Parametrix and the Kohn-Rossi Laplacian on contact Riemannian manifolds

Masayoshi NAGASE

Abstract

On a contact Riemannian manifold which is compact and not assumed to be integrable, we intend to construct a parametrix for the Kohn-Rossi Laplacian. In particular, we will explicitly express the inverse of its principal part. Beals-Greiner constructed it in the case where the manifold is integrable. Our study depends heavily on theirs. We have some tools useful for the study in the non-integrable case, by means of which their results are extended to the general case and furthermore the inverse can be revealed more clearly.

0 Introduction

In this paper, on a contact Riemannian manifold M which is compact and not assumed to be integrable, we intend to construct a parametrix Q for the Kohn-Rossi Laplacian $\Box_H$ (cf. (1.6), (1.7)): that is, the operators $Q\Box_H - I$, $\Box_H Q - I$ have C^∞-kernels. In particular, we will explicitly express the inverse operator of the principal part $\underline{\Box_H}$ (cf. (1.9)). The operator $\Box_H$ is not elliptic and consequently the standard elliptic theory does not work well for the study. We wish to investigate its parametrix as a stepping-stone to a close study of such a troublesome but important operator.

Beals-Greiner ([2, Chap.4]) constructed it in the case where M is integrable. Our study depends heavily on theirs. Fortunately we have some tools useful for the study in the non-integrable case, by means of which their results are extended to the general case and furthermore the inverse $(\underline{\Box_H})^{-1}$ can be revealed more clearly. In addition, the terms of the symbol $\sigma(Q)$, which is written as an expansion $\sum_{k\geq 2}\sigma_{-k}(Q)$, can be expressed explicitly up to an arbitrarily low degree, though the quantity of (elementary) calculation increases rather rapidly.

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We will prove the main theorems in §3 and §4, which propose the explicit formulas for the symbol and the kernel of $(\underline{\Box_H})^{-1}$ and mention some properties of $\Box_H$ derived from the existence of a global parametrix. §1 and §2 are devoted to the explanation of our tools. In §1 a general contact Riemannian manifold and the hermitian Tanno connection ${}^\sharp\nabla$, etc., on it ([4]) will be reviewed. The connection ${}^\sharp\nabla$ coincides with the well-known Tanaka-Webster connection (e.g. [3]) in the integrable case, and the author thinks that, as the Tanaka-Webster one fits for the study in the integrable case, so must do the connection ${}^\sharp\nabla$ in the general case. In fact, he applied it to several problems in the general contact Riemannain case ([5], [6], [7], [8], [9], etc.), and the study in this paper is one of such considerations. The formulas in the main theorems are described by means of geodesics, normal coordinates, parallel transportations, etc., with respect to ${}^\sharp\nabla$. In §2, referring to [2, §11] we will review the concept of y-coordinates x^y associated to the ${}^\sharp\nabla$-normal coordinates x. The new coordinates x^y play an important role in investigating the inverse $(\underline{\Box_H})^{-1}$. It is our idea to consider only the case where the coordinates x are ${}^\sharp\nabla$-normal ones. With no consideration to the use of connection Beals-Greiner ([2]) adopted the coordinates x unrestrictedly, so that their formulas have some vague parts even though restricted to the integrable case.

Last, we want to mention briefly another approach to the Laplacian $\Box_H$: In [4], the author studied the heat kernel $e^{-t\Box_H}$. We proved its unique existence and showed that its pointwise trace can be expanded into $\operatorname{tr} e^{-t\Box_H}(P^0, P^0) \sim \sum_{k\geq 0} t^{-(n+1)+k} a_k(P^0)$ when $t \to 0$, and all the coefficients are described as certain universal polynomials built from the curvature and the torsion of the hermitian Tanno connection. Further, by using only a basic knowledge of calculus, one can describe the polynomials explicitly up to an arbitrarily high order. We may incidentally remark that the results in §4 of the paper can be deduced also from those in [4].

Together with [4], this paper thus deepens our understanding of the Kohn-Rossi Laplacian.

1 Contact Riemannian manifold and the Kohn-Rossi Laplacian

Let $M = (M; e^0, e_0, J, g)$ be a $(2n+1)$-dimensional contact Riemannian manifold. Here e^0 is a contact 1-form and e_0 is the unique vector field satisfying $e_0 \rfloor e^0 := e^0(e_0) = 1$, $e_0 \rfloor de^0 := de^0(e_0, \ ) = 0$, and (J, g) is a pair of $(1,1)$-tensor field and Riemannian metric satisfying $g(e_0, X) = e^0(X)$, $g(X, JY) = -de^0(X, Y) := -X(e^0(Y)) - Y(e^0(X)) + e^0([X, Y])$ and $J^2X = -X + e^0(X)e_0$.

Referring to [4] and [9], first we will review briefly some basic properties of the hermitian Tanno connection denoted by ${}^\sharp\nabla$ ([4]), which is a tool crucial for our study. We set $H = \ker e^0$, $H_\pm = \{X \in \mathbb{C}H \mid JX = \pm i\, X\}$ ($\mathbb{C}H := H \otimes \mathbb{C}$). Without the assumption that J is integrable (i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$), we will equip M with the connection, which is characterized by the following conditions:

$$\begin{gathered} {}^\sharp\nabla e^0 = 0, \quad {}^\sharp\nabla g = 0, \quad {}^\sharp\nabla J = 0, \\ \pi_+ T({}^\sharp\nabla)(Z, W) = 0 \quad (Z \in H_+,\ W \in \mathbb{C}TM), \end{gathered}$$

where $T({}^\sharp\nabla)$ is the torsion tensor and $\pi_+ : \mathbb{C}TM = \mathbb{C}e_0 \oplus H_+ \oplus H_- \to H_+$ is the natural projection (cf. [4, Lemma 1.1], [6, §2]). We notice that it coincides with the Tanaka-Webster connection ([3, §1.2]) provided that J is integrable. On a small neighborhood $U = U_{\mathbb{P}}$ of a given point $\mathbb{P}$, we always consider a unitary frame $e^{\mathbb{C}}_\bullet = (e^{\mathbb{C}}_0, e^{\mathbb{C}}_1, \ldots, e^{\mathbb{C}}_n, e^{\mathbb{C}}_{\bar 1}, \ldots, e^{\mathbb{C}}_{\bar n})$ of $\mathbb{C}TU$ ($e^{\mathbb{C}}_0 := e_0$, $e^{\mathbb{C}}_{\bar\alpha} = \overline{e^{\mathbb{C}}_\alpha} \in H_-$, $g(e^{\mathbb{C}}_\alpha, e^{\mathbb{C}}_{\bar\beta}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) which is ${}^\sharp\nabla$-parallel along all the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$. Its dual frame is denoted by $e^\bullet_{\mathbb{C}} = (e^0_{\mathbb{C}}, e^1_{\mathbb{C}}, \ldots, e^n_{\mathbb{C}}, e^{\bar 1}_{\mathbb{C}}, \ldots, e^{\bar n}_{\mathbb{C}})$ (hence, $e^0_{\mathbb{C}} = e^0$). We take the associated orthonormal frames $e_\bullet = (e_0, e_1, \cdots, e_{2n})$, $e^\bullet = (e^0, e^1, \cdots, e^{2n})$ with respect to the underlying Riemannian metric g, i.e.,

$$(1.1)\qquad \begin{gathered} e_\alpha = \frac{e^{\mathbb{C}}_\alpha + e^{\mathbb{C}}_{\bar\alpha}}{\sqrt{2}}, \quad e_{n+\alpha} = \frac{e^{\mathbb{C}}_{\bar\alpha} - e^{\mathbb{C}}_\alpha}{\sqrt{-2}}, \quad e^\alpha = \frac{e^\alpha_{\mathbb{C}} + e^{\bar\alpha}_{\mathbb{C}}}{\sqrt{2}}, \quad e^{n+\alpha} = \frac{e^\alpha_{\mathbb{C}} - e^{\bar\alpha}_{\mathbb{C}}}{\sqrt{-2}}, \\ g = e^0_{\mathbb{C}} \otimes e^0_{\mathbb{C}} + \sum_{1\le\alpha\le n} (e^\alpha_{\mathbb{C}} \otimes e^{\bar\alpha}_{\mathbb{C}} + e^{\bar\alpha}_{\mathbb{C}} \otimes e^\alpha_{\mathbb{C}}) = \sum_{0\le j\le 2n} e^j \otimes e^j. \end{gathered}$$

Further, let $x_\bullet = {}^t(x_0, x_1 \ldots, x_{2n})$ be the ${}^\sharp\nabla$-normal coordinates centered at $\mathbb{P}$ with $\partial/\partial x_j = e_j$ at $0 = \mathbb{P}$, and $x^{\mathbb{C}}_\bullet = {}^t(x^{\mathbb{C}}_0, x^{\mathbb{C}}_1 \ldots, x^{\mathbb{C}}_n, x^{\mathbb{C}}_{\bar 1} \ldots, x^{\mathbb{C}}_{\bar n})$ be the complexified one. Also the frames $(\partial/\partial x^{\mathbb{C}}_\bullet) = (\partial/\partial x^{\mathbb{C}}_0, \partial/\partial x^{\mathbb{C}}_1, \cdots, \partial/\partial x^{\mathbb{C}}_{\bar 1}, \cdots)$, $(dx^{\mathbb{C}}_\bullet) = (dx^{\mathbb{C}}_0, dx^{\mathbb{C}}_1, \cdots, dx^{\mathbb{C}}_{\bar 1}, \cdots)$ are similarly defined, that is,

$$(1.2)\qquad x^{\mathbb{C}}_0 = x_0, \quad x^{\mathbb{C}}_\alpha = \frac{x_\alpha + i x_{n+\alpha}}{\sqrt{2}}, \quad \frac{\partial}{\partial x^{\mathbb{C}}_0} = \frac{\partial}{\partial x_0}, \quad \frac{\partial}{\partial x^{\mathbb{C}}_\alpha} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_\alpha} - i \frac{\partial}{\partial x_{n+\alpha}}\Big),$$

etc. Hereafter the Greek indices α, β, ... will vary from 1 to n, and so will do the block Latin indices A, B, ... in $\{0, 1, \ldots, n, \bar 1, \ldots, \bar n\}$. We notice ${}^\sharp\nabla e^{\mathbb{C}} = 0$, ${}^\sharp\nabla\Gamma(H_\pm) \subset \Gamma(H_\pm)$ and we have the following.

