IKEDA-WATANABE'S CONNECTION, BROWNIAN MOTION AND NAVIER-STOKES EQUATION ON RIEMANNIAN MANIFOLDS

SHIZAN FANG AND ZHONGMIN QIAN

Abstract. We consider the Navier-Stokes equation on a Riemannian manifold with the Ricci curvature bounded below. In stochastic analysis, a non-degenerate diffusion process on a Riemannian manifold was obtained by rolling Brownian motion with respect to a suitable metric compatible linear connection, which was introduced by N. Ikeda and S. Watanabe about 40 years ago. To each solution of the Navier-Stokes equation, we associate such a connection and compute the related time-dependent Ricci curvature, which allow us to obtain a link with the strain tensor and the helicity density in a simple formula in the case of dimension 3.

[1]

1. Introduction

The Navier-Stokes equation (in short, NS) in a domain U of $\mathbb{R}^n$ is a system of partial differential equations

$$\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p_t = 0, \quad \nabla \cdot u_t = 0, \quad u|_{t=0} = u_0, \tag{1.1}$$

which describes the evolution of the velocity u_t of an incompressible viscous fluid with kinematic viscosity $\nu > 0$, where the boundary ∂U of U is assumed to be bounded and smooth enough, and $u(x,t) = \phi(x,t)$ for $x \in \partial U$ and $t > 0$ for a smooth enough function $\phi(\cdot, t)$ defined on the boundary. This equation applies to all incompressible viscous fluids including turbulence, the phenomena which are almost as varied as in the realm of life, cf. U. Frisch [25]. Two types of transport processes are involved in this equation: one is the diffusion effect described by $\nu\,\Delta u_t$, another is the non-linear convection $(u_t \cdot \nabla)\, u_t$. The Reynolds number

$$Re = \frac{\text{convective effet}}{\text{diffusion effet}}$$

plays a central role in describing the mechanics of the fluid motion. In particular the phenomenon of turbulence appears as the Reynolds number Re becomes larger than the critical value.

Date: July 19, 2024.

[1]2020 Mathematics Subject Classification. Primary 35Q30; Secondary 58J65, 60H30.

The vorticity ξ_t of u_t (when $n=3$) is given by $\xi_t = \nabla \times u_t$, which may have some advantageous in the description of the dynamics of the fluid motion. If u_t is a solution to Navier-Stokes equation (1.1), then ξ_t satisfies the vorticity transport equation

$$\frac{d\xi_t}{dt} + \nabla_{u_t}\xi_t - \nu\Delta\xi_t = \nabla^s_{\xi_t} u_t \tag{1.2}$$

where $\nabla^s u_t$, the symmetric part of ∇u_t, which measures the rate of the strain. The helicity density h_t, defined by $h_t = u_t \cdot \xi_t$, is a fundamental quantity in the study of laminar and turbulent flows, as a measure of the degree how the vortex lines of a fluid flow are tangled and intertwined (see [38]). The information of h_t is helpful to understand the regularity of u_t, see for example [27, 7]. If (u_t, ξ_t) is a solution to (1.2), then together with the Poisson equation: $\Delta u = -\nabla \times \xi$, u is determined by the Biot-Savart formula. In [40], Olshanskii and Rebholz proved that the NS equation is equivalent to the following equation involving vorticity and helicity density:

$$\frac{\partial \xi_t}{\partial t} - \nu\,\Delta\xi_t + 2\nabla^s_{u_t}\xi_t - \nabla h_t = 0. \tag{1.3}$$

More precisely if (u_t, ξ_t, h_t) is a solution to (1.3), and (u_t, ξ_t) are linked by Biot-Savart formula, then u_t is a solution to (1.1); furthermore, a numerical scheme based on (1.3) was proposed in [40].

In differential geometry, the Ricci curvature (R_{jk}) is served to describe how a shape is deformed along geodesics. More precisely, in a geodesic normal chart of a Riemannian manifold M, the Riemannian metric is written in

$$g_{ij} = \delta_{ij} + O(|x|^2),$$

and the density $\sqrt{\det g}$ of the volume measure (with respect to the Lebesgue measure in a local chart) has an expansion

$$\sqrt{\det g} = 1 - \frac{1}{6}R_{jk}x^j x^k + O(|x|^3). \tag{1.4}$$

If the Ricci curvature $\mathrm{Ric}(\xi,\xi)$ is positive along the vector ξ, the small cone about ξ has smaller volume than in flat case. The Ricci curvature plays a key role not only in geometric analysis (for example in the study of the Ricci flow equation), but also in many scientific areas such as in mathematical physics (for example in the study of Einstein field equation). In the study of Laplace-Beltrami operator Δ on M, the lower bound of Ric gives information on the long time behavior of the Brownian motion on M. For an elliptic operator L on M, there is a metric g and a vector field Z, so that $L = \Delta + Z$. The Ricci curvature has been generalized to L by Bakry and Emery, so that the geometric analysis associated with an elliptic operator L has been developed in the past decades. In Probability theory, during 1970's, the Cartan's development has been successfully established for Brownian motion paths (see [17, 36]); in [29], Ikeda and Watanabe showed that the non-degenerated diffusion process associated to an elliptic operator L could be constructed in the same way, by replacing the Levi-Civita connection by a suitable metric compatible linear connection, called the Ikeda-Watanabe connection.

The purpose of this work is to explore the geometry aspects coded in the NS equation (1.1). To a solution u_t of (1.1), its associated Ikeda-Watanabe connection is defined and its related time-dependent Ricci curvature $\widehat{\mathrm{Ric}}^t$ is determined.

There is a huge literature on Navier-Stokes equations on $\mathbb{R}^n$, and one may refer to [26, 33] for nice expositions and to [10] for the well-posedness of the Navier-Stokes equation, for example.

In this paper, for convenience, we consider the following NS equation on a Riemannian manifold M:

$$\partial_t u_t + \nabla_{u_t} u_t + \nu \square u_t = -\nabla p_t, \quad \mathrm{div}(u_t) = 0, \quad u|_{t=0} = u_0, \tag{1.5}$$

where p_t denotes the pressure and $\square$ is the de Rham-Hodge operator. The reason we choose here $\square$ is that it preserves the class of vector fields of divergence free (see [21, 20]) which is convenient in many computations below.

In literature, other type of Navier-Stokes equations on a manifold have been formulated, but we will not enter the detail and refer the reader to [15, 41, 37, 44]. Variational principles for Navier-Stokes equation in the spirit of [5] have been established recently in [11, 2, 3, 4] in terms of a class of incompressible Brownian martingales.

The organisation of the paper is as follows. In Section 2, we will present the framework of stochastic development, introduce the Ikeda-Watanabe connection, also compute the associated Ricci curvature. In Section 3, we will present the vorticity form of NS equations on Riemannian manifolds, as well as their probabilistic representation. In section 4, we will prove the main theorem in its general form on Riemannian manifolds. In section 5, we prove the existence of weak solution of NS on a Riemannian manifold with Ricci curvature bounded below. Section 6 is devoted to the proof of Proposition 3.2. Finally, in section 7, we give some remarks on results obtained in [41].

2. Stochastic development, Ikeda-Watanabe connection

Let M be a Riemannian manifold of n dimensions. An element r in the orthonormal frame bundle $O(M)$ is an isometry from $\mathbb{R}^n$ onto $T_{\pi(r)}M$ where $\pi : O(M) \to M$ is the canonical projection. More precisely, an element $r \in O(M)$ is composed of (x, r), where $x = \pi(x, r)$ and r is an isometry from $\mathbb{R}^n$ onto T_xM. For the sake of simplicity, we read r as $(\pi(r), r)$, but we sometimes have to distinguish them. The Levi-Civita connection on M gives rise to n canonical horizontal vector fields $\{A_1, \ldots, A_n\}$ on $O(M)$, defined by the identity that $d\pi(r) \cdot A_i(r) = r\varepsilon_i$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the canonical basis of $\mathbb{R}^n$.

A vector field v on M can be lift to a horizontal vector field V on $O(M)$ such that $d\pi(r)V(r) = v_{\pi(r)}$. Let $\mathcal{X}(M)$ denote the space of vector fields on M.

Given a time-independent vector field v on M there is a metric compatible connection Γ^v defined as the following. Let $\{x_t,\ t \geq 0\}$ be the diffusion process with its infinitesimal generator $L = \frac{1}{2}\Delta_M + v$. This diffusion may be constructed in the following way. Let $(r_t)_{t\geq 0}$ be the solution to the Stratanovich stochastic differential equation (SDE) on $O(M)$:

$$dr_t = \sum_{i=1}^n A_i(r_t) \circ dW_t^i + V(r_t)\, dt, \tag{2.1}$$

where $t \to (W_t^1, \ldots, W_t^n)$ is a standard Brownian motion on $\mathbb{R}^n$. Let $x_t = \pi(r_t)$. We assume that x_t has the life-time $\zeta = \infty$ almost surely.

In Chapter V of [29], Ikeda and Watanabe introduced a metric compatible connection Γ^v so that the above diffusion process $\{x_t; t \geq 0\}$ can be constructed by rolling without slip the standard Brownian motion on $\mathbb{R}^n$ with respect to the connection Γ^v. More precisely let $\{B_1, \ldots, B_n\}$ be the canonical horizontal vector fields on $O(M)$ with respect to Γ^v, consider SDE on $O(M)$:

$$d\tilde{r}_w(t) = \sum_{i=1}^n B_i(\tilde{r}_w(t)) \circ dW_t^i, \quad \tilde{r}_w(0) = r.$$

Then the generator of the diffusion process $t \to \tilde{x}_t(w) = \pi(\tilde{r}_w(t))$ is $\frac{1}{2}\Delta_M + v$. In fact, the construction of Γ^v is such that

$$\frac{1}{2}\sum_{j=1}^n \mathcal{L}_{B_j}^2 (f \circ \pi) = \big((\frac{1}{2}\Delta_M + v)f\big) \circ \pi, \quad f \in C_c^2(M). \tag{2.2}$$

This connection Γ^v was defined locally in [29]. On a local chart U, $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ is a local basis of tangent spaces T_xM with $x \in U$, and $v = \sum_{i=1}^n v^i \partial/\partial x_i$. Let $\Gamma_{ij}^{0,k}$ be the Christoffel symbols of Levi-Civita connection. According to ([29], p.271), the Christoffel symbols Γ_{ij}^k of Γ^v is defined by (see also [1]),

$$\Gamma_{ij}^k = \Gamma_{ij}^{0,k} - \frac{2}{n-1}\Big(\delta_{ki}\sum_{\ell=1}^n g_{j\ell}v^\ell - g_{ij}v^k\Big). \tag{2.3}$$

Accordingly, the second term appearing on the right-hand side of (2.3) is the tensor field of Ikeda-Watanabe's connection. The global formulation for Γ^v is stated as the following proposition which will be used later on, but its proof of course follows (2.3) immediately.

