POLYNOMIAL INVARIANTS OF VIRTUAL DOODLES AND FLAT VIRTUAL KNOTS

JOONOH KIM AND KYOUNG-TARK KIM

Abstract. In this paper, we introduce new polynomial invariants of virtual doodles obtained by combining a state model with an integer labeling to flat virtual knot diagrams. Using our invariants we show that the doodle diagram $d_{4.1}$ is different from the doodle diagram $d_{4.4}$.

1. Introduction

A virtual link diagram is a link diagram in $\mathbb{R}^2$ possibly having some encircled crossings without over/under information. Such an encircled crossing is called a virtual crossing (see Fig. 1).

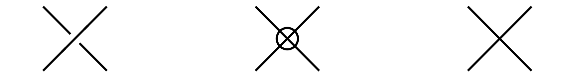

Figure 1. Crossing types

Two virtual link diagrams are said to be equivalent if they are related by a finite sequence of generalized Reidemeister moves described in Fig. 2. A virtual link is defined as an equivalence class of virtual link diagrams [4].

A flat virtual link diagram is a virtual link diagram in $\mathbb{R}^2$ in which the over/under information at each real crossing is ignored. In particular, a 1-component flat virtual link diagram is called a flat virtual knot diagram. A crossing without over/under information is called a flat crossing (see Fig. 1). Two flat virtual link diagrams are said to be equivalent if they are related by a finite sequence of generalized flat Reidemeister moves described in Fig. 3, which are flattened versions of the generalized Reidemeister moves. A flat virtual link is defined as an equivalence class of flat virtual link diagrams. Hence, a flat virtual knot is an equivalence class of flat virtual knot diagrams.

Doodles were first introduced in [1]. The original definition of a doodle was a collection of embedded circles in the sphere S^2 with no triple or higher intersection points. Khovanov [8] extended the idea to allow

2020 Mathematics Subject Classification. Primary: 57K12, Secondary: 57K16.
Key words and phrases. virtual link; flat virtual doodle; polynomial invariant.

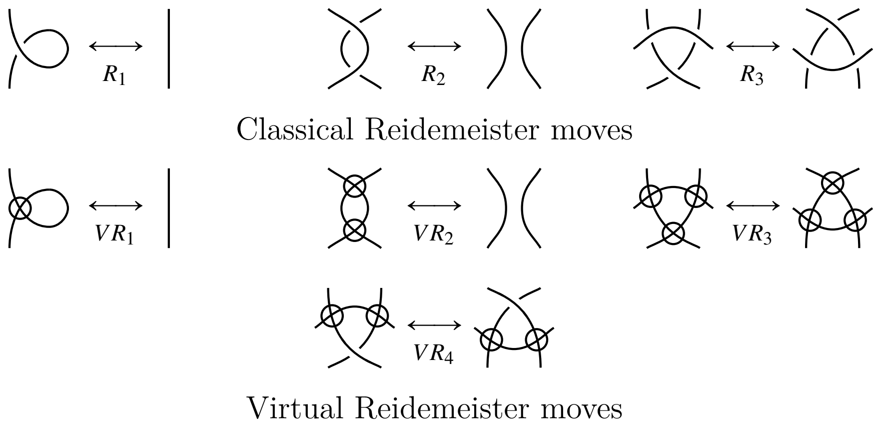

Figure 2. Generalized Reidemeister moves

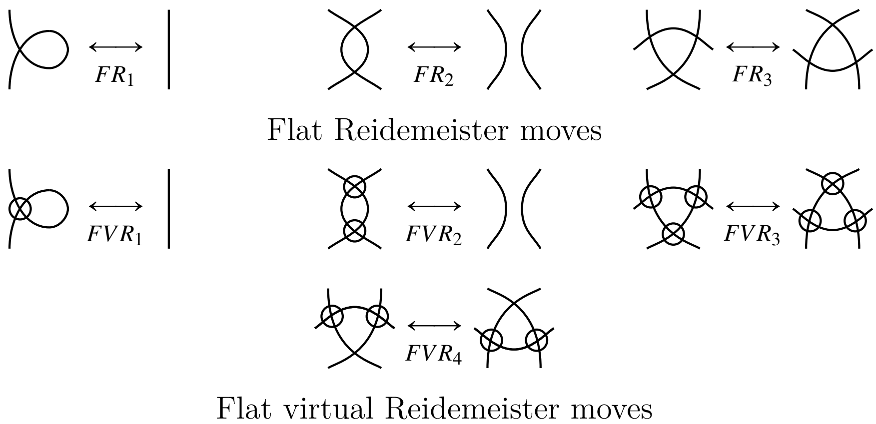

Figure 3. Generalized flat Reidemeister moves

each component to be an immersed circle in S^2. In [3], Bartholomew, Fenn, N. Kamada, and S. Kamada extend the range of doodles to immersed circles in closed orientated surfaces of any genus.

A virtual doodle diagram can be regarded as a flat virtual link diagram in S^2. So, a virtual doodle diagram with just one component may be a flat virtual knot diagram. Two virtual doodle diagrams are called equivalent if they are related by a finite sequence of the generalized flat Reidemeister moves except the move FR_3 in Fig. 3, called doodle moves. A virtual doodle is defined as an equivalence class of virtual doodle diagrams [3].

Let D be a flat virtual link diagram. D is said to be oriented if each component of D is oriented. Equivalence of two oriented flat virtual link diagrams is similarly defined as in the case of unoriented diagrams except that orientations are taken into consideration. An oriented flat virtual link is defined to be an equivalence class of oriented flat virtual link diagrams.

The study of virtual knots is closely related to that of flat virtual knots. Indeed, for a given virtual knot diagram, we obtain a flat virtual knot diagram by ignoring the over/under information at each real crossing. Thus, some invariants of virtual knots are modified ones of flat virtual knots [5, 6, 7].

Invariants of virtual doodles are hardly known as research has only recently been carried out. In [3], Bartholomew, Fenn, N. Kamada, and S. Kamada introduced an invariant defined in terms of colorings of virtual diagrams, specifically flat virtual knot diagrams. In [7], the author presented an invariant using the flat virtual knot invariant introduced in [5].

The purpose of this paper is to provide new polynomial invariants for virtual doodles and flat virtual knots. This paper is organized as follows: In Section 2, we present our method for constructing a new polynomial invariant of flat virtual knots. In Section 3 and Section 4, we demonstrate the invariance of this polynomial. Finally, in Section 5, we apply the method described in Section 2 to provide a polynomial invariant for virtual doodles.

In the following sections, we assume that diagrams, virtual links and virtual doodles are oriented.

Acknowledgments

The author, Joonoh Kim, is supported by NRF-2021R1I1A1A01050446. The author, Kyoung-Tark Kim, is supported by NRF-2021R1I1A3050400.

The authors are very grateful to the reviewer for providing detailed explanations that address the oversights and errors in the paper.

2. The polynomial $P_D(t)$ for a flat virtual knot diagram D

Let D be a flat virtual knot diagram and $C(D)$ the set of all flat crossings of D. Suppose that $|C(D)| \geqslant 2$. We first fix $p \in C(D)$, which we call a base point, and locally assign integers ± 1 to two arcs near p as in Fig. 4.

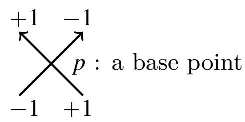

Figure 4. Local $\{\pm 1\}$-assignment near a base point

Now consider $q \in C(D)\backslash\{p\}$. As shown in the leftmost diagram in Fig. 5, let $D(q)$ be the diagram obtained from D by smoothing the flat crossing q. The diagram $D(q)$ has two components. We say that $D(q)$ is compatible with p if p is a mixed crossing of $D(q)$, i.e., p is a crossing of distinct two components of $D(q)$; see Fig. 6.