Proposition 1.1 *The connection forms* $\omega({}^\sharp\nabla)^A_B$ *with*

$${}^\sharp\nabla e^{\mathbb{C}}_\beta = \sum e^{\mathbb{C}}_\alpha \cdot \omega({}^\sharp\nabla)^\alpha_\beta, \quad {}^\sharp\nabla e^{\mathbb{C}}_{\bar\beta} = \sum e^{\mathbb{C}}_{\bar\alpha} \cdot \omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta}, \quad \omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta} = -\omega({}^\sharp\nabla)^\beta_\alpha$$

and the transition functions $V_\bullet(x^{\mathbb{C}})$, $V^\bullet(x^{\mathbb{C}})$ *defined by*

$$\begin{gathered} e^{\mathbb{C}}_\bullet = (\partial/\partial x^{\mathbb{C}}_\bullet) \cdot V_\bullet, \quad e^\bullet_{\mathbb{C}} = (dx^{\mathbb{C}}_\bullet) \cdot V^\bullet, \quad \textit{hence, } V_\bullet = ({}^t V^\bullet)^{-1} \\ (\textit{i.e., } e^{\mathbb{C}}_A = \textstyle\sum V_{BA}\, \partial/\partial x^{\mathbb{C}}_B, \textit{ etc.}) \end{gathered}$$

are expanded as

$$\begin{aligned}(1.3)\qquad &\omega({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x^{\mathbb{C}}_A)(x^{\mathbb{C}})\\ &= -\sum_{\ell=1}^{\infty}\frac{\ell}{(\ell+1)!}\sum x^{\mathbb{C}}_{A_1}\cdots x^{\mathbb{C}}_{A_\ell}\frac{\partial^{\ell-1}F({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x^{\mathbb{C}}_A,\partial/\partial x^{\mathbb{C}}_{A_1})}{\partial x^{\mathbb{C}}_{A_2}\cdots\partial x^{\mathbb{C}}_{A_\ell}}(0),\end{aligned}$$

$$\begin{aligned}(1.4)\qquad V^{BA}(x^{\mathbb{C}}) &= \delta^{BA}+\sum_{\ell=1}^{\infty}\frac{\ell}{(\ell+1)!}\sum x^{\mathbb{C}}_{A_1}\cdots x^{\mathbb{C}}_{A_\ell}\frac{\partial^{\ell-1}T({}^\sharp\nabla)^A_{A_1}(\partial/\partial x^{\mathbb{C}}_B)}{\partial x^{\mathbb{C}}_{A_2}\cdots\partial x^{\mathbb{C}}_{A_\ell}}(0)\\ &\quad+\sum_{\ell=2}^{\infty}\frac{\ell-1}{(\ell+1)!}\sum x^{\mathbb{C}}_{A_1}\cdots x^{\mathbb{C}}_{A_\ell}\frac{\partial^{\ell-2}F({}^\sharp\nabla)^A_{A_1}(\partial/\partial x^{\mathbb{C}}_{A_2},\partial/\partial x^{\mathbb{C}}_B)}{\partial x^{\mathbb{C}}_{A_3}\cdots\partial x^{\mathbb{C}}_{A_\ell}}(0).\end{aligned}$$

$$\begin{pmatrix} F({}^\sharp\nabla)^A_B(X,Y):=g(F({}^\sharp\nabla)(X,Y)e^{\mathbb{C}}_B,e^{\mathbb{C}}_{\bar A}),\\ T({}^\sharp\nabla)^A_B(Y):=g(T({}^\sharp\nabla)(e^{\mathbb{C}}_B,Y),e^{\mathbb{C}}_{\bar A})\end{pmatrix}$$

where we put $F({}^\sharp\nabla)(X,Y)=[{}^\sharp\nabla_X,{}^\sharp\nabla_Y]-{}^\sharp\nabla_{[X,Y]}$, $T({}^\sharp\nabla)(X,Y)={}^\sharp\nabla_XY-{}^\sharp\nabla_YX-[X,Y]$. *The transition functions* $v_\bullet(x)$, $v^\bullet(x)$ *defined by*

$$e_\bullet=(\partial/\partial x_\bullet)\cdot v_\bullet,\quad e^\bullet=(dx_\bullet)\cdot v^\bullet,\quad \textit{hence, } v_\bullet=({}^tv^\bullet)^{-1}$$

are also expanded as

$$\begin{aligned}(1.5)\qquad v^{ji}(x) &= \delta^{ji}+\sum_{\ell=1}^{\infty}\frac{\ell}{(\ell+1)!}\sum x_{i_1}\cdots x_{i_\ell}\frac{\partial^{\ell-1}T({}^\sharp\nabla)^i_{i_1}(\partial/\partial x_j)}{\partial x_{i_2}\cdots\partial x_{i_\ell}}(0)\\ &\quad+\sum_{\ell=2}^{\infty}\frac{\ell-1}{(\ell+1)!}\sum x_{i_1}\cdots x_{i_\ell}\frac{\partial^{\ell-2}F({}^\sharp\nabla)^i_{i_1}(\partial/\partial x_{i_2},\partial/\partial x_j)}{\partial x_{i_3}\cdots\partial x_{i_\ell}}(0).\end{aligned}$$

$$\begin{pmatrix} F({}^\sharp\nabla)^i_j(X,Y):=g(F({}^\sharp\nabla)(X,Y)e_j,e_i),\\ T({}^\sharp\nabla)^i_j(Y):=g(T({}^\sharp\nabla)(e_j,Y),e_i)\end{pmatrix}$$

Proof. The expansions (1.3), (1.4) were shown in [4, Proposition 2.4] (cf. [1, Appendix II]). As for (1.5): The equalities

$$x^{\mathbb{C}}_\alpha\otimes\frac{\partial}{\partial x^{\mathbb{C}}_\alpha}+x^{\mathbb{C}}_{\bar\alpha}\otimes\frac{\partial}{\partial x^{\mathbb{C}}_{\bar\alpha}}=x_\alpha\otimes\frac{\partial}{\partial x_\alpha}+x_{n+\alpha}\otimes\frac{\partial}{\partial x_{n+\alpha}},$$

etc., and (1.4) yield

$$\begin{aligned}V^{BA} &= \delta^{BA}+\sum_{\ell=1}^{\infty}\frac{\ell}{(\ell+1)!}\sum x_{i_1}\cdots x_{i_\ell}\frac{\partial^{\ell-1}g(T({}^\sharp\nabla)(e_{i_1},\partial/\partial x^{\mathbb{C}}_B),e^{\mathbb{C}}_{\bar A})}{\partial x_{i_2}\cdots\partial x_{i_\ell}}(0)\\ &\quad+\sum_{\ell=2}^{\infty}\frac{\ell-1}{(\ell+1)!}\sum x_{i_1}\cdots x_{i_\ell}\frac{\partial^{\ell-2}g(F({}^\sharp\nabla)(\partial/\partial x_{i_2},\partial/\partial x^{\mathbb{C}}_B)e_{i_1},e^{\mathbb{C}}_{\bar A})}{\partial x_{i_3}\cdots\partial x_{i_\ell}}(0),\end{aligned}$$

and we have

$$e^\bullet_{\mathbb{C}}=e^\bullet\cdot\begin{pmatrix}1 & \underset{(0,\beta)\text{-entry}}{0} & \underset{(0,\bar\beta)\text{-entry}}{0}\\ \underset{(\alpha,0)\text{-entry}}{0} & \frac{E}{\sqrt2} & \frac{E}{\sqrt2}\\ \underset{(\bar\alpha,0)\text{-entry}}{0} & \frac{iE}{\sqrt2} & \frac{-iE}{\sqrt2}\end{pmatrix}=:e^\bullet\cdot\widetilde{E},$$

$$
\begin{aligned}
v^{\bullet} &= \widetilde{E}\, V^{\bullet} \widetilde{E}^{-1} \\
&= \begin{pmatrix}
V^{00} & \underset{(0,\,\beta)\text{-entry}}{\frac{V^{0\beta}+V^{0\bar\beta}}{\sqrt{2}}} & \underset{(0,\,n+\beta)\text{-entry}}{\frac{-iV^{0\beta}+iV^{0\bar\beta}}{\sqrt{2}}} \\
\underset{(\alpha,\,0)\text{-entry}}{\frac{V^{\alpha 0}+V^{\bar\alpha 0}}{\sqrt{2}}} & \frac{(V^{\alpha\beta}+V^{\bar\alpha\bar\beta})+(V^{\alpha\bar\beta}+V^{\bar\alpha\beta})}{2} & \frac{-i(V^{\alpha\beta}-V^{\bar\alpha\bar\beta})+i(V^{\alpha\bar\beta}-V^{\bar\alpha\beta})}{2} \\
\underset{(n+\alpha,\,0)\text{-entry}}{\frac{iV^{\alpha 0}-iV^{\bar\alpha 0}}{\sqrt{2}}} & \frac{i(V^{\alpha\beta}-V^{\bar\alpha\bar\beta})+i(V^{\alpha\bar\beta}-V^{\bar\alpha\beta})}{2} & \frac{(V^{\alpha\beta}+V^{\bar\alpha\bar\beta})-(V^{\alpha\bar\beta}+V^{\bar\alpha\beta})}{2}
\end{pmatrix}.
\end{aligned}
$$

Hence, by straightforward computation, we obtain the expansion (1.5). ∎

Now, we put

$$
\begin{aligned}
&\wedge_H^{0,*} T^*M = \{\omega \in \wedge^* \mathbb{C}T^*M \mid X \rfloor \omega = 0 \ (X \in \mathbb{R}e_0 \cup H_+)\}, \\
&\Omega^{0,*}M = \Gamma(\wedge_H^{0,*} T^*M)
\end{aligned}
$$

and set $\bar\partial_H = \Pi_{(0,*+1)} \circ d : \Omega^{0,*}M \to \Omega^{0,*+1}M$, where d is the usual exterior differentiation and $\Pi_{(0,*+1)}$ denotes the natural projection $\Omega^{*+1}M := \Gamma(\wedge^{*+1}\mathbb{C}T^*M) \to \Omega^{0,*+1}M$. Its formal adjoint is denoted by $\bar\partial_H^*$ and the formally self-adjoint operator

$$
\Box_H = \Box_{H,q} := \bar\partial_H^* \bar\partial_H + \bar\partial_H \bar\partial_H^* : \Omega^{0,q}M \to \Omega^{0,q}M \tag{1.6}
$$

is called **the Kohn-Rossi Laplacian**. It is known (cf. [4, Proposition 1.3]) that, on U, they can be expressed as follows:

$$
\begin{aligned}
\bar\partial_H &= \sum e_{\mathbb{C}}^{\bar\alpha} \wedge {}^\sharp\nabla_{e^{\mathbb{C}}_{\bar\alpha}}, \quad \bar\partial_H^* = -\sum e_{\mathbb{C}}^{\bar\alpha} \vee {}^\sharp\nabla_{e^{\mathbb{C}}_{\alpha}}, \\
\Box_H &= -\sum \left({}^\sharp\nabla_{e^{\mathbb{C}}_{\alpha}} {}^\sharp\nabla_{e^{\mathbb{C}}_{\bar\alpha}} - {}^\sharp\nabla_{{}^\sharp\nabla_{e^{\mathbb{C}}_{\alpha}} e^{\mathbb{C}}_{\bar\alpha}} \right) - \sqrt{-1}\, q\, {}^\sharp\nabla_{e^{\mathbb{C}}_0} \\
&\qquad - \sum F({}^\sharp\nabla)^{\gamma}_{\delta}(e^{\mathbb{C}}_{\bar\alpha}, e^{\mathbb{C}}_{\beta}) \cdot e_{\mathbb{C}}^{\bar\alpha} \wedge e_{\mathbb{C}}^{\bar\beta} \vee e_{\mathbb{C}}^{\bar\gamma} \wedge e_{\mathbb{C}}^{\bar\delta} \vee,
\end{aligned} \tag{1.7}
$$

where $e_{\mathbb{C}}^{\bar\alpha}\wedge$ is the exterior production of $e_{\mathbb{C}}^{\bar\alpha}$ and $e_{\mathbb{C}}^{\bar\alpha}\vee := e^{\mathbb{C}}_{\bar\alpha} \rfloor$ is the interior one of $e^{\mathbb{C}}_{\bar\alpha}$. (We notice that, even if $e^{\mathbb{C}}_{\bullet}$, etc., are just unitary, the formulas hold.) Here we want to state that hereafter our study will specialize solely in the case

$$
0 < q < n \tag{1.8}
$$

for reasons that will become apparent.

