EFFICIENT KNOT INVARIANTS FROM QUANDLES

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Abstract. We investigate the efficiencies and the relations of quandle invariants for knots. For example, we see that for any finitely generated connected quandle X, there exists a knot diagram which admits a surjective X-coloring. Also, we show the equivalence of shadow cocycle invariants and 3-cocycle invariants and the independence of homotopy invariants and non-abelian cocycle invariants.

1. Introduction

Since quandles were introduced in [11] and [12], various knot invariants have been defined and used to investigate knots and links, e.g., quandle cocycle invariants [1] and quandle homotopy invariants [13]. Invariants from quandles can be calculated from diagrams and are so powerful that the fundamental invariant, knot quandles, distinguishes the oriented knots (in a weak sense).

However, one has rarely cared about the efficiencies of such invariants. For example, it is well known that a surjective coloring of a knot by a quandle X exists only if X is connected, but is the converse true? Also, the quandle homotopy invariant can take any value of $\pi_2(B^Q X)$, but does this hold also when restricted to colorings of knots? In the case of a group G, a condition for a pair $(m, \ell) \in G^2$ to admit a G-representation sending a meridian-longitude pair to (m, ℓ) was given in [10], but a similar result for quandles is not known.

In this paper, we show the efficiency of quandle invariants for knots and study the relationships between them. In particular, Theorem 4.1 shows that the shadow homology invariant on (X, X) of a quandle X is equivalent to the third homology invariant. Furthermore, we prove the independence of homotopy invariants and non-abelian invariants under some obvious restrictions (Theorem 5.8). We should remark that homology invariants have the universalities over cocycle invariants and hence we can easily rewrite results on homology invariants in terms of cocycle invariants.

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2. Quandles and knot invariants

A *quandle* is a set X equipped with a binary operation $*$ satisfying

- (Q1) we have $x * x = x$ for any $x \in X$;
- (Q2) the map $s_x : X \ni a \mapsto a * x \in X$ is bijective for any $x \in X$; and
- (Q3) we have $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

In this paper, we denote $s_y^{-1}(x)$ by $x \mathbin{\overline{*}} y$. For a subset S of X, the quandle *generated* by S is the minimum subquandle containing S.

The *associated group* $\mathrm{As}(X)$ of a quandle X is defined by

$$\mathrm{As}(X) = \langle X \mid e_x e_y = e_y e_{x*y} \text{ for } x, y \in X \rangle,$$

where e_x denotes the generator corresponds to $x \in X$. We call a set with a right action of $\mathrm{As}(X)$ an X-*set*. For example, we can regard X itself as an X-set by setting $a \cdot e_x = a * x$ for $a, x \in X$. An orbit of this action on X is called a *connected component*, and the quandle X is said to be *connected* if the action is transitive.

Let us review homology theory on quandles briefly; see, e.g., [1, 14] for details. Let X be a quandle and Y an X-set. Let $C_n^R(X, Y)$ be the free abelian group generated by $Y \times X^n$. Boundary operators $\partial : C_n^R(X, Y) \to C_{n-1}^R(X, Y)$ are defined to make $(C_\bullet^R(X, Y), \partial)$ a chain complex, and subgroups $C_n^D(X, Y)$ of $C_n^R(X, Y)$ defined by $C_0^D(X, Y) = C_1^D(X, Y) = 0$ and

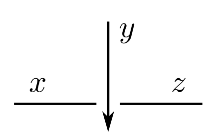
$$C_n^D(X, Y) = \mathrm{span}_{\mathbb{Z}}\{(y, x_1, \ldots, x_n) \in C_n^R(X, Y) \mid x_i = x_{i+1} \text{ for some } i\}$$

for $n \geq 2$ form a subcomplex. Denote the quotient $C_n^R(X, Y)/C_n^D(X, Y)$ by $C_n^Q(X, Y)$ and define *quandle homology groups* $H_\bullet^Q(X, Y)$ of (X, Y) to be the homology groups of the quotient complex $(C_\bullet^Q(X, Y), \partial)$. In this paper, we omit the coefficient groups of homology groups when they are $\mathbb{Z}$. If the X-set Y is a single point with the trivial action, we omit the symbol Y; i.e., we denote $H_\bullet^Q(X, \{\mathrm{pt.}\})$ by $H_\bullet^Q(X)$, simply.

3. Surjective colorings

In this paper, knots and links are considered in S^3 and always oriented.

Let X be a quandle. An X-*coloring* of a link diagram D is a map $\mathcal{C}$ which assigns an element $\mathcal{C}(x) \in X$ to each arc x of D satisfying

$\mathcal{C}(x) * \mathcal{C}(y) = \mathcal{C}(z)$ at any crossing 

of D.