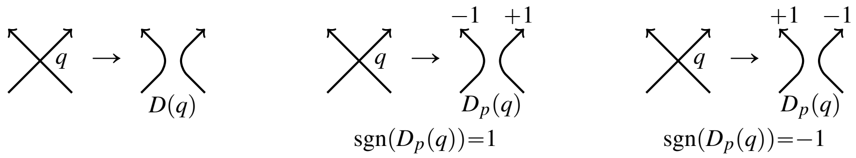

Figure 5. Sign of the diagram $D_p(q)$

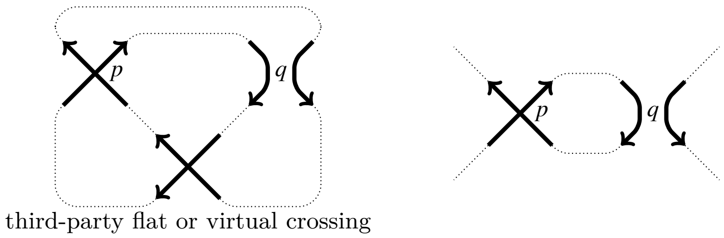

Figure 6. Examples of compatibility and incompatibility

For $p \in C(D)$, we define

$$C_p(D) = \{q \in C(D)\backslash\{p\} : D(q) \text{ is compatible with } p\}.$$

If the diagram $D(q)$ is compatible with the base point p, then the labels ± 1 are given to the two components of $D(q)$ in such a way that the assignment at p is as in Fig. 4. The diagram $D(q)$ together with the labeled components is denoted by $D_p(q)$.

The sign of the diagram $D_p(q)$, which we denote by $\mathrm{sgn}(D_p(q))$, is defined as ± 1 depending on the situation depicted in Fig. 5. For $c \in C(D_p(q))$, we define $\mathrm{ind}(D_p(q); c)$ by the rule depicted in Fig. 7.

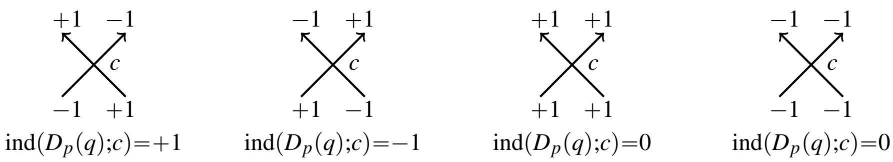

Figure 7. Indices for $c \in C(D_p(q))$

For $q \in C_p(D)$, we define the linking index $\mathrm{lk}(D_p(q))$ by

$$\mathrm{lk}(D_p(q)) = \sum_{c \in C(D_p(q))} \mathrm{ind}(D_p(q); c).$$

Now, we are in a position to give the following definition.

Definition 1. For a flat virtual knot diagram D, we define a polynomial $P_D(t) \in \mathbb{Z}[t]$ as follows:

$$P_D(t) = \begin{cases} 0 & \text{if } |C(D)| \leqslant 1, \\ \displaystyle\sum_{p \in C(D)} \sum_{q \in C_p(D)} \mathrm{sgn}(D_p(q)) \cdot t^{|\mathrm{lk}(D_p(q))|} & \text{otherwise.} \end{cases}$$

Remark 2. Let $-D$ be the inverse oriented diagram of D. Then, according to Definition 1, we have $P_{-D}(t) = -P_D(t)$ because the indices in Fig. 4 and Fig. 7 are preserved for $-D$, while the signs in Fig. 5 are reversed for $-D$.

Example 3. Let D be the following flat virtual knot diagram as in Fig. 8.

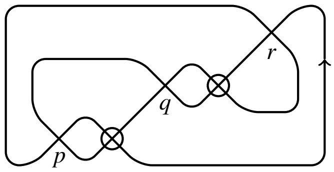

Figure 8. Example, the diagram D

Note that $C(D) = \{p,q,r\}$, $C_p(D) = \{q,r\}$, $C_q(D) = \{p\}$, and $C_r(D) = \{p\}$. Thus

$$\begin{aligned}P_D(t) &= \operatorname{sgn}(D_p(q))t^{|\mathrm{lk}(D_p(q))|} + \operatorname{sgn}(D_p(r))t^{|\mathrm{lk}(D_p(r))|} + \operatorname{sgn}(D_q(p))t^{|\mathrm{lk}(D_q(p))|} \\ &\quad + \operatorname{sgn}(D_r(p))t^{|\mathrm{lk}(D_r(p))|} \\ &= (+1)t^{|+1|} + (+1)t^{|+1|} + (-1)t^{|+2|} + (-1)t^{|+2|} = -2t^2 + 2t.\end{aligned}$$

3. Invariance of $P_D(t)$ under the moves FR_1 and FR_2

In this section, we will show that $P_D(t)$ is invariant under the moves FR_1 and FR_2. Throughout this and the next section, the symbol D stands for an oriented flat virtual knot diagram.

Lemma 4. Suppose $|C(D)| \geqslant 3$. Let E be the diagram obtained from D by applying a move FR_1 eliminating a flat crossing r of D. If $p \in C(E) = C(D)\backslash\{r\}$ and $q \in C_p(E)$, then $\operatorname{sgn}(D_p(q)) = \operatorname{sgn}(E_p(q))$ and $\mathrm{lk}(D_p(q)) = \mathrm{lk}(E_p(q))$.

Proof. It is obvious that $\mathrm{ind}(D_p(q);r) = 0$. Since the move FR_1 eliminating r fixes all crossings except r, we see that $\mathrm{ind}(D_p(q);c) = \mathrm{ind}(E_p(q);c)$ for $c \in C(D_p(q))\backslash\{r\}$. Hence, we obtain

$$\begin{aligned}\mathrm{lk}(D_p(q)) &= \sum_{c\in C(D_p(q))} \mathrm{ind}(D_p(q);c) = \mathrm{ind}(D_p(q);r) + \sum_{c\in C(D_p(q))\backslash\{r\}} \mathrm{ind}(D_p(q);c) \\ &= \sum_{c\in C(E_p(q))} \mathrm{ind}(E_p(q);c) = \mathrm{lk}(E_p(q)).\end{aligned}$$

We also have $\operatorname{sgn}(D_p(q)) = \operatorname{sgn}(E_p(q))$ for the same reason as above. □

Proposition 5. The polynomial $P_D(t)$ is invariant under the move FR_1.

Proof. Let E be the diagram obtained from D by applying a move FR_1 eliminating a flat crossing r of D. For any $p \in C(D)$, we put

$$P_{(D,p)}(t) = \sum_{q\in C_p(D)} \operatorname{sgn}(D_p(q))t^{|\mathrm{lk}(D_p(q))|}. \tag{3.1}$$

Since $C_r(D) = \emptyset$, we have $P_{(D,r)}(t) = 0$ by definition. Hence, we obtain

$$P_D(t) = P_{(D,r)}(t) + \sum_{p \in C(D) \setminus \{r\}} P_{(D,p)}(t) = \sum_{p \in C(D) \setminus \{r\}} P_{(D,p)}(t).$$

Now, we have three cases. First, we consider the case where $|C(D)| = 1$. The definition of the polynomial gives $P_D(t) = 0 = P_E(t)$.

Next, we consider the case where $|C(D)| = 2$. Let $C(D) = \{r, p\}$. Since $C_p(D) = \emptyset$, we have $P_{(D,p)} = 0$ and hence $P_D(t) = 0$. On the other hand, we obtain $P_E(t) = 0$ because $|C(E)| = 1$.

Finally, we consider the case where $|C(D)| \geqslant 3$. Since $r \notin C_p(D)$ for $p \in C(D) \setminus \{r\}$, Lemma 4 gives the equality $P_{(D,p)}(t) = P_{(E,p)}(t)$, which leads to $P_D(t) = P_E(t)$. This completes the proof. $\square$

Lemma 6. Suppose that $|C(D)| \geqslant 2$, and that D has a 2-gon whose vertices are flat crossings r and s as depicted in Fig. 9. Then, for $p \in C(D) \setminus \{r, s\}$, either $\{r, s\} \cap C_p(D) = \emptyset$ or $\{r, s\} \subseteq C_p(D)$ holds. Furthermore, if $\{r, s\} \subseteq C_p(D)$, then $\mathrm{sgn}(D_p(r)) = -\mathrm{sgn}(D_p(s))$ and $\mathrm{lk}(D_p(r)) = \mathrm{lk}(D_p(s))$.

