The remarks on the asymptotics of extreme values of discrete random variables

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Abstract. The paper investigates the asymptotic of almost surely extreme values of discrete random variables. The answers are given to some questions that remained open. Applications to the geometric distribution and the Poisson distribution are presented. The example of the birth and death process is also studied.

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1 Introduction and main results

Let us consider the sequence $\xi, \xi_1, \xi_2, \dots$ of independent identically distributed random variables (i.i.d.r.v.). And let

$$\bar{\xi}_n = \max_{1\le i\le n} \xi_i.$$

Random variable $\bar{\xi}_n$ has been quite thoroughly investigated (see [7]-[9], [13]). It should be noted that the absolute majority of works only the case of weak convergence of the distribution $\bar{\xi}_n$ was studied.

We are mostly interested in the asymptotic behavior of the extreme values of $\bar{\xi}_n$ almost surely (a.s.). Moreover, only a discrete case will be studied. It is assumed that r.v. ξ has a distribution (i, p_i), $i \ge 0$, so, further we assume that

$$\mathbf{P}(\xi = i) = p_i > 0, \quad \sum_{i=0}^{\infty} p_i = 1.$$

This topic has been discussed in a number of works ([1]-[3], [12], [15]-[17]) (the fact that the asymptotics of extreme values of r.v. in continuous and discrete cases can differ significantly has long been known, see, for example, [3]).

To formulate the following result, we introduce some necessary notation.

For r.v. ξ with distribution (i, p_i), $i \geq 0$, put:

$$R(n) = -\ln \mathbf{P}(\xi \geq n) = -\ln \left(\sum_{i \geq n} p_i \right),$$

$$r(n) = R(n) - R(n-1).$$

Let us define the following functions for sufficiently large $t > 0$:

$$L_0(t) = t, \quad L_m(t) = \ln L_{m-1}(t), \quad m \in \mathbb{N}.$$

It was noticed quite a while ago that asymptotic behavior of $\bar{\xi}_n$ in discrete case is closely related to sequence

$$a(n) = \max \left(k \geq 0 : \sum_{i \geq k} p_i \geq \frac{1}{n} \right). \tag{1}$$

Further for the sequence $(r(n))$ we define its extension to the function $r : (0, \infty) \to \mathbb{R}$ by setting $r(x) = r(\lceil x \rceil)$, ($\lceil x \rceil$ - the least integer $\geq x$).

Let

$$R(x) = \int_0^x r(y)dy.$$

The function R is a piecewise linear extension of the sequence $R(n)$.

Given a function $H : \mathbb{R} \to \mathbb{R}$ we denote by H^{-1} its generalized inverse defined by

$$H^{-1}(y) = \inf \{x \in \mathbb{R} : H(x) \geq y\}, \quad y \in \mathbb{R}.$$

Put

$$\alpha_m(t) = \sum_{i=1}^{m} L_i(t), \quad a_m(t) = R^{-1}(\alpha_m(t)),$$

$$d(t) = R^{-1}(L_1(t) - L_3(t)).$$

We will also assume that the following condition is satisfied:
$\forall x > 0$

$$\lim_{t \to \infty} \frac{r(tx)}{r(t)} = x^{\rho}, \quad \rho > -1. \tag{2}$$

Note that for the geometric distribution and the Poisson distribution, the condition (2) is fulfilled at $\rho = 0$.

As it turned out, the asymptotic behavior of $\bar{\xi}_n$ depends significantly on the parameter ρ. Therefore, it seems appropriate to consider three cases:

A) $\quad 0 < \rho < \infty$;

B) $\quad -1 < \rho < 0$;
C) $\quad \rho = 0$.
First we present some well-known results of cases A), B), C).

Theorem 1. *(Case A), [15], [16]) Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $a(n)$ defined by equality (1) and the condition (2) is fulfilled when $0 < \rho < \infty$.*
Then

$$\mathbf{P}\left(\limsup_{n\to\infty}(\bar{\xi}_n - a(n)) = 1\right) = 1, \tag{3}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}(\bar{\xi}_n - a(n)) = -1\right) = 1, \tag{4}$$

and

$$\mathbf{P}(\bar{\xi}_n = a(n) \quad i.o.) = 1,$$

("i.o." - abbreviation "infinitely often").

It should be noted that in articles [15], [16] the equalities close to (3) and (4) were obtained in a more general situation.

Theorem 2. *(Case B), [1]) Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $m \geq 1$ some fixed integer and the condition (2) is fulfilled when $-1 < \rho < 0$.*
Then

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(n))(\bar{\xi}_n - a_m(n))}{L_{m+1}(n)} = 1\right) = 1, \tag{5}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(\bar{\xi}_n - d(n))}{2L_3(n)} = -1\right) = 1. \tag{6}$$

In fact, Theorem 2 is not clearly formulated in [1], but it simply follows from it. Indeed, if the function $r(x)$ satisfies condition (2), then it is regularly varying at $+\infty$ with index ρ, $r \in RV_\rho$. Then also $R \in RV_{\rho+1}$, $R^{-1} \in RV_{1/(\rho+1)}$ ([4], Proposition 1.5.8, Theorem 1.5.12) and $r(R^{-1}) \in RV_{\rho/(\rho+1)}$.
Therefore $r(a_1(t)) = r(R^{-1}(\ln t)) = (\ln t)^{\rho/(1+\rho)}g(\ln t)$, where function $g(t)$ slowly varying to infinity. From the last equality and Theorem 2 of article [1], we immediately obtain equalities (5), (6).