Proposition 2.1. *Let ∇^v be the covariant derivative with respect to the connection Γ^v, and ∇^0 with respect to the Levi-Civita connection. Then*

$$\nabla^v_X Y = \nabla^0_X Y - \frac{2}{n-1} K_v(X,Y) \tag{2.4}$$

and the torsion of the connection Γ^v

$$T^v(X,Y) = -\frac{2}{n-1} K_v(X,Y) \tag{2.5}$$

for any $X, Y \in \mathcal{X}(M)$, where

$$K_v(X,Y) = \langle Y, v\rangle X - \langle X, Y\rangle v. \tag{2.6}$$

Therefore T^v is skew-symmetric (TSS), that is $\langle T^v(X,Y), Z\rangle = -\langle T^v(Z,Y), X\rangle$ for all $X, Y, Z \in \mathcal{X}(M)$ if and only if $v = 0$.

Proof. Using (2.4) and the fact $\nabla^0_X Y - \nabla^0_Y X - [X,Y] = 0$, we have

$$T^v(X,Y) = -\frac{2}{n-1}\big(K_v(X,Y) - K_v(Y,X)\big) = \frac{-2}{n-1}\Big(\langle Y, v\rangle X - \langle X, v\rangle Y\Big),$$

that is (2.5). Let X, Y, Z be vector fields such that $\langle T^v(X,Y), Z\rangle + \langle T^v(Z,Y), X\rangle = 0$, then

$$2\langle Y, v\rangle\langle X, Z\rangle = \langle X, v\rangle\langle Y, Z\rangle + \langle Z, v\rangle\langle Y, X\rangle.$$

Taking $Y = v$ and $X = Z$ in above equality, we get

$$|v|^2|X|^2 = \langle X, v\rangle^2.$$

If $v \neq 0$, taking X orthogonal to v yields a contradiction. □

Proposition 2.2. *Let Ric^0 (resp. Ric^v) be the Ricci curvature defined by the connection ∇^0 (resp. ∇^v). Then*

$$\mathrm{Ric}^v(X) = \mathrm{Ric}^0(X) - \frac{4(n-2)}{(n-1)^2} K_v(X,v) + \frac{2(n-2)}{n-1}\nabla^0_X v + \frac{2}{n-1}\mathrm{div}(v)\, X \tag{2.7}$$

for every $X \in \mathcal{X}(M)$.

Proof. Recall that T^v is the torsion of the connection ∇^v, which is a tensor field of type $(1,2)$. Hence

$$\begin{aligned}\nabla^v_X\nabla^v_Y Z &= \nabla^0_X\nabla^v_Y Z + T^v(X, \nabla^v_Y Z) = \nabla^0_X\Big(\nabla^0_Y Z + T^v(Y,Z)\Big) + T^v(X, \nabla^v_Y Z)\\ &= \nabla^0_X\nabla^0_Y Z + (\nabla^0_X T^v)(Y,Z) + T^v(\nabla^0_X Y, Z) + T^v(Y, \nabla^0_X Z) + T^v(X, \nabla^v_Y Z).\end{aligned}$$

By exchanging X and Y we also have

$$\nabla^v_Y\nabla^v_X Z = \nabla^0_Y\nabla^0_X Z + (\nabla^0_Y T^v)(X,Z) + T^v(\nabla^0_Y X, Z) + T^v(X, \nabla^0_Y Z) + T^v(Y, \nabla^v_X Z).$$

On the other hand, by definition

$$\nabla^v_{[X,Y]}Z = \nabla^0_{[X,Y]}Z + T^v([X,Y],Z).$$

Therefore, by putting the previous three equations we deduce that the curvature tensor

$$R^v(X,Y)Z = \nabla^v_X\nabla^v_Y Z - \nabla^v_Y\nabla^v_X Z - \nabla^v_{[X,Y]}Z$$

admits the following expression

$$\begin{aligned} R^0(X,Y)Z &+ (\nabla^0_X T^v)(Y,Z) - (\nabla^0_Y T^v)(X,Z) + T^v(\nabla^0_X Y - \nabla^0_Y X, Z)\\ &- T^v(Y, T^v(X,Z)) + T^v(X, T^v(Y,Z)) - T^v([X,Y],Z). \end{aligned}$$

Let $\{e_1,\ldots,e_n\}$ be a local frame field of TM. Then $\mathrm{Ric}^v(X) = \sum_{i=1}^n R^v(X,e_i)e_i$, so that the previous identity can be rewritten

$$\mathrm{Ric}^v(X) = \mathrm{Ric}^0(X) + I_1 - I_2 + I_3 - I_4,$$

where

$$I_1 = \sum_{i=1}^n T^v(X, T^v(e_i,e_i)), \quad I_2 = \sum_{i=1}^n T^v(e_i, T^v(X,e_i)),$$

$$I_3 = \sum_{i=1}^n (\nabla^0_X T^v)(e_i,e_i), \quad I_4 = \sum_{i=1}^n (\nabla^0_{e_i} T^v)(X,e_i).$$

Remark that $\sum_{i=1}^n K_v(e_i,e_i) = -(n-1)v$. Since $T^v(X,Y) = -\frac{2}{n-1}K_v(X,Y)$, by an elementary computation

$$I_1 = \frac{4}{(n-1)^2}\sum_{i=1}^n K_v(X, K_v(e_i,e_i)) = -\frac{4(n-1)}{(n-1)^2}K_v(X,v)$$

and

$$I_2 = \frac{4}{(n-1)^2}\sum_{i=1}^n K_v(e_i, K_v(X,e_i)) = -\frac{4}{(n-1)^2}K_v(X,v).$$

To handle the other two terms, we first observe that

$$(\nabla^0_X T^v)(Y,Z) = -\frac{2}{n-1}K_{\nabla^0_X v}(Y,Z)$$

and

$$(\nabla^0_Y T^v)(X,Z) = -\frac{2}{n-1}K_{\nabla^0_Y v}(X,Z).$$

Therefore

$$I_3 = -\frac{2}{n-1}\sum_{i=1}^n K_{\nabla^0_X v}(e_i, e_i) = 2\nabla^0_X v.$$

$$I_4 = -\frac{2}{n-1}\sum_{i=1}^n K_{\nabla^0_{e_i} v}(X, e_i) = -\frac{2}{n-1}\mathrm{div}(v)\, X + \frac{2}{n-1}\nabla^0_X v.$$

Putting these equations together (2.7) follows immediately. □

After having computed the curvature tensor of the connection ∇^v, we may work out its Ricci curvature accordingly. However, since the dual connection of Γ^v is not metric, we prefer to use the so-called *intrinsic Ricci tensor* instead. The intrinsic Ricci tensor is introduced by B. Driver in [13], which is used in stochastic analysis on the path space of Riemannian manifolds (see also [9, 23, 28, 35]). A Weitzenböck formula for a connection which is not necessarily torsion skew-symmetric has been established in [18].

Definition 2.3. *The intrinsic Ricci tensor is defined by*

$$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^v(X) + \sum_{i=1}^n (\nabla^v_{e_i} T^v)(X, e_i) \tag{2.8}$$

for $X \in \mathcal{X}(M)$, where $\{e_i;\ i = 1, \ldots, n\}$ is a local orthonormal frame field of the tangent bundle.

Theorem 2.4. *If the dimension $n = 3$, then $\widehat{\mathrm{Ric}^v}$ admits the following simple expression:*

$$\widehat{\mathrm{Ric}^v} = \mathrm{Ric}^0 + 2v \otimes v + 2\nabla^{0,s} v, \tag{2.9}$$

where $\nabla^{0,s} v$ denotes the symmetric part of $\nabla^0 v$.

Proof. By (2.5), we have

$$T^v\big(\nabla^v_{e_i} X, e_i\big) = -\frac{2}{n-1}\Big(\langle e_i, v\rangle \nabla^v_{e_i} X - \langle \nabla^v_{e_i} X, v\rangle\, e_i\Big),$$

$$T^v\big(X, \nabla^v_{e_i} e_i\big) = -\frac{2}{n-1}\Big(\langle \nabla^v_{e_i} e_i, v\rangle X - \langle X, v\rangle\, \nabla^v_{e_i} e_i\Big).$$

Since $\nabla^v_{e_i}\Big(T^v(X, e_i)\Big) = \Big((\nabla^v_{e_i} T^v)(X, e_i) + T^v\big(\nabla^v_{e_i} X, e_i\big) + T^v\big(X, \nabla^v_{e_i} e_i\big)\Big)$, a little bit computation leads to

$$(\nabla^v_{e_i} T^v)(X, e_i) = -\frac{2}{n-1}\Big(\langle e_i, \nabla^v_{e_i} v\rangle\, X - \langle X, \nabla^v_{e_i} v\rangle\, e_i\Big).$$

Put

$$(\nabla^v_{e_i} T^v)(X, e_i) = -\frac{2}{n-1}\Big(\langle e_i, \nabla^0_{e_i} v\rangle\, X - \langle X, \nabla^0_{e_i} v\rangle\, e_i\Big) + J_i,$$

where

$$J_i = \frac{4}{(n-1)^2}\Big(\langle e_i, K_v(e_i,v)\rangle X - \langle X, K_v(e_i,v)\rangle\, e_i\Big).$$

While it is easy to see that

$$\sum_{i=1}^n J_i = \frac{4}{(n-1)^2}\Big((n-1)|v|^2 X - K_v(X,v)\Big).$$

Therefore the sum $\displaystyle\sum_{i=1}^n (\nabla^v_{e_i} T^v)(X,e_i)$ is equal to

$$-\frac{2}{n-1}\Big(\mathrm{div}(v)\, X - \sum_{i=1}^n \langle X, \nabla^0_{e_i} v\rangle\, e_i\Big) + \frac{4}{(n-1)^2}\Big((n-1)|v|^2 X - K_v(X,v)\Big).$$

Since $n=3$, the above formula yields that

$$\sum_{i=1}^3 (\nabla^v_{e_i} T^v)(X,e_i) = -\mathrm{div}(v)\, X + \sum_{i=1}^3 \langle X, \nabla^0_{e_i} v\rangle\, e_i + 2|v|^2 X - K_v(X,v). \tag{2.10}$$