Figure 9. 2-gons

Proof. Since $D_p(r)$ is equivalent to $D_p(s)$, $r \in C_p(D)$ if and only if $s \in C_p(D)$. Hence, either $\{r, s\} \subseteq C_p(D)$ or $\{r, s\} \cap C_p(D) = \emptyset$ holds.

Now, suppose $\{r, s\} \subseteq C_p(D)$. Since the components of $D_p(r)$ and $D_p(s)$ are labeled in a neighborhood N as depicted in Fig. 10, it is clear that $\mathrm{sgn}(D_p(r)) = -\mathrm{sgn}(D_p(s))$, where $\varepsilon \in \{\pm 1\}$. Since the splices at r and s fix all flat crossings in $C_p(D) \setminus \{r, s\}$, we see that, for $c \in C(D_p(r)) \setminus \{s\} = C(D_p(s)) \setminus \{r\}$, $\mathrm{ind}(D_p(r); c) = \mathrm{ind}(D_p(s); c)$. Moreover, note from Fig. 10 that $\mathrm{ind}(D_p(r); s) = \mathrm{ind}(D_p(s); r)$. Hence, we obtain

$$\begin{aligned}
\mathrm{lk}(D_p(r)) &= \sum_{c \in C(D_p(r))} \mathrm{ind}(D_p(r); c) = \mathrm{ind}(D_p(r); s) + \sum_{c \in C(D_p(r)) \setminus \{s\}} \mathrm{ind}(D_p(r); c) \\
&= \mathrm{ind}(D_p(s); r) + \sum_{c \in C(D_p(s)) \setminus \{r\}} \mathrm{ind}(D_p(s); c) = \sum_{c \in C(D_p(s))} \mathrm{ind}(D_p(s); c) \\
&= \mathrm{lk}(D_p(s)).
\end{aligned}$$

$\square$

Lemma 7. Suppose that $|C(D)| \geqslant 2$, and that D has a 2-gon whose vertices are flat crossings r and s as in the left diagram in Fig. 9. Let $X = C_r(D) \cap C_s(D)$. Then, the following hold:

(i) $X \cap \{r, s\} = \emptyset$, $C_r(D) = X \cup \{s\}$, and $C_s(D) = X \cup \{r\}$.

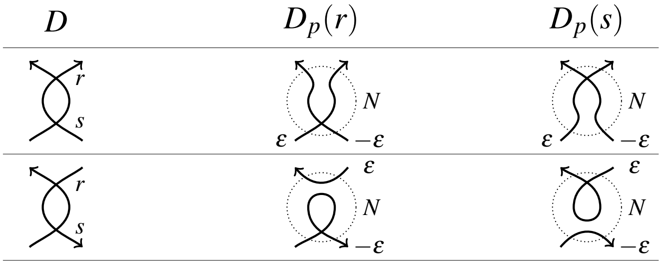

Figure 10. $D_p(r)$ and $D_p(s)$ (where $\varepsilon \in \{\pm 1\}$)

(ii) For $q \in X$, $\mathrm{sgn}(D_r(q)) = -\mathrm{sgn}(D_s(q))$ and $\mathrm{lk}(D_r(q)) = -\mathrm{lk}(D_s(q))$.

(iii) $\mathrm{sgn}(D_r(s)) = 1$, $\mathrm{sgn}(D_s(r)) = -1$, and $\mathrm{lk}(D_r(s)) = \mathrm{lk}(D_s(r))$.

Proof. (i) Since $r \notin C_r(D)$ and $s \notin C_s(D)$, we have $X \cap \{r,s\} = \varnothing$. It is clear that $s \in C_r(D)$ and $r \in C_s(D)$. For $p \in C(D) \backslash \{r,s\}$, r is a mixed crossing of $D(p)$ if and only if s is a mixed crossing of $D(p)$, whence $p \in C_r(D)$ if and only if $p \in C_s(D)$. Thus, either $p \in X$ or $p \notin C_r(D) \cup C_s(D)$ holds. We claim that $C_r(D) \backslash X = \{s\}$. It is clear that $C_r(D) \backslash X \supseteq \{s\}$. Let $p \in C_r(D) \backslash X$. Suppose that $p \in C(D) \backslash \{r,s\}$. Since $p \notin X$, it follows from the previous argument that $p \notin C_r(D) \cup C_s(D)$ which contradicts to $p \in C_r(D)$. Thus, $p \in \{r,s\}$. Since $r \notin C_r(D)$, we have $p = s$ and $C_r(D) \backslash X = \{s\}$. Since $X \subseteq C_r(D)$, we see that $C_r(D) = X \cup \{s\}$. The proof for $C_s(D) = X \cup \{r\}$ is similar.

(ii) Note that $D_r(q)$ and $D_s(q)$ are the same 2-component diagram except their labels. We put $D_r(q) = E_r \cup F_r$ and $D_s(q) = E_s \cup F_s$ where we may assume that, when disregarding the labels, E_r and F_r are the same diagrams as E_s and F_s, respectively. If we denote the labels given to E_r, F_r, E_s, and F_s by $l(E_r)$, $l(F_r)$, $l(E_s)$, and $l(F_s)$, respectively, then we see that $l(E_r) = l(F_s)$ and $l(F_r) = l(E_s)$, that is, $D_r(q)$ and $D_s(q)$ have opposite labels. Hence, $\mathrm{sgn}(D_r(q)) = -\mathrm{sgn}(D_s(q))$. Since $D_r(q)$ and $D_s(q)$ have opposite labels, the equality $\mathrm{ind}(D_r(q);c) = \mathrm{ind}(D_s(q);c)$ holds for $c \in C(D_r(q)) = C(D_s(q))$. Thus, we have

$$\mathrm{lk}(D_r(q)) = \sum_{c \in C(D_r(q))} \mathrm{ind}(D_r(q);c) = \sum_{c \in C(D_s(q))} (-\mathrm{ind}(D_s(q);c)) = -\mathrm{lk}(D_s(q)).$$

(iii) The diagrams of Fig. 10 indicate that $\mathrm{sgn}(D_r(s)) = 1$ and $\mathrm{sgn}(D_s(r)) = -1$. Since $D_r(s)$ and $D_s(r)$ are identical as a labeled diagram, we have $\mathrm{ind}(D_r(s);r) = \mathrm{ind}(D_s(r);s)$. Since $C(D_r(s)) \backslash \{r\} = C(D_s(r)) \backslash \{s\}$, we also have $\mathrm{ind}(D_r(s);c) = \mathrm{ind}(D_s(r);c)$ for $c \in$

$C(D_r(s))\backslash\{r\} = C(D_s(r))\backslash\{s\}$. The previous facts give

$$\begin{aligned}
\mathrm{lk}(D_r(s)) &= \sum_{c\in C(D_r(s))} \mathrm{ind}(D_r(s);c) = \mathrm{ind}(D_r(s);r) + \sum_{c\in C(D_r(s))\backslash r} \mathrm{ind}(D_r(s);c) \\
&= \mathrm{ind}(D_s(r);s) + \sum_{c\in C(D_s(r))\backslash s} \mathrm{ind}(D_s(r);c) = \sum_{c\in C(D_s(r))} \mathrm{ind}(D_s(r);c) \\
&= \mathrm{lk}(D_s(r))
\end{aligned}$$

□

Lemma 8. Suppose that $|C(D)| \geqslant 2$, and that D has a 2-gon whose vertices are flat crossings r and s as in the right diagram in Fig. 9. Then, the following hold:

(i) $C_r(D) = C_s(D)$ and $C_r(D) \cap \{r,s\} = \varnothing$.
(ii) For $q \in C_r(D)$, $\mathrm{sgn}(D_r(q)) = -\mathrm{sgn}(D_s(q))$ and $\mathrm{lk}(D_r(q)) = -\mathrm{lk}(D_s(q))$.

Proof. (i) Since r and s are self crossings of $D(s)$ and $D(r)$ respectively, we have $r \notin C_s(D)$ and $s \notin C_r(D)$, where a flat crossing is said to be self if the flat crossing is formed by only one component. For $p \in C(D)\backslash\{r,s\}$, r is a mixed crossing of $D(p)$ if and only if s is a mixed crossing of $D(p)$, whence $p \in C_r(D)$ if and only if $p \in C_s(D)$. Therefore, we have $C_r(D) = C_s(D)$ and $C_r(D) \cap \{r,s\} = \varnothing$.