Theorem 3. *(Case C), [1]) Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $m \geq 1$ some fixed integer and the condition (2) is fulfilled when $\rho = 0$.*
(i) If

$$\frac{r(a_1(n))}{L_{m+1}(n)} \to 0, \quad n \to \infty, \tag{7}$$

then the equality (5) holds.

(ii) If

$$\frac{L_2(n)r(a_1(n))}{L_3(n)} \to 0, \quad n \to \infty,$$

then the equality (6) holds.

As it is seen from the results above, for cases A and B we have a complete picture of the asymptotic behavior of the r.v. $\bar{\xi}_n$. In the boundary case C the situation is more complicated. Although the case of the Poisson distribution was analyzed in articles [15], [16], a number of related problems remain open. For example, under the condition

$$\frac{L_2(n)r(a_1(n))}{L_3(n)} \to \infty, \quad n \to \infty, \tag{8}$$

in article [1] the following relation was established:

$$\mathbf{P}\left(\liminf_{n\to\infty}(\bar{\xi}_n - d(n)) = \kappa\right) = 1, \tag{9}$$

where $\kappa \in [-1, 0]$. Here by κ we denote nonrandom constant.
Unfortunately, the constant κ was not found even in the important case of geometric distribution.

The same applies to the value of κ_1 in equality

$$\mathbf{P}\left(\limsup_{n\to\infty}(\bar{\xi}_n - a_m(n) = \kappa_1\right) = 1,$$

under the condition

$$\frac{r(a_1(n))}{L_{m+1}(n)} \to \infty, \quad n \to \infty. \tag{10}$$

In this article, we will try to find answers to these questions.

The main results of the work are the following theorems.

Theorem 4. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, the function r(x) is monotonic and satisfies conditions (8) and (2) when $\rho = 0$.*

(i) If

$$\sum_{n\geq 1} \frac{1}{R(n)} = \infty, \tag{11}$$

then

$$\mathbf{P}\left(\liminf_{n\to\infty}(\bar{\xi}_n - \lfloor d(n)\rfloor) = -1\right) = 1. \tag{12}$$

(ii) If the series $\sum_{n\geq 1} \frac{1}{R(n)}$ converges, then

$$\mathbf{P}\left(\liminf_{n\to\infty}(\bar{\xi}_n - \lfloor d(n)\rfloor) = 0\right) = 1,$$

where $\lfloor x\rfloor$ - the least integer $\leq x$.

Theorem 5. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $m \geq 1$ some fixed integer. Let condition (2) be satisfied for $\rho = 0$ and condition (10). Then*

(i) If $m = 1$ and $r(n)$ monotonically increases, then equality of (3) of Theorem 1 holds.

(ii) If $m \geq 2$ and series

$$\sum_{n\geq 1} \frac{1}{R(n)L_1(R(n))\ldots L_{m-2}(R(n))} \tag{13}$$

converges, then

$$\mathbf{P}\left(\limsup_{n\to\infty}(\bar{\xi}_n - \lfloor a_m(n)\rfloor) = 0\right) = 1. \tag{14}$$

(iii) If $m \geq 2$ and the series (13) diverges, then

$$\mathbf{P}\left(\limsup_{n\to\infty}(\bar{\xi}_n - \lfloor a_m(n)\rfloor) = 1\right) = 1.$$

In the next two sections, the proofs of Theorems 4 and 5 are going to be given. And at the end of the article, the examples of applications of the results obtained to the geometric distribution and the Poisson distribution are considered. In addition, birth and death process with linear growth and immigration will be considered.

2 Proof of Theorem 4

Let's start with (i). Since r.v. $\bar{\xi}_n$ takes integer values, then in order to prove (12) it suffices to show that

$$\mathbf{P}(\bar{\xi}_n \leq \lfloor d(n)\rfloor - 1 \quad i.o.) = 1, \tag{15}$$

$$\mathbf{P}(\bar{\xi}_n < \lfloor d(n)\rfloor - 1 \quad i.o.) = 0. \tag{16}$$

But from the relation (9) we obtain

$$\mathbf{P}(\liminf_{n\to\infty}(\bar{\xi}_n - \lfloor d(n)\rfloor) = \hat{\kappa}) = 1,$$

where $\hat{\kappa} \in \{-1, 0, 1\}$, and equality (16) is also fulfilled. Hence it is clear that it remains to prove the equality (15).

Here we use an important result of the article [12], which we formulate in the form of a lemma.

Lemma 1 ([12]). *Let $(\xi_n), n \geq 1$ be a sequence of independent copies of a random variable ξ. Further, let $(u_n), n \geq 1$ be a nondecreasing sequence of real numbers such that*

$$\mathbf{P}(\xi > u_n) \to 0 \quad \textit{and} \quad n\mathbf{P}(\xi > u_n) \to \infty, \quad n \to \infty.$$

Then the probability

$$\mathbf{P}\left(\bar{\xi}_n \le u_n \quad i.o.\right)$$

equals zero or one according to whether the series

$$\sum_{n=1}^{\infty} \mathbf{P}(\xi > u_n) \exp\left(-n\mathbf{P}(\xi > u_n)\right)$$

converges or diverges.

We also have the following implications:

- *if* $\lim_{n\to\infty} \mathbf{P}(\xi > u_n) = c > 0$, *then* $\mathbf{P}\left(\bar{\xi}_n \le u_n \quad i.o.\right) = 0$;
- *if* $\liminf_{n\to\infty} n\mathbf{P}(\xi > u_n) < \infty$, *then* $\mathbf{P}\left(\bar{\xi}_n \le u_n \quad i.o.\right) = 1$.

