REMARK ON THE ALEXANDER POLYNOMIALS OF PERIODIC KNOTS

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To the memory of Tamao Ozaki

Abstract. Let p be an odd prime number. We will show that the degree $p-1$ monic Alexander polynomials of p-periodic knots are uniquely determined by p.

Furthermore we will give an upper bound for the number of the degree $p-1$ Alexander polynomials of p-periodic knots under a certain condition on their leading coefficients.

1. Introduction

Let p be an odd prime number and K a non-trivial knot in S^3 which has period p, namely, there exists an orientation preserving homeomorphism $T : S^3 \longrightarrow S^3$ such that $T(K) = K$, the order of T is p, and the fixed point set of T is an unknot disjoint from K. Then the Alexander polynomial $\Delta_K(t)$ of such a knot must have some distinguished properties as Murasugi [3] has revealed.

In this short note, we will show that the degree $p-1$ monic Alexander polynomials of the p-periodic knots are uniquely determined by p. .

The proof will be done by applying number theory to Murasugi's work [3] on a necessary condition on $\Delta_K(t)$ for periodic knots K.

For a knot K in S^3, we denote by $\Delta_K(t) \in \mathbb{Z}[t]$ the Alexander polynomial of K normalized so that $\Delta_K(0) \neq 0$ and the leading coefficient is positive.

Our main results are the following:

Theorem 1. *Let p be an odd prime number and $K \subseteq S^3$ a non-trivial periodic knot of period p. If $\Delta_K(t)$ is monic and has degree $p-1$, then we have*

$$\Delta_K(t) = \sum_{n=0}^{p-1} (-1)^n t^n = t^{p-1} - t^{p-2} + \cdots - t + 1.$$

For the case where degree $p-1$ Alexander polynomials of p-periodic knots have general leading coefficients, we will show the following:

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Theorem 2. *Let p be an odd prime number and define $\Pi(p)$ to be the set of all the p-periodic knots in S^3 whose Alexander polynomial has degree $p-1$. Also, for a finite set S of prime numbers which does not contain p, we define the set $\mathcal{P}(p,S) \subseteq \mathbb{Z}[t]$ to be the collection of all the $\Delta_K(t)$'s such that $K \in \Pi(p)$ and that all the prime factors of the leading coefficient of $\Delta_K(t)$ are contained in S. Then $\mathcal{P}(p,S)$ is finite and*

$$\#\mathcal{P}(p,S) \leq 3 \cdot 7^{2(p-1)+2\#S(\mathbb{Q}(\zeta_p))},$$

where $S(\mathbb{Q}(\zeta_p))$ denotes the set of all the primes of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$ lying over the prime numbers in S.

Remark 1. (1) If K is fibered, then $\Delta_K(t)$ is monic.
(2) The polynomial $\sum_{n=0}^{p-1}(-1)^n t^n$ in Theorem 1 is in fact the Alexander polynomial of the torus knot $T(2,p)$, which is p-periodic.
(3) For a non-trivial knot K of prime period p, if the leading coefficient of $\Delta_K(t)$ is prime to p, then $\deg \Delta_K(t) \geq p-1$ (See Davis and Livingston[2, Cor. 4.2]).

2. Proof of Theorem 1.

Let T be an orientation preserving homeomorphism of S^3 of order p such that $T(K) = K$ and it acts on K fixed point freely, and that the set $B \subseteq S^3$ of the fixed points of T is an unknot. Then the quotient space S^3/T is homeomorphic to S^3, and we let $\overline{K}$ and $\overline{B}$ be the quotient knots of K and B in S^3/T, respectively. We write $\Delta_{\overline{K}\cup\overline{B}}(t,u) \in \mathbb{Z}[t,u]$ for the two-variable Alexander polynomial of the link $\overline{K}\cup\overline{B}$ normalized so that $\Delta_{\overline{K}\cup\overline{B}}(t,u) \notin t\mathbb{Z}[t,u] \cup u\mathbb{Z}[t,u]$ (Note that $\Delta_{\overline{K}\cup\overline{B}}(t,u)$ is defined up to ± 1). Let λ denote the linking number of K and B.