(ii) By the similar reason as in the proof of Lemma 7 (ii), observe that $D_r(q)$ and $D_s(q)$ have opposite labels. It follows that $\mathrm{sgn}(D_r(q)) = -\mathrm{sgn}(D_s(q))$ and $\mathrm{lk}(D_r(q)) = -\mathrm{lk}(D_s(q))$. □

Lemma 9. Suppose that $|C(D)| \geqslant 3$, and that E is the diagram obtained from D by applying a move FR_2 eliminating two flat crossings r and s of D. Let $p \in C(E) = C(D)\backslash\{r,s\}$ and $q \in C_p(E)$. Then, we have $\mathrm{sgn}(D_p(q)) = \mathrm{sgn}(E_p(q))$ and $\mathrm{lk}(D_p(q)) = \mathrm{lk}(E_p(q))$.

Proof. Since the move FR_2 fixes the outside of a neighborhood N as depicted in Fig. 10, it follows that $\mathrm{sgn}(D_p(q)) = \mathrm{sgn}(E_p(q))$ and $\mathrm{ind}(D_p(q);c) = \mathrm{ind}(E_p(q);c)$ for $c \in C(E) = C(D)\backslash\{r,s\}$. It is also easy to see that $\mathrm{ind}(D_p(q);r) = -\mathrm{ind}(D_p(q);s)$. By the previous facts, we obtain

$$\begin{aligned}
\mathrm{lk}(D_p(q)) &= \sum_{c\in C(D_p(q))} \mathrm{ind}(D_p(q);c) \\
&= \mathrm{ind}(D_p(q);r) + \mathrm{ind}(D_p(q);s) + \sum_{c\in C(D_p(q))\backslash\{r,s\}} \mathrm{ind}(D_p(q);c) \\
&= \sum_{c\in C(E_p(q))} \mathrm{ind}(E_p(q);c) = \mathrm{lk}(E_p(q)).
\end{aligned}$$

□

For an oriented flat virtual knot diagram D with a 2-gon, whose vertices are flat crossings r and s as in Fig. 9, we put

$$P_{(D,r,s)}(t) = P_{(D,r)}(t) + P_{(D,s)}(t),$$

where we use Eq. (3.1) in the right hand side. Then, we have

$$P_D(t) = P_{(D,r,s)}(t) + \sum_{p \in C(D) \setminus \{r,s\}} P_{(D,p)}(t).$$

Lemma 10. Suppose that $|C(D)| \geqslant 2$, and that D has a 2-gon whose vertices are flat crossings r and s as in the left diagram in Fig. 9. Then, we have $P_{(D,r,s)}(t) = 0$.

Proof. Let $X = C_r(D) \cap C_s(D)$. If $X \neq \varnothing$, then Lemma 7 implies that

$$\begin{aligned} P_{(D,r,s)}(t) &= \left(\operatorname{sgn}(D_r(s)) t^{|\operatorname{lk}(D_r(s))|} + \sum_{p \in X} \operatorname{sgn}(D_r(p)) t^{|\operatorname{lk}(D_r(p))|} \right) \\ &\quad + \left(\operatorname{sgn}(D_s(r)) t^{|\operatorname{lk}(D_s(r))|} + \sum_{p \in X} \operatorname{sgn}(D_s(p)) t^{|\operatorname{lk}(D_s(p))|} \right) \\ &= \left(\operatorname{sgn}(D_r(s)) t^{|\operatorname{lk}(D_r(s))|} + \operatorname{sgn}(D_s(r)) t^{|\operatorname{lk}(D_s(r))|} \right) \\ &\quad + \left(\sum_{p \in X} \operatorname{sgn}(D_r(p)) t^{|\operatorname{lk}(D_r(p))|} + \sum_{p \in X} \operatorname{sgn}(D_s(p)) t^{|\operatorname{lk}(D_s(p))|} \right) \\ &= 0 \end{aligned}$$

If $X = \varnothing$, then Lemma 7 also shows that

$$P_{(D,r,s)}(t) = \operatorname{sgn}(D_r(s)) t^{|\operatorname{lk}(D_r(s))|} + \operatorname{sgn}(D_s(r)) t^{|\operatorname{lk}(D_s(r))|} = 0. \qquad \square$$

Lemma 11. Suppose that $|C(D)| \geqslant 2$, and that D has a 2-gon whose vertices are flat crossings r and s as in the right diagram in Fig. 9. Then, we have $P_{(D,r,s)}(t) = 0$.

Proof. Lemma 8 gives

$$P_{(D,r,s)}(t) = \sum_{q \in C_r(D)} \left(\operatorname{sgn}(D_r(q)) + \operatorname{sgn}(D_s(q)) \right) t^{|\operatorname{lk}(D_r(q))|} = 0. \qquad \square$$

If a flat virtual knot diagram is oriented, then it exhibits two types of the move FR_2, which depend on the orientations of the two arcs involved in the move. If the move is realized by two arcs with the same direction as in the left figure of Fig. 9, then it is called type A. The move realized by two arcs with opposite directions as in the right figure of Fig. 9 is called type B.

Lemma 12. The polynomial $P_D(t)$ is invariant under the move FR_2 of type A.

Proof. Suppose $|C(D)| \geqslant 2$. Let E be the diagram obtained from D by applying a move FR_2 of type A eliminating flat crossings r and s of D. We have three cases based on the number $|C(D)|$.

First, we consider the case where $|C(D)| = 2$, i.e., $C(D) = \{r,s\}$. It is easy to see that $C_r(D) = \{s\}$ and $C_s(D) = \{r\}$. Lemma 10 gives $P_D(t) =$

$P_{(D,r,s)}(t) = 0$. Since $|C(E)| = 0$, we have $P_E(t) = 0$ by Definition 1. Hence, it holds that $P_D(t) = P_E(t)$.

Next, we consider the case where $|C(D)| = 3$. Suppose that $C(D) = \{r, s, p\}$. We claim that $P_{(D,p)}(t) = 0$. Note that $C_p(D) = \varnothing$ or $C_p(D) = \{r, s\}$ by Lemma 6. If $C_p(D) = \varnothing$, then it is clear that $P_{(D,p)}(t) = 0$. If $C_p(D) = \{r, s\}$, then Lemma 6 also gives

$$P_{(D,p)}(t) = \operatorname{sgn}(D_p(r))t^{|\operatorname{lk}(D_p(r))|} + \operatorname{sgn}(D_p(s))t^{|\operatorname{lk}(D_p(s))|} = 0.$$

The previous claim and Lemma 10 yield that $P_D(t) = P_{(D,p)}(t) + P_{(D,r,s)}(t) = 0$. On the other hand, since $|C(E)| = 1$, $P_E(t) = 0$ by Definition 1. Thus, it holds that $P_D(t) = P_E(t)$.

Finally, we consider the case where $|C(D)| \geqslant 4$. Note that $C(E) = C(D) \backslash \{r, s\}$. Since $P_{(D,r,s)}(t) = 0$ by Lemma 10, it is sufficient to show that $P_{(D,p)}(t) = P_{(E,p)}(t)$ for $p \in C(E)$, which leads to $P_D(t) = P_E(t)$. By Lemma 6 we have $\{r, s\} \cap C_p(D) = \varnothing$ or $\{r, s\} \subseteq C_p(D)$. If $\{r, s\} \cap C_p(D) = \varnothing$, then $C_p(D) = C_p(E)$ whence Lemma 9 gives

$$\begin{aligned} P_{(D,p)}(t) &= \sum_{q \in C_p(D)} \operatorname{sgn}(D_p(q))t^{|\operatorname{lk}(D_p(q))|} = \sum_{q \in C_p(E)} \operatorname{sgn}(D_p(q))t^{|\operatorname{lk}(D_p(q))|} \\ &= \sum_{q \in C_p(E)} \operatorname{sgn}(E_p(q))t^{|\operatorname{lk}(E_p(q))|} = P_{(E,p)}(t). \end{aligned}$$

If $\{r, s\} \subseteq C_p(D)$, then $C_p(D) = C_p(E) \cup \{r, s\}$ and Lemma 6 gives

$$\operatorname{sgn}(D_p(r))t^{|\operatorname{lk}(D_p(r))|} + \operatorname{sgn}(D_p(s))t^{|\operatorname{lk}(D_p(s))|} = 0.$$

Hence, we obtain $P_{D,p}(t) = P_{E,p}(t)$ by Lemma 9 as in the previous case. This completes the proof. $\square$

Lemma 13. The polynomial $P_D(t)$ is invariant under the move FR_2 of type B.