On the other hand, by (2.7), for $n=3$,

$$\mathrm{Ric}^v(X) = \mathrm{Ric}^0(X) - K_v(X,v) + \nabla^0_X v + \mathrm{div}(v)\, X, \tag{2.11}$$

and

$$\sum_{i=1}^3 \langle X, \nabla^0_{e_i} v\rangle\, e_i + \nabla^0_X v = \sum_{i=1}^3 \Big(\langle X, \nabla^0_{e_i} v\rangle + \langle \nabla^0_X v, e_i\rangle\Big)\, e_i = 2\nabla^{0,s}_X v.$$

By summing up (2.10) and (2.11), we then obtain

$$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^0(X) + 2|v|^2 X - 2K_v(X,v) + 2\nabla^{0,s}_X v.$$

Now remarking that $|v|^2 X - K_v(X,v) = \langle X, v\rangle v$, we therefore deduce that

$$\widehat{\mathrm{Ric}^v}(X) = \mathrm{Ric}^0(X) + 2\langle X, v\rangle v + 2\nabla^{0,s}_X v$$

for any vector field X and (2.9) holds. □

3. Vorticity and its probabilistic representation

Let's first recall the definition of the Hodge Laplacian $\square$ on vector fields. There exists a one-to-one correspondence between the space of vector fields $\mathcal{X}(M)$ and that of differential 1-forms $\Lambda^1(M)$. On a local chart U, $\big\{\partial/\partial x_1, \ldots, \partial/\partial x_n\big\}$ is a basis of the tangent space T_xM and $\{dx^1, \ldots, dx^n\}$ a dual basis of the co-tangent space T_x^*M. The inner product in T_xM (as well as the one for tensor bundle) is denoted by $\langle\ ,\ \rangle$, while the duality between T_x^*M and T_xM is denoted by $(\ ,\ )$. Let $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j\rangle$. Let $u = \sum_{i=1}^n u_i \partial/\partial x_i$ be a vector field. Then the associated differential form $\tilde{u}$ is given by

$$\tilde{u} = \sum_{i=1}^n \Big(\sum_{j=1}^n g_{ij} u_j\Big) dx^i.$$

Similarly given a differential 1-form $\omega = \sum_{j=1}^n \omega_j dx^j$, its corresponding vector field $\omega^\#$ has the expression

$$\omega^\# = \sum_{i=1}^n \Big(\sum_{\ell=1}^n g^{i\ell}\omega_\ell\Big) \frac{\partial}{\partial x_i},$$

where $g^{ij} = \langle dx^i, dx^j\rangle$. Note that (g^{ij}) is the inverse matrix of (g_{ij}). It can be verified that for $A \in \mathcal{X}(M)$ and $\omega \in \Lambda^1(M)$, $(\omega, A) = \langle \omega^\#, A\rangle = \langle \omega, \tilde{A}\rangle$. Let $\Box = dd^* + d^*d$ be the De Rham-Hodge operator on differential forms, where d^* is the adjoint operator of the exterior derivative d. Then $\Box A = (\Box\tilde{A})^\#$. Moreover

$$\int_M (\Box\omega, A)\, dx = \int_M \langle \Box\omega, \tilde{A}\rangle\, dx = \int_M \langle \omega, \Box\tilde{A}\rangle\, dx = \int_M (\omega, \Box A)\, dx,$$

if A or ω has a compact support, and M has an empty boundary. Here dx denotes the Riemannian-Lebesgue measure on M. The Bochner-Weitzenböck reads as

$$\Box A = -\Delta A + \mathrm{Ric}(A), \quad A \in \mathcal{X}(M) \tag{3.1}$$

where ΔA denotes the trace Laplacian of a vector field A. For a $(1,1)$-type tensor $T : \mathcal{X}(M) \to \mathcal{X}(M)$, we denote by $T^\# : \Lambda^1(M) \to \Lambda^1(M)$ its adjoint

$$(T^\#\omega, A) = (\omega, T(A)), \quad A \in \mathcal{X}(M). \tag{3.2}$$

Let u_t be a (smooth) solution to the Navier-Stokes equation on a Riemannian manifold M without boundary:

$$\partial_t u_t + \nabla_{u_t} u_t + \nu\Box u_t = -\nabla p_t, \quad \mathrm{div}(u_t) = 0,\ u|_{t=0} = u_0. \tag{3.3}$$

The second equation which says that u_t is diverence-free for each t, may be stated that $d^*\tilde{u}_t = 0$. The vorticity, denoted by ω_t, of u_t (for every t) is defined to be $\omega_t = *\tilde{\omega}_t$, where $\tilde{\omega}_t = d\tilde{u}_t$ is the exterior derivative of the corresponding one form of the vector field u_t and $*$ is the Hodge star operator. According to T.Taylor [44], if the dimension $n = 3$ and M is a domain of the Euclidean space of three dimensions, then our definition reduces to the usual curl of u_t.

For simplicity, we may also use the following convention. Let β be a differential p-form and $T : \mathcal{X}(M) \to \mathcal{X}(M)$ a tensor of type $(1,1)$. Define the p-form $\beta \triangleleft T$ by, for

$X_1, \ldots, X_p$,

$$(\beta \triangleleft T)(X_1, \ldots, X_p) = \beta\big(T(X_1), X_2, \ldots, X_p\big) + \ldots + \beta\big(X_1, \ldots, X_{p-1}, T(X_p)\big). \tag{3.4}$$

If β is a 2-form and $T = \nabla u$, then for $X, Y \in \mathcal{X}(M)$,

$$(\beta \triangleleft \nabla u)(X, Y) = \beta(\nabla_X u, Y) + \beta(X, \nabla_Y u). \tag{3.5}$$

If β is a 1-form and $T = \nabla u$, then for $X \in \mathcal{X}(M)$,

$$(\beta \triangleleft \nabla u)(X) = \beta(\nabla_X u).$$

Proposition 3.1. *Let* $\dim(M) = 3$ *and* ω_t *the vorticity of* u_t*, where* u_t *is a solution to Navier-Stokes equation* (3.3)*. Then* ω_t *satisfies the following equation*

$$\partial_t \omega_t + \nabla_{u_t} \omega_t + \nu \Box \omega_t = \omega_t \triangleleft (\nabla^s u_t). \tag{3.6}$$

Proof. Eq. (3.6) is called the vorticity transport equation, which can be obtained directly by applying the exterior derivative d both sides of the first equation in (3.3) and using the fact that $d^* \tilde{u}_t = 0$, see [44] for example. □

If we consider u_t and the vorticity stretching factor $\nabla^s u_t$ in (3.6) as given variables, then the vorticity transport equation appears as a linear parabolic equation, which can be treated by using PDE theory and stochastic analysis too. For example it is possible to derive the vorticity ω_t via the Feynman-Kac formula and therefore express the vorticity ω_t in terms of the velocity, the tensor-of-strain and functional integration implicitly. Such a formulation for ω_t has been obtained for the Euclidean case, and useful in the study of the Navier-Stokes equation.

Our next task is to derive such a functional integral representation for the vorticity ω_t. The complication in our case is that the vorticity transport equation (3.6) is a linear parabolic equation on a manifold, so the classical Feynman-Kac formula can not be applied directly. Therefore, we need to rewrite the vorticity transport equation (3.6) into a parabolic equation on a flat space. This can be done by lifting the vorticity transport equation (3.6) to the orthonormal frames $O(M)$, cf. [36, 29] in which the heat equations of tensor fields are treated.

Let ω be a differential 1-form. Define

$$F^i_\omega(r) = (\omega_{\pi(r)}, r\varepsilon_i) = (\pi^*\omega, A_i)_r, \quad i = 1, \ldots, n, \tag{3.7}$$

for $r \in O(M)$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of the Euclidean space $\mathbb{R}^n$, $\pi^*\omega$ is the pull-back of ω by the bundle projection $\pi : O(M) \mapsto M$. Then

$$(\mathcal{L}_{A_j} F^i_\omega)(r) = (\nabla_{r\varepsilon_j} \omega, r\varepsilon_i) = (\nabla\omega, r\varepsilon_j \otimes r\varepsilon_i) \tag{3.8}$$

for $i, j \leq n$, where the second duality takes place in $T_{\pi(r)}M \otimes T_{\pi(r)}M$. In fact, suppose $s \mapsto r(s) \in O(M)$ is the smooth curve such that $r(0) = r, r'(0) = A_j(r)$.

Let $\xi_s = \pi(r(s))$. Then $//_s^{-1} := r \circ r(s)^{-1}$ is the parallel translation from $T_{\xi_s}M$ onto T_xM along the curve $\xi_\cdot$, so that

$$F_\omega^i(r(s)) = (\omega_{\xi_s}, r(s)\varepsilon_i) = (//_s^{-1}\omega_{\xi_s}, r\varepsilon_i).$$

Eq.(3.8) now follows immediately by differentiating both sides of the previous equation with respect to s at $s=0$. Similarly one verify that $(\mathcal{L}_{A_j}^2 F_\omega^i)(r) = (\nabla_{r\varepsilon_j}\nabla\omega, r\varepsilon_j \otimes r\varepsilon_i)$. In particular

$$\Delta_{O(M)}F_\omega^i := \sum_{j=1}^n \mathcal{L}_{A_j}^2 F_\omega^i = (\Delta\omega, r\varepsilon_i) = F_{\Delta\omega}^i(r).$$

This equation for the trace Laplacian of ω can be obtained similarly for a general tensor field, cf. [6, 8, 9, 16, 18, 29, 31, 36, 42] for example. Thanks to this equality for the trace Laplacian, we are now in a position to lift the vorticity transport equation to the principal bundle $O(M)$. Let U_t be the horizontal lift of u_t to $O(M)$ for each t. Then $U_t(r) = \sum_{j=1}^n \langle u_t(\pi(r)), r\varepsilon_j\rangle A_j(r)$. According to (3.8),

$$(\mathcal{L}_{U_t}F_\omega^i)(r) = \sum_{j=1}^n \langle u_t, r\varepsilon_j\rangle (\mathcal{L}_{A_j}F_\omega^i)(r) = \langle \nabla_{u_t}\omega, r\varepsilon_i\rangle = F_{\nabla_{u_t}\omega}^i(r).$$

In order to handle the vorticity transport equation, we need to handle term involving the tensor-of-strain. To this end write $\phi_t = \omega_t \triangleleft \nabla^s u_t$. Then

$$F_{\phi_t}^i(r) = (\phi_t, r\varepsilon_i) = \omega_t(\nabla_{r\varepsilon_i}^s u_t) = \sum_{j=1}^n \langle \nabla_{r\varepsilon_i}^s u_t, r\varepsilon_j\rangle\, (\omega_t, r\varepsilon_j) = \sum_{j=1}^n \langle \nabla_{r\varepsilon_i}^s u_t, r\varepsilon_j\rangle F_{\omega_t}^j.$$