It follows from Murasugi [3] that

$$\Delta_K(t) = t^{-k}\Delta_{\overline{K}}(t)\prod_{i=1}^{p-1}\Delta_{\overline{K}\cup\overline{B}}(t,\zeta_p^i) \tag{1}$$

with some $0 \leq k \in \mathbb{Z}$ and a primitive p-th root of unity ζ_p. Also, by using Murasugi's congruence [3], we have

$$\Delta_K(t) \equiv \pm t^j \left(\frac{t^\lambda - 1}{t-1}\right)^{p-1} \Delta_{\overline{K}}(t)^p \pmod{p}$$

for some $j \in \mathbb{Z}$. Hence we derive $\lambda = 2$ and $\Delta_{\overline{K}}(t) = 1$. Indeed, since $\Delta_{\overline{K}}(t)$ is a factor of monic $t^k\Delta_K(t)$ in $\mathbb{Z}[t]$ by (1), we see that $\Delta_{\overline{K}}(t)$

is also monic. This implies $\deg(\Delta_{\overline{K}}(t)) = \deg(\Delta_{\overline{K}}(t) \mod p) = 0$ and $\lambda = 2$ (in fact we conclude $\pm t^j = 1$) because $\deg(\Delta_K(t) \mod p) = \deg(\Delta_K(t)) = p-1$ and $t \nmid \left(\left(\frac{t^\lambda - 1}{t-1} \right)^{p-1} \Delta_{\overline{K}}(t)^p \mod p \right)$ in $\mathbb{Z}/p\mathbb{Z}[t]$. Hence we may assume that

$$\Delta_{\overline{K}\cup\overline{B}}(t, \zeta_p^i) = t^d(g(\zeta_p^i)t - h(\zeta_p^i)) \tag{2}$$

with $0 \leq d \in \mathbb{Z}$ such that $(p-1)d = k$ and some $g(u), h(u) \in \mathbb{Z}[u]$ for $1 \leq i \leq p-1$ by (1).

Because $\Delta_K(t)$ is monic, we see that $\eta_1 := g(\zeta_p)$ is a unit of the ring $\mathbb{Z}[\zeta_p]$ by the relation $\prod_{i=1}^{p-1} g(\zeta_p^i) = 1$, which comes from (1). Also, since the constant term of $\Delta_K(t)$ is equal to 1, we find that $\eta_2 := h(\zeta_p)$ is also a unit of $\mathbb{Z}[\zeta_p]$. Hence if we put $\varepsilon := \eta_1^{-1}\eta_2 \in \mathbb{Z}[\zeta_p]^\times$, then we have

$$\Delta_K(t) = \prod_{i=1}^{p-1} g(\zeta_p^i) \prod_{i=1}^{p-1} (t - g(\zeta_p^i)^{-1} h(\zeta_p^i)) = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} (t - \sigma(\varepsilon)). \tag{3}$$

On the other hand, it follows from the second Torres condition (See [4]) that

$$\Delta_{\overline{K}\cup\overline{B}}(t^{-1}, \zeta_p^{-1}) = t^a \zeta_p^b \Delta_{\overline{K}\cup\overline{B}}(t, \zeta_p) \tag{4}$$

for some $a, b \in \mathbb{Z}$. Then we find from (2) and (4) that $a = -1 - 2d$ and

$$h(\zeta_p^{-1}) = -\zeta_p^b g(\zeta_p), \ g(\zeta_p^{-1}) = -\zeta_p^b h(\zeta_p).$$

Therefore, if we denote by $J \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ the complex conjugation, we have

$$J(\varepsilon) = J(h(\zeta_p))/J(g(\zeta_p)) = h(\zeta_p^{-1})/g(\zeta_p^{-1}) = g(\zeta_p)/h(\zeta_p) = \varepsilon^{-1}. \tag{5}$$