Proof. The proof is similar to that of Lemma 12, using Lemma 8 and Lemma 11, with the exception that there is a distinction in the case of $|C(D)| = 2$ for two sets, $C_r(D)$ and $C_s(D)$. $\square$

The following is an immediate consequence from Lemmas 12 and 13.

Proposition 14. The polynomial $P_D(t)$ is invariant under the move FR_2.

4. Invariance of $P_D(t)$ under the moves FR_3 and FVR_4

In this section, we will show that $P_D(t)$ is invariant under the moves FR_3 and FVR_4. There are two oriented versions of the triangle move FR_3 as illustrated in Fig. 11. It is known that FR_3 of type B can be realized by a finite sequence of the moves FR_3 of type A and FR_2. Therefore, in the sequel, we will focus on demonstrating that the polynomial $P_D(t)$ remains invariant under the move FR_3 of type A.

Figure 11. Two oriented types of the move FR_3

Figure 12. The settings in D and E

Lemma 15. Let E be the diagram obtained from D through a single move FR_3 of type A. Assuming that the settings in D and E are as depicted in Fig. 12, the following statements hold:

(i) $r_0 \in C_{r_1}(D)$ if and only if $r_0 \in C_{r_2}(D)$; in this case, $\mathrm{sgn}(D_{r_1}(r_0)) = -1$, $\mathrm{sgn}(D_{r_2}(r_0)) = 1$, and $\mathrm{lk}(D_{r_1}(r_0)) = -\mathrm{lk}(D_{r_2}(r_0))$.

(ii) $s_0 \in C_{s_1}(E)$ if and only if $s_0 \in C_{s_2}(E)$; in this case, $\mathrm{sgn}(E_{s_1}(s_0)) = 1$, $\mathrm{sgn}(E_{s_2}(s_0)) = -1$, and $\mathrm{lk}(E_{s_1}(s_0)) = -\mathrm{lk}(E_{s_2}(s_0))$.

(iii) $r_2 \in C_{r_1}(D)$ if and only if $s_2 \in C_{s_0}(E)$; in this case, $\mathrm{sgn}(D_{r_1}(r_2)) = \mathrm{sgn}(E_{s_0}(s_2)) = 1$ and $\mathrm{lk}(D_{r_1}(r_2)) = \mathrm{lk}(E_{s_0}(s_2))$.

(iv) $r_1 \in C_{r_2}(D)$ if and only if $s_1 \in C_{s_0}(E)$; in this case, $\mathrm{sgn}(D_{r_2}(r_1)) = \mathrm{sgn}(E_{s_0}(s_1)) = -1$ and $\mathrm{lk}(D_{r_2}(r_1)) = \mathrm{lk}(E_{s_0}(s_1))$.

(v) $r_1 \in C_{r_0}(D)$ if and only if $s_1 \in C_{s_2}(E)$; in this case, $\mathrm{sgn}(D_{r_0}(r_1)) = \mathrm{sgn}(E_{s_2}(s_1)) = 1$ and $\mathrm{lk}(D_{r_0}(r_1)) = \mathrm{lk}(E_{s_2}(s_1))$.

(vi) $r_2 \in C_{r_0}(D)$ if and only if $s_2 \in C_{s_1}(E)$; in this case, $\mathrm{sgn}(D_{r_0}(r_2)) = \mathrm{sgn}(E_{s_1}(s_2)) = -1$ and $\mathrm{lk}(D_{r_0}(r_2)) = \mathrm{lk}(E_{s_1}(s_2))$.

Proof. To begin with, we show the claim (i). Since r_1 and r_2 form a 2-gon in $D(r_0)$, it follows that $r_0 \in C_{r_1}(D)$ if and only if $r_0 \in C_{r_2}(D)$. Since $D_{r_1}(r_0)$ and $D_{r_2}(r_0)$ are the same diagram with opposite labels, we have $\mathrm{lk}(D_{r_1}(r_0)) = -\mathrm{lk}(D_{r_2}(r_0))$. From the labels at the base points, we see that $\mathrm{sgn}(D_{r_1}(r_0)) = -1$ and $\mathrm{sgn}(D_{r_2}(r_0)) = 1$.

Next, we prove the claim (ii). We find that s_1 and s_2 form a 2-gon in $D(s_0)$. We also find that $E_{s_1}(s_0)$ and $E_{s_2}(s_0)$ are the same diagram with opposite labels. Hence, the same reason as in the previous case give the claim (ii).

Furthermore, we give the proof of the claim (iii). $D(r_2)$ is isotopic to $E(s_2)$. Since r_1 and s_0 are the corresponding crossings, it is easy to see that $r_2 \in C_{r_1}(D)$ if and only if $s_2 \in C_{s_0}(E)$. Since $D_{r_1}(r_2)$ is isotopic to $E_{s_0}(s_2)$, we easily obtain the remaining claims.

Finally, we clarify the claims (iv), (v) and (vi). These claims are analogous to the claim (iii). Hence, the proofs for them are also similar to that of the claim (iii), completing the proof. □

Now, whenever $q \notin C_p(D)$ for $p \in C(D)$ and $q \in C(D) \backslash \{p\}$, we adopt the convention $\mathrm{sgn}(D_p(q)) = 0$ and $\mathrm{lk}(D_p(q)) = 0$. From this convention, it is evident to see that the polynomial $P_D(t)$ in Definition 1 can be written by

$$P_D(t) = \sum_{p \in C(D)} \sum_{q \in C(D) \backslash \{p\}} W(D_p(q);t), \tag{4.1}$$

where $W(D_p(q;t)) = \mathrm{sgn}(D_p(q)) \cdot t^{|\mathrm{lk}(D_p(q))|}$.

Let E be the diagram from D by applying a move FR_3 of type A. Suppose that D and E differ only in one place as in Fig. 11. Let $R = \{r_0, r_1, r_2\} \subseteq C(D)$ and $S = \{s_0, s_1, s_2\} \subseteq C(E)$ (cf. Fig. 12), and $T = C(D) \backslash R = C(E) \backslash S$. Since $C(D) \backslash \{p\} = T \cup (R \backslash \{p\})$ for $p \in R$ and $C(D) \backslash \{p\} = R \cup (T \backslash \{p\})$ for $p \in T$, we have

$$\begin{aligned} P_D(t) &= \sum_{p \in R} \left(\sum_{q \in C(D) \backslash \{p\}} W(D_p(q);t) \right) + \sum_{p \in T} \left(\sum_{q \in C(D) \backslash \{p\}} W(D_p(q);t) \right) \\ &= \sum_{p \in R} \left(\sum_{q \in R \backslash \{p\}} W(D_p(q);t) \right) + \sum_{p \in R} \left(\sum_{q \in T} W(D_p(q);t) \right) \\ &\quad + \sum_{p \in T} \left(\sum_{q \in R} W(D_p(q);t) \right) + \sum_{p \in T} \left(\sum_{q \in T \backslash \{p\}} W(D_p(q);t) \right). \end{aligned}$$

We also obtain

$$\begin{aligned} P_E(t) &= \sum_{p \in S} \left(\sum_{q \in S \backslash \{p\}} W(D_p(q);t) \right) + \sum_{p \in S} \left(\sum_{q \in T} W(D_p(q);t) \right) \\ &= \sum_{p \in T} \left(\sum_{q \in S} W(D_p(q);t) \right) + \sum_{p \in T} \left(\sum_{q \in T \backslash \{p\}} W(D_p(q);t) \right). \end{aligned}$$

Then, we have the following.