Let $K(t,r) = (K_{ij}(t,r))$, where $K_{ij}(t,r) = \langle \nabla_{r\varepsilon_i}^s u_t(\pi(r)), r\varepsilon_j\rangle$ for $i,j \le n$. Then it is easy to see that $F_{\phi_t}(r) = K(t,r)F_{\omega_t}(r)$. It remains to deal with the Hodge Laplacian $\square\omega$. Applying the Bochner-Weitzenböck formula to ω, $\square\omega = -\Delta\omega + \mathrm{Ric}^{\#}\omega$. Let $\mathrm{ric}_r = r^{-1}\mathrm{Ric}_{\pi(r)}r$ be the equi-invariant representation of Ric on $O(M)$. Then $F_{\mathrm{Ric}^{\#}\omega} = \mathrm{ric}\, F_\omega$. Finally we may lift the vorticity transport equation to the principal bundle $O(M)$ by applying the scalarization F on both sides of (3.6), which gives rise to the corresponding vorticity transport equation on $O(M)$:

$$\frac{d}{dt}F_{\omega_t} = \nu\Delta_{O(M)}F_{\omega_t} - \mathcal{L}_{U_t}F_{\omega_t} + (K(t,\cdot) - \nu\,\mathrm{ric})F_{\omega_t}. \tag{3.9}$$

A functional integration representation formula for the vorticity ω_t can be obtained by using Feynman-Kac formula to (3.9), if the underlying diffusion we are going to use is non-explosive. Therefore some technical assumptions have to be imposed on the geometry of the manifold (and in fact on the regularity of the vorticity ω_t as well). From now on, we will work with a manifold whose Ricci curvature is bounded from below. That is, there is a constant $\kappa \in \mathbb{R}$ such that

$$\mathrm{Ric} \ge -\kappa. \tag{3.10}$$

First of all, as we have pointed out already, we shall prove that the concerned diffusion process do not explode at a finite time. For this purpose, we consider a family of vector fields $\{v_t(x);\ t \geq 0\}$ on M, which satisfies the following conditions: $(t,x) \mapsto v_t(x)$ is continuous, for each $t \geq 0$, $v_t \in C^{1+\alpha}$ for some $\alpha > 0$, and $\mathrm{div}(v_t) = 0$. Let V_t be the horizontal lift of v_t to $O(M)$. Then $\mathrm{div}(V_t) = \mathrm{div}(v_t) \circ \pi$ (cf. [22], page 595), and therefore $\mathrm{div}(V_t) = 0$.

Consider the following Stratonovich type stochastic differential equation on $O(M)$:

$$dr_t = \sum_{k=1}^{n} A_k(r_t) \circ dW_t^k + V_t(r_t)dt, \quad r_{|t=0} = r_0. \tag{3.11}$$

Denote by $r_t(w, r_0)$ the solution to (3.11), and $\zeta(w, r_0)$ its life-time. Let

$$\Sigma(t,w) = \{r_0 \in O(M);\ \zeta(w,r_0) > t\}.$$

Then for each $t > 0$ given, almost surely $\Sigma(t,w)$ is an open subset of $O(M)$ and $r_0 \to r_t(w,r_0)$ is a local diffeomorphism on $\Sigma(t,w)$ (cf. [31]). Let $r_t(r_0) = r_t(w,r_0)$ for simplicity. The Jacobian J_{r_t} of $r_0 \to r_t(r_0)$ is equal to 1, and according to [31], the Jacobian $J_{r_t^{-1}}$ of inverse map r_t^{-1} is given by

$$J_{r_t^{-1}} = \exp\Big(-\int_0^t \sum_{k=1}^{n} \mathrm{div}(A_k)(r_s(r_0)) \circ dW_s^k - \int_0^t \mathrm{div}(V_s(r_s(r_0))\, ds\Big) = 1.$$

For any $\varphi \in C_c(O(M))$, almost surely,

$$\int_{O(M)} \varphi(r_t(r_0))\, \mathbf{1}_{\Sigma(t,w)}(r_0)\, dr_0 = \int_{O(M)} \varphi(r_0) \mathbf{1}_{r_t(\Sigma(t,w))}(r_0)\ dr_0, \tag{3.12}$$

where dr_0 is the Liouville measure on $O(M)$ ([42], page 185) such that $\pi_\#(dr_0) = dx_0$.

Let $d_M(x,y)$ be the Riemannian distance on M between x and y. Fix a reference point $x_M \in M$, consider

$$\rho(r) = d_M(\pi(r), x_M).$$

It is known that for each x_0 given, $x \to d_M(x, x_0)$ is smooth out of $C_{x_0} \cup \{x_0\}$, where C_{x_0} is the cut-locus of x_0. It is known that C_{x_0} is negligible with respect to dx. Hence ρ is smooth out of $\pi^{-1}(C_{x_M} \cup \{x_M\})$. According to [42], page 197, on the complement of $\pi^{-1}(C_{x_0} \cup \{x_0\})$,

$$\frac{1}{2}\Delta_{O(M)} d_M(\pi(\cdot), x_0) \leq \frac{n-1}{2d_M(\pi(\cdot), x_0)} + \frac{1}{2}\sqrt{n\kappa}, \tag{3.13}$$

and $|\nabla_x d_M(x,x_0)| = 1$. Therefore on the complement of $\pi^{-1}(C_{x_0} \cup \{x_0\})$, it holds that

$$|\mathcal{L}_{V_t} d_M(\pi(\cdot), x_0)| \leq |V_t|. \tag{3.14}$$

The lower bound of $\frac{1}{2}\Delta_{O(M)}\rho$ is more delicate, however according to a result on page 90 in [28], for $\pi(r) \in B(x_M, R)\backslash(C_{x_0} \cup \{x_0\})$,

$$\frac{1}{2}\Delta_{O(M)} d_M(\pi(\cdot), x_0) \geq \frac{n-1}{2\rho} - \frac{1}{2}\sqrt{n(n-1)K_R}, \tag{3.15}$$

where K_R is the upper bound of sectional curvature on the big ball $B(x_M, R)$.

Proposition 3.2. *Suppose the Ricci curvature is bounded from below* (3.10) *and suppose*

$$\int_0^T \int_M |v_s(x)|^2 \, dx ds < \infty. \tag{3.16}$$

Then there is a non-decreasing process $\hat{L}_t \geq 0$ *and a Brownian motion* $\{\beta_t;\ t \geq 0\}$ *on* $\mathbb{R}$ *such that for almost surely initial* r_0,

$$\rho(r_t) - \rho(r_0) = \beta_t + \int_0^t \big((\frac{1}{2}\Delta_{O(M)} + \mathcal{L}_{V_s})\rho\big)(r_s)\ ds - \hat{L}_t, \quad t < \zeta(w, r_0). \tag{3.17}$$

Proof. The proof will be postphoned in Section 6. □

Theorem 3.3. *Assume that* (3.10) *and* (3.16) *hold. Then for almost* r_0, $\zeta(w, r_0) = \infty$ *almost surely.*

Proof. We have, by (3.17),

$$\rho(r_{t\wedge\zeta})^2 \leq \rho(r_0)^2 + t\wedge\zeta + 2\int_0^{t\wedge\zeta} \rho(r_s) d\beta_s + 2\int_0^{t\wedge\zeta} \rho(r_s)\,(L_s\rho)(r_s)\, ds,$$

where $\mathcal{L}_s = \frac{1}{2}\Delta_{O(M)} + \mathcal{L}_{V_s}$. Using (3.13) and (3.14), there is constants $C > 0$ such that

$$\begin{aligned}\mathbb{E}(\rho(r_{t\wedge\zeta})^2) &\leq \rho(r_0)^2 + C\int_0^t \mathbb{E}\Big(\big(2\rho(r_s)(L_s\rho)(r_s) + 1\big)\mathbf{1}_{(s<\zeta)}\Big)\, ds \\ &\leq \rho(r_0)^2 + 2C\int_0^t \mathbb{E}\Big((1+\rho(r_s))(1+|V_s(r_s)|)\mathbf{1}_{(s<\zeta)}\Big)\, ds.\end{aligned}$$

By hypothesis (3.10), there is a constant $c_0 > 0$ such that $\mathrm{vol}(B(x_0, \delta)) \leq e^{c_0\delta}$, and therefore for a constant $\lambda_0 > 0$,

$$C_M = \int_{O(M)} \exp(-\lambda_0\, d_M^2(\pi(r_0), x_0))\, dr_0 < +\infty.$$

Define the probability measure $d\mu$ on $O(M)$ by

$$d\mu(r_0) = \frac{1}{C_M}\exp(-\lambda_0\, d_M^2(\pi(r_0), x_0))\, dr_0. \tag{3.18}$$

Then

$$
\begin{aligned}
&\int_{O(M)} \mathbb{E}(\rho(r_{t\wedge\zeta})^2)\, d\mu \\
&\leq \int_{O(M)} \rho(r_0)^2 d\mu + 2C \int_0^t \int_{O(M)} \mathbb{E}\Big((1+\rho(r_s))(1+|V_s(r_s)|)\mathbf{1}_{(s<\zeta)}\Big)\, d\mu ds \\
&\leq \int_{O(M)} \rho(r_0)^2 d\mu + 4C\Big(\int_0^t \int_{O(M)} \mathbb{E}\Big((1+\rho(r_{s\wedge\zeta})^2)\Big)\, d\mu ds\Big)^{1/2} \times \\
&\qquad\qquad \times \Big(\int_0^t \int_{O(M)} \mathbb{E}\Big((1+|V_s(r_s)|)^2\mathbf{1}_{(s<\zeta)}\Big)\, d\mu ds\Big)^{1/2}.
\end{aligned}
$$

Note that

$$
\int_0^t \int_{O(M)} \mathbb{E}\Big((1+|V_s(r_s)|)^2\mathbf{1}_{(s<\zeta)}\Big)\, d\mu ds \leq 2\Big(T + \frac{1}{C_M}\int_0^T \int_M |v_s(x)|^2\, dx ds\Big).
$$

Set $\psi(t) = \displaystyle\int_{O(M)} \mathbb{E}\big(\rho(r_{t\wedge\zeta})^2\big)\, d\mu$ and

$$
C(T,v) = 4C\sqrt{2}\sqrt{T + \frac{1}{C_M}||v||^2_{L^2([0,T]\times M)}}. \tag{3.19}
$$

Remarking that $\sqrt{\xi} \leq 1+\xi$ for $\xi \geq 0$, above two inequalities imply that

$$
\psi(t) \leq \Big(\int_{O(M)} \rho(r_0)^2\, d\mu + C(T,v)\Big) + C(T,v)\int_0^t \psi(s)\, ds.
$$