We need the following fact from the theory of cyclotomic fields (see [5, Prop.1.5]):

Lemma 1. *For any $\varepsilon \in \mathbb{Z}[\zeta_p]^\times$, there exist $r \in \mathbb{Z}$ and $\varepsilon_0 \in \mathbb{Z}[\zeta_p + \zeta_p^{-1}]^\times$ such that*

$$\varepsilon = \zeta_p^r \varepsilon_0.$$

By Lemma 1, our $\varepsilon = \eta_1^{-1}\eta_2$ is expressed as $\varepsilon = \zeta_p^r \varepsilon_0$ by some $r \in \mathbb{Z}$ and $\varepsilon_0 \in \mathbb{Z}[\zeta_p + \zeta_p^{-1}]^\times$. Then we obtain

$$\varepsilon^{-1} = J(\varepsilon) = J(\zeta_p)^r J(\varepsilon_0) = \zeta_p^{-r} \varepsilon_0 = \zeta_p^{-2r} \varepsilon,$$

by (5), from which we derive

$$\varepsilon = \pm \zeta_p^r.$$

Because $\Delta_K(\zeta_p^r) \equiv \Delta_K(1) = \pm 1 \not\equiv 0 \pmod{(\zeta_p - 1)}$ and $\Delta_K(-1) \equiv \Delta_K(1) = \pm 1 \not\equiv 0 \pmod 2$, we conclude that $\varepsilon = -\zeta_p^r$ with some $r \in \mathbb{Z}$ prime to p, and

$$\Delta_K(t) = t^{p-1} - t^{p-2} + \cdots - t + 1$$

by (3). Thus we have proved Theorem 1. □

3. Proof of Theorem 2

First we will give the following theorem to prove Theorem 2:

Theorem 3. *Let $F/\mathbb{Q}$ be a finite extension field and S a finite set of prime numbers. Define $\mathbf{P}(S) \subseteq \mathbb{Z}[t]$ to be the set of all the palindromic polynomials $f(t)$ in $\mathbb{Z}[t]$ such that $f(1) = \pm 1$ and that all the prime factors of the leading coefficient of $f(t)$ are contained S. Furthermore put*

$$\mathcal{R}(F, S) := \{\alpha \in F \mid f(\alpha) = 0 \text{ for a certain } f(t) \in \mathbf{P}(S)\}.$$

Then we have

$$\#\mathcal{R}(F, S) \leq 3 \cdot 7^{[F:\mathbb{Q}]+2\#(S(F)\cup\infty(F))},$$

where $S(F)$ and $\infty(F)$ denote the set of the primes of F lying over the prime numbers in S and that of the archimedean primes of F, respectively.

Proof. Let $\alpha \in \mathcal{R}(F, S)$ be any element. Then there exists $f(t) \in \mathbf{P}(S)$ such that $f(\alpha) = 0$. We have

$$f(t) = a\prod_{i=1}^{d}(t - \alpha_i) \in \mathbb{Z}[t]$$

for a certain $a \in \mathbb{Z}$ whose prime factors are contained in S, and $\alpha_i \in \overline{\mathbb{Q}}$ with $\alpha_1 = \alpha$.

Let v be any additive valuation of $\overline{\mathbb{Q}}$ associated to a non-archimedean prime of $\overline{\mathbb{Q}}$ lying over a prime number outside S. Since $a\alpha_i$ is integral over $\mathbb{Z}$ and $v(a) = 0$, we see

$$v(\alpha_i) = v(a\alpha_i) \geq 0. \tag{6}$$

Because $f(t)$ is a palindromic polynomial, we have

$$\prod_{i=1}^{d}\alpha_i = \pm 1,$$

from which we derive

$$\sum_{i=1}^{d} v(\alpha_i) = 0. \tag{7}$$

Then it follows from (6) and (7) that $v(\alpha_i) = 0$ for $1 \leq i \leq d$, which means that $\alpha_1 = \alpha \in F$ is an $S(F)$-unit of F.