Next, referring to [2, §10], let us introduce the symbol spaces. We put

$$
\mathcal{F}_m^H(U) = \{ f \in C^\infty(\pi^*_{T^*U \setminus \{0\}} \mathrm{End}(\wedge_H^{0,q} T^*U)) \mid f(\mathbb{Q}, \lambda \cdot T) = \lambda^m f(\mathbb{Q}, T) \},
$$

where $\lambda \cdot T$ is the Heisenberg dilation of $T = (T^0, T^H) \in T^*U = \mathbb{R}e^0 \oplus H^*$ by $\lambda > 0$, i.e., $\lambda \cdot T := (\lambda^2 T^0, \lambda T^H)$. By using the ${}^\sharp\nabla$-parallel transportation along the ${}^\sharp\nabla$-geodesics to $\mathbb{P}$, we trivialize the bundles on U as

$$
\begin{aligned}
&T^*U \;\cong\; U \times T^*_{\mathbb{P}}U \;\cong\; U \times \mathbb{R}^{2n+1}, \quad e^{\bullet}(x) \cdot \sigma \leftrightarrow (x, e^{\bullet}(0) \cdot \sigma) \leftrightarrow (x, \sigma), \\
&\wedge_H^{0,q} T^*U \;\cong\; U \times \wedge_H^{0,q} T^*_{\mathbb{P}}U \;\cong\; U \times \mathbb{C}^{\binom{n}{q}}
\end{aligned}
$$

and put

$$\mathcal{F}_H^m(U) = \Big\{ f \in C^\infty(\pi^*_{T^*U}\mathrm{End}(\wedge_H^{0,q}T^*U)) \mid \\ \text{there exist } f_k \in \mathcal{F}_k^H(U)\ (k \le m) \text{ such that } f \sim \sum_{k\le m} f_k \Big\},$$

where "$f \sim \sum_{k\le m} f_k$" means that, for each multi-indices A, B and each $N > 0$, there exists a locally bounded function $c_{ABN}(x) > 0$ such that

$$\Big|\partial_x^A \partial_\sigma^B \Big(f - \sum_{k>m-N} f_k \Big)(x,\sigma)\Big| \le c_{ABN}(x)\, |\sigma|_H^{m-|B|_H-N} \quad (|\sigma|_H \ge 1).$$

$$\Big(|\sigma|_H := \{|\sigma_0|^2 + \sum_{j\ge1} |\sigma_j|^4\}^{1/4},\ |B|_H := 2B_0 + \sum_{j\ge 1} B_j = B_0 + |B| \Big)$$

Now, we consider another trivialization

$$T^*U \ \cong\ U \times T^*_{\mathbb{P}}U \ \cong\ U \times \mathbb{R}^{2n+1}, \quad (dx_\bullet)_x \cdot \xi \leftrightarrow (dx_\bullet)_0 \cdot \xi \leftrightarrow (x,\xi)$$

and regard the elements of $C^\infty(\pi^*_{T^*U}\mathrm{End}(\wedge_H^{0,q}T^*U))$ as the cross-sections of the bundle over $U \times \mathbb{R}^{2n+1}$ ($\ni (x,\xi)$), which are, hence, denoted by $\mathbf{q}(x,\xi)$. We set

$$(dx_\bullet)_x \cdot \xi_\bullet = e^\bullet(x) \cdot \sigma_\bullet(x,\xi), \quad \text{hence, } \sigma_\bullet(x,\xi) = {}^t v_\bullet(x) \cdot \xi_\bullet$$

and put

$$\mathcal{S}^H_m(U) = \{ \mathbf{q} \in C^\infty(\pi^*_{T^*U\setminus\{0\}}\mathrm{End}(\wedge_H^{0,q}T^*U)) \mid \\ \text{there exists } f \in \mathcal{F}^H_m(U) \text{ such that } \mathbf{q}(x,\xi) = f(x,\sigma(x,\xi)) \},$$

$$\mathcal{S}_H^m(U) = \{ \mathbf{q} \in C^\infty(\pi^*_{T^*U}\mathrm{End}(\wedge_H^{0,q}T^*U)) \mid \\ \text{there exists } f \in \mathcal{F}_H^m(U) \text{ such that } \mathbf{q}(x,\xi) = f(x,\sigma(x,\xi)) \}.$$

(Refer to [2, Proposition(10.46)] which remarks on the choice of the frames $e^\bullet$.) As usual we put $\mathcal{S}_H^\infty(U) = \bigcup_m \mathcal{S}_H^m(U)$, $\mathcal{S}_H^{-\infty}(U) = \bigcap_m \mathcal{S}_H^m(U)$, etc. A pseudodifferential operator P acting on $C^\infty(\wedge_H^{0,q}T^*U)$ whose usual symbol $\sigma(P)$ belongs to $\mathcal{S}_H^m(U)$ is then called an **H-pseudodifferential operator (on U) of degree** m. Those of H-pseudodifferential operators acting on $C^\infty(U)$ will be denoted also by $\mathcal{S}_H^m(U)$, etc., if no confusion occurs. The symbol spaces $\mathcal{S}_H^m(M)$, etc., and H-pseudodifferential operators on M of degree m are then defined in an ordinary manner.

The Laplacian $\Box_H$ is obviously an H-differential operator of degree 2 and the principal part $\underline{\Box_H}$, i.e., $\sigma(\underline{\Box_H}) \in \mathcal{S}_2^H(M)$, is expressed on U as

$$\underline{\Box_H}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}} \cdot f_{\mathbb{J}}\Big) = \frac{1}{2} \sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}} \cdot \Box_{\mathbb{J}}(f_{\mathbb{J}}) \tag{1.9}$$

$$
\begin{aligned}
&:= \frac{1}{2}\sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}}\cdot\Big\{-2\sum e^{\mathbb{C}}_{\alpha}e^{\mathbb{C}}_{\bar\alpha} - \sqrt{-1}\,2q\,e^{\mathbb{C}}_0\Big\}(f_{\mathbb{J}})\\
&= \frac{1}{2}\sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}}\cdot\Big\{-\sum (e^{\mathbb{C}}_{\alpha}e^{\mathbb{C}}_{\bar\alpha} + e^{\mathbb{C}}_{\bar\alpha}e^{\mathbb{C}}_{\alpha}) - \sqrt{-1}\,\lambda\,e^{\mathbb{C}}_0\Big\}(f_{\mathbb{J}}) \quad (\lambda := -(n-2q))\\
&= \frac{1}{2}\sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}}\cdot\Big\{-\sum_{j=1}^{2n} (e_j)^2 - \sqrt{-1}\,\lambda\,e_0\Big\}(f_{\mathbb{J}}),
\end{aligned}
$$

where we set $\mathbb{J} = ((1 \le)\, j_1 < \cdots < j_q\, (\le n))$ and $e^{\mathbb{J}}_{\mathbb{C}} = e^{j_1}_{\mathbb{C}} \wedge \cdots \wedge e^{j_q}_{\mathbb{C}}$. Our main interest centers around the operator.

2 On the y-coordinates and the y-group structure

Let us regard the small ${}^\sharp\nabla$-normal coordinate neighborhood $(U_{\mathbb{P}}, x)$ (given in §1) naturally as a small neighborhood of 0 in $(\mathbb{R}^{2n+1}, x)$. Then, [4, §3] says that $\mathbb{R}^{2n+1}$ has a contact Riemannian structure and the associated hermitian Tanno connection ${}^\sharp\nabla$ which satisfy the following: their restrictions to $U_{\mathbb{P}}$ coincide with the given ones, and the coordinates x of $(\mathbb{R}^{2n+1}, x)$ is the global ${}^\sharp\nabla$-normal ones on $\mathbb{R}^{2n+1}$. In §2 and §3, our study will be advanced on the contact Riemannian manifold $U = \mathbb{R}^{2n+1}$. For example, $e^{\mathbb{C}}_{\bullet}$ is, hence, a globally defined frame of $\mathbb{C}TU = \mathbb{C}T\mathbb{R}^{2n+1}$ which is ${}^\sharp\nabla$-parallel along all the ${}^\sharp\nabla$-geodesics from 0.

Referring to [2, §11], given a point $y \in \mathbb{R}^{2n+1}\,(= U)$, we start with reviewing the y-coordinates, the y-group structure, etc., of $\mathbb{R}^{2n+1}$. The new coordinates centered at y defined by

$$x^y = {}^t(x^y_0, \ldots, x^y_{2n}) = v_{\bullet}(y)^{-1}(x - y) \tag{2.1}$$

are called **the y-coordinates with respect to the frame $e_{\bullet}$**, which are uniquely determined by the conditions:

$$x^y = C(y)(x-y), \qquad e_{\bullet}(y) = (\partial/\partial x^y_{\bullet})\Big|_{x^y_{\bullet}=0}$$

with some matrix $C(y)$. We have

$$
\begin{aligned}
&(\partial/\partial x_{\bullet})_x = (\partial/\partial x^y_{\bullet})_{x^y}\cdot v_{\bullet}(y)^{-1},\\
&e_{\bullet}(x) = (\partial/\partial x^y_{\bullet})_{x^y}\cdot v_{\bullet}(y; x^y), \quad v_{\bullet}(y; x^y) := v_{\bullet}(y)^{-1}v_{\bullet}(x)
\end{aligned}
$$

and, if we denote the symbols of the operators $i^{-1}\dfrac{\partial}{\partial x^y_j}\Big|_{x^y}$ and $i^{-1}e_j|_{y+v_{\bullet}(y)x^y}$ by ξ^y_j and $\sigma_j(y; x^y, \xi^y)$, respectively, then have

$$\xi^y_{\bullet} = {}^t v_{\bullet}(y)\cdot \xi_{\bullet}, \quad \sigma_{\bullet}(y; x^y, \xi^y) = {}^t v_{\bullet}(y; x^y)\cdot \xi^y_{\bullet}.$$

Next, let us set

$$b_\bullet(y) = (b_{kj}(y))_{1\le k,j\le 2n}, \quad b_{kj}(y) := \frac{\partial v_{0j}(y;x^y)}{\partial x^y_k}\Big|_{x^y=0} = \sum_{i,\ell} v_{\ell k}(y)\frac{\partial v_{ij}}{\partial x_\ell}(y)v^{i0}(y). \tag{2.2}$$

Then, by Proposition 1.1, we have

$$b_{n+\beta,\beta}(0) = \frac{1}{2}, \quad b_{\beta,n+\beta}(0) = -\frac{1}{2}, \quad b_{kj}(0) = 0 \ \ \text{(otherwise)},$$

and the Euclidean space $(\mathbb{R}^{2n+1}, x^y)$ with the group structure

$$\begin{gathered} x^y \cdot z^y = {}^t((x^y\cdot z^y)_0, (x^y\cdot z^y)_1, \ldots), \\ (x^y\cdot z^y)_0 := x^y_0 + z^y_0 + \sum_{j,k=1}^{2n} b_{kj}(y)x^y_k z^y_j, \quad (x^y\cdot z^y)_j := x^y_j + z^y_j \ \ (j\ge 1) \end{gathered} \tag{2.3}$$

is called **the y-group**.

