad \beta \mapsto |f_\beta(1)|, \quad \beta \mapsto |f_\beta(c_\beta)|$$

are continuous. Therefore, there exist the positive values:

$$S^+(X) := \sup\{L > 0 \mid c_\beta \notin \overline{U_\beta} \text{ for any } \beta \in I^+(L)\},$$

$$S^-(X) := \sup\{L > 0 \mid c_\beta \notin \overline{U_\beta} \text{ for any } \beta \in I^-(L)\},$$

where

$$I^+(L) := \{\beta = -2+X+iY \in \tilde{Q}_r \mid 0 < Y < L\},$$

$$I^-(L) := \{\beta = -2+X+iY \in \tilde{Q}_r \mid -L < Y < 0\}.$$

Suppose that $S^+(X) < T(X)$. Then, as in the proofs of Lemma 5.1, Lemma 6.1, and Lemma 6.2, $f_\beta : U_\beta \to D_\beta$ is univalent, and there exists a constant $K' \ge 1$ independent of β such that ∂U_β is a K'-quasicircle for any $\beta \in I^+(S^+(X))$. Let

$$\beta(X) := -2+X+iS^+(X).$$

It follows from the argument in the proof of Lemma 5.3 that $\partial U_{\beta(X)}$ is a $(K')^2$-quasicircle and $f_{\beta(X)} : U_{\beta(X)} \to D_{\beta(X)}$ is univalent. Therefore, we have $c_{\beta(X)} \notin \overline{U_{\beta(X)}}$. Since $H(X, Y) < 0$ for $W(X) < Y < T(X)$ from (7.3), we obtain $S^+(X) \leq W(X)$. Let $p_\beta \in \overline{U_\beta}$ be the preimage of $f_\beta(c_\beta)$ by f_β for $\beta \in I^+(S^+(X)) \cup \{\beta(X)\}$. Note that $p_\beta \neq c_\beta$ and $p_\beta \in U_\beta$ for $\beta \in I^+(S^+(X))$. From the construction, we see that $S^+(X) = W(X)$ and $p_\beta \to p_{\beta(X)} \in \partial U_{\beta(X)}$ as $\beta \to \beta(X)$. Moreover, since the multiplicity of c_β is unchanged for all $\beta \in \tilde{Q}_r$, it follows that $p_{\beta(X)} \neq c_{\beta(X)}$. This implies that $c_\beta \notin \overline{U_\beta}$ for all $\beta \in I^+(S^+(X)) \cup \{\beta(X)\}$. This contradicts the definition of $S^+(X)$, and hence $S^+(X) \geq T(X)$. Similarly, we can show that $S^-(X) \geq T(X)$. Since $0 < X \leq k(r) + 2$ is arbitrary, $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and ∂U_β is a piecewise smooth Jordan curve containing exactly one critical point of f_β for any $\beta \in \tilde{Q}_r$. As in the proof of Lemma 6.2, for some constant $K \geq 1$ independent of β, ∂U_β is a K-quasicircle for any $\beta \in \tilde{Q}_r$. □

Remark 7.3. Let $W(X)$ be as in the proof of Lemma 7.2. Obviously, the map

$$W : (0, k(r) + 2] \to \mathbb{R}, \quad X \mapsto W(X)$$

is continuous. Note that $W(X) \to 0$ as $X \to 0$. In addition, it follows from the proof of Lemma 7.2 that ∂D_β contains $f_\beta(c_\beta)$ and ∂U_β does not contain c_β for $\beta = -2 + X \pm iW(X)$.

It follows from Lemma 6.1, Lemma 6.2, Lemma 7.2, and the quasiconformal surgery technique in Section 5 that:

Lemma 7.4. *There exists a continuous mapping*

$$\mathcal{S} : Q_r \to \mathbb{C} \setminus \{0, -1\}$$

such that:

(1) *If $\alpha \in \mathcal{S}(C_r)$, then $\partial \triangle_\alpha$ contains exactly two critical points;*

(2) *If $\alpha \in \mathcal{S}(\tilde{Q}_r \cup I_r)$, then $\partial \triangle_\alpha$ contains exactly one critical point* 1*;*

(3)
$$\mathcal{S}(-2) = -2.$$

Remark 7.5. From the construction, the three sets $\mathcal{S}(C_r)$, $\mathcal{S}(\tilde{Q}_r \cup I_r)$, and $\{-2\}$ are mutually disjoint.

Moreover, it follows from Remark 7.3, Lemma 7.4, and (7.1), (7.2), and (7.3) in the proof of Lemma 7.2 that:

Lemma 7.6. *Let $\mathcal{S}$ be as in Lemma 7.4. Then there exist uncountable sets $\Omega_{2,1}$, $\Omega_{2,2}$, and $\Omega_{2,3}$ in $\mathcal{S}(\tilde{Q}_r)$ such that:*

(1)
$$\mathcal{S}(\tilde{Q}_r) = \Omega_{2,1} \cup \Omega_{2,2} \cup \Omega_{2,3};$$

(2) *If $\alpha \in \Omega_{2,1}$, then $g_\alpha(c_\alpha) \in \triangle_\alpha$;*

(3) *If $\alpha \in \Omega_{2,2}$, then $g_\alpha(c_\alpha) \in \partial \triangle_\alpha$;*

(4) *If $\alpha \in \Omega_{2,3}$, then $g_\alpha(c_\alpha) \notin \overline{\triangle_\alpha}$;*

(5)
$$\Omega_{2,2} \subset \partial \Omega_{2,1} \cap \partial \Omega_{2,3}, \quad \Omega_3 := \mathcal{S}(C_r) \subset \partial \Omega_{2,3}.$$

Remark 7.7. Obviously, the three sets $\Omega_{2,1}$, $\Omega_{2,2}$, and $\Omega_{2,3}$ are mutually disjoint. The set $\mathcal{S}(\tilde{Q}_r)$ may contain some open set.

Proof of the Main Theorem (iv). By Lemma 7.4 and Lemma 7.6, we can also take

$$\Omega_2 := \mathcal{S}(\tilde{Q}_r).$$

Furthermore, this construction of $\Omega_2 := \mathcal{S}(\tilde{Q}_r)$ shows the claim. □

8. Concluding remarks

In this paper, we deal with the one parameter family $\{g_\alpha\}_{\alpha\in\mathbb{C}\setminus\{0,-1\}}$. By the Main Theorem, g_α has the Siegel disk $\triangle_\alpha$ (centered at the origin) bounded by a quasicircle containing critical points for uncountably many α. However, there are many parameters α left. We ask the following questions:

Question 2. *Are $\triangle_\alpha$ bounded by quasicircles containing at least one critical point of g_α for all $\alpha \in \mathbb{C} \setminus \{0,-1\}$?*

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