In fact, the *fundamental quandle* Q_L of the link L represented by D is defined and there exists a one-to-one correspondence between the set of the X-colorings of D and the set of the quandle homomorphisms from Q_L to X; see, e.g., [4, 14] for details. We shall say that an X-coloring $\mathcal{C}$ is *surjective* if it is surjective as the quandle homomorphism, i.e., if X is generated by the colors $\mathcal{C}(x)$ of the arcs x.

Proposition 3.1. *Let X be a quandle. There exist a link (resp. knot) diagram D and a surjective X-coloring on D if and only if X is finitely generated (resp. finitely generated and connected).*

Proof. The "only-if" part immediately follows from the fact that the fundamental quandle is finitely generated and if the link is a knot it is connected.

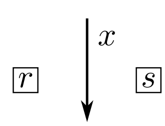
To see the "if" part, let $S = \{x_1, \ldots, x_n\}$ be a finite generating set of X. We take a trivial n-component link diagram D_0 and let $\mathcal{C}_0$ be the X-coloring which assigns x_i to the i-th component. By the definition, $\mathcal{C}_0$ is surjective.

Further, we assume X to be connected. For $i = 2, \ldots, n$, we take a ribbon r_i connecting the first and the i-th components as follows. We first take $g \in \mathrm{As}(X)$ so that $x_1 \cdot g = x_i$. Since X is generated by S, we can express g as $e_{x_{j_1}}^{\epsilon_1} \cdots e_{x_{j_k}}^{\epsilon_k}$, where $\epsilon_1, \ldots, \epsilon_k = \pm 1$. We define r_i by starting at the first component, passing through the j_1-, ..., j_k-th components in the directions of $\epsilon_1, \ldots, \epsilon_k$ in this order, and ending at the i-th component. We assume $r_2, \ldots, r_n$ to be disjoint. Define a knot K by taking the ribbon-sum. By the definition of the ribbons, the corresponding diagram D of K admits a coloring $\mathcal{C}$ extending $\mathcal{C}_0$. Since the image of $\mathcal{C}$ contains S, $\mathcal{C}$ is surjective. $\square$

4. 3-cocycle invariants vs. shadow 2-cocycle invariants

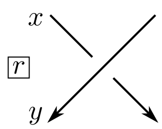
Let X be a quandle and Y an X-set. Let us consider an X-coloring $\mathcal{C}$ on a link diagram D and a map $\mathcal{R} : \{\text{regions of } D\} \to Y$. If $\mathcal{R}$ satisfies a condition

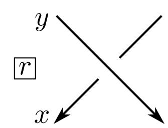
$$\mathcal{R}(r) \cdot e_{\mathcal{C}(x)} = \mathcal{R}(s) \quad \text{for any arc}$$



then the pair $(\mathcal{C}, \mathcal{R})$ is called a *shadow (X, Y)-coloring* of D. We should remark that the whole region coloring $\mathcal{R}$ is uniquely determined by the coloring $\mathcal{C}$ of arcs and the color $\mathcal{R}(r)$ of a single region r. Regarding X itself as an X-set, we simply call a shadow (X, X)-coloring a *shadow X-coloring.*

Let $\mathcal{S} = (\mathcal{C}, \mathcal{R})$ be a shadow (X, Y)-coloring. For each crossing point p, we define a weight $\Phi_{X,Y}(\mathcal{S}, p) \in C_2^Q(X, Y)$ by

$\Phi_{X,Y}\Big(\mathcal{S},$  $\Big) = (\mathcal{R}(r), \mathcal{C}(x), \mathcal{C}(y)),$

$\Phi_{X,Y}\Big(\mathcal{S},$  $\Big) = -(\mathcal{R}(r), \mathcal{C}(x), \mathcal{C}(y)).$

Then, the sum of the weights over all crossings can be checked to be a cycle and we denote the homology class by $\Phi_{X,Y}(\mathcal{S})$, i.e.,

$$\Phi_{X,Y}(\mathcal{S}) = \left[\sum_p \Phi_{X,Y}(\mathcal{S}, p)\right] \quad \in H_2^Q(X, Y).$$

In particular, denote $\Phi_{X,\{\mathrm{pt.}\}}(\mathcal{S})$ by $\Phi_2(\mathcal{C}) \in H_2^Q(X)$ and $\Phi_{X,X}(\mathcal{S})$ by $\Phi_X(\mathcal{S}) \in H_2^Q(X, X)$. Furthermore, we should remark that the quotient map $C_2^Q(X, X) \to C_3^Q(X)$ induces a homomorphism $p_* : H_2^Q(X, X) \to H_3^Q(X)$; let us denote $p_*\Phi_X(\mathcal{S})$ by $\Phi_3(\mathcal{S}) \in H_3^Q(X)$.