We say that an operator Q acting on $C^\infty(\mathbb{R}^{2n+1}, x^y)$ is **y-invariant** if $L^*_{x^y}\circ Q = Q\circ L^*_{x^y}$ for every x^y, where L_{x^y} is the left translation by x^y. For each j the unique y-invariant vector field e^y_j which agrees with $\partial/\partial x^y_j$ at y is given by

$$(e^y_j f)(x^y) = \frac{d}{ds} f(x^y\cdot {}^t(0,\ldots,0,\overset{j}{\check{s}},0.\ldots,0)^y)\Big|_{s=0} \qquad (f\in C^\infty(\mathbb{R}^{2n+1}, x^y)). \tag{2.4}$$

The y-invariant frame $e^y_\bullet = (e^y_0, e^y_1, \ldots, e^y_{2n})$ with $e^y_\bullet(0) = (\partial/\partial x^y_\bullet)|_{x^y=0}$ thus obtained is expressed as

$$e^y_\bullet(x^y) = (\partial/\partial x^y_\bullet)\cdot v^y_\bullet(x^y),$$

$$v^y_\bullet(x^y) := \begin{pmatrix} 1 & \underset{(0,\beta)\text{-entry}}{\sum_{k\ge1} b_{k\beta}(y)x^y_k} & \underset{(0,n+\beta)\text{-entry}}{\sum_{k\ge1} b_{k,n+\beta}(y)x^y_k} \\ \underset{(\alpha,0)\text{-entry}}{0} & E & O \\ \underset{(n+\alpha,0)\text{-entry}}{0} & O & E \end{pmatrix},$$

$$\text{i.e.,}\ \ e^y_0 = \frac{\partial}{\partial x^y_0}, \quad e^y_j = \frac{\partial}{\partial x^y_j} + \sum_{k=1}^{2n} b_{kj}(y)x^y_k\frac{\partial}{\partial x^y_0} \ \ (j\ge 1),$$

and, if we denote the symbol of the operator $i^{-1}e^y_j$ by $\sigma^y_j(x^y,\xi^y)$, then we have

$$\sigma^y_\bullet(x^y,\xi^y) = {}^t v^y_\bullet(x^y)\cdot\xi^y_\bullet$$

$$\text{i.e.,}\ \ \sigma^y_0(x^y,\xi^y) = \xi^y_0, \quad \sigma^y_j(x^y,\xi^y) = \xi^y_j + \sum_{k=1}^{2n} b_{kj}(y)x^y_k\,\xi^y_0.$$

Its dual frame $e^{y,\bullet}$ is, hence, expressed as

$$\begin{gathered} e^{y,\bullet}(x^y) = (dx^y_\bullet)\cdot v^{y,\bullet}(x^y), \quad v^{y,\bullet}(x^y) := {}^t v^y_\bullet(x^y)^{-1}, \\ \text{i.e.,}\ \ e^{y,0} = dx^y_0 - \sum_{k,j=1}^{2n} b_{kj}(y)x^y_k dx^y_j, \quad e^{y,j} = dx^y_j \ \ (j\ge 1). \end{gathered} \tag{2.5}$$

Here, let us investigate the matrix $b_\bullet$ (cf. (2.2)) closely.

Lemma 2.1 *If we set*

$$b_{\bullet:m}(y) = (b_{kj:m}(y))_{1\le k,j\le 2n} \quad (0 \le m \le 2n),$$
$$b_{kj:m}(y) := \frac{\partial v_{mj}(y;x^y)}{\partial x^y_k}\Big|_{x^y=0} = \sum_{i,\ell} v_{\ell k}(y)\frac{\partial v_{ij}}{\partial x_\ell}(y)v^{im}(y),$$

then we have

$$[e_k, e_j] = \sum_{m=0}^{2n}\{b_{kj:m} - b_{jk:m}\}e_m \quad (k, j \ge 1). \tag{2.6}$$

Further, if we set

$$\begin{aligned} B_{\beta\alpha:m}(y) &= \frac{1}{2}\Big\{(b_{\beta\alpha:m}(y) - b_{n+\beta,n+\alpha:m}(y)) - i(b_{n+\beta,\alpha:m}(y) + b_{\beta,n+\alpha:m}(y))\Big\},\\ B_{\beta\bar\alpha:m}(y) &= \frac{1}{2}\Big\{(b_{\beta\alpha:m}(y) + b_{n+\beta,n+\alpha:m}(y)) - i(b_{n+\beta,\alpha:m}(y) - b_{\beta,n+\alpha:m}(y))\Big\},\\ B_{\bar\beta\bar\alpha:m}(y) &= \overline{B_{\beta\alpha:m}(y)}, \quad B_{\bar\beta\alpha:m}(y) = \overline{B_{\beta\bar\alpha:m}(y)}, \end{aligned} \tag{2.7}$$

then we have

$$[e^{\mathbb{C}}_\beta, e^{\mathbb{C}}_\alpha] = \sum_{m=0}^{2n}\{B_{\beta\alpha:m} - B_{\alpha\beta:m}\}e_m, \quad [e^{\mathbb{C}}_\beta, e^{\mathbb{C}}_{\bar\alpha}] = \sum_{m=0}^{2n}\{B_{\beta\bar\alpha:m} - B_{\bar\alpha\beta:m}\}e_m. \tag{2.8}$$

Proof. Since

$$[e_k, e_j] = [\sum v_{\ell k}\frac{\partial}{\partial x_\ell}, \sum v_{ij}\frac{\partial}{\partial x_i}] = \sum\Big\{v_{\ell k}\frac{\partial v_{ij}}{\partial x_\ell} - v_{\ell j}\frac{\partial v_{ik}}{\partial x_\ell}\Big\}v^{im}e_m,$$

(2.6) is valid. (2.8) follows from (2.6) easily. ∎

Proposition 2.2 *We have $b_{\beta\alpha} = b_{\beta\alpha:0}$, etc., and accordingly let us put $B_{\beta\alpha} = B_{\beta\alpha:0}$, etc. Then we have*

$$B_{\beta\alpha} = B_{\alpha\beta}, \quad B_{\bar\beta\alpha} = B_{\alpha\bar\beta} + i\delta_{\beta\alpha}, \tag{2.9}$$

$$b_{\beta\alpha} = b_{\alpha\beta}, \quad b_{n+\beta,n+\alpha} = b_{n+\alpha,n+\beta}, \quad b_{n+\beta,\alpha} = b_{\alpha,n+\beta} + \delta_{\beta\alpha}. \tag{2.10}$$

Remark. [2, (21.7)] says $b_{\beta\alpha} = b_{\alpha\beta}$ because of the integrability of J. But, in fact, (2.10) holds even if J is not integrable.

Proof. [4, (1.8)] implies that (even if J is not integrable) the coefficients of e_0 in the expansions of $[e^{\mathbb{C}}_\beta, e^{\mathbb{C}}_\alpha]$, $[e^{\mathbb{C}}_\beta, e^{\mathbb{C}}_{\bar\alpha}]$ are equal to 0, $-i\delta_{\beta\alpha}$, respectively. That is, we obtain (2.9). (2.10) follows from (2.9) and (2.7). ∎

Even more, we have the following: In the same way as the definitions of $x^{\mathbb{C}}_\bullet$, $(\partial/\partial x^{\mathbb{C}}_\bullet)$, $(dx^{\mathbb{C}}_\bullet)$ (cf. (1.2)) and $e^{\mathbb{C}}_\bullet$, $e^\bullet_{\mathbb{C}}$ (cf. (1.1)) , we define $x^{y,\mathbb{C}}_\bullet$, $(\partial/\partial x^{y,\mathbb{C}}_\bullet)$, $(dx^{y,\mathbb{C}}_\bullet)$ and $e^{y,\mathbb{C}}_\bullet$, $e^{y,\bullet}_{\mathbb{C}}$.

Lemma 2.3 *We have*

$$e^{y,\mathbb{C}}_{\bullet} = (\partial/\partial x^{y,\mathbb{C}}_{\bullet}) \cdot v^{y,\mathbb{C}}_{\bullet}(x^{y,\mathbb{C}}), \quad e^{y,\bullet}_{\mathbb{C}} = (dx^{y,\mathbb{C}}_{\bullet}) \cdot v^{y,\bullet}_{\mathbb{C}}(x^{y,\mathbb{C}}), \quad v^{y,\bullet}_{\mathbb{C}} = {}^{t}(v^{y,\mathbb{C}}_{\bullet})^{-1},$$

$$v^{y,\mathbb{C}}_{\bullet} := \begin{pmatrix} 1 & \underset{(0,\,\beta)\text{-}entry}{\sum_{A\neq 0} B_{A\beta}(y) x^{y,\mathbb{C}}_{A}} & \underset{(0,\,\bar\beta)\text{-}entry}{\sum_{A\neq 0} B_{A\bar\beta}(y) x^{y,\mathbb{C}}_{A}} \\ \underset{(\alpha,\,0)\text{-}entry}{0} & E & O \\ \underset{(\bar\alpha,\,0)\text{-}entry}{0} & O & E \end{pmatrix}.$$

Now, (2.5) and (2.10) yield

$$\begin{aligned} de^{y,0} &= d(dx^{y}_{0} - \sum_{k,j\geq 1} b_{kj}(y) x^{y}_{k} dx^{y}_{j}) = \sum_{k,j\geq 1} (b_{jk}(y) - b_{kj}(y)) dx^{y}_{k} \wedge dx^{y}_{j} \\ &= \sum dx^{y}_{\alpha} \wedge dx^{y}_{n+\alpha} = \sum e^{y,\alpha} \wedge e^{y,n+\alpha}, \end{aligned}$$

so that

$$e^{y,0} \wedge (de^{y,0})^{n} = n!\,(-1)^{n(n-1)/2}\, e^{y,0} \wedge e^{y,1} \wedge e^{y,2} \wedge \cdots \wedge e^{y,2n},$$

that is, $e^{y,0}$ is a contact form. Let us set

$$\begin{aligned} &H^{y} := \ker e^{y,0} = \langle e^{y}_{1}, \ldots, e^{y}_{2n} \rangle, \quad \text{hence, } T\mathbb{R}^{2n+1} = \langle e^{y}_{0} \rangle \oplus H^{y}, \\ &J^{y} : T\mathbb{R}^{2n+1} \to T\mathbb{R}^{2n+1}, \quad J^{y}(e^{y}_{0}) = 0,\ J^{y}(e^{y}_{\alpha}) = e^{y}_{n+\alpha},\ J^{y}(e^{y}_{n+\alpha}) = -e^{y}_{\alpha}, \\ &\mathbb{C}H^{y} = H^{y}_{+} \oplus H^{y}_{-} := \langle e^{y,\mathbb{C}}_{1}, \ldots, e^{y,\mathbb{C}}_{n} \rangle \oplus \langle e^{y,\mathbb{C}}_{\bar 1}, \ldots, e^{y,\mathbb{C}}_{\bar n} \rangle \quad \text{hence, } J^{y}\Big|_{H^{y}_{\pm}} = \pm i, \\ &g^{y} = e^{y,0}_{\mathbb{C}} \otimes e^{y,0}_{\mathbb{C}} + \sum (e^{y,\alpha}_{\mathbb{C}} \otimes e^{y,\bar\alpha}_{\mathbb{C}} + e^{y,\bar\alpha}_{\mathbb{C}} \otimes e^{y,\alpha}_{\mathbb{C}}) = \sum_{0\leq j\leq 2n} e^{y,j} \otimes e^{y,j}. \end{aligned}$$

Then $M^{y} := (\mathbb{R}^{2n+1}; x^{y}_{\bullet}, e^{y,\bullet}, e^{y}_{\bullet}, J^{y}, g^{y})$ is, hence, a contact Riemannian manifold. We denote the hermitian Tanno connection and the pseudo-Hermitian torsion by ${}^{\sharp}\nabla^{y}$ and τ^{y} (i.e., $\tau^{y}(X) := T({}^{\sharp}\nabla^{y})(X, e^{y}_{0})$). Since Lemma 2.3 and (2.9) imply

$$[e^{y,\mathbb{C}}_{\alpha}, e^{y,\mathbb{C}}_{\beta}] = 0, \quad [e^{y,\mathbb{C}}_{\alpha}, e^{y,\mathbb{C}}_{\bar\beta}] = -i\delta_{\alpha\beta}\, e^{y,\mathbb{C}}_{0}, \quad [e^{y,\mathbb{C}}_{0}, e^{y,\mathbb{C}}_{\beta}] = 0,$$

we have the following.