The Gronwall lemma then yields that

$$
\int_{O(M)} \mathbb{E}(\rho(r_{t\wedge\zeta})^2)\, d\mu \ \leq \Big(\int_{O(M)} \rho(r_0)^2\, d\mu + C(T,v)\Big) \exp(C(T,v)).
$$

The result follows. □

We are now in a position to establish the main result of this section. Let $T > 0$ be fixed. Assume that u_t is a solution to (3.3) such that $(t,x) \mapsto u_t(x)$ is continuous and for each $t \geq 0$, $u_t \in C^{1+\alpha}$ with $\alpha > 0$. Consider the following SDE on $O(M)$,

$$
\begin{cases}
dr_{s,t}(r,w) = \sqrt{2\nu}\displaystyle\sum_{i=1}^n A_i(r_{s,t}(r,w)) \circ dW_t^i - U_{T-t}(r_{s,t}(r,w))\, dt, \quad s < t < T, \\
r_{s,s}(r,w) = r.
\end{cases} \tag{3.20}
$$

Let $v_t(x) = u_{T-t}(x)$. Then by Theorem 3.3, SDE (3.20) does not explode at a finite time. Let $Q_{s,t}(w)$ be solution to the resolvent equation

$$\frac{d}{dt}Q_{s,t}(w) = Q_{s,t}(w)J_{T-t}(r_{s,t}(r,w)), \quad s < t < T,\ Q_{s,s}(w) = Id \tag{3.21}$$

where

$$J_t(r) = K(t,r) - \nu\,\mathrm{ric}_r. \tag{3.22}$$

Theorem 3.4. *Under the notations and the assumptions stated above, if*

$$\mathbb{E}\Big(sup_{s\leq t\leq T}|Q_{s,t}F_{\omega_{T-t}}(r_{s,t})|\Big) < +\infty, \tag{3.23}$$

then the following functional integration representation holds:

$$F_{\omega_t} = \mathbb{E}\Big(Q_{T-t,T}F_{\omega_0}(r_{T-t,T})\Big). \tag{3.24}$$

Proof. For the sake of simplicity, we denote $r_{s,t} = r_{s,t}(r,w)$ and set $F(t,r) = F_{\omega_t}(r)$. Applying Itô's formula to $Q_{s,t}F(T-t, r_{s,t})$ for d_t with $t \in (s,T)$, we have

$$\begin{aligned}
&d_t\Big(Q_{s,t}\,F(T-t,r_{s,t})\Big) = d_tQ_{s,t}\,F(T-t,r_{s,t}) + Q_{s,t}\,d_t\big(F(T-t,r_{s,t})\big)\\
&= Q_{s,t}J_{T-t}(r_{s,t})F(T-t,r_{s,t}) + \sqrt{2\nu}\,Q_{s,t}\,\sum_{i=1}^n(\mathcal{L}_{A_i}F)(T-t,r_{s,t})\,dW_t^i\\
&+ Q_{s,t}\Big(-(\partial_tF)(T-t,r_{s,t}) + \nu\,(\Delta_{O(M)}F)(T-t,r_{s,t}) - (\mathcal{L}_{U_{T-t}}F)(T-t,r_{s,t})\Big)\,dt\\
&= \sqrt{2\nu}\,Q_{s,t}\,\sum_{i=1}^n(\mathcal{L}_{A_i}F)(T-t,r_{s,t})\,dW_t^i,
\end{aligned}$$

where the last equality is due to Equation (3.9). It follows that

$$Q_{s,t}\,F(T-t,r_{s,t}) - F(T-s,r) = \sqrt{2\nu}\,\sum_{i=1}^n\int_s^t Q_{s,\tau}\,(\mathcal{L}_{A_i}F)(T-\tau,r_{s,\tau})\,dW_\tau^i.$$

Under Condition (3.23), the local martingale defined by the right hand side of above equality becomes a true martingale. Taking expectation on the two sides gives

$$\mathbb{E}\Big(Q_{s,t}\,F(T-t,r_{s,t}\Big) = F(T-s,r).$$

Now let $t = T$, then $\mathbb{E}\Big(Q_{s,T}\,F(0,r_{s,T})\Big) = F(T-s,r)$. Replacing s by $T-t$, we get representation formula (3.24). □

4. Intrinsic Ricci tensors for Navier-Stokes equations

Since a solution u_t of the Navier-Stokes equation is in general a time-dependent vector field on M, the associated SDE (3.20) in Section 3 is much more complicated than SDE (2.1), which in turn gives rise to a family of Ikeda-Watanabe connections as defined in Section 2. On the other hand we may consider the following SDE on the principal bundle $O(M)$:

$$dr_w(t) = \sqrt{2\nu}\sum_{i=1}^n B_i(r_w(t))\circ dW_t^i, \quad r_w(0) = r,$$

whose infinitesimal generator is given by

$$\nu\sum_{i=1}^n \mathcal{L}_{B_i}^2(f\circ\pi) = \Big((\nu\Delta_M + 2\nu v)f\Big)\circ\pi,$$

where v is a time-dependent vector field given by

$$v = -\frac{1}{2\nu}u_t. \tag{4.1}$$

The Ricci curvature associated with the Ikeda-Watanabe connection $\Gamma^{v(t,)}$ is denoted by Ric^t for every t if no confusion may arise. According to (2.9)

$$\widehat{\mathrm{Ric}^t} = \mathrm{Ric}^0 + \frac{1}{2\nu^2}u_t\otimes u_t - \frac{1}{\nu}\nabla^{0,s}u_t, \tag{4.2}$$

where Ric^0 is the Ricci curvature of the manifold M.

Proposition 4.1. *Suppose the dimension $n = 3$.*

(i) It holds that

$$\mathrm{div}(\widehat{\mathrm{Ric}^t}) = \mathrm{div}(\mathrm{Ric}^0) + \frac{1}{2\nu^2}\nabla_{u_t}u_t - \frac{1}{\nu}\mathrm{Ric}^0 u_t. \tag{4.3}$$

(ii) Let $\widehat{\mathrm{Scal^t}}$ denote the associated scalar curvature: $\widehat{\mathrm{Scal^t}} = \sum_{i=1}^n\langle\widehat{\mathrm{Ric}^t}e_i,\ e_i\rangle$ for any orthonormal basis (e_i) of T_xM. Then

$$\widehat{\mathrm{Scal^t}} = \mathrm{Scal}^0 + \frac{1}{2\nu^2}|\mathrm{u_t}|^2. \tag{4.4}$$

Proof. (i) Since $\mathrm{div}(u_t) = 0$, $\mathrm{div}(u_t\otimes u_t) = \nabla_{u_t}u_t$, and

$$\nabla u_t = \nabla^s u_t + \nabla^{sk}u_t.$$

We claim that

$$\mathrm{div}(\nabla^{sk}u_t) = -\square u_t.$$

In fact, if $X \in \mathcal{X}(M)$ with a compact support in M, then

$$\begin{aligned}\int_M \langle \mathrm{div}(\nabla^{sk} u_t), X\rangle\, dx &= -\int_M \langle \nabla^{sk} u_t, \nabla X\rangle\; dx = -\int_M \langle \nabla^{sk} u_t, \nabla^{sk} X\rangle\; dx \\ &= -\int_M \langle d\tilde{u}_t, d\tilde{X}\rangle\, dx = -\int_M \langle d^* d\tilde{u}_t, \tilde{X}\rangle\, dx \\ &= -\int_M \langle \Box \tilde{u}_t, \tilde{X}\rangle\, dx.\end{aligned}$$

Therefore

$$\mathrm{div}(\nabla^s u_t) = \Delta u_t + \Box u_t = \mathrm{Ric}^0 u_t.$$

(ii) Since

$$\sum_{i=1}^n \langle \nabla^{0,s}_{e_i} u_t, e_i\rangle = \mathrm{div}(u_t) = 0,$$

Eq.(4.4) follows immediately from (4.2). ☐

The following result captures the vorticity dynamics in time and in space in terms of the various curvatures we have introduced.

Theorem 4.2. *Let M be a Riemannian manifold having Ricci tensor bounded below, of the dimension $n = 3$. Suppose u_t together with ω_t is a regular solution to Eq.*(3.6)*. Then the following identity holds:*

$$\frac{1}{2}\frac{d}{dt}\int_M |\omega_t|^2\, dx + \nu\int_M |\nabla^0 \omega_t|^2\, dx = \frac{1}{2\nu}\int_M (\omega_t, u_t)^2\, dx - \nu\int_M (\widehat{\mathrm{Ric}^t}^{\#}\omega_t, \omega_t)\; dx. \tag{4.5}$$

Proof. By definition of $\mathcal{L}_{u_t}$ we have

$$\int_M \langle \nabla_{u_t}\omega_t, \omega_t\rangle\, dx = \frac{1}{2}\int_M \mathcal{L}_{u_t}|\omega_t|^2\, dx = 0,$$

and therefore, by Eq.(3.6),

$$\frac{1}{2}\frac{d}{dt}\int_M |\omega_t|^2\, dx + \nu\int_M |\nabla^0\omega_t|^2\, dx = -\nu\int_M \langle \mathrm{Ric}^0\,\omega_t, \omega_t\rangle\, dx + \int_M \langle \omega_t \triangleleft \nabla^s u_t, \omega_t\rangle\, dx. \tag{4.6}$$

On the other hand, according to Eq.(4.2), for any vector field A,

$$(\widehat{\mathrm{Ric}^t}^{\#}\omega_t, A) = (\omega_t, \mathrm{Ric}^0 A) + \frac{1}{2\nu^2}(\omega_t, u_t)\langle u_t, A\rangle - \frac{1}{\nu}(\omega_t, \nabla^{0,s}_A u_t),$$

and according to (3.5)

$$(\omega_t, \nabla^{0,s}_A u_t) = (\omega_t \triangleleft \nabla^{0,s} u_t)(A).$$

Hence

$$\widehat{\mathrm{Ric}^t}^{\#}\omega_t = \mathrm{Ric}^{0,\#}\omega_t + \frac{1}{2\nu^2}(\omega_t, u_t)\tilde{u}_t - \frac{1}{\nu}\omega_t \triangleleft \nabla^{0,s} u_t. \tag{4.7}$$

Expressing the right hand side of (4.6) in term of $\widehat{\mathrm{Ric}^t}^{\#}$, Eq.(4.7) then implies that

$$\langle \widehat{\mathrm{Ric}^t}^{\#}\, \omega_t,\ \omega_t\rangle = \langle \mathrm{Ric}^0\, \omega_t, \omega_t\rangle + \frac{1}{2\nu^2}(\omega_t, u_t)^2 - \frac{1}{\nu}\langle \omega_t \lhd \nabla^{0,s} u_t,\ \omega_t\rangle.$$

Thus

$$-\nu\,\langle \mathrm{Ric}^0\, \omega_t, \omega_t\rangle + \langle \omega_t \lhd \nabla^{0,s} u_t,\ \omega_t\rangle. = -\nu\,\langle \widehat{\mathrm{Ric}^t}^{\#}\, \omega_t,\ \omega_t\rangle + \frac{1}{2\nu}(\omega_t, u_t)^2.$$

By substituting this term in the right hand side of (4.6), we get (4.5). □

5. Existence of weak solutions

Navier-Stokes equations on a compact Riemannian manifold is studied in the monograph by M. Taylor [44]. In this section, we will deal with the case of Riemannian manifold with the Ricci curvature bounded below.