If in Lemma 1 we choose $u_n = \lfloor d(n) \rfloor - 1$, then it is obvious when $n \to \infty$

$$\mathbf{P}(\xi > \lfloor d(n) \rfloor - 1) \to 0$$

and

$$n\mathbf{P}(\xi > \lfloor d(n) \rfloor - 1) = n\mathbf{P}(\xi \ge \lfloor d(n) \rfloor) \ge n \exp(-R(d(n))) = L_2(n) \to \infty.$$

Thus, according to Lemma 1, the equality (15) is true if and only if the series

$$S = \sum_{n=1}^{\infty} \mathbf{P}(\xi > \lfloor d(n) \rfloor - 1) \exp\left(-n\mathbf{P}(\xi > \lfloor d(n) \rfloor - 1)\right)$$

diverges.

Let's rewrite the series S as follows

$$S = \sum_{n=1}^{\infty} \mathbf{P}(\xi \ge \lfloor d(n) \rfloor) \exp\left(-n\mathbf{P}(\xi \ge \lfloor d(n) \rfloor)\right) = \sum_{k\ge 0} \alpha_k S_k, \tag{17}$$

where

$$\alpha_k = \mathbf{P}(\xi \ge k), \quad S_k = \sum_{\lfloor d(n) \rfloor = k} \exp(-n\alpha_k).$$

Next, we find a lower bound for S_k. Put

$$\begin{aligned} n_k = \min(n \ge 3 : d(n) \ge k) &= \min(n \ge 3 : L_1(n) - L_3(n) \ge R(k)) \\ &= \min\left(n \ge 3 : \frac{n}{L_2(n)} \ge \exp(R(k))\right). \end{aligned} \tag{18}$$

Since $\exp(-t\alpha_k)$ is a decreasing function, then

$$\begin{aligned} S_k &= \sum_{n=n_k}^{n_{k+1}-1} \exp(-n\alpha_k) \ge \int_{n_k}^{n_{k+1}} \exp(-t\alpha_k) dt \\ &= \frac{1}{\alpha_k} \exp(-n_k\alpha_k)(1 - \exp(-\alpha_k(n_{k+1} - n_k))). \end{aligned} \tag{19}$$

In the next step, we find an approximate formula for the integer n_k. For this, we introduce the function $\phi(x) = x/L_2(x)$. In interval $[e^2, \infty)$ the function $\phi(x)$ is continuous and increases monotonically. And therefore, for $x \in [(e^2)/\ln 2, \infty)$ there exists its inverse $\phi^{-1}(x)$. Let's set it by formula

$$\phi^{-1}(x) = xL_2(x)(1 + \psi(x)). \tag{20}$$

Then we have

$$x = \phi(\phi^{-1}(x)) = \frac{xL_2(x)(1 + \psi(x))}{L_2(x) + \ln\left(1 + \frac{L_3(x)+\ln(1+\psi(x))}{L_1(x)}\right)}$$

or

$$\psi(x) = \frac{1}{L_2(x)} \ln\left(1 + \frac{L_3(x) + \ln(1 + \psi(x))}{L_1(x)}\right).$$

From the last equality, elementary considerations deduce the following asymptotic relation:

$$\psi(x) \to 0 \quad \text{when} \quad x \to \infty.$$

This implies the following approximation for the function $\psi(x)$:

$$\psi(x) = \frac{L_3(x)(1 + o(1))}{L_1(x)L_2(x)}. \tag{21}$$

Adding together the equalities (18), (20), (21), we get

$$n_k = \lceil \phi^{-1}(\exp(R(k))) \rceil = \exp(R(k))L_1(R(k)) \left(1 + \frac{L_2(R(k))(1 + o(1))}{R(k)L_1(R(k))}\right) + \theta, \tag{22}$$

where $0 \leq \theta < 1$.

Since $r(k) \geq 0$, then

$$\alpha_k(n_{k+1} - n_k) \geq \alpha_k \exp(R(k))L_1(R(k)) \left(\exp(r(k+1)) - 1 - \frac{L_2(R(k))(1 + o(1))}{R(k)L_1(R(k))}\right)$$

$$\geq L_1(R(k)) \left(r(k+1) - \frac{L_2(R(k))(1 + o(1))}{R(k)L_1(R(k))}\right).$$

Next, we show that under the conditions of Theorem 4

$$L_1(R(k))\, r(k) \to \infty \quad \text{when} \quad k \to \infty. \tag{23}$$

Thus, from the last inequalities and (23), we obtain

$$\alpha_k(n_{k+1} - n_k) \to \infty. \tag{24}$$

Before proceeding further, let's make sure that the asymptotic relation (23) is correct. It is clear that it is enough to consider the case when the function $r(x)$ is decreasing.

Let us first remind that the function $r(x)$ is slowly varying at infinity. Therefore, according to Karamat's theorem (see[4], Proposition 1.5.8, Theorem 1.5.12) when $x \to \infty$

$$R(x) = xr(x)(1 + o(1)), \quad R^{-1}(x) = x\hat{r}(x), \tag{25}$$

where $\hat{r}(x)$ also is slowly varying.

In addition $\forall \epsilon > 0$ for sufficiently large x

$$x^{-\epsilon} \le r(x) \le x^{\epsilon}$$

(see [6], chapter 8).