On the other hand, because $a(1-\alpha_i) = a - a\alpha_i$ is also integral over $\mathbb{Z}$, we obtain

$$v(1-\alpha_i) = v(a(1-\alpha_i)) \geq 0.$$

Also, since $f(1) = \pm 1$, we have

$$a\prod_{i=1}^{d}(1-\alpha_i) = \pm 1,$$

from which we derive

$$\sum_{i=1}^{d} v(1-\alpha_i) = v(\pm 1) = 0.$$

Hence $v(1-\alpha_i) = 0$ for $1 \leq i \leq d$, which means that $1-\alpha = 1-\alpha_1$ is also an $S(F)$-unit of F.

Now we apply the following result from analytic number theory given by Evertse [1]:

Theorem 4 (Evertse [1])**.** *Let F be a finite extension of $\mathbb{Q}$ and T a finite set of non-archimedean primes of F. Then the number of solutions (X, Y) of the equation*

$$X + Y = 1$$

in the T-unit group of F is at most $3 \cdot 7^{[F:\mathbb{Q}]+2\#(T\cup\infty(F))}$.

As we have seen in the above, $(\alpha, 1-\alpha)$ is a solution of the equation $X + Y = 1$ in the $S(F)$-unit group of F. Hence, it follows from Theorem 4 that the number of such α's is at most $3 \cdot 7^{[F:\mathbb{Q}]+2\#(S(F)\cup\infty(F))}$. Therefore we obtain

$$\#\mathcal{R}(F, S) \leq 3 \cdot 7^{[F:\mathbb{Q}]+2\#(S(F)\cup\infty(F))}.$$

□

Now we will derive Theorem 2 from Theorem 3. Assume that $\Delta_K(t) \in \mathcal{P}(p, S)$. Then, as in the proof of Theorem 1, we find that

$$\Delta_K(t) = \prod_{i=1}^{p-1}(g(\zeta_p^i)t - h(\zeta_p^i)) \tag{8}$$

for some polynomials $g(u), h(u) \in \mathbb{Z}[u]$, since the leading coefficient of $\Delta_K(t)$ is prime to p by $p \notin S$ and $\deg \Delta_K(t) = p-1$.

Hence all the $p-1$ roots of $\Delta_K(t)$ are contained in $\mathcal{R}(\mathbb{Q}(\zeta_p), S)$, since $\Delta_K(t) \in \mathbf{P}(S)$. Hence it follows from (8) that

$$\Delta_K(t) = a \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} (t - \sigma(\alpha))$$

holds for a certain $\alpha \in \mathcal{R}(\mathbb{Q}(\zeta_p), S)$ and $a = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} (1-\sigma(\alpha))^{-1}$.

Therefore, applying Theorem 3, we have

$$\#\mathcal{P}(p, S) \leq \#\mathcal{R}(\mathbb{Q}(\zeta_p), S) \leq 3 \cdot 7^{2(p-1)+2\#(S(\mathbb{Q}(\zeta_p)))}$$

because $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p-1$ and $\#\infty(\mathbb{Q}(\zeta_p)) = \frac{p-1}{2}$. □

Example 1. Assume that $p = 3$ and $S = \{2, 7\}$. Then we see that $S(\mathbb{Q}(\zeta_3)) = \{2\mathbb{Z}[\zeta_3], (2+\sqrt{-3})\mathbb{Z}[\zeta_3], (2-\sqrt{-3})\mathbb{Z}[\zeta_3]\}$. Hence we have

$$\#\mathcal{P}(3, \{2, 7\}) \leq 3 \cdot 7^{2\cdot(3-1)+2\cdot 3} = 3 \cdot 7^{10}$$

by Theorem 2.

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