Proposition 2.4 *The contact Riemannian manifold M^{y} is integrable and we have*

$$\omega({}^{\sharp}\nabla^{y})^{\alpha}_{\beta} = 0, \quad \tau^{y} = 0.$$

The Kohn-Rossi Laplacian acting on $\Omega^{0,q}M^{y}$ (cf. (1.7), (1.9)) *is, hence, expressed as*

$$\begin{aligned} &\Box^{y}_{H}(\sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot f_{\mathbb{J}}) = \frac{1}{2} \sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot \Box^{y}_{\mathbb{J}}(f_{\mathbb{J}}) \\ &= \frac{1}{2} \sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot \Big\{ -2 \sum e^{y,\mathbb{C}}_{\alpha} e^{y,\mathbb{C}}_{\bar\alpha} - \sqrt{-1}\, 2q\, e^{y,\mathbb{C}}_{0} \Big\}(f_{\mathbb{J}}) \\ &= \frac{1}{2} \sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot \Big\{ -\sum_{j=1}^{2n} (e^{y}_{j})^{2} - \sqrt{-1}\, \lambda\, e^{y}_{0} \Big\}(f_{\mathbb{J}}) \qquad (\lambda = -(n-2q)). \end{aligned}$$

3 The model operator $\Box_H^y$ and its inverse

The operator $\Box_H^y$ on M^y will approximate $\Box_H$ near y, and is called **the model operator**. Its element $\Box_{\mathbb{J}}^y$ acting on $C^\infty(\mathbb{R}^{2n+1}, x^y)$, called **the model operator** too, is expressed also as

$$\Box_{\mathbb{J}}^y = -\sum_{j=1}^{2n}(e_j^y)^2 - i\Big(\sum_{\alpha\in\mathbb{J}} ie_{\mathbb{C}}^{y,0}([e_\alpha^{y,\mathbb{C}}, e_{\overline{\alpha}}^{y,\mathbb{C}}]) - \sum_{\alpha\notin\mathbb{J}} ie_{\mathbb{C}}^{y,0}([e_\alpha^{y,\mathbb{C}}, e_{\overline{\alpha}}^{y,\mathbb{C}}])\Big)e_0^y.$$

Since $0 < q < n$ (cf. (1.8)), we have

$$\left|\sum_{\alpha\notin\mathbb{J}} ie_{\mathbb{C}}^{y,0}([e_\alpha^{y,\mathbb{C}}, e_{\overline{\alpha}}^{y,\mathbb{C}}]) - \sum_{\alpha\in\mathbb{J}} ie_{\mathbb{C}}^{y,0}([e_\alpha^{y,\mathbb{C}}, e_{\overline{\alpha}}^{y,\mathbb{C}}])\right| < \left|\sum_{\alpha} ie_{\mathbb{C}}^{y,0}([e_\alpha^{y,\mathbb{C}}, e_{\overline{\alpha}}^{y,\mathbb{C}}])\right|.$$

That is, the Levi form $\mathcal{L}^y$ defined by

$$\mathcal{L}^y(Z, \overline{W}) = -ide^{y,0}(Z, \overline{W}) = ie_{\mathbb{C}}^{y,0}([Z, \overline{W}]) \quad (Z, W \in H_+^y)$$

satisfies the condition $Y(q)$ (cf. [2, Definition (21.34)]). Hence, we know (cf. [2, Theorem (21.35)]) that the model operators $\Box_{\mathbb{J}}^y$, $\Box_H^y$ have the inverse operators $(\Box_{\mathbb{J}}^y)^{-1}$, $(\Box_H^y)^{-1}$. In this section we will investigate the inverse operators closely and express their symbols and kernels explicitly.

Let us change the coordinates x^y into the new ones x'^y, which we call **the normal y-coordinates**, by the transformation (cf. [2, (1.18), (1.19)])

$$\begin{aligned} &\psi : (\mathbb{R}^{2n+1}, x^y) \to (\mathbb{R}^{2n+1}, x'^y),\\ &x^y \mapsto x'^y = {}^t(x_0^y - \frac{1}{2}\sum_{j,k=1}^{2n} q_{kj}(y)x_k^y x_j^y,\ x_1^y, \dots, x_{2n}^y),\\ &q_{kj}(y) = q_{jk}(y) := \frac{1}{2}(b_{kj}(y) + b_{jk}(y)). \end{aligned} \tag{3.1}$$

It will simplify our investigation. In fact, the y-group structure (2.3) of $(\mathbb{R}^{2n+1}, x^y)$ induces the new one of $(\mathbb{R}^{2n+1}, x'^y)$

$$\begin{aligned} &x'^y \cdot z'^y = {}^t((x'^y \cdot z'^y)_0, (x'^y \cdot z'^y)_1, \dots),\\ &(x'^y \cdot z'^y)_0 := x_0'^y + z_0'^y + \sum_{j,k=1}^{2n} c_{kj}(y)x_k'^y z_j'^y, \quad (x'^y \cdot z'^y)_j := x_j'^y + z_j'^y \quad (j \geq 1),\\ &c_{kj}(y) = -c_{jk}(y) := \frac{1}{2}(b_{kj}(y) - b_{jk}(y)) \quad (\text{hence,}\ \ b_{kj} = q_{kj} + c_{kj}) \end{aligned} \tag{3.2}$$

with

$$c_\bullet(y) := (c_{kj}(y)) = \begin{pmatrix} O & -a \\ a & O \end{pmatrix}, \quad a = \begin{pmatrix} a_1 & & O \\ & \ddots & \\ O & & a_n \end{pmatrix}, \quad a_j = \frac{1}{2}, \tag{3.3}$$

which is fairly simpler than the original one. In [2] the coordinates x'^y were called the skew-symmetric ones and, by another transformation, x'^y were changed into the normal ones ([2, (1.26)-(1.29)]). But obviously x'^y are already such normal ones in our case.

On $(\mathbb{R}^{2n+1}, x'^y)$, we consider the y-invariant frame $e'^y_\bullet$ with $e'^y_\bullet(0) = (\partial/\partial x'^y_\bullet)|_{x^y=0}$ (cf. (2.4)), which is expressed as

$$e'^y_\bullet(x'^y) = (\partial/\partial x'^y_\bullet) \cdot v'^y_\bullet(x'^y), \tag{3.4}$$

$$v'^y_\bullet(x'^y) := \begin{pmatrix} 1 & \underset{(0,\,\beta)\text{-entry}}{\sum_{k\geq 1} c_{k\beta}(y) x'^y_k} & \underset{(0,\,n+\beta)\text{-entry}}{\sum_{k\geq 1} c_{k,n+\beta}(y) x'^y_k} \\ \underset{(\alpha,\,0)\text{-entry}}{0} & E & O \\ \underset{(n+\alpha,\,0)\text{-entry}}{0} & O & E \end{pmatrix}.$$

Further, if we denote the symbols of $i^{-1}\dfrac{\partial}{\partial x'^y_j}$ and $i^{-1}e'^y_j$ by ξ'^y and $\sigma'^y_j(x'^y, \xi'^y)$ respectively, then we have

$$\sigma'^y_\bullet(x'^y, \xi'^y) = {}^t v'^y_\bullet(x'^y) \cdot \xi'^y_\bullet. \tag{3.5}$$

Now, let us set

$$\Box'^y_{\mathbb{J}} := \psi_* \Box^y_{\mathbb{J}} = -\sum_{j=1}^{2n} (e'^y_j)^2 - \sqrt{-1}\,\lambda\, e'^y_0 \tag{3.6}$$

and, first, introduce an explicit expression of the symbol of the inverse operator $(\Box'^y_{\mathbb{J}})^{-1}$.

Theorem 3.1 (On the symbol of $(\Box'^y_{\mathbb{J}})^{-1}$) *If $\xi'^y \neq 0$, then*

$$\begin{aligned} \sigma((\Box'^y_{\mathbb{J}})^{-1})(x'^y, \xi'^y) &= \mathbf{q}'(y; x'^y, \xi'^y) \\ &= \widetilde{\mathbf{q}}'(y; \sigma'^y(x'^y, \xi'^y)) := \int_0^\infty e^{-\lambda \xi'^y_0 s} G(\sigma'^y(x'^y, \xi'^y), s)\, ds \end{aligned} \tag{3.7}$$

with

$$G(\xi'^y, s) = \begin{cases} \Big(\dfrac{1}{\cosh(|\xi'^y_0| s)}\Big)^n \exp\Big(-\displaystyle\sum_{j=1}^{2n} (\xi'^y_j)^2 \cdot \frac{\tanh(|\xi'^y_0| s)}{|\xi'^y_0|}\Big) & : \xi'^y_0 \neq 0, \\ \exp\Big(-\displaystyle\sum_{j=1}^{2n} (\xi'^y_j)^2 s\Big) & : \xi'^y_0 = 0, \end{cases} \tag{3.8}$$

where the integrand $e^{-\lambda \xi'^y_0 s} G(\sigma'^y(x'^y, \xi'^y), s)$ is C^0 on $\mathbb{R}^{2n+1} \times [0, \infty)$, C^∞ on $(\mathbb{R}^{2n+1} - \{0\}) \times [0, \infty)$ and is rapidly decreasing with respect to s. Notice that $\sigma'^y_0(x'^y, \xi'^y) = \xi'^y_0$, and if it vanishes, then $\sigma'^y_H(x'^y, \xi'^y) = \xi'^y_H$, where we set $\xi'^y_H = {}^t(\xi'^y_1, \dots, \xi'^y_{2n})$, etc.

Since

$$((\Box_{\mathbb{J}}^{y})^{-1}u)(x^y) = ((\psi^*(\Box_{\mathbb{J}}^{\prime y})^{-1})u)(x^y) = ((\Box_{\mathbb{J}}^{\prime y})^{-1}\psi_* u)(\psi(x^y)),$$

we have the following.

Corollary 3.2 (On the symbol of $(\Box_{\mathbb{J}}^{y})^{-1}$) *We set*

$$\mathbf{q} = \psi^*\mathbf{q}', \quad i.e., \ \mathbf{q}(y;x^y,\xi^y) = \mathbf{q}'(y;\psi(x^y,\xi^y)) = \mathbf{q}'(y;x'^y,\xi'^y).$$

If $\xi^y \neq 0$ (i.e., $\xi'^y \neq 0$), then we have

$$\sigma((\Box_{\mathbb{J}}^{y})^{-1})(x^y,\xi^y) = \mathbf{q}(y;x^y,\xi^y).$$

The expression of $(\Box_H^y)^{-1}$ in terms of the symbol is, hence,

$$(3.9) \qquad \begin{aligned} (\Box_H^y)^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}} \cdot f_{\mathbb{J}}\Big)(x^y) &= 2\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}(x^y)} \cdot (\Box_{\mathbb{J}}^{y})^{-1}(f_{\mathbb{J}})(x^y) \\ &= 2\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}(x^y)} \cdot \frac{1}{(2\pi)^{2n+1}} \int e^{i\langle x^y,\xi^y\rangle} \mathbf{q}(y;x^y,\xi^y)\, \widehat{f}_{\mathbb{J}}(\xi^y)\, d\xi^y. \end{aligned}$$

In fact, $\mathbf{q}(y;x^y,\xi^y) \in \mathcal{S}_{-2}^{H}$ in the last line must be changed into an element of $\mathcal{S}_H^{-2}$ which is equal to the original one apart from a neighborhood of $\xi^y = 0$.

Next, we will introduce an explicit expression of the kernel of the inverse operator $(\Box_{\mathbb{J}}^{\prime y})^{-1}$.