From the above estimates it follows that:

$$r(x)L_1(R(x)) = r(x)(\ln x + \ln r(x) + o(1)) \ge r(x)((1 - \epsilon)\ln x + o(1)).$$

And therefore (23) it is true if

$$r(x)\ln x \to \infty, \quad x \to \infty. \tag{26}$$

Let's write the condition (8) as follows: when $x \to \infty$

$$\frac{L_2(x)r(a_1(x))}{L_3(x)} = \frac{L_2(x)r(R^{-1}(\ln x))}{L_3(x)} = \frac{L_2(x)r(\ln x\hat{r}(\ln x))}{L_3(x)} \to \infty. \tag{27}$$

Since $r(x)$ is decreasing, then taking into account (25) we have for sufficiently large x and $0 < \varepsilon < 1$:

$$1 = \frac{1}{x}R(R^{-1}(x)) = \hat{r}(x)r(x\hat{r}(x))(1 + o(1)) \le \hat{r}(x)r(x^{1-\varepsilon})(1 + o(1)).$$

As a consequence of the last estimate, we obtain for some $C > 0$, $\forall x > x_0$:

$$\hat{r}(x) \ge C > 0.$$

Therefore, for sufficiently large x:

$$r(\ln x\hat{r}(\ln x)) \le r(C\ln x).$$

Hence and the relation (27) we obtain: for $y = \ln x \to \infty$

$$\frac{L_1(y)r(y)}{L_2(y)} \sim \frac{L_2(x)r(C\ln x)}{L_3(x)} \ge \frac{L_2(x)r(\ln x\hat{r}(\ln x))}{L_3(x)} \to \infty.$$

This means that (26) is fulfilled, so the asymptotic relations (23), (24) are true.

Now we can evaluate the series S from below. In accordance with (19) and (24) when $k \to \infty$

$$S_k \geq \frac{1}{\alpha_k} \exp(-\alpha_k n_k)(1 + o(1)).$$

And further (see (17),(22))

$$\begin{aligned} S &\geq \sum_{k \geq 0} \exp(-\alpha_k n_k)(1 + o(1)) \\ &= \sum_{k \geq 0} \exp(-\alpha_k \exp(R(k)) L_1(R(k))(1 + \chi_k) - \alpha_k \theta)(1 + o(1)) \\ &= \sum_{k \geq 0} \frac{1 + o(1)}{(R(k))^{1+\chi_k}} \exp(-\alpha_k \theta), \end{aligned} \tag{28}$$

where

$$0 \leq \theta \leq 1, \quad R(k) = kr(k)(1 + o(1)), \quad \chi_k = \frac{L_2(R(k))(1 + o(1))}{R(k) L_1(R(k))}.$$

But when $k \to \infty$

$$\ln(R(k))^{\chi_k} = \chi_k(\ln k + \ln r(k) + o(1)) \to 0,$$

accordingly

$$(R(k))^{\chi_k} \to 1.$$

From the last relation (28) and condition (11) it immediately follows that the series S diverges, and therefore equalities (15) and (12) are true.

(ii) is proven according to a similar scheme. We can establish that the convergence of the series $\sum_{n \geq 1} \frac{1}{R(n)}$ implies the equality

$$\mathbf{P}\left(\bar{\xi}_n \leq \lfloor d(n) \rfloor - 1 \quad \text{i.o.}\right) = 0, \tag{29}$$

but

$$\mathbf{P}\left(\bar{\xi}_n \leq \lfloor d(n) \rfloor \quad \text{i.o.}\right) = 1. \tag{30}$$

As we already know from Lemma 1, the equality (29) is equivalent to the convergence of the series S from (17).

Further, instead of the lower estimate of the value of S_k, the following upper estimate is used

$$S_k = \sum_{n=n_k}^{n_{k+1}-1} \exp(-n\alpha_k) \leq \int_{n_k-1}^{n_{k+1}-1} \exp(-t\alpha_k) dt = \frac{\exp(\alpha_k)}{\alpha_k} \exp(-n_k \alpha_k)(1 - \exp(-\alpha_k(n_{k+1} - n_k))).$$

Since the condition (11) was not used when proving the asymptotic ralation (24), then

$$S_k \leq \frac{\exp(\alpha_k)}{\alpha_k} \exp(-\alpha_k n_k)(1 + o(1)).$$

Here and formula (22) for series S we can give the following upper bound

$$\begin{aligned} S &\leq e\sum_{k\geq 0}\exp(-\alpha_k n_k)(1+o(1)) \\ &\leq e\sum_{k\geq 0}\exp(-\alpha_k \exp(R(k))L_1(R(k))(1+\chi_k)-\alpha_k\theta)(1+o(1)) \\ &\leq e\sum_{k\geq 0}\frac{1+o(1)}{(R(k))^{1+\chi_k}}, \end{aligned} \tag{31}$$

where $\chi_k \geq 0$ is defined above.

From the estimates given above in (i) and (31), we conclude that the convergence of the series S follows from the convergence of the series $\sum_{n\geq 1}\frac{1}{R(n)}$.

It remains to prove equality (30). Again, we will use Lemma 1, according to which the equality (30) is true if

$$\liminf_{n\to\infty} n\mathbf{P}\left(\xi > \lfloor d(n)\rfloor\right) < \infty. \tag{32}$$

Let us choose a subsequence (n_k) given by formula (22) and for which $\lfloor d(n_k)\rfloor = k$. Put

$$g(k) = n_k\mathbf{P}\left(\xi > \lfloor d(n_k)\rfloor\right) = n_k\mathbf{P}\left(\xi \geq k+1\right)$$

and suppose that $g(k)\to\infty$ when $k\to\infty$.