Lemma 16. The following statements hold:

(i) $\displaystyle \sum_{p \in R} \left(\sum_{q \in R \backslash \{p\}} W(D_p(q);t) \right) = \sum_{p \in S} \left(\sum_{q \in S \backslash \{p\}} W(D_p(q);t) \right).$

(ii) $\displaystyle \sum_{p \in R} \left(\sum_{q \in T} W(D_p(q);t) \right) = \sum_{p \in S} \left(\sum_{q \in T} W(D_p(q);t) \right).$

(iii) $\displaystyle \sum_{p \in T} \left(\sum_{q \in R} W(D_p(q);t) \right) = \sum_{p \in T} \left(\sum_{q \in S} W(D_p(q);t) \right).$

Proof. (i) Using Eq. (4.1) and Lemma 15, we obtain

$$\begin{aligned}
\sum_{p\in R}\left(\sum_{q\in R\setminus\{p\}} W(D_p(q);t)\right) &= \left(\operatorname{sgn}(D_{r_1}(r_0))t^{|\operatorname{lk}(D_{r_1}(r_0))|} + \operatorname{sgn}(D_{r_2}(r_0))t^{|\operatorname{lk}(D_{r_2}(r_0))|}\right) \\
&\quad + \operatorname{sgn}(D_{r_1}(r_2))t^{|\operatorname{lk}(D_{r_1}(r_2))|} + \operatorname{sgn}(D_{r_2}(r_1))t^{|\operatorname{lk}(D_{r_2}(r_1))|} \\
&\quad + \operatorname{sgn}(D_{r_0}(r_1))t^{|\operatorname{lk}(D_{r_0}(r_1))|} + \operatorname{sgn}(D_{r_0}(r_2))t^{|\operatorname{lk}(D_{r_0}(r_2))|} \\
&= 0 + \operatorname{sgn}(E_{s_0}(s_2))t^{|\operatorname{lk}(E_{s_0}(s_2))|} + \operatorname{sgn}(E_{s_0}(s_1))t^{|\operatorname{lk}(E_{s_0}(s_1))|} \\
&\quad + \operatorname{sgn}(E_{s_2}(s_1))t^{|\operatorname{lk}(E_{s_2}(s_1))|} + \operatorname{sgn}(E_{s_1}(s_2))t^{|\operatorname{lk}(E_{s_1}(s_2))|} \\
&= \left(\operatorname{sgn}(E_{s_1}(s_0))t^{|\operatorname{lk}(E_{s_1}(s_0))|} + \operatorname{sgn}(E_{s_2}(s_0))t^{|\operatorname{lk}(E_{s_2}(s_0))|}\right) \\
&\quad + \operatorname{sgn}(E_{s_0}(s_2))t^{|\operatorname{lk}(E_{s_0}(s_2))|} + \operatorname{sgn}(E_{s_0}(s_1))t^{|\operatorname{lk}(E_{s_0}(s_1))|} \\
&\quad + \operatorname{sgn}(E_{s_2}(s_1))t^{|\operatorname{lk}(E_{s_2}(s_1))|} + \operatorname{sgn}(E_{s_1}(s_2))t^{|\operatorname{lk}(E_{s_1}(s_2))|} \\
&= \sum_{p\in S}\left(\sum_{q\in S\setminus\{p\}} W(D_p(q);t)\right).
\end{aligned}$$

(ii) For each $i\in\{0,1,2\}$ and $q\in T$, it is easy to show that $q\in C_{r_i}(D)$ if and only if $q\in C_{s_i}(E)$, in which case, we have $\operatorname{sgn}(D_{r_i}(q)) = \operatorname{sgn}(E_{s_i}(q))$ and $\operatorname{lk}(D_{r_i}(q)) = \operatorname{lk}(E_{s_i}(q))$. Thus, we obtain

$$\begin{aligned}
\sum_{p\in R}\left(\sum_{q\in T} W(D_p(q);t)\right) &= \sum_{i=0}^{2}\left(\sum_{q\in T}\operatorname{sgn}(D_{r_i}(q))t^{|\operatorname{lk}(D_{r_i}(q))|}\right) \\
&= \sum_{i=0}^{2}\left(\sum_{q\in T}\operatorname{sgn}(E_{s_i}(q))t^{|\operatorname{lk}(E_{s_i}(q))|}\right) \\
&= \sum_{p\in S}\left(\sum_{q\in T} W(D_p(q);t)\right).
\end{aligned}$$

(iii) For each $p\in T$ and $i\in\{0,1,2\}$, it is also easy to verify that $r_i\in C_p(D)$ if and only if $s_i\in C_p(E)$, in which case, we have $\operatorname{sgn}(D_p(r_i)) = \operatorname{sgn}(E_p(s_i))$ and $\operatorname{lk}(D_p(r_i)) = \operatorname{lk}(E_p(s_i))$. Therefore, we have

$$\begin{aligned}
\sum_{p\in T}\left(\sum_{q\in R} W(D_p(q);t)\right) &= \sum_{p\in T}\left(\sum_{i=0}^{2}\operatorname{sgn}(D_p(r_i))t^{|\operatorname{lk}(D_p(r_i))|}\right) \\
&= \sum_{p\in T}\left(\sum_{i=0}^{2}\operatorname{sgn}(E_p(s_i))t^{|\operatorname{lk}(E_p(s_i))|}\right) \\
&= \sum_{p\in T}\left(\sum_{q\in R} W(D_p(q);t)\right). \qquad \square
\end{aligned}$$

The following is an immediate consequence of Lemma 16.

Proposition 17. The polynomial $P_D(t)$ is invariant under the move FR_3.

Let E be the diagram obtained from D by a single move FVR_4. Let r be the flat crossing of D for the move FVR_4 and s the corresponding crossing of E. We put $X = C(D)\backslash\{r\} = C(E)\backslash\{s\}$. Then, we have

$$\begin{aligned} P_D(t) &= \sum_{q\in C(D)\backslash\{r\}} W(D_r(q);t) + \sum_{p\in C(D)\backslash\{r\}} \left(\sum_{q\in C(D)\backslash\{p\}} W(D_p(q);t) \right) \\ &= \sum_{q\in X} W(D_r(q);t) + \sum_{p\in X} \left(W(D_p(r);t) + \sum_{q\in X\backslash\{p\}} W(D_p(q);t) \right) \\ &= \sum_{q\in X} W(D_r(q);t) + \sum_{p\in X} W(D_p(r);t) + \sum_{p\in X} \left(\sum_{q\in X\backslash\{p\}} W(D_p(q);t) \right). \end{aligned}$$

we also have

$$P_E(t) = \sum_{q\in X} W(E_s(q);t) + \sum_{p\in X} W(E_p(s);t) + \sum_{p\in X} \left(\sum_{q\in X\backslash\{p\}} W(E_p(q);t) \right).$$

Then, we have the following

Lemma 18. The following statements hold:

(i) $\displaystyle\sum_{q\in X} W(D_r(q);t) = \sum_{q\in X} W(E_s(q);t)$.

(ii) $\displaystyle\sum_{p\in X} W(D_p(r);t) = \sum_{p\in X} W(E_p(s);t)$.

(iii) $\displaystyle\sum_{p\in X} \left(\sum_{q\in X\backslash\{p\}} W(D_p(q);t) \right) = \sum_{p\in X} \left(\sum_{q\in X\backslash\{p\}} W(E_p(q);t) \right)$.

Proof. Observe first that, depending on the orientation, $D(r)$ is isotopic to $E(s)$ or $D(r)$ is obtainable from $E(s)$ by using the move FVR_2.