Proposition 5.1. *In the smooth case, it holds*

$$\frac{1}{2}\frac{d}{dt}\int_M |u_t|^2\, dx + \nu \int_M |\nabla u_t|^2\, dx = -\nu \int_M \langle \mathrm{Ric}\, u_t, u_t\rangle\, dx. \tag{5.1}$$

Proof. Since M is a closed manifold

$$\int_M \langle \nabla_{u_t} u_t, u_t\rangle\, dx = \frac{1}{2}\int_M \mathcal{L}_{u_t} |u_t|^2\, dx = 0$$

and $\int_M \langle \nabla p, u_t\rangle\, dx = 0$. By using equation (3.3), we get

$$\frac{1}{2}\frac{d}{dt}\int_M |u_t|^2\, dx + \nu \int_M \langle \square u_t, u_t\rangle\, dx = 0.$$

Therefore (5.1) follows the Bochner-Weitzenböck formula immediately. □

Proposition 5.2. *Assume that* (3.10) *holds, that is,* $\mathrm{Ric} \geq -\kappa$. *Then the following a priori estimate holds*

$$\frac{1}{2}||u_t||_2^2 + \nu \int_0^t ||\nabla u_s||_2^2\, ds \leq \frac{1}{2}||u_0||_2^2\, \exp(2\nu t \kappa^+), \tag{5.2}$$

where $\kappa^+ = \sup\{\kappa, 0\}$.

Proof. According to (5.1), the following energy inequality holds:

$$\frac{1}{2}\frac{d}{dt}\int_M |u_t|^2\, dx + \nu \int_M |\nabla u_t|^2\, dx \leq \nu\kappa \int_M |u_t|^2\, dx \leq \nu\kappa^+ \int_M |u_t|^2\, dx.$$

Let $\psi(t) = \frac{1}{2}||u_t||_2^2 + \nu \int_0^t ||\nabla u_s||_2^2\, ds$. Then ψ satisfies inequality

$$\psi(t) \leq \frac{1}{2}||u_0||_2^2 + 2\nu\kappa^+ \int_0^t \psi(s)\, ds$$

and (5.2) follows from the Gronwall lemma immediately □

In what follows, we will establish the existence of weak solutions in Leray sense over any $[0,T]$ and

$$u \in L^2([0,T], H^1(M)) \cap L^\infty([0,T], L^2(M)).$$

To this end, we will use the heat semi-group $\mathbf{T}_t = e^{-t\Box/2}$ to regularize vector fields. Let v be a continuous vector field on M with compact support and define $\mathbf{T}_t v = (\mathbf{T}_t \tilde{v})^\#$. Then $\mathbf{T}_t v$ solves the heat equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\Box\right)(\mathbf{T}_t v) = 0.$$

By ellipticity of $\Box$ (see for example [46]), $(t,x) \to (\mathbf{T}_t v)(x)$ is smooth. It was shown in [22] that

$$\mathrm{div}(\mathbf{T}_t v) = \mathbf{T}_t^M(\mathrm{div}(v)),$$

where $\mathbf{T}_t^M$ denotes heat semi-group on functions. Hence $\mathbf{T}_t$ preserves the space of divergence free vector fields. It is well-known that

$$|\mathbf{T}_t v| \leq e^{t\kappa + /2}\ \mathbf{T}_t^M |v|. \tag{5.3}$$

It follows that for $1 \leq p \leq +\infty$, $||\mathbf{T}_t v||_p \leq e^{t\kappa+/2}\,||v||_p$, and for $1 \leq p < +\infty$, $\mathbf{T}_t v \to v$ in L^p.

Consider a family of smooth functions $\varphi_\varepsilon \in C_c^\infty(M)$ with compact support such that

$$0 \leq \varphi_\varepsilon \leq 1, \quad \varphi_\varepsilon(x) = 1 \text{ for } x \in B(x_M, 1/\varepsilon) \quad \text{and} \quad \sup_{\varepsilon>0} ||\nabla\varphi_\varepsilon||_\infty < +\infty, \tag{5.4}$$

where x_M is a fixed point of M. For $\varepsilon > 0$, we define

$$F_\varepsilon(u) = -\mathbf{T}_\varepsilon \mathbf{P}\big(\varphi_\varepsilon\, \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\big) - \nu \mathbf{T}_\varepsilon \Box \mathbf{T}_\varepsilon u, \quad u \in L^2(M)$$

where $\mathbf{P}$ is the orthogonal projection from $L^2(M)$ to the subspace of vector fields of divergence free. We have

$$||\mathbf{T}_\varepsilon \mathbf{P}\big(\varphi_\varepsilon\, \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\big)||_2 \leq e^{\varepsilon\kappa^+/2} ||\mathbf{P}\big(\varphi_\varepsilon\, \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\big)||_2 \leq e^{\varepsilon\kappa^+/2} ||\nabla_{\varphi_\varepsilon \mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)||_2.$$

Since φ_ε is of compact support, we have

$$||\nabla_{\varphi_\varepsilon \mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)||_2 \leq ||\varphi_\varepsilon \mathbf{T}_\varepsilon u||_\infty\ ||\nabla(\varphi_\varepsilon \mathbf{T}_\varepsilon u)||_2. \tag{5.5}$$

Again due to compact support of φ_ε, when $n = 3$, by Sobolev's embedding theorem, there is a constant $\beta(\varepsilon) > 0$ such that

$$||\varphi_\varepsilon \mathbf{T}_\varepsilon u||_\infty \leq \beta(\varepsilon)\, ||\varphi_\varepsilon \mathbf{T}_\varepsilon u||_{H^2}.$$

For the general case, it is sufficient to bound the uniform norm by the norm of H^m with $m > \frac{n}{2}$.

Recall that Weitzenböck formula for p-differential forms reads as follows [29, 19]:

$$\square = -\Delta + \mathcal{R}_p^{\#}, \tag{5.6}$$

where $\Delta\phi = \mathrm{Trace}(\nabla\nabla\phi)$ for a p-form ϕ, and $\mathcal{R}_p^{\#} : \Lambda^p(M) \to \Lambda^p(M)$ is a tensor, called Weitzenböck curvature. For $p = 1$, $\mathcal{R}_1 = \mathrm{Ric}^{\#}$ is Ricci tensor. For the following result, the lower bound of $\mathcal{R}_2^{\#}$ is needed. Assume that

$$\mathcal{R}_p \geq -\kappa_p, \quad \kappa \in \mathbb{R}. \tag{5.7}$$

Proposition 5.3. *Let $T > 0$. Suppose (5.7) holds for $p = 1, 2$. Then there are constants β_1, β_2 such that*

$$||\square \mathbf{T}_\varepsilon u||_2 \leq \frac{\beta_1}{\varepsilon}||u||_2, \quad ||\nabla \mathbf{T}_\varepsilon u||_2 \leq \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \varepsilon > 0. \tag{5.8}$$

Proof. This follows immediately from the Bismut formulae obtained in [19, 14]. For a detailed proof, see [24]. □

By Proposition 5.3, there are constants $\beta(\varepsilon) > 0, \tilde{\beta}(\varepsilon) > 0$ such that

$$||\varphi_\varepsilon \mathbf{T}_\varepsilon u||_\infty \leq \beta(\varepsilon)\, ||u||_2, \quad ||\mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u||_2 \leq \tilde{\beta}(\varepsilon)\, ||u||_2. \tag{5.9}$$

Combining (5.5) and (5.9), there are two constants $\beta_1(\varepsilon) > 0$ and $\beta_2(\varepsilon) > 0$ such that

$$||F_\varepsilon(u)||_2 \leq \beta_1(\varepsilon)\, ||u||_2^2 + \beta_2(\varepsilon)||u||_2,$$

and F_ε is locally Lipschitz. By theory of ordinary differential equation, there is a unique solution u^ε to

$$\frac{du_t^\varepsilon}{dt} = F_\varepsilon(u_t^\varepsilon), \quad u_0^\varepsilon = u_0 \in L^2, \quad \mathrm{div}(u_t^\varepsilon) = 0, \tag{5.10}$$

up to the explosion time τ.

Theorem 5.4. *Suppose that $||\mathrm{Ric}||_\infty < \infty$, and $\mathcal{R}_2$ is bounded below. Then for any $T > 0$, there is a weak solution $u \in L^2([0, T], H^1)$ to Navier-Stokes equation (3.3) such that the following energy inequality holds:*

$$\frac{1}{2}||u_t||_2^2 + \nu \int_0^t ||\nabla u_s||_2^2\, ds \leq \frac{1}{2}||u_0||_2^2 \, \exp(2\nu t\kappa^+),$$

where κ is lower bound of Ric.