Then (see ralations (22), (28))

$$g(k) = (\exp(R(k))L_1(R(k))(1+\chi_k)+\theta)\exp(-R(k+1))$$

$$= \exp(-r(k+1))L_1(R(k))(1+\chi_k)+o(1).$$

In these equalities, let's move on to logarithms

$$\ln g(k) = -r(k+1) + L_2(R(k)) + o(1) \to \infty, \quad k\to\infty.$$

Finally, using formula (25), we get

$$r(k+1) = L_2(k) - \ln g(k) + o(1).$$

Thus with sufficiently large k

$$r(k+1) \leq L_2(k).$$

But the last inequality and relation (25) contradict the convergence of series $\sum_{n\geq 1}\frac{1}{R(n)}$. Therefore, the condition (32) and the equality (30) are fulfilled. □

3 Proof of Theorem 5

Let's start with (i). According to Teorem 1 of article [15] the equalities

$$\mathbf{P}(\bar{\xi}_n = a(n) + 1 \quad i.o.) = 1, \tag{33}$$

$$\mathbf{P}(\bar{\xi}_n > a(n) + 1 \quad i.o.) = 0, \tag{34}$$

are true if

$$\sum_{n \geq 1} \exp(-r(n)) < \infty. \tag{35}$$

It is not difficult to see that equality (3) is a simple consequence of (33), (34). Therefore, it is enough to establish the implication: (10) $\Rightarrow$ (35).

Let us choose arbitrary $C > 1$. Then under condition (10) for sufficiently large x

$$r(a_1(x)) = r(R^{-1}(L_1(x))) \geq C L_2(x),$$

or when $y = L_1(x)$

$$r(R^{-1}(y)) \geq C L_1(y),$$

and then if $R^{-1}(y) = z$

$$r(z) \geq C L_1(R(z)).$$

Here we use formula (25) again

$$r(z) \geq C L_1(z r(z)(1 + o(1))) = C(L_1(z) + L_1(r(z)) + o(1)) \geq C L_1(z). \tag{36}$$

It is obvious that the last inequalities ensure the convergence of the series (35). It should also be noted that $\lfloor a_1(n) \rfloor = a(n)$.

Next, we proceed to the proof of (ii). We obtain the relation (14) as a corollary of the following equalities:

$$\mathbf{P}(\bar{\xi}_n \geq \lfloor a_m(n) \rfloor \quad i.o.) = 1, \tag{37}$$

$$\mathbf{P}(\bar{\xi}_n \geq \lfloor a_m(n) \rfloor + 1 \quad i.o.) = 0. \tag{38}$$

Their proof is based on the following lemma.

Lemma 2 (Corollary 4.3.1 in [8]). *Let $(\xi_k), k \geq 1$ be a sequence of independent copies of a random variable ξ with cumulative distribution function F and let $(u_n), n \geq 1$ be a nondecreasing sequence of real numbers. Then the probability*

$$\mathbf{P}(\bar{\xi}_n \geq u_n \quad i.o.)$$

equals zero or one according to whether the series

$$\sum_{n=1}^{\infty} (1 - F(u_n))$$

converges or diverges.

Let's start with equality (37). In Lemma 2, put $u_n = \lfloor a_m(n) \rfloor$. And we get

$$\begin{aligned} S = \sum_{n=1}^{\infty}(1 - F(u_n)) &= \sum_{n=1}^{\infty}\exp(-R(\lfloor a_m(n) \rfloor)) \geq \sum_{n=1}^{\infty}\exp(-R(a_m(n))) \\ &= \sum_{n=1}^{\infty}\exp(-\alpha_m(n)) = \sum_{n=1}^{\infty}\frac{1}{nL_1(n)\dots L_{m-1}(n)} = \infty. \end{aligned}$$

According to Lemma 2, (37) follows from this.

The proof of equality (38) will also be based on Lemma 2, but we choose $u_n = \lfloor a_m(n) \rfloor + 1$. We will write the series S as following

$$S = \sum_{n=1}^{\infty}(1 - F(u_n)) = \sum_{n=1}^{\infty}\exp(-R(\lfloor a_m(n) \rfloor + 1)) = \sum_{k\geq 0}\exp(-R(k+1))\sum_{n\in I_k}1, \qquad (39)$$

where $I_k = \{n : \lfloor a_m(n) \rfloor = k\}$.

We will show that the convergence of the series (13) implies the convergence of the series (39). Let us consider the set I_k in more detail

$$\begin{aligned} I_k &= \{n : k \leq a_m(n) < k+1\} = \{n : R(k) \leq \alpha_m(n) < R(k+1)\} \\ &= \{n : \exp(R(k)) \leq \varphi(n) < \exp(R(k+1))\}, \end{aligned}$$

where $\varphi(x) = xL_1(x)\dots L_{m-1}(x)$ is a continuous and monotonically increasing function for sufficiently large x.

Here it becomes clear that in order to estimate the value $\sum_{n\in I_k}1$ we need to find an approximate formula for the inverse function $\varphi^{-1}(x)$. Just as in case of equalities (25), for sufficiently large x we have $\varphi^{-1}(x) = x\hat{L}(x)$, where function $\hat{L}(x)$ is slowly varying at infinity which we write as

$$\hat{L}(x) = \frac{1}{L_1(x)\dots L_{m-1}(x)(1 + g(x))}.$$

It appears,

$$g(x) \to 0 \quad \text{when} \quad x \to \infty. \qquad (40)$$

Indeed

$$x = \varphi(\varphi^{-1}(x)) = \frac{xL_1(\varphi^{-1}(x))\dots L_{m-1}(\varphi^{-1}(x))}{L_1(x)\dots L_{m-1}(x)(1+g(x))}. \qquad (41)$$

In addition for $k \geq 1$, $y = \varphi^{-1}(x)$ and $x \to \infty$

$$\frac{L_k(\varphi^{-1}(x))}{L_k(x)} = \frac{L_k(y)}{L_k(\varphi(y))} = \frac{L_{k-1}(L_1(y))}{L_{k-1}(L_1(y) + \ldots + L_{m-1}(y))} \to 1.$$

The last asymptotic relations together with equality (41) give (40).