(i) Suppose that r and s are base points of D and E, respectively. For $q \in X$, it is clear that $q \in C_r(D)$ if and only if $q \in C_s(E)$. In this case, we have $\mathrm{sgn}(D_r(q)) = \mathrm{sgn}(E_s(q))$ and $\mathrm{lk}(D_r(q)) = \mathrm{lk}(E_s(q))$. Thus, we have

$$\begin{aligned} \sum_{q\in X} W(D_r(q);t) &= \sum_{q\in X} \mathrm{sgn}(D_r(q))t^{|\mathrm{lk}(D_r(q))|} \\ &= \sum_{q\in X} \mathrm{sgn}(E_s(q))t^{|\mathrm{lk}(E_s(q))|} = \sum_{q\in X} W(E_s(q);t). \end{aligned}$$

(ii) For $p \in X$, it is also evident that $r \in C_p(D)$ if and only if $s \in C_p(E)$. Note that, in this case, we have $\mathrm{sgn}(D_p(r)) = \mathrm{sgn}(E_p(s)$ and $\mathrm{lk}(D_p(r)) =$

$\mathrm{lk}(E_p(s))$. Thus, we obtain

$$\begin{aligned}\sum_{p\in X} W(D_p(r);t) &= \sum_{p\in X} \mathrm{sgn}(D_p(r))t^{|\mathrm{lk}(D_p(r))|}\\ &= \sum_{p\in X} \mathrm{sgn}(E_p(s))t^{|\mathrm{lk}(E_p(s))|} = \sum_{p\in X} W(E_p(s);t).\end{aligned}$$

(iii) This is clear. □

The following is an immediate consequence of Lemma 18.

Proposition 19. The polynomial $P_D(t)$ is invariant under the move FVR_4.

Theorem 20. For a flat virtual knot K, the polynomial $P_K(t)$ is an invariant for K, i.e., it is invariant under the generalized flat Reidemeister moves.

Proof. Clearly, it is invariant under the moves FVR_1, FVR_2, and FVR_3. By Propositions 5, 14, 17 and 19, it is invariant under the moves FR_1, FR_2, FR_3, and FVR_4. □

5. An invariant of virtual doodles

In this section, based on the invariant $P_D(t)$ for flat virtual knots, we introduce an invariant of virtual doodles with one component.

Let D be a virtual doodle diagram with one component and $p \in C(D)$. By smoothing p we obtain two component diagram D_p, which is labeled ± 1 as in Fig. 13.

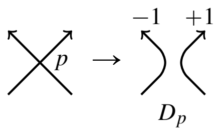

Figure 13. The two component labeled diagram D_p

For $p \in C(D)$, we define $\sigma_D(p)$ by

$$\sigma_D(p) = \sum_{c\in C(D_p)} \mathrm{ind}(D_p;c),$$

where $\mathrm{ind}(D_p;c)$ is defined similarly as in Fig. 7. Now, we give the main definition in this section.

Definition 21. Let D be a virtual doogle diagram. We define the polynomial $R_D(t,u) \in \mathbb{Z}[t,u]$ by

$$R_D(t,u) = \begin{cases} 0 & \text{if } |C(D)| \leqslant 1,\\ \displaystyle\sum_{p\in C(D)} u^{|\sigma_D(p)|}\left(\sum_{q\in C_p(D)} \mathrm{sgn}(D_p(q))\cdot t^{|\mathrm{lk}(D_p(q))|}\right) & \text{otherwise.}\end{cases}$$

Remark 22. We also have $R_{-D}(t,u) = -R_D(t,u)$ for the inverse diagram $-D$ of D.

Proposition 23. The polynomial $R_D(t,u)$ is invariant under the move FR_1.

Proof. Let E be the diagram obtained from D by applying a move FR_1 eliminating $r \in C(D)$. If $|C(D)| = 1$, then the definition gives $R_D(t,u) = 0 = R_E(t,u)$. Using Eq. (3.1), we put for $p \in C(D)$

$$R_{(D,p)}(t,u) = u^{|\sigma_D(p)|}P_{(D,p)}(t).$$

Since $P_{(D,r)}(t) = 0$ (cf. the proof of Proposition 5), we have $R_{(D,r)}(t,u) = 0$ and

$$R_D(t,u) = R_{(D,r)}(t,u) + \sum_{p \in C(D)\setminus\{r\}} R_{(D,p)}(t,u) = \sum_{p \in C(D)\setminus\{r\}} R_{(D,p)}(t,u).$$

Assume that $|C(D)| \geqslant 2$. If $|C(D)| = 2$, say $C(D) = \{r,p\}$, then $C_p(D) = \varnothing$ and hence $P_{(D,p)}(t) = 0$. Thus, $R_D(t,u) = R_{(D,p)}(t,u) = 0$. On the other hand, $R_E(t,u) = 0$, since $|C(E)| = 1$.

Now, suppose that $|C(D)| \geqslant 3$. Since $P_{(D,p)}(t) = P_{(E,p)}(t)$ for $p \in C(D)\setminus\{r\} = C(E)$ as in the proof of Proposition 5, we have $R_{(D,p)}(t,u) = R_{(E,p)}(t,u)$. This leads the desired result $R_D(t,u) = R_E(t,u)$. □

Proposition 24. The polynomial $R_D(t,u)$ is invariant under the move FR_2.

Proof. First, we consider the move FR_2 of type A. Suppose $|C(D)| \geqslant 2$ and let E be the diagram obtained from D by applying a move FR_2 of type A eliminating $r,s \in C(D)$. We put $R_{(D,r,s)}(t,u) = R_{(D,r)}(t,u) + R_{(D,s)}(t,u)$.

First, suppose that $|C(D)| = 2$. Since $|C(E)| = 0$, $R_E(t,u) = 0$ by definition. It is sufficient to show that $R_D(t,u) = 0$. Let $C(D) = \{r,s\}$. Since D_r and D_s are equivalent diagrams with opposite labels, we have $\sigma_D(r) = -\sigma_D(s)$ whence $|\sigma_D(r)| = |\sigma_D(s)|$. Since $C_r(D) = \{s\}$ and $C_s(D) = \{r\}$ as in the proof of Lemma 12, we obtain $R_D(t,u) = R_{(D,r,s)}(t,u) = 0$ by using Lemma 7 (iii).

Next, suppose that $|C(D)| = 3$. Since $|C(E)| = 1$, $R_E(t,u) = 0$ by definition. It is sufficient to show that $R_D(t,u) = 0$. Let $C(D) = \{r,s,p\}$. Since $P_{(D,p)}(t) = 0$ as in the proof of Lemma 12, we have $R_{(D,p)}(t,u) = 0$. Furthermore, since $|\sigma_D(r)| = |\sigma_D(s)|$, we can deduce $R_{(D,r,s)}(t,u) = 0$ by performing a similar computation as presented in the proof of Lemma 10. Thus, we have $R_D(t,u) = R_{(D,p)}(t,u) + R_{(D,r,s)}(t,u) = 0$.

Finally, suppose that $|C(D)| \geqslant 4$. Since $C(E) = C(D)\setminus\{r,s\}$, we have

$$R_D(t,u) = R_{(D,r,s)}(t,u) + \sum_{p \in C(E)} R_{(D,p)}(t,u)$$

and

$$R_E(t,u) = \sum_{p \in C(E)} R_{(E,p)}(t,u).$$

We only have to show the following two claims.

Claim 1. $R_{(D,r,s)}(t,u) = 0$.

Claim 2. $R_{(D,p)}(t,u) = R_{(E,p)}(t,u)$ for $p \in C(E)$.

Proof of Claim 1. This is similar to the previous case where $|C(D)| = 3$.

Proof of Claim 2. By Lemma 6 we have $\{r,s\} \cap C_p(D) = \emptyset$ or $\{r,s\} \subseteq C_p(D)$. If $\{r,s\} \cap C_p(D) = \emptyset$, then $C_p(D) = C_p(E)$. Since it is evident that $\sigma_D(p) = \sigma_E(p)$, we have by Lemma 9

$$\begin{aligned} R_{(D,p)}(t,u) &= u^{|\sigma_D(p)|} \sum_{q \in C_p(D)} \mathrm{sgn}(D_p(q)) t^{|\mathrm{lk}(D_p(q))|} \\ &= u^{|\sigma_E(p)|} \sum_{q \in C_p(E)} \mathrm{sgn}(E_p(q)) t^{|\mathrm{lk}(E_p(q))|} = R_{(E,p)}(t,u). \end{aligned}$$

If $\{r,s\} \subseteq C_p(D)$, then $C_p(D) = C_p(E) \cup \{r,s\}$. We have by Lemma 6

$$u^{|\sigma_p(D)|} \mathrm{sgn}(D_p(r)) t^{|\mathrm{lk}(D_p(r))|} + u^{|\sigma_p(D)|} \mathrm{sgn}(D_p(s)) t^{|\mathrm{lk}(D_p(s))|} = 0.$$

Hence, we also obtain $R_D(t,u) = R_E(t,u)$ by Lemma 9.