Proposition 3.3 (cf. [2, Theorem (5.9)], (3.2.1)-(3.2.2)) *There exists a unique tempered distribution $\mathbf{k}'(y;\,)$ on $(\mathbb{R}^{2n+1},x'^y)$, i.e., $\mathbf{k}'(y;\,) \in \mathcal{S}'(\mathbb{R}^{2n+1},x'^y)$, such that it is locally integrable, of C^∞ except at 0, and satisfies*

$$\mathbf{k}'(y;\,)\circ\delta_t = t^{-2n}\mathbf{k}'(y;\,) \quad (t>0,\ \delta_t x'^y := {}^t(t^2x_0'^y, tx_1'^y,\dots,tx_{2n}'^y))$$

and, last,

$$(3.10) \quad ((\Box_{\mathbb{J}}^{\prime y})^{-1}f)(x'^y) = \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y;(z'^y)^{-1}\cdot x'^y)\, f(z'^y)\, dz'^y \quad (f \in C_c^\infty(\mathbb{R}^{2n+1},x'^y)).$$

The kernel $\mathbf{k}'(y;\,)$ is expressed as follows.

Theorem 3.4 (On the kernel of $(\Box_{\mathbb{J}}^{\prime y})^{-1}$) *Assume $x'^y \neq 0$ and consider the integral path in $\mathbb{C}$*

$$I = \begin{cases} I_0 = (-\infty,\infty) & : x_H'^y := {}^t(x_1'^y,\dots,x_{2n}'^y) \neq 0, \\ I_\varepsilon = (-\infty + i\varepsilon\cdot\operatorname{sgn} x_0'^y,\ \infty + i\varepsilon\cdot\operatorname{sgn} x_0'^y) & : x_0'^y \neq 0, \end{cases}$$

where we fix ε *with* $0 < \varepsilon < \pi/2$*. Then, we have*

$$\mathbf{k}'(y; x'^y) = (2\pi)^{-1}\Gamma(n) \int_I A(s)\, e^{-\lambda s}\Big(\gamma(x'^y_H, s) - i x'^y_0 s\Big)^{-n} ds, \tag{3.11}$$

$$A(s) := (4\pi)^{-n}\Big(\frac{s}{\sinh s}\Big)^n, \quad \gamma(x'^y_H, s) := \frac{1}{4}|x'^y_H|^2 s \coth s.$$

The integrand is integrable. In the case $x'^y_0 \neq 0$ *the integral on the path* I_ε *does not depend on the choice of* ε*, and, in the case* $x'^y_H \neq 0$ *and* $x'^y_0 \neq 0$*, the integral on* I_0 *coincides with that on* I_ε*.*

Remark. As for the integral (3.11) with $I = I_0$ (and $x'^y_H \neq 0$): If we set $x'^y_H = 0$ forcibly, the integrand is not integrable because $\gamma(x'^y_H, s) - i x'^y_0 s = -i x'^y_0 s$. To regularize the integral in the case $x'^y_H = 0$ (and, hence, $x'^y_0 \neq 0$) we deform the integral path I_0 to get the integral (3.11) with $I = I_\varepsilon$ (and $x'^y_0 \neq 0$).

Corollary 3.5 (On the kernel of $(\Box^y_{\mathbb{J}})^{-1}$) *Set*

$$\mathbf{k}(y;) = \psi^* \mathbf{k}'(y;), \quad \textit{i.e.}, \ \mathbf{k}(y; x^y) = \mathbf{k}'(y; \psi(x^y)) = \mathbf{k}'(y; x'^y),$$

then we have

$$\begin{aligned}((\Box^y_{\mathbb{J}})^{-1} f)(x^y) &= \int_{\mathbb{R}^{2n+1}} \mathbf{k}(y; (z^y)^{-1} \cdot x^y)\, f(z^y)\, dz^y \\ &= \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y; (z'^y)^{-1} \cdot x'^y)\, (\psi_* f)(z'^y)\, dz'^y.\end{aligned}$$

The expression of $(\Box^y_H)^{-1}$ *in terms of the kernel is, hence,*

$$\begin{aligned}(\Box^y_H)^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot f_{\mathbb{J}}\Big)(x^y) &= 2\sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}(x^y)} \cdot (\Box^y_{\mathbb{J}})^{-1}(f_{\mathbb{J}})(x^y) \\ &= 2\sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}(x^y)} \cdot \int_{\mathbb{R}^{2n+1}} \mathbf{k}(y; (z^y)^{-1} \cdot x^y)\, f_{\mathbb{J}}(z^y)\, dz^y.\end{aligned} \tag{3.12}$$

3.1 On the proof of Theorem 3.1

Let us prove Theorem 3.1. The smooth function $G(\xi'^y, s)|_{\xi'^y_0 \neq 0}$ can be extended uniquely to the smooth one (3.8) (on $\xi'^y \neq 0$), which is certainly C^0 on $\mathbb{R}^{2n+1} \times [0, \infty)$ and C^∞ on $(\mathbb{R}^{2n+1} - \{0\}) \times [0, \infty)$. In the case $\xi'^y_0 = 0$ it is obvious that the integrand of (3.7) is rapidly decreasing, but, in the case $\xi'^y_0 \neq 0$ it will not so obvious. For example, we will show

$$\big|G(\xi'^y, s)\big| \leq C \exp\Big(-n|\xi'^y_0|s\Big) \quad : \xi'^y_0 \neq 0. \tag{3.1.1}$$

Since $0 < q < n$, we have

$$-\lambda\xi_0'^y - n|\xi_0'^y| < 0.$$

Hence, if (3.1.1) holds, then, also in the case $\xi_0'^y \neq 0$, the integrand is exponentially decreasing when $s \to \infty$. In this way, for the proof it will suffice to focus only on the case $\xi_0'^y \neq 0$. In the following, thus we assume $\xi_0'^y \neq 0$.

The merit to adopt the transformation (3.1) is that $e_\bullet'^y$ and $\sigma_\bullet'^y = \sigma_\bullet'^y(x'^y, \xi'^y)$ are expressed simply as

$$e_0'^y = \frac{\partial}{\partial x_0'^y}, \quad e_j'^y = \frac{\partial}{\partial x_j'^y} + \frac{1}{2}x_{n+j}'^y\frac{\partial}{\partial x_0'^y}, \quad e_{n+j}'^y = \frac{\partial}{\partial x_{n+j}'^y} - \frac{1}{2}x_j'^y\frac{\partial}{\partial x_0'^y},$$
$$\sigma_0'^y = \xi_0'^y, \quad \sigma_j'^y = \xi_j'^y + \frac{1}{2}x_{n+j}'^y\xi_0'^y, \quad \sigma_{n+j}'^y = \xi_{n+j}'^y - \frac{1}{2}x_j'^y\xi_0'^y$$

(cf. (3.2), (3.3), (3.4), (3.5)). Referring to (3.6) the symbol of $\square_{\mathbb{J}}'^y$ is expressed as

$$\begin{aligned}\sigma(\square_{\mathbb{J}}'^y)(x'^y, \xi'^y) &= \mathbf{p}'(y; x'^y, \xi'^y)\\ &= \widetilde{\mathbf{p}}'(y; \sigma'^y(x'^y, \xi'^y)) := \sum_{j=1}^{2n}\sigma_j'^y(x'^y, \xi'^y)^2 + \lambda\sigma_0'^y(x'^y, \xi'^y)\end{aligned}$$

and the operator $\square_{\mathbb{J}}'^y$ is, hence, y-invariant, so that its inverse operator is also y-invariant and its symbol can be expressed as

$$\sigma((\square_{\mathbb{J}}'^y)^{-1})(x'^y, \xi'^y) = \mathbf{q}'(y; x'^y, \xi'^y) = \widetilde{\mathbf{q}}'(y; \sigma'^y(x'^y, \xi'^y)).$$

To investigate $\widetilde{\mathbf{q}}'$, we have only to follow the argument in [2, §4]. Let us put $\mathbf{q}(\xi) = \widetilde{\mathbf{q}}'(y; \xi)$, $\mathbf{p}(\xi) = \widetilde{\mathbf{p}}'(y; \xi)$. Notice that if we set $\xi = \sigma'^y(x'^y, \xi'^y)$ then $\xi_0 = \sigma_0'^y(x'^y, \xi'^y) = \xi_0'^y$. Therefore, the assumption $\xi_0'^y \neq 0$ induces $\xi_0 \neq 0$, and we have

$$\begin{aligned}1 = \sigma((\square_{\mathbb{J}}'^y)^{-1}\square_{\mathbb{J}}'^y)(x, \xi)\Big|_{x=0} &= \sum_\alpha \frac{1}{\alpha!}\Big(\partial_\xi^\alpha(\mathbf{q}(\sigma(x, \xi)))\cdot D_x^\alpha(\mathbf{p}(\sigma(x, \xi)))\Big)\Big|_{x=0}\\ &= \Big\{\mathbf{p}(\xi) - \sum_{j=1}^{2n}\frac{1}{4}\xi_0^2(\frac{\partial}{\partial\xi_j})^2\Big\}\mathbf{q}(\xi) = \Big\{\sum_{j=1}^{2n}\xi_j^2 + \lambda\xi_0 - \sum_{j=1}^{2n}\frac{1}{4}\xi_0^2(\frac{\partial}{\partial\xi_j})^2\Big\}\mathbf{q}(\xi).\end{aligned}$$

That is, under the assumption $\xi_0 \neq 0$, $\mathbf{q}(\xi)$ satisfies the equation

$$\sum_{j=1}^{2n}\Big\{\xi_j^2 - \frac{1}{4}\xi_0^2\Big(\frac{\partial}{\partial\xi_j}\Big)^2\Big\}\mathbf{q}(\xi) + \lambda\xi_0\,\mathbf{q}(\xi) = 1,$$

which is formally solved in [2, (4.17), (4.23)] to give

$$\mathbf{q}(\xi) = \int_0^\infty e^{-\lambda\xi_0 s}G(\xi, s)\,ds, \tag{3.1.2}$$

where $G(\xi, s)$ is the function (3.8) in the case $\xi_0'^y \neq 0$ but with ξ'^y replaced by ξ. And we have the estimate (cf. (3.1.1))

$$|G(\xi, s)| \leq 2^n \exp\Big(-\frac{1}{2}\sum_{j=1}^{2n}|\xi_0|s\Big) = 2^n \exp\Big(-n|\xi_0|s\Big),$$

etc. That is, in the case $0 < q < n$ the integrand of (3.1.2), or (3.7) is rapidly decreasing. In this way, Theorem 3.1 can be justified (cf. [2, (4.26)-(4.28)]).

3.2 On the proof of Theorem 3.4

We refer to the argument in [2, §5]. By formal computation we have

$$\begin{aligned}
(3.2.1)\quad ((\Box_{\mathbb{J}}'^y)^{-1}f)(x'^y) &= \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y; x'^y, x'^y - z'^y) f(z'^y) dz'^y \\
&:= \int_{\mathbb{R}^{2n+1}} \Big\{(2\pi)^{-2n-1}\int_{\mathbb{R}^{2n+1}} e^{i\langle x'^y - z'^y, \xi'^y\rangle}\mathbf{q}'(y; x'^y, \xi'^y) d\xi'^y\Big\} f(z'^y) dz'^y, \\
((\Box_{\mathbb{J}}'^y)^{-1}f)(x'^y) &= (L^*_{x'^y}((\Box_{\mathbb{J}}'^y)^{-1}f))(0) = ((\Box_{\mathbb{J}}'^y)^{-1}(L^*_{x'^y}f))(0) \\
&= \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y; 0, -z'^y)\,(L^*_{x'^y}f)(z'^y)\,dz'^y = \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y; 0, -z'^y)\, f(x'^y z'^y)\,dz'^y \\
&= \int_{\mathbb{R}^{2n+1}} \mathbf{k}'(y; 0, (z'^y)^{-1})\, f(x'^y z'^y)\,dz'^y.
\end{aligned}$$

As for the last equality, notice that $-z'^y = (z'^y)^{-1}$ because the matrix $c_\bullet(y)$ at (3.3) is skew-symmetric. Setting

$$\begin{aligned}
(3.2.2)\qquad \mathbf{k}'(y; x'^y) := \mathbf{k}'(y; 0, x'^y) &= (2\pi)^{-2n-1}\int_{\mathbb{R}^{2n+1}} e^{i\langle x'^y, \xi'^y\rangle}\mathbf{q}'(y; 0, \xi'^y)\,d\xi'^y \\
&= (2\pi)^{-2n-1}\int_{\mathbb{R}^{2n+1}} e^{i\langle x'^y, \xi'^y\rangle}\,\widetilde{\mathbf{q}}'(y; \xi'^y)\,d\xi'^y,
\end{aligned}$$

thus formally we obtain the formula (3.10).