So

$$\varphi^{-1}(x) = \frac{x}{L_1(x)\dots L_{m-1}(x)(1+g(x))} = \frac{x(1+o(1))}{L_1(x)\dots L_{m-1}(x)}.$$

Put

$$n_k = \lceil \varphi^{-1}(\exp(R(k)))\rceil = \frac{\exp(R(k))}{R(k)L_1(R(k))\dots L_{m-2}(R(k))(1+g(\exp(R(k)))} + \theta,$$

where $0 \le \theta < 1$. Moreover

$$\sum_{n\in I_k} 1 = n_{k+1} - n_k$$

and

$$\frac{n_k}{n_{k+1}} = \exp(-r(k+1))\frac{R(k)L_1(R(k))\dots L_{m-2}(R(k))}{R(k+1)L_1(R(k+1))\dots L_{m-2}(R(k+1))}.$$

Since at $i = 0, 1, \dots, m-2$ and $k \to \infty$

$$\frac{L_i(R(k+1))}{L_i(R(k))} \to 1$$

and in accordance with condition (10) $r(k) \to \infty$, then

$$\frac{n_k}{n_{k+1}} = o(1) \quad \text{when} \quad k \to \infty.$$

The last ratio allow us to estimate the series S from (39):

$$\begin{aligned} S &= \sum_{k\ge 0} \exp(-R(k+1)) \sum_{n\in I_k} 1 = \sum_{k\ge 0} \exp(-R(k+1)) n_{k+1}\left(1 - \frac{n_k}{n_{k+1}}\right) \qquad (42)\\ &= \sum_{k\ge 0} \frac{1+o(1)}{R(k+1)L_1(R(k+1))\dots L_{m-2}(R(k+1))}. \end{aligned}$$

Thus, the convergence of the series S is equivalent to the convergence of the last series in (42).

To obtain equality (38), it remains to apply Lemma 2.

(iii). If $m \ge 2$ and the series (13) diverges, then series S also diverges from (42). Therefore, by Lemma 2

$$\mathbf{P}(\bar{\xi}_n \ge \lfloor a_m(n) \rfloor + 1 \quad i.o.) = 1.$$

To complete the proof, it remains to establish the equality

$$\mathbf{P}(\bar{\xi}_n \ge \lfloor a_m(n) \rfloor + 2 \quad i.o.) = 0. \qquad (43)$$

First, we note that under condition (10) for an arbitrary $C > 1$ and for sufficiently large x

$$r(x) \ge CL_m(x). \qquad (44)$$

Indeed for sufficiently large x

$$r(a_1(x)) = r(R^{-1}(L_1(x))) \geq CL_{m+1}(x).$$

Next, we should simply repeat the considerations given in the proof of inequalities (36).

It is clear that the proof of equality (43) will be based on Lemma 2. We choose $u_n = \lfloor a_m(n) \rfloor + 2$. Then the series S will look like

$$S = \sum_{n=1}^{\infty}(1 - F(u_n)) = \sum_{n=1}^{\infty}\exp(-R(\lfloor a_m(n) \rfloor + 2)) = \sum_{k\geq 0}\exp(-R(k+2))(n_{k+1} - n_k). \quad (45)$$

In addition, as stated in the (ii)

$$n_{k+1} - n_k = \frac{\exp(R(k+1))(1+o(1))}{R(k+1)L_1(R(k+1))\dots L_{m-2}(R(k+1))}.$$

From here and estimates (44) and (45) we have

$$\begin{aligned} S &= \sum_{k\geq 0}\frac{\exp(-r(k+2))(1+o(1))}{R(k+1)L_1(R(k+1))\dots L_{m-2}(R(k+1))} \\ &\leq \sum_{k\geq 0}\frac{(1+o(1))}{R(k+1)L_1(R(k+1))\dots L_{m-2}(R(k+1))(L_{m-1}(k+1))^C}. \end{aligned}$$

If we add here the equality

$$R(x) = xr(x)(1+o(1))$$

(see (25)), then we come to the conclusion: series S converges. According to Lemma 2, this means that equality (43) is true. □

4 Examples

Let's consider some examples of application of the Theorems given above.

Example 1. (Geometric distribution) .

Let $0 < q < 1$,

$$\mathbf{P}(\xi = i) = p_i = q(1-q)^i, \quad i \geq 0.$$

Then

$$\mathbf{P}(\xi \geq i) = (1-q)^i = \exp(-\gamma i), \quad \gamma = \ln\frac{1}{1-q},$$

that is

$$R(n) = \gamma n, \quad r(n) = \gamma, \quad a_m(n) = \frac{1}{\gamma}\,\alpha_m(n).$$

It is obvious that condition (2) is fulfilled for the example when $\rho = 0$ and condition (7). Hence according to Theorem 3

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{\gamma\bar{\xi}_n - \alpha_m(n)}{L_{m+1}(n)} = 1\right) = 1.$$

In addition, it is easy to verify that conditions (8), (11) of Theorem 4 are also true and $d(n) = (L_1(n) - L_3(n))/\gamma$. And therefore by Theorem 4 we have

$$\mathbf{P}\left(\liminf_{n\to\infty}\left(\bar{\xi}_n - \left\lfloor \frac{L_1(n) - L_3(n)}{\gamma} \right\rfloor\right) = -1\right) = 1.$$

Example 2. (Poisson distribution).