The proof for the move FR_2 of type B is similar to that for the previous case. □

Note that $R_D(t,u)$ is invariant under the moves FVR_1, FVR_2, FVR_3, and FVR_4 because $\sigma_D(p)$ is clearly invariant under these moves. Since D can be regarded as a flat virtual knot diagram, it is also clear that

$$P_D(t) = R_D(t,1).$$

Let d be a virtual doodle with just one component and D its diagram. We define the polynomial $R_d(t,u)$ by $R_D(t,u)$. Then, we have the following.

Theorem 25. For a virtual doodle d with just one component, the polynomial $R_d(t,u)$ is an invariant for d, i.e., it is invariant under the generalized flat Reidemeister moves except the move FR_3.

Example 26. We consider the virtual doodles $d_{4.1}$ and $d_{4.4}$ in [3], which have diagrams as in Fig. 14. We denote by D and E the diagrams of $d_{4.1}$ and $d_{4.4}$ in Fig. 14, respectively.

Then we have

$$C_{r_0}(D) = \{r_2\}, \quad C_{r_1}(D) = \{r_2, p\}, \quad C_{r_2}(D) = \{r_0, r_1, p\}, \quad C_p(D) = \{r_1, r_2\},$$

and

$$C_{s_0}(E) = \{s_1\}, \quad C_{s_1}(E) = \{s_0, p\}, \quad C_{s_2}(E) = \{p\}, \quad C_p(E) = \{s_1, s_2\}.$$

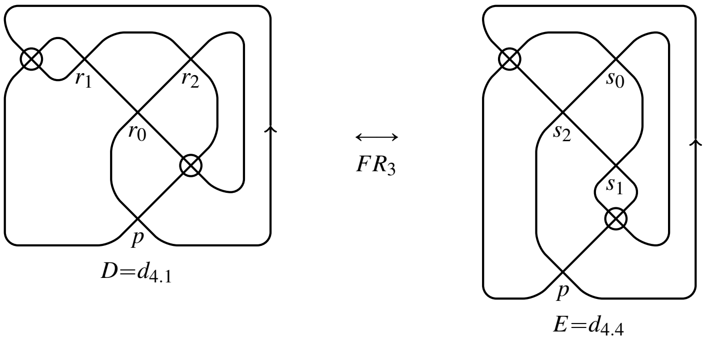

Figure 14. $d_{4.1}$ and $d_{4.4}$ in [3]

Thus we compute $R_{d_{4.1}}(t,u)$ as

$$\begin{aligned}
R_{d_{4.1}}(t,u) &= u^{|\sigma_D(r_0)|}\mathrm{sgn}(D_{r_0}(r_2))t^{|\mathrm{lk}(D_{r_0}(r_2))|} + u^{|\sigma_D(r_1)|}\mathrm{sgn}(D_{r_1}(r_2))t^{|\mathrm{lk}(D_{r_1}(r_2))|}\\
&\quad + u^{|\sigma_D(r_1)|}\mathrm{sgn}(D_{r_1}(p))t^{|\mathrm{lk}(D_{r_1}(p))|} + u^{|\sigma_D(r_2)|}\mathrm{sgn}(D_{r_2}(r_0))t^{|\mathrm{lk}(D_{r_2}(r_0))|}\\
&\quad + u^{|\sigma_D(r_2)|}\mathrm{sgn}(D_{r_2}(r_1))t^{|\mathrm{lk}(D_{r_2}(r_1))|} + u^{|\sigma_D(r_2)|}\mathrm{sgn}(D_{r_2}(p))t^{|\mathrm{lk}(D_{r_2}(p))|}\\
&\quad + u^{|\sigma_D(p)|}\mathrm{sgn}(D_p(r_1))t^{|\mathrm{lk}(D_p(r_1))|} + u^{|\sigma_D(p)|}\mathrm{sgn}(D_p(r_2))t^{|\mathrm{lk}(D_p(r_2))|}\\
&= u^{|0|}(-1)t^{|-1|} + u^{|0|}(+1)t^{|+1|} + u^{|0|}(-1)t^{|+2|} + u^{|+1|}(+1)t^{|+1|} + u^{|+1|}(-1)t^{|0|}\\
&\quad + u^{|+1|}(-1)t^{|+2|} + u^{|-2|}(+1)t^{|0|} + u^{|-2|}(+1)t^{|+1|}\\
&= -t^2 + ut - u - ut^2 + u^2 + u^2t.
\end{aligned}$$

On the other hand, we compute $R_{d_{4.4}}(t,u)$ as

$$\begin{aligned}
R_{d_{4.4}}(t,u) &= u^{|\sigma_E(s_0)|}\mathrm{sgn}(E_{s_0}(s_1))t^{|\mathrm{lk}(E_{s_0}(s_1))|} + u^{|\sigma_E(s_1)|}\mathrm{sgn}(E_{s_1}(s_0))t^{|\mathrm{lk}(E_{s_1}(s_0))|}\\
&\quad + u^{|\sigma_E(s_1)|}\mathrm{sgn}(E_{s_1}(p))t^{|\mathrm{lk}(E_{s_1}(p))|} + u^{|\sigma_E(s_2)|}\mathrm{sgn}(E_{s_2}(p))t^{|\mathrm{lk}(E_{s_2}(p))|}\\
&\quad + u^{|\sigma_E(p)|}\mathrm{sgn}(E_p(s_1))t^{|\mathrm{lk}(E_p(s_1))|} + u^{|\sigma_E(p)|}\mathrm{sgn}(E_p(s_2))t^{|\mathrm{lk}(E_p(s_2))|}\\
&= u^{|+1|}(-1)t^{|0|} + u^{|0|}(+1)t^{|+1|} + u^{|0|}(-1)t^{|+2|} + u^{|+1|}(-1)t^{|+2|}\\
&\quad + u^{|-2|}(+1)t^{|0|} + u^{|-2|}(+1)t^{|+1|}\\
&= -u + t - t^2 - ut^2 + u^2 + u^2t.
\end{aligned}$$

Note that $R_{d_{4.1}}(t,u) \neq R_{d_{4.4}}(t,u)$. Observe that

$$P_{d_{4.1}}(t) = R_{d_{4.1}}(t,1) = R_{d_{4.4}}(t,1) = P_{d_{4.4}}(t).$$

Since a virtual doodle can be regarded as a flat virtual link, the equality $P_{d_{4.1}}(t) = P_{d_{4.4}}(t)$ also follows from the fact that $d_{4.1}$ and $d_{4.4}$ are related by a move FR_3.

References

[1] R. Fenn, P. Taylor, Introducing doodles, Topology of low dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, 37-43.

[2] A. Bartholomew, R. Feen, N. Kamada, S. Kamada, Doodles on Surfaces, arXiv:1612.08473v2 [math.GT] 17 Jan 2018.
[3] A. Bartholomew, R. Feen, N. Kamada, S. Kamada, Coloring and doubled colorings of virtual doodles, Topology Appl., Vol. 264 (2019), 290-299.
[4] L. Kauffman, Virtual knot theory, European. J. Combin., Vol. 20 (1999), 663-690.
[5] J. Kim, The affine index polynomial invariant of flat virtual knots, J. Knot Theory Ramifications, Vol. 23, No. 14 (2014).
[6] J. Kim, Polynomial invariants of flat virtual links using invariant of virtual links, J. Knot Theory Ramifications, Vol. 24, No. 6, 1550036 (2015).
[7] J. Kim, Polynomial invariants of virtual doodles and virtual knots, J. Knot Theory Ramifications, Vol. 29, No. 12, 2050073 (2020).
[8] M. Khovanov, Doodle group, Trans. Amer. Math. Soc., Vol. 349 (1997), 2297-2315.

Email address: `string1821@daum.net`

Institute of Mathematical Science, Pusan National University, 2 Busan University Road 63, Geumjeong-gu, Busan, Republic of Korea

Email address, Corresponding author: `kyoungtarkkim@gmail.com`

Institute for Future Education, Busan National University of Education, 24, Gyodae-ro, Yeonje-gu, Busan, Republic of Korea