Proof. Rewriting (5.10) in the following explicit form, for $t < \tau$,

$$\frac{du_t^\varepsilon}{dt} + \mathbf{T}_\varepsilon \mathbf{P}\big(\varphi_\varepsilon\, \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon)\big) + \nu \mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u_t^\varepsilon = 0.$$

Note that

$$\int_M \langle \mathbf{T}_\varepsilon \mathbf{P}\big(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon)\big),\ u_t^\varepsilon\rangle\, dx = \int_M \langle \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon)\big),\ \varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon\rangle\, dx$$
$$= \int_M \mathcal{L}_{\mathbf{T}_\varepsilon u_t^\varepsilon} |\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon|^2\ dx = 0.$$

Since $\mathrm{div}(\mathbf{T}_\varepsilon u_t^\varepsilon) = 0$, and

$$\int_M \langle \mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u_t^\varepsilon,\ u_t^\varepsilon\rangle\ dx = \int_M |\nabla \mathbf{T}_\varepsilon u_t^\varepsilon|^2\, dx + \int_M \langle \mathrm{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon),\ \mathbf{T}_\varepsilon u_t^\varepsilon\rangle\ dx,$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\int_M |u_t^\varepsilon|^2\, dx + \nu \int_M |\nabla \mathbf{T}_\varepsilon u_t^\varepsilon|^2\, dx = -\nu \int_M \langle \mathrm{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon),\ \mathbf{T}_\varepsilon u_t^\varepsilon\rangle\ dx$$
$$\leq -\nu\kappa \int_M |\mathbf{T}_\varepsilon u_t^\varepsilon|^2\, dx,$$

or in another form

$$\frac{1}{2}||u_t^\varepsilon||_2^2 + \nu \int_0^t |||\nabla \mathbf{T}_\varepsilon u_s^\varepsilon||_2^2\, ds \leq \frac{1}{2}||u_0||_2^2 + \nu\kappa^+ \int_0^t ||\mathbf{T}_\varepsilon u_s^\varepsilon||_2^2\, ds. \tag{5.11}$$

According to (5.3), above inequality implies that

$$\frac{1}{2}||u_t^\varepsilon||_2^2 \leq \frac{1}{2}||u_0||_2^2 + \nu\kappa^+ e^{\varepsilon\kappa^+} \int_0^t ||u_s^\varepsilon||_2^2\, ds,$$

and therefore, by using Gronwall lemma, for $t < \tau$, we have

$$\frac{1}{2}||u_t^\varepsilon||_2^2 \leq \frac{1}{2}||u_0||_2^2\ \exp(t\nu\kappa^+ e^{\varepsilon\kappa^+}).$$

It follows that $\tau = \infty$. Again, according to (5.3) and (5.11),

$$\frac{1}{2}||\mathbf{T}_\varepsilon u_t^\varepsilon||_2^2 + \nu e^{\varepsilon\kappa^+} \int_0^t |||\nabla \mathbf{T}_\varepsilon u_s^\varepsilon||_2^2\, ds \leq \frac{1}{2} e^{\varepsilon\kappa^+}||u_0||_2^2 + \nu\kappa^+ e^{\varepsilon\kappa^+} \int_0^t ||\mathbf{T}_\varepsilon u_s^\varepsilon||_2^2\, ds.$$

Therefore Gronwall lemma yields, for $\varepsilon \leq 1$, that

$$\frac{1}{2}||\mathbf{T}_\varepsilon u_t^\varepsilon||_2^2 + \nu e^{\varepsilon\kappa^+} \int_0^t |||\nabla \mathbf{T}_\varepsilon u_s^\varepsilon||_2^2\, ds \leq \frac{e^{\kappa^+}}{2}||u_0||_2^2\ \exp(t\nu\kappa^+ e^{\kappa^+}). \tag{5.12}$$

Let $T > 0$. By (5.12), the family $\big\{\mathbf{T}_\varepsilon u_\cdot^\varepsilon;\ \varepsilon \in (0,1]\big\}$ is bounded in $L^2([0,T], H^1)$ as well in $L^\infty([0,T], L^2)$. Then there is a sequence ε_n and a $u \in L^2([0,T], H^1) \cap L^\infty([0,T], L^2)$ such that $\mathbf{T}_{\varepsilon_n} u^{\varepsilon_n}$ converges weakly to u in $L^2([0,T], H^1)$ and $*$-weakly in $L^\infty([0,T], L^2)$. Now standard arguments allow to prove that u is a weak solution (3.3). The boundedness of Ric is needed while passing to the limit of the term $\displaystyle\int_M \langle \mathrm{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon),\ v_t\rangle\ dx.$

□

6. Proof of Proposition 3.2

We give here a complete proof of Proposition 3.2. According to the proof of Theorem 3.5.1 in [28], and emphasize the steps we have to modify.

Proof. Let i_x be the injectivity radius at x and suppose that

$$i_M = \inf\{i_x;\ x \in M\} > 0. \tag{6.1}$$

This means that the ball $B(x, i_M)$ does not meet the cut-locus C_x of x. We prepare what we will need for proving (3.17).

Let $x \in B(x_0, i_M/2)^c$ which maybe is closed to or in C_{x_0}. Let $\gamma_x : [0, \eta(x)] \to M$ be a distance-minimizing geodesic connecting x_0 and x, parameterized by length. Then $\gamma_x(i_M/4) \notin C_x$ or $x \notin C_{\gamma_x(i_M/4)}$. Put $y = \gamma_x(i_M/4)$. Then $d_M(x_0, x) = d_M(x_0, y) + d_M(y, x)$. Since C_y is closed, there is $\varepsilon_0 > 0$ such that

$$B(x, \varepsilon_0) \cap C_y = \emptyset.$$

We suppose that such ε_0 is valid for all x (in fact, we will restrict ourselves in a compact set). Let $\varepsilon < \varepsilon_0 \wedge \frac{i_M}{8}$, and define

$$D_\varepsilon = \big\{x \in M;\ d_M(x, C_{x_M}) < \varepsilon\big\}.$$

We claim that

$$D_\varepsilon \subset B(x_M, i_M/2)^c. \tag{6.2}$$

In fact, if there exists $x \in D_\varepsilon$ such that $d_M(x, x_M) < i_M/2$; there is $z \in C_{x_M}$ such that $d_M(x, z) < \varepsilon$; then $d_M(x_M, z) \le d_M(x_M, x) + d_M(x, z) < i_M$ which contradicts the definition of i_M. Let γ_x be the geodesic considered above. Then $x \notin C_y$ with $y = \gamma_x(i_M/4)$.

Now introduce the stopping times σ_q by $\sigma_0 = 0$ and

$$\sigma_q = \inf\big\{t > \sigma_{q-1};\ d_M(\pi(r_t), \pi(r_{\sigma_{q-1}})) = \varepsilon\big\}.$$

Let $t > 0$ and set $t_q = t \wedge \sigma_q$. Then

$$\rho(r_t) - \rho(r_0) = \sum_{q=1}^{+\infty} \Big(\rho(r_{t_q}) - \rho(r_{t_{q-1}})\Big). \tag{6.3}$$

(i) If $\pi(r_{t_{q-1}}) \notin D_\varepsilon$, then for $s \in [t_{q-1}, t_q]$, $\pi(r_s) \notin C_{x_M}$. Applying Itô formula, we have

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) = \sum_{k=1}^{n} \int_{t_{q-1}}^{t_q} (\mathcal{L}_{A_k}\rho)(r_s)\, dW_s^k + \int_{t_{q-1}}^{t_q} (L_s\rho)(r_s)\, ds, \tag{6.4}$$

where $L_s = \dfrac{1}{2}\Delta_{O(M)} + \mathcal{L}_{V_s}$

(ii) Set $x_q = \pi(r_{t_q})$. If $x_{q-1} \in D_\varepsilon$, then by discussion at beginning, there is y_{q-1} on a distance-minimizing geodesic γ connecting x_M and x_{q-1} such that $d_M(x_M, y_{q-1}) = \dfrac{i_M}{4}$ and $x_{q-1} \notin C_{y_{q-1}}$ and for $s \in [t_{q-1}, t_q]$,

$$d_M(\pi(r_s), x_{q-1}) \le \varepsilon < \varepsilon_0.$$

Therefore $\pi(r_s) \notin C_{y_{q-1}}$. Let $\rho_q^*(r) = d_M(\pi(r), y_{q-1})$. Applying Itô formula to ρ_q^*, we have

$$\rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}) = \sum_{k=1}^n \int_{t_{q-1}}^{t_q} (\mathcal{L}_{A_k}\rho_q^*)(r_s)\, dW_s^k + \int_{t_{q-1}}^{t_q} (L_s\rho_q^*)(r_s)\, ds.$$

On one hand

$$d_M(x_M, x_{q-1}) = d_M(x_M, y_{q-1}) + d_M(x_{q-1}, y_{q-1}) \quad \text{or} \quad \rho(r_{t_{q-1}}) = \frac{i_M}{4} + \rho_q^*(r_{t_{q-1}}),$$

and on the other hand

$$d_M(x_M, x_q) \le d_M(x_M, y_{q-1}) + d_M(x_q, y_{q-1}) \quad \text{or} \quad \rho(r_{t_q}) \le \frac{i_M}{4} + \rho_q^*(r_{t_q}).$$

It follows that

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) \le \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}).$$

Therefore there exists $\hat{L}_q \ge 0$ such that

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) = \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}) - \hat{L}_q.$$

Define

$$\tau_R = \inf\{t > 0,\ d_M(x_M, \pi(r_t)) > R\}.$$

As did in [28], page 95, we get

$$\rho(r_{t\wedge\tau_R}) - \rho(r_0) = \beta_{t\wedge\tau_R} + \int_0^{t\wedge\tau_R} (L_s\rho)(r_s)\, ds - \hat{L}_\varepsilon(t\wedge\tau_R) + R_\varepsilon(t\wedge\tau_R),$$

where

$$\hat{L}_\varepsilon(t) = \sum_{q=1}^{+\infty} \hat{L}_q \mathbf{1}_{D_\varepsilon}\pi((r_{t_{q-1}}))$$

which converges to $\hat{L}(t)$ as $\varepsilon \to 0$. The term $R_\varepsilon(t) = m_\varepsilon(t) + b_\varepsilon(t)$ with $m_\varepsilon(t)$ the same as in [28], page 95, so that

$$\mathbb{E}(|m_\varepsilon(t)|^2) \le 4\int_0^t \mathbb{E}(\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s)))\, ds.$$

Therefore for any compact subset $K \subset B(x_M, R)$,

$$\begin{aligned}
&\int_{\pi^{-1}(K)} \mathbb{E}(|m_\varepsilon(t\wedge\tau_R)|^2)\,dr_0 \le 4\int_0^t \int_{\pi^{-1}(K)} \mathbb{E}(\mathbf{1}_{D_{2\varepsilon}}(\pi(r_{s\wedge\tau_R})))\,dr_0 ds \\
&\to 4\int_0^t \int_{\pi^{-1}(K)} E(\mathbf{1}_{C_{x_M}}(\pi(r_{s\wedge\tau_R})))\,dr_0 ds \le 4\int_0^t\int_M \mathbf{1}_{C_{x_M}}(x)dxds = 0.
\end{aligned}$$

The term $b_\varepsilon(t)$ has to be modified such that

$$b_\varepsilon(t) = \sum_{q=1}^{+\infty}\Big[\int_{t_{q-1}}^{t_q}\Big(L_s\rho_q^*(r_s) - L_s\rho(r_s)\Big)\,ds\Big]\mathbf{1}_{D_\varepsilon}(\pi(r_{t_{q-1}})).$$

By (3.13) and (3.15), we have to control the term $1/\rho$. For $x_{q-1}\in D_\varepsilon$ and for $s\in[t_{q-1}, t_q]$,

$$d_M(x_M, x_s) \ge d_M(x_M, x_{q-1}) - d_M(x_{q-1}, x_s) \ge \frac{i_M}{2} - \varepsilon \ge \frac{3i_M}{8},$$

and

$$d_M(y_{q-1}, x_s) \ge d_M(x_M, x_s) - d_M(x_M, y_{q-1}) \ge \frac{3i_M}{8} - \frac{i_M}{4} = \frac{i_M}{8}.$$