Let

$$\mathbf{P}(\xi = i) = p_i = \frac{\lambda^i}{i!}\exp(-\lambda), \quad i \geq 0, \quad \lambda > 0.$$

It is known [15], [16] that the following asymptotic relations are true for the Poisson distribution with parameter $\lambda > 0$:

$$R(n) = \left(n + \frac{1}{2}\right)\ln n - n(\ln\lambda + 1) - \lambda + \frac{1}{2}\ln 2\pi + o(1), \tag{46}$$

$$r(n) = \ln n + O(1), \tag{47}$$

$$a_1(n) = \frac{\ln n}{L_2(n)}\left(1 + \frac{L_3(n) + \ln\lambda + 1 + o(1)}{L_2(n)}\right). \tag{48}$$

It is not difficult to check that for $m \geq 2$ the condition (10) of Theorem 5 is fulfilled. Indeed, from (47) and (48) we get:

$$\frac{r(a_1(n))}{L_{m+1}(n)} = \frac{L_2(n) - L_3(n) + O(1)}{L_{m+1}(n)} \to \infty, \quad n \to \infty.$$

It is also easy to check that for $m \geq 2$ the condition (8) of Theorem 4 is true. As agreed (46)

$$\sum_{n\geq 1} \frac{1}{R(n)L_1(R(n))} \sim \sum_{n\geq 1} \frac{1}{nL_1^2(n)} < \infty,$$

then for $m \geq 3$ the series (13) converges.

Therefore, according to (ii) of Theorem 5 for $m \geq 3$

$$\limsup_{n\to\infty}(\bar{\xi}_n - \lfloor a_m(n) \rfloor) = 0 \quad \text{a.s.} \tag{49}$$

Case $m = 2$ is also simple

$$\sum_{n\geq 1} \frac{1}{R(n)} \sim \sum_{n\geq 1} \frac{1}{nL_1(n)} = \infty.$$

Therefore, the series (13) diverges. According to (iii) of Theorem 5 we have

$$\limsup_{n\to\infty}(\bar{\xi}_n - \lfloor a_2(n) \rfloor) = 1 \quad \text{a.s.}$$

Next, we apply Theorem 4. As mentioned above, we make sure that condition (11) of Theorem 4 is true. Then

$$\liminf_{n\to\infty}(\bar{\xi}_n - \lfloor d(n) \rfloor) = -1 \quad \text{a.s.} \tag{50}$$

Remark 1. *Previously, the asymptotic behavior of extreme values $\bar{\xi}_n$ a.s. for the case of Poisson distribution have been studied in articles [15], [16]. The formulation of the corresponding results, for example in [15], is somewhat different from the above equalities (49)-(50). Of course, the statements obtained in [15] regarding the Poisson distribution are close to equalities (49)- (50). But the direct proof of their equivalence does not seem simpler than the proof of the equalities (49)-(50) themselves.*

Example 3. (Birth and death processes).
Let $X(t)$ be the birth and death process with parameters

$$\lambda_n = \lambda n + a, \quad \mu_n = \mu n, \quad \lambda > 0, \mu > 0, a > 0, \quad n = 0, 1, 2, \dots \tag{51}$$

(see $[11, \text{chapter } 7, \S 6]$).

Such process is called a process with linear growth and immigration. If state n describes the size of the population at a certain moment in time, then the probability of transition to state $n+1$ in a short period of time δ is equal to $(\lambda n + a)\delta + o(\delta)$, and the probability of transition $n \to n-1$ is given by $\mu n\delta + o(\delta)$. Coefficient a can be interpreted as the infinitesimal intensity of population growth due to immigration.

Assume that the following condition is fulfilled

$$\rho = \frac{\lambda}{\mu} < 1. \tag{52}$$

It is not difficult to verify that under condition (52) $X(t)$ will be a regenerative process of a special type with moments of regeneration $S_0 = 0, S_1, S_2, \dots$, here S_k is the first moment of entering state 0 after the $k-th$ exit from it. Moreover

$$M_T = \mathbf{E}T_k = \frac{1}{\lambda_0 p_0} = \frac{1}{a p_0},$$

$$p_0 = \left(\sum_{k=0}^{\infty} \theta_k\right)^{-1}, \quad \theta_0 = 1, \quad \theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \quad k \geq 1,$$

where $T_k = S_k - S_{k-1}$ is the duration of k-th regeneration cycle [10]. We are interested in asymptotic behavior a.s. of extreme values of the population:

$$\bar{X}(t) = \sup_{0 \leq s < t} X(s), \quad t \geq 0.$$

Let

$$q(n) = \mathbf{P}(\bar{X}(T_1) \geq n) = \exp(-R(n)).$$

It is known that

$$q(n) = \frac{1/\rho - 1}{C_0} \rho^n n^{a/\lambda}(1 + o(1)),$$

where

$$C_0 = \lim_{n\to\infty} n^{a/\lambda} \prod_{i=1}^{n} \left(1 - \frac{1}{1 + i\lambda/a}\right),$$

(see [14]).

This implies that

$$R(n) = -\ln q(n) = n \ln \frac{1}{\rho} - \frac{a}{\lambda} \ln n - C_1 + o(1), \quad C_1 = \ln \frac{1/\rho - 1}{C_0} \tag{53}$$

and

$$R(n) - R(n-1) = r(n) = \ln \frac{1}{\rho} + o(1). \tag{54}$$

The law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf was established for the process $\bar{X}(t)$ in [14]. Here we strengthen this statement as follows.