Next, the integrand of (3.11) has the following property.

Lemma 3.2.1 *The function $A(s)(\gamma(x_H'^y, s) - ix_0'^y s)^{-n}$ of $s\,(\in \mathbb{C})$ is meromorphic and has no poles in*

$$(3.2.3)\qquad \begin{aligned}
&\Big\{s \,\Big|\, |\mathrm{Im}\, s| < \frac{\pi}{2}\Big\} &&: x_H'^y \neq 0 \;\&\; x_0'^y = 0, \\
&\Big\{s \,\Big|\, 0 \leq \mathrm{Im}\, s\cdot \mathrm{sgn}\, x_0'^y < \frac{\pi}{2}\Big\} &&: x_H'^y \neq 0 \;\&\; x_0'^y \neq 0, \\
&\Big\{s \,\Big|\, s \neq \pm k\pi\ (k = 0, 1, 2, \ldots)\Big\} &&: x_H'^y = 0 \;\&\; x_0'^y \neq 0,
\end{aligned}$$

and is analytic on the integral path I. Further, there are $C > 0$, $c > 0$ such that, on I,

$$(3.2.4)\qquad |A(s)\,e^{-\lambda s}| \leq C\exp\Big((|\lambda| - (n - \frac{1}{2}))\,|\mathrm{Re}\, s|\Big) \leq C\exp\Big(-|\mathrm{Re}\, s|\Big),$$

$$(3.2.5)\qquad \mathrm{Re}(\gamma(x_H'^y, s) - ix_0'^y s) > c > 0.$$

Proof. We put $x = x'^y$, $\xi = \xi'^y$, etc., for short. As for (3.2.4)**:** The second inequality follows from $0 < q < n$. Let us show the first one. On $I = I_0$, for $|s|$ large, we have

$$\frac{s}{\sinh s} = \frac{|s|}{\sinh |s|} = \frac{2|s|}{e^{|s|} - e^{-|s|}} \leq \frac{2|s|}{e^{|s|} - e^{|s|}/2}$$
$$= 4|s|e^{-|s|} \leq 8n\, e^{|s|/2n} e^{-|s|} = 8n\, e^{-(1-\frac{1}{2n})|s|},$$

so that, on $I = I_0$,

$$|A(s)\, e^{-\lambda s}| \leq C \exp\Big(|\lambda s| - (1 - \frac{1}{2n})n|s|\Big) = C \exp\Big((|\lambda| - (n - \frac{1}{2}))|s|\Big).$$

On $I = I_\varepsilon$, for $|s|$ large, we have

$$\Big|\frac{s}{\sinh s}\Big| = \Big|\frac{u+iv}{\sinh(u+iv)}\Big| = \Big|\frac{u+iv}{\sinh u \cdot \cos v + i \cosh u \cdot \sin v}\Big|$$
$$= \frac{|u+iv|}{(\sinh^2 u \cdot \cos^2 v + \cosh^2 u \cdot \sin^2 v)^{1/2}} \leq \frac{|u+iv|}{(\sinh^2 u \cdot \cos^2 v + \sinh^2 u \cdot \sin^2 v)^{1/2}}$$
$$= \frac{|u+iv|}{\sinh |u|} = \frac{|\mathrm{Re}\, s + i\varepsilon|}{\sinh(|\mathrm{Re}\, s|)} \leq \frac{|\mathrm{Re}\, s| + \varepsilon}{\sinh(|\mathrm{Re}\, s|)} \leq \frac{2|\mathrm{Re}\, s|}{\sinh(|\mathrm{Re}\, s|)}.$$

Thus, on $I = I_\varepsilon$, we obtain the above estimate with s replaced by $\mathrm{Re}\, s$. As for (3.2.5): The function $s \coth s$ is meromorphic and has no poles at $s \neq \pm ik\pi$ $(k \in \mathbb{N})$ and we know

$$\mathrm{Re}(s \coth s) > 0 \quad (-\frac{\pi}{2} < \mathrm{Im}\, s < \frac{\pi}{2}) \tag{3.2.6}$$

and

$$\mathrm{Re}(\gamma(x_H, s) - ix_0 s) = \sum_{j=1}^{2n} \frac{x_j^2}{4} \mathrm{Re}(s \coth s) + x_0 \mathrm{Im}\, s. \tag{3.2.7}$$

Now, by (3.2.6), on $I = I_0$ (& $x_H \neq 0$) we have $\mathrm{Re}(s \coth s) > c > 0$ with some $c > 0$, and on $I = I_\epsilon$ (& $x_0 \neq 0$) we have $\mathrm{Re}(s \coth s) > 0$ and

$$x_0\, \mathrm{Im}\, s = x_0\, \mathrm{Im}(t + i\varepsilon \cdot \mathrm{sgn}\, x_0) = x_0\, \varepsilon \cdot \mathrm{sgn}\, x_0 = \varepsilon |x_0| > 0.$$

Thus we obtain the estimation (3.2.5). As for (3.2.3): (3.2.6), (3.2.7) say that $\mathrm{Re}(\gamma(x_H, s) - ix_0 s) > 0$ as long as

$$(x_0 \mathrm{Im}\, s,\, x_H) \neq (0, 0, \ldots, 0), \quad x_0\, \mathrm{Im}\, s \geq 0, \quad |\mathrm{Im}\, s| < \frac{\pi}{2}.$$

Hence, $(\gamma(x_H, s) - ix_0 s)^{-n}$ has no poles in

$$\Big\{s \,\Big|\, |\mathrm{Im}\, s| < \frac{\pi}{2}\Big\} \;:\; x'^y_H \neq 0 \;\&\; x'^y_0 = 0,$$
$$\Big\{s \,\Big|\, 0 \leq \mathrm{Im}\, s \cdot \mathrm{sgn}\, x'^y_0 < \frac{\pi}{2}\Big\} \;:\; x'^y_H \neq 0 \;\&\; x'^y_0 \neq 0,$$
$$\Big\{s \,\Big|\, s \neq 0\Big\} \;:\; x'^y_H = 0 \;\&\; x'^y_0 \neq 0.$$

Since $A(s)$ has no poles in $\{s \mid s \neq \pm k\pi \ (k = 1, 2, \ldots)\}$, certainly the function $A(s)(\gamma(x_H, s) - ix_0 s)^{-n}$ has no poles in (3.2.3). ∎

Let us prove Theorem 3.4.

Proof of Theorem 3.4. Again, we put $x = x'^y$, $\xi = \xi'^y$, etc., for short. First, let us prove the formula (3.11) with "$I = I_0$ and $x_H \neq 0$". Theorem 3.1 implies that, in the case $\xi_0 \neq 0$, we have

$$\widetilde{\mathbf{q}}'(y;\xi) = \int_0^\infty e^{-\lambda\xi_0 s} G(\xi, s)\, ds = \int_0^\infty |\xi_0|^{-1} e^{-\mu s} G(\xi, |\xi_0|^{-1}s)\, ds,$$

$$G(\xi, |\xi_0|^{-1}s) = \Big(\frac{1}{\cosh s}\Big)^n \exp\Big(-\sum_{j=1}^{2n} \frac{\xi_j^2}{|\xi_0|} \tanh s\Big),$$

where we set $\mu = \lambda \cdot \operatorname{sgn} \xi_0$. In the following, we want to compute

$$\begin{aligned}\mathbf{k}'(y;x) &= \int_0^\infty \Big\{(2\pi)^{-2n-1} \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}^{2n}} e^{i\langle x,\xi\rangle} |\xi_0|^{-1} e^{-\mu s} G(\xi, |\xi_0|^{-1}s)\, d\xi\Big\} ds \\ &= \int_0^\infty \Big\{(2\pi)^{-1} \int_{\mathbb{R}\setminus\{0\}} e^{ix_0\xi_0} |\xi_0|^{-1} e^{-\mu s} \\ &\qquad \Big((2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{i\langle x_H,\xi_H\rangle} G(\xi, |\xi_0|^{-1}s)\, d\xi_H\Big) d\xi_0\Big\} ds.\end{aligned}$$

First, we have

$$\begin{aligned}&(2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{i\langle x_H,\xi_H\rangle} G(\xi, |\xi_0|^{-1}s)\, d\xi_H \\ &= \Big(\frac{1}{\cosh s}\Big)^n \cdot (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{i\langle x_H,\xi_H\rangle} \exp\Big(-\sum_{j=1}^{2n} \frac{\xi_j^2}{|\xi_0|} \tanh s\Big)\, d\xi_H \\ &= (4\pi)^{-n} \Big(\frac{|\xi_0|}{\sinh s}\Big)^n \exp\Big(-\frac{1}{4} \sum_{j=1}^{2n} |\xi_0| x_j^2 \coth s\Big).\end{aligned}$$

Hence, referring to the formula [2, (5.22)(5.23)], we have

$$\begin{aligned}&(2\pi)^{-1} \int_{\mathbb{R}\setminus\{0\}} e^{ix_0\xi_0} |\xi_0|^{-1} e^{-\mu s} \Big((2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{i\langle x_H,\xi_H\rangle} G(\xi, |\xi_0|^{-1}s)\, d\xi_H\Big) d\xi_0 \\ &= (4\pi)^{-n} \Big(\frac{1}{\sinh s}\Big)^n (2\pi)^{-1} \int_{-\infty}^\infty e^{ix_0\xi_0} |\xi_0|^{n-1} \\ &\qquad \exp\Big(-\lambda s \cdot \operatorname{sgn}\xi_0 - |\xi_0| \frac{1}{4} \sum_{j=1}^{2n} x_j^2 \coth s\Big)\, d\xi_0 \\ &= (4\pi)^{-n} \Big(\frac{1}{\sinh s}\Big)^n (2\pi)^{-1} \Gamma(n) \Big\{e^{-\lambda s} \big(\frac{1}{4} \sum_{j=1}^{2n} x_j^2 \coth s - ix_0\big)^{-n} \\ &\qquad + e^{\lambda s} \big(\frac{1}{4} \sum_{j=1}^{2n} x_j^2 \coth s + ix_0\big)^{-n}\Big\}\end{aligned}$$

$$= (4\pi)^{-n}\Big(\frac{s}{\sinh s}\Big)^n (2\pi)^{-1}\Gamma(n)\Big\{e^{-\lambda s}(\frac{1}{4}\sum_{j=1}^{2n} sx_j^2\coth s - ix_0 s)^{-n} + e^{\lambda s}(\frac{1}{4}\sum_{j=1}^{2n} sx_j^2\coth s + ix_0 s)^{-n}\Big\}$$

and

$$\begin{aligned}\mathbf{k}'(y;x) &= (2\pi)^{-1}\Gamma(n)\Big\{\int_0^\infty (4\pi)^{-n}\Big(\frac{s}{\sinh s}\Big)^n e^{-\lambda s}(\frac{1}{4}\sum_{j=1}^{2n} sx_j^2\coth s - ix_0 s)^{-n} ds \\ &\qquad + \int_0^\infty (4\pi)^{-n}\Big(\frac{s}{\sinh s}\Big)^n e^{\lambda s}(\frac{1}{4}\sum_{j=1}^{2n} sx_j^2\coth s + ix_0 s)^{-n} ds\Big\} \\ &= (2\pi)^{-1}\Gamma(n)\int_{-\infty}^\infty (4\pi)^{-n}\Big(\frac{s}{\sinh s}\Big)^n e^{-\lambda s}\Big(\frac{1}{4}\sum_{j=1}^{2n} sx_j^2\coth s - ix_0 s\Big)^{-n} ds.\end{aligned}$$

Here the integrability is guaranteed by (3.2.4), (3.2.5). Thus the formula (3.11) with "$I = I_0$ and $x_H \neq 0$" was obtained.