Therefore, according to (3.14), since $x_s = \pi(r_s)\in D_{2\varepsilon}$, there exists a constant $\alpha > 0$ such that

$$\int_{t_{q-1}}^{t_q}\Big|\Big(L_s\rho_q^*(r_s) - L_s\rho(r_s)\Big)\Big|\,ds\mathbf{1}_{D_\varepsilon}(\pi(r_{t_{q-1}})) \le \alpha\int_{t_{q-1}}^{t_q}(1+|V_s(r_s)|)\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s))\,ds.$$

It follows that

$$\mathbb{E}(|b_\varepsilon(t)|) \le \alpha\,\mathbb{E}\Big(\int_0^t (1+|V_s(r_s)|)\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s))\,ds\Big). \tag{6.5}$$

Now let $d\mu(r_0)$ be the probability measure defined in (3.18), integrating with respect to it, we get

$$\begin{aligned}
&\int_0^t\int_{\pi^{-1}(K)} \mathbb{E}\Big((1+|V_s(r_s)|)\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s))\mathbf{1}_{(s<\tau_R)}\Big)\,d\mu(r_0)ds \\
&\to \int_0^t\int_{\pi^{-1}(K)} \mathbb{E}\Big((1+|V_s(r_s)|)\mathbf{1}_{C_{x_M}}(\pi(r_s))\mathbf{1}_{(s<\tau_R)}\Big)\,d\mu(r_0)ds \\
&\le \sqrt{t}\,\Big(\int_0^t\int_M |v_s(x)|^2\mathbf{1}_{C_{x_M}}(x)\,dxds\Big)^{1/2} = 0,
\end{aligned}$$

under the hypothesis (3.19). The proof of Proposition 3.2 is complete. □

7. Final remarks

1) The condition

$$\mathrm{Ric}(x) \geq -C\left(1 + d_M^2(x)\right)$$

is sufficient to insure the non-explosion of the Brownian motion on M, but seems too weak to guarantee the good behavior of the heat semi-group on differential forms. For example, the key upper bound (5.3) is not established.

2) On manifolds, there are several choices for Laplacian operator on vectors. On [15], D.G. Ebin and J.E. Marsden introduced the so-called deformation operator, denoted by $\hat{\square}$, which admits expression

$$\hat{\square}A = -\Delta A - \mathrm{Ric}(A) \quad \text{if} \quad \mathrm{div}(A) = 0.$$

Here the Ricci tensor has an opposite sign, in contrast with the De Rham-Hodge operator $\square$. In the paper [41], V. Pierfelice considered NS equation with $\hat{\square}$. Major geometric assumptions in [41] are the non-positiveness of sectional curvature and for two positive constants c_1 and c_2,

$$-c_1 \leq \mathrm{Ric} \leq -c_2.$$

The well-posedness of mild solutions in the sense of Kato-Fujita has been established.

ACKNOWLEDGEMENTS. The two authors are grateful to the referee for his careful reading and useful suggestions.

References

[1] H. Airault, P. Malliavin, Integration by parts formulas and dilatation vector fields on elliptic probability space, *Prob. Theory Rel. Fields*, 106 (1996), 447-494.
[2] M. Arnaudon, A.B. Cruzeiro, Lagrangian Navier-Stokes diffusions on manifolds: variational principle and stability, *Bull. Sci. Math.*, 136 (8) (2012), 857–881.
[3] M. Arnaudon, A.B. Cruzeiro, Stochastic Lagrangian flows on some compact manifolds, *Stochastics*, 84 (2012), 367–381.
[4] M. Arnaudon, A.B. Cruzeiro, S. Fang, Generalized stochastic Lagrangian paths for the Navier–Stokes equation, *Ann. Sc. Norm. Super. Pisa*, CI. Sci., 18 (2018), 1033–1060.
[5] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l' hydrodynamique des fluides parfaits, *Ann. Inst. Fourier*, 16 (1966), 316–361.
[6] D. Bakry, Etude des transformations de Riesz dans les variétés à courbure de Ricci minorée, *Sém. Probab.*, XXI, Lect. Notes in Math. 1247 (1987), 137-172.
[7] L.C. Berselli, D. Cordoba, On the regularity of the solutions to 3D Navier-Stokes equations: a remark on the role of the helicity, *Comptes Rendus Math.*, 347 (2009), 613-618.
[8] J.M. Bismut, *Mécanique aléatoire*, Lect. Notes in Maths, 866, Springer-Verlag, 1981.
[9] J.M. Bismut, *Large deviations and Malliavin calculus*, Birkhäuser, Prog. Math. 45 (1984).

[10] J. Y. Chemin, I. Gallagher, On the global wellposedeness of the 3-D Navier-Stokes equations with large initial data, *Ann. sci. École Norm. Sup.*, 39 (2006), 679-698.

[11] F. Cipriano, A.B. Cruzeiro, Navier-Stokes equations and diffusions on the group of homeomorphisms of the torus, *Comm. Math. Phys.* 275 (2007), 255–269.

[12] P. Constantin, G. Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations, *Comm. Pure Appl. Math.*, 61 (2008), 330–345.

[13] B. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact manifold, *J. Funct. Anal.*, 109 (1992), 272-376.

[14] B. Driver, A. Thalmaier, Heat equation derivative formulas for vector bundles, *J. Funct. Analysis*, 183 (2001), 42-108.

[15] D.G. Ebin, J.E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* 92 (1970), 102–163.

[16] K.D. Elworthy, *Stochastic differential equations on manifolds*, London Math. Soc. Lect. Note, 70, Cambridge university Press, 1982.

[17] J. Eells, K. D. Elworthy, Stochastic dynamical system, *Control theory and topics in functional analysis*, III, Intern. atomic energy agency, Vienna, 1976, 179-185.

[18] K.D. Elworthy, Y. Le Jan, X.M. Li, *On the geometry of diffusion operators and stochastic flows*, Lecture Notes in Mathematics, 1720, Springer-Verlag, 1999.

[19] K. D. Elworthy, X.M. Li, Bismut formulae for differential forms, *C. R. Acad. Sci. Paris*, 327 (1998), 87-92.

[20] S. Fang, Nash embedding, shape operator and Navier-Stokes equation on a Riemannian manifold, *Acta Math. Appl. Sin.*, Engl. Ser. 36 (2020), no. 2, 237-252.

[21] S. Fang, D. Luo, Constantin and Iyer's representation formula for the Navier-Stokes equations on manifolds, *Potential Analysis*, 48 (2018), 181–206.

[22] S. Fang, H. Li and D. Luo, Heat semi-group and generalized flows on complete Riemannian manifolds, *Bull. Sci. Math.*, 135 (2011), 565-600.

[23] S. Fang, P. Malliavin, Stochastic analysis on the path space of a Riemannian manifold, *J. Funct. Anal.*, 118 (1993), 249-274.

[24] S. Fang, Z. Qian, Vorticity, Helicity, Intrinsic geometry for Navier-Stokes equations, hal-02312072, arXiv: 1910.05175.2019.

[25] U. Frisch *Turbulence, the legacy of A.N. Kolmogorov*, Cambridge University Press, 1995.

[26] I. Gallagher, Le problème de Cauchy pour les équations de Navier-Stokes, *Facettes mathématiques de la mécanique des fluides*, 31-64, Edition Ecole Polytechnique, 2010.

[27] D. Holm, Y. Kimura, Zero-helicity Lagrangian kinematics of three-dimensional advection, *Physics of Fluids A: Fluid Dynamics 3*, 1033 (1991).

[28] E. Hsu, *Stochastic Analysis on Manifolds*, Graduate Studies in Math., 38 (2002), AMS.

[29] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Math. Library, 24, 1981.

[30] M. H. Kobayashi, On the Navier-Stokes equations on manifolds with curvature, *J Eng Math*, (2008) 60:55–68

[31] H. Kunita, *Stochastic flows and Stochastic differential equations*, Cambridge University Press, 1990.

[32] J. Li, Z. Qian, The vortex dynamics in incompressible viscous turbulent flows, arXive : 2101.00375v2.

[33] Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow*, Revised English Edition, Translated from the Russian by R. A. Silverman, Gordon and Breach (1963).

[34] Xiang-dong Li, On the strong L^p-Hodge decomposition over complete Riemannian manifolds, *J. Funct. Ana*, 257 (2009), 3617-3646.

[35] T. Lyons, Z. Qian, A class of vector fields on path spaces, *J. Funct. Anal.*, 145 (1997), 205-223.
[36] P. Malliavin, Formule de la moyenne, calcul des perturbations et théorie d'annulation pour les formes harmoniques, *J. Funt. Analysis*, 17 (1974), 274-291.
[37] M. Mitrea, M. Taylor, Navier-Stokes equations on Lipschitz domains in Riemannian manifolds, *Math. Ann.*, 321 (2001), 955–987.
[38] H. Keith Moffatt, Helicity and singular structure in fluid dynamics, *PNAS*, March 11, 2014, vol. 111, no 10, 3663-3670.
[39] T. Nagasawa, Navier-Stokes flow on Riemannian manifolds, *Nonlinear Analysis theory, Method and Applications*, 30 (1997), 825-832.
[40] Maxim A. Olshanskii, Leo G. Rebholz, Velocity-vorticity-helicity formulation and a solver for the Navier-Stokes equations, *Journal of Computational Physics*, 229 (2010) 4291–4303.
[41] V. Pierfelice, The incompressible Navier-Stokes equations on non-compact manifolds, *J. Geom. Anal.*, 27 (2017), 577–617.
[42] D. Stroock, *An introduction to the analysis of paths on a Riemannian manifold*, Math. Surveys and Monographs, 74. American Mathematical Society, Providence, RI, 2000.
[43] D. Stroock, S.R.S. Varadhan, *Multidimensional diffusion processes*, Grund. Math. Wissenschaften, 233 (1979), Springer.
[44] M. Taylor, *Partial Differential Equations III: Nonlinear Equations, Nonlinear equations*, Vol. 117, Applied Mathematical Sciences, Springer New York second edition (2011).
[45] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.
[46] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate texts in Math., 94 (1983), Springer-Verlag.

Shizan Fang: Institut de Mathématiques de Bourgogne,
UMR 5584 CNRS, Université de Bourgogne,
F-21078 Dijon, France

e-mail: Shizan.Fang@u-bourgogne.fr

Zhongmin Qian: Mathematical Institute,
University of Oxford, Oxford OX2 6GG, UK

e-mail: zhongmin.qian@maths.ox.ac.uk