Let $X(t)$ be the birth and death process for parameters λ_n, μ_n, that are given by equalities (51) and are fulfilled by conditions (52).

Then

$$\mathbf{P}\left(\limsup_{t\to\infty} \frac{\bar{X}(t) \ln \frac{1}{\rho} - \alpha_m(t) - \frac{a}{\lambda} L_2(t)}{L_{m+1}(t)} = 1\right) = 1, \tag{55}$$

$$\mathbf{P}\left(\liminf_{t\to\infty} \left(\bar{X}(t) - \left[L^*\left(\frac{t}{M_T}\right)\right]\right) = -1\right) = 1, \tag{56}$$

where

$$L^*(t) = L_1(t) - L_3(t) + \frac{a}{\lambda}\left(L_2(t) - L_2\left(\frac{1}{\rho}\right)\right) + C_1 + o(1),$$

C_1 is given by the equality (53), M_T and p_0 are defined above.

To obtain equality (55), we use Theorem 23 and (i) of Corollary 2 from [2]. Thus, in the conditions of example 3, we have

$$\mathbf{P}\left(\limsup_{t\to\infty} \frac{r(a_1(t))(\bar{X}(t) - a_m(\frac{t}{M_T}))}{L_{m+1}(t)} = 1\right) = 1. \tag{57}$$

Considering equalities (54), (57), it remains to find a simple asymptotic formula for the function $a_m(t)$.

The function $R(x)$ is a piecewise linear extension of the sequence $(R(n))$. By construction, it is absolutely continuous and increasing. Therefore

$$R(a_m(t)) = R(R^{-1}(\alpha_m(t)) = \alpha_m(t). \tag{58}$$

In addition, the function $R(x)$ in the whole points is given by equation (53). Then, choosing $\vartheta = a_m(t) - \lfloor a_m(t) \rfloor$, we get

$$\begin{aligned} R(a_m(t)) &= R(\lfloor a_m(t) \rfloor) + \vartheta(R(\lceil a_m(t) \rceil) - R(\lfloor a_m(t) \rfloor)) \\ &= R(\lfloor a_m(t) \rfloor) + \vartheta \left(\ln \frac{1}{\rho} + o(1) \right) \\ &= a_m(t) \ln \frac{1}{\rho} - \frac{a}{\lambda} \ln \lfloor a_m(t) \rfloor - C_1 + o(1). \end{aligned} \tag{59}$$

Hence and (58) the following relation follows:

$$\ln a_m(t) = L_2(t) - L_2 \left(\frac{1}{\rho} \right) + o(1).$$

Next, we substitute this expression for $\ln a_m(t)$ into equation (59). Putting the last equalities together, we get

$$a_m(t) = \left(\ln \frac{1}{\rho} \right)^{-1} \left(\alpha_m(t) + \frac{a}{\lambda} \left(L_2(t) - L_2 \left(\frac{1}{\rho} \right) \right) + C_1 + o(1) \right).$$

This equality and (57) complete the proof of (55).

Let us turn to relation (56). Denote by $\bar{Y}_n$ the extreme value of the process X(t) for n regeneration cycles:

$$\bar{Y}_n = \bar{X}(T_1 + \ldots T_n).$$

It is clear from equality (53) that condition (11) of Theorem 4 is satisfied. Therefore

$$\mathbf{P} \left(\liminf_{n \to \infty} (\bar{Y}_n - \lfloor d(n) \rfloor) = -1 \right) = 1, \tag{60}$$

$d(n) = R^{-1}(L_1(n) - L_3(n))$.

Let $N(t)$ denote the counting process for the sequence $(T_1 + \ldots + T_n)$

$$N(t) = \max(n \geq 0 : T_1 + \ldots + T_n < t), \quad t \geq 0.$$

It is clear that when t changes from 0 to ∞, the process $N(t)$ runs through all natural numbers a.s. And therefore, we can substitute the process $N(t)$ instead of n into equality (60). We get

$$\mathbf{P} \left(\liminf_{t \to \infty} (\bar{Y}_{N(t)} - \lfloor d(N(t)) \rfloor) = -1 \right) = 1, \tag{61}$$

The next step, we use the well-known result of the renewal theory [5]:

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{M_T} \quad a.s.,$$

that is by $t \to \infty$

$$\ln N(t) = \ln \frac{t}{M_T} + o(1) \quad a.s.$$

From the last equality we get

$$R(d(N(t))) = L_1(N(t)) - L_3(N(t)) = L_1\left(\frac{t}{M_T}\right) - L_3\left(\frac{t}{M_T}\right) + o(1).$$

Next, we apply the asymptotic equalities (53), (54) once again. Just as in the case of the $a_m(t)$ function, simple calculations give

$$\left| d(N(t)) - L^*(\frac{t}{M_T}) \right| = o(1), \quad \left| d(N(t)+1) - L^*\left(\frac{t}{M_T}\right) \right| = o(1),$$

where the function $L^*(t)$ is defined in equality (56).

Note that equality (61) remains true even when the process $N(t)$ is replaced by $N(t)+1$. The relation (56) follows from this, because

$$\bar{Y}_{N(t)} \le \bar{X}(t) \le \bar{Y}_{N(t)+1}.$$

Remark 2. *The result close to equality (55) is given in Corollary* 3 *of [2]. Unfortunately, it contains one inaccuracy (when proving it, the authors confused the functions* $\alpha_m(t)$ *and* $L_m(t)$*, defined above).*

The same remark also applies to Corollary 4 *from [2] (it should be replaced by* $L_m(t)$ *to* $\alpha_m(t)$*).*

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