As was indicated in Remark of the theorem the above formula cannot cover the case $x_0 \neq 0$ and $x_H = 0$, and we want to draw the formula (3.11) with "$I = I_\varepsilon$ and $x_0 \neq 0$". If $x_0 \neq 0$ and $x_H \neq 0$, then the integrand $A(s)\, e^{-\lambda s}\Big(\gamma(x_H, s) - ix_0 s\Big)^{-n}$ of the integral (3.11) is analytic on $\Big\{s \;\Big|\; 0 \le \operatorname{Im} s \cdot \operatorname{sgn} x_0 < \frac{\pi}{2}\Big\}$ (cf. (3.2.3)). Hence, by the ordinary theory of analytic function, we know that, in the case $x_0 \neq 0$ and $x_H \neq 0$, the integral (3.11) on the integral path $I = I_0$ coincides with that on $I = I_\varepsilon$. The latter one (with "$I = I_\varepsilon$, $x_0 \neq 0$ and $x_H \neq 0$") can be extended naturally to the desired one (with "$I = I_\varepsilon$ and $x_0 \neq 0$"). It will be obvious now that the formula (3.11) thus obtained fulfils the requirement stated in the theorem. ∎

4 Parametrix and some global properties of $\Box_H$

Returning to the setting in §1, let us construct a global parametrix of $\Box_H = \Box_{H,q}$. First, we will construct a parametrix of the principal part $\underline{\Box_H}$ on U.

Given a symbol $\mathbf{q}(x, \xi) \in \mathcal{S}_H^\infty(U)$, in the y-coordinates (x^y, ξ^y) it is written as

$$\mathbf{q}(y; x^y, \xi^y) := \mathbf{q}(y + v_\bullet(y)x^y, v^\bullet(y)\xi^y).$$

Conversely, a symbol $\mathbf{q}(y; x^y, \xi^y)$ in the y-ones can be written in the usual ones (x, ξ) as

$$\mathbf{q}(x, \xi) := \mathbf{q}(x; 0, \sigma(x, \xi)).$$

Indeed, let φ be the transformation of $\mathbb{R}^{2n+1}$ defined by $\varphi(x^y) = y + v_\bullet(y)x^y$ (cf. (2.1)), then we have

$$\mathbf{q}(\varphi(x^y); 0, \sigma(\varphi(x^y), v^\bullet(y)\xi^y)) = \mathbf{q}(\psi(x^y), v^\bullet(y)\xi^y)$$

$$= \mathbf{q}(y; \psi^{-1}\psi(x^y), \psi^* v^\bullet(y)\xi^y) = \mathbf{q}(y; x^y, {}^t v_\bullet(y)\, v^\bullet(y)\xi^y)$$
$$= \mathbf{q}(y; x^y, \xi^y).$$

Now, together with Corollaries 3.2 and 3.5, it implies the following.

Theorem 4.1 (On a parametrix of $\Box_H$ on U) *Let $\mathbf{q}(y; x^y, \xi^y)$ be the symbol given in Corollary 3.2, and let $\mathbf{k}(y; x^y)$ be the kernel given in Corollary 3.5. Then the inverse operator $(\Box_H)^{-1}$ is expressed as*

$$(4.1) \qquad (\underline{\Box_H})^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}} \cdot f_{\mathbb{J}}\Big)(y) = 2\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}(y)} \cdot \frac{1}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n+1}} e^{i\langle y,\xi\rangle} \mathbf{q}(y; 0, \sigma(y,\xi))\, \widehat{f_{\mathbb{J}}}(\xi)\, d\xi,$$

$$(4.2) \qquad (\underline{\Box_H})^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}} \cdot f_{\mathbb{J}}\Big)(y) = 2\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}(y)} \cdot \int_{\mathbb{R}^{2n+1}} \det v^\bullet(y)\, \mathbf{k}(y; (z^y)^{-1})\, f_{\mathbb{J}}(z)\, dz$$

with $\sigma((\underline{\Box_H})^{-1}) \in \mathcal{S}_H^{-2}(U; \mathrm{End}(\wedge_H^{0,q} T^* U))$, and $\Box_H$ on U has a two-sided parametrix Q_U given by

$$Q_U = (\underline{\Box_H})^{-1}(I + R_U^H + \cdots + (R_U^H)^k + \cdots)$$
$$\textit{with } \ \Box_H(\underline{\Box_H})^{-1} := I - R_U^H,\ \sigma(R_U^H) \in \mathcal{S}_H^{-1}(U; \mathrm{End}(\wedge_H^{0,q} T^* U)).$$

As for the expansion $\sigma(Q_U) = \sum_{k\geq 2} \sigma_{-k}(Q_U)$, $\sigma_{-k}(Q_U) \in \mathcal{S}_{-k}^H(U; \mathrm{End}(\wedge_H^{0,q} T^* U))$, the terms can be expressed explicitly up to an arbitrarily low degree by using Proposition 1.1.

Proof. As for (4.1) : Since the inverse operator $(\Box_{\mathbb{J}}^y)^{-1}$ has the symbol $\mathbf{q}(y; x^y, \xi^y)$ (cf. Corollary 3.2), the above argument and [2, Theorem (18.4)] imply that the inverse $(\Box_{\mathbb{J}})^{-1}$ has the symbol $\mathbf{q}(y; 0, \sigma(y,\xi))$. Thus we obtain the formula (4.1). As for (4.2) : First we have

$$(4.3) \qquad (\underline{\Box_H})^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{\mathbb{J}}} \cdot f_{\mathbb{J}}\Big)(y) = (\Box_H^y)^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}} \cdot f_{\mathbb{J}}^y\Big)(0)$$
$$(f_{\mathbb{J}}^y(z^y) := f_{\mathbb{J}}(y + v_\bullet(y) z^y)).$$

Indeed, by (2.1) and (3.9), we have

$$\widehat{f_{\mathbb{J}}^y}(\xi^y) = \int e^{-i\langle z^y, \xi^y\rangle} f_{\mathbb{J}}^y(z^y) dz^y = \int e^{-i\langle {}^t v^\bullet(y)(z-y), {}^t v_\bullet(y)\, \xi_\bullet\rangle} f_{\mathbb{J}}(z) \det {}^t v^\bullet(y)\, dz$$
$$= e^{i\langle y,\xi\rangle} \det {}^t v^\bullet(y) \int e^{-i\langle z,\xi\rangle} f_{\mathbb{J}}(z)\, dz = e^{i\langle y,\xi\rangle} \det {}^t v^\bullet(y)\, \widehat{f_{\mathbb{J}}}(\xi),$$

$$(\Box_H^y)^{-1}\Big(\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}} \cdot f_{\mathbb{J}}^y\Big)(0) = 2\sum_{|\mathbb{J}|=q} \overline{e_{\mathbb{C}}^{y,\mathbb{J}}(0)} \cdot \frac{1}{(2\pi)^{2n+1}} \int e^{i\langle 0, \xi^y\rangle} \mathbf{q}(y; 0, \xi^y)\, \widehat{f_{\mathbb{J}}^y}(\xi^y)\, d\xi^y$$

$$= 2 \sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}(y)} \cdot \frac{1}{(2\pi)^{2n+1}} \int \mathbf{q}(y;0,\sigma(y,\xi)) e^{i\langle y,\xi\rangle} \det {}^t v^{\bullet}(y) \, \widehat{f_{\mathbb{J}}}(\xi) \det {}^t v_{\bullet}(y) \, d\xi$$

$$= 2 \sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}(y)} \cdot \frac{1}{(2\pi)^{2n+1}} \int e^{i\langle y,\xi\rangle} \mathbf{q}(y;0,\sigma(y,\xi)) \, \widehat{f_{\mathbb{J}}}(\xi) \, d\xi = (\underline{\Box_H})^{-1} (\sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}} \cdot f_{\mathbb{J}})(y).$$

Thus we obtain (4.3). Hence, referring to (3.12), we have

$$(\Box^y_H)^{-1} (\sum_{|\mathbb{J}|=q} \overline{e^{y,\mathbb{J}}_{\mathbb{C}}} \cdot f^y_{\mathbb{J}})(0) = 2 \sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}(y)} \cdot \int \mathbf{k}(y;(z^y)^{-1}) \, f^y_{\mathbb{J}}(z^y) \, dz^y$$

$$= 2 \sum_{|\mathbb{J}|=q} \overline{e^{\mathbb{J}}_{\mathbb{C}}(y)} \cdot \int \mathbf{k}(y;(z^y)^{-1}) \, f_{\mathbb{J}}(z) \det {}^t v^{\bullet}(y) \cdot dz.$$

The remaining part will be obvious by the standard argument. ∎

A two-sided global parametrix Q of $\Box_H$ is constructed as follows: According to Theorem 4.1, on a small open set U we construct a parametrix Q_U with $\sigma(Q_U) \in \mathcal{S}^{-2}_H(U; \mathrm{End}(\wedge^{0,q}_H T^*U))$ which is properly supported (cf. [2, (9.21)]). We take a finite number of such pairs $\{(U, Q_U)\}$, where $\{U\}$ is an open covering of M. Let $\{\phi_U\}$ be a C^∞-partition of unity subordinate to $\{U\}$. Then the operator Q defined by

$$Qf = \sum_U Q_U(\phi_U f)$$

is a two-sided global parametrix with $\sigma(Q) \in \mathcal{S}^{-2}_H(M; \mathrm{End}(\wedge^{0,q}_H T^*M))$.

By the standard Fredholm theory (cf. [2, Theorem (19.16)]), now we know that the L^2-extension

$$\Box_H : L^2(M; \wedge^{0,q}_H T^*M) \to L^2(M; \wedge^{0,q}_H T^*M)$$

has the following properties:

$$\dim \ker \Box_H < \infty, \quad \ker \Box_H \subset C^\infty(M; \wedge^{0,q}_H T^*M),$$

and range $\Box_H$ is closed and

$$\operatorname{codim} \operatorname{range} \Box_H < \infty.$$

Further, the associated projections

$$\Pi_1 : L^2(M; \wedge^{0,q}_H T^*M) = \ker \Box_H \oplus (\ker \Box_H)^\perp \ \to \ (\ker \Box_H)^\perp,$$

$$\Pi_2 : L^2(M; \wedge^{0,q}_H T^*M) = \operatorname{range} \Box_H \oplus (\operatorname{range} \Box_H)^\perp \ \to \ \operatorname{range} \Box_H$$

and the continuous operator $\mathcal{N}_H : L^2(M; \wedge^{0,q}_H T^*M) \to L^2(M; \wedge^{0,q}_H T^*M)$ satisfying

$$\Box_H \mathcal{N}_H = \Pi_2 \ (\text{on } L^2(M; \wedge^{0,q}_H T^*M)), \quad \mathcal{N}_H \Box_H = \Pi_1 \ (\text{on } \operatorname{dom} \Box_H),$$

which is called the partial inverse of $\Box_H$ and is unique modulo smoothing operators, are all H-pseudodifferential operators. We know that the operator $\mathcal{N}_H$, which is of degree -2, is in fact a parametrix of $\Box_H$.

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Department of Mathematics, Graduate School of Science and Engineering,
Saitama University, Saitama-City, Saitama 338-8570, Japan
E-mail address: mnagase@rimath.saitama-u.ac.jp