The homomorphism $H_2^Q(X, X) \to H_2^Q(X)$ induced by forgetting the first coordinate takes $\Phi_X(\mathcal{S})$ to $\Phi_2(\mathcal{C})$, and [6] defines a shifting chain homomorphism $H_3^Q(X) \to H_2^Q(X)$ which sends $\Phi_3(\mathcal{S})$ to $\Phi_2(\mathcal{C})$: Both Φ_X and Φ_3 have the universalities over Φ_2. By the definition p_* takes Φ_X to Φ_3, and here we show that this preserves the information of the invariant:

Theorem 4.1. *Let X be a quandle. Then, there exists a homomorphism $q : H_3^Q(X) \to H_2^Q(X, X)$ such that $q(\Phi_3(\mathcal{S})) = \Phi_X(\mathcal{S})$ holds for any shadow X-coloring $\mathcal{S}$ of any link diagram.*

In order to prove Theorem 4.1, let us recall quandle spaces. For a qundle X and an X-set Y, the *rack space* [5] $B_Y X$ is defined as the quotient of $\bigsqcup_{n=0}^{\infty}[0, 1]^n \times Y \times X^n$ by a certain relation $\sim_R$, where the sets X and Y are equipped with the discrete topologies; see the original paper [5] for details. We consider the equivalence relation $\sim_Q$ on $B_Y X$ generated by

$$\begin{aligned}&(t_1, \dots, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \dots, t_n; y; x_1, \dots, x_{i-1}, x_i, x_i, x_{i+2}, \dots, x_n)\\ &\sim_Q (t_1, \dots, t_{i-1}, t'_i, t'_{i+1}, t_{i+2}, \dots, t_n; y; x_1, \dots, x_{i-1}, x_i, x_i, x_{i+2}, \dots, x_n)\end{aligned}$$

for $y \in Y, x_j \in X$, and $t_j, t'_i, t'_{i+1} \in [0, 1]$ such that $t_i + t_{i+1} = t'_i + t'_{i+1}$, and let $B_Y^Q X$ denote the quotient space $B_Y X/\sim_Q$, which is called the *quandle space*. We denote $B_{\{\mathrm{pt.}\}}X$ by BX and $B^Q_{\{\mathrm{pt.}\}}X$ by $B^Q X$; we should remark that for general Y, $B_Y^Q X$ is a covering space of $B^Q X$.

In the case where $Y = X$, we can define a continuous free action of $\mathbb{R}/\mathbb{Z}$ on $B^Q_X X$ by

$$t \cdot (t_1, \ldots, t_n; x; x_1, \ldots, x_n) = (t, t_1, \ldots, t_n; x; x, x_1, \ldots, x_n).$$

Let us denote the quotient space by $\hat{B}X$ and call it the *extended quandle space*. Quandle spaces $B^Q X$ and $\hat{B}X$ were originally introduced in [13] and [15], and the definitions here follows [9] and [8].

For an X-coloring $\mathcal{C}$ of a link diagram D, a map $\xi : S^2 \to B^Q X$ is defined and the homotopy class $\Xi(\mathcal{C}) \in \pi_2(B^Q X)$ of ξ is called the *quandle homotopy invariant*, which is invariant under Reidemeister moves of the diagram. For a shadow (X, Y)-coloring $\mathcal{S} = (\mathcal{C}, \mathcal{R})$, let $\Xi(\mathcal{S}) \in \pi_2(B^Q_Y X, r)$, where $r \in Y$ is the color of the unbounded region and also denotes the corresponding vertex of $B^Q_Y X$, be the homotopy class of the lift $\tilde{\xi} : S^2 \to B^Q_Y X$ of ξ. Similarly, the *shadow homotopy invariant* $\hat{\Xi}(\mathcal{S}) \in \pi_2(\hat{B}X, r)$ is defined for a shadow X-coloring $\mathcal{S}$ with $r \in X$ being the color of the unbounded region. We refer the original papers [13] and [15] for details.

We regard $\mathbb{Z} \times X$ as an X-set by setting $(a, x) \cdot e_y = (a + 1, x * y)$. In [8], we showed a commutative diagram

$$\begin{array}{ccc} B^Q_{\mathbb{Z}\times X} X & \xrightarrow[\cong]{\tilde{s}\times(p\circ p_Q)} & \mathbb{R} \times \hat{B}X \\ \downarrow & & \downarrow \\ B^Q_X X & \xrightarrow[\cong]{s\times p} & S^1 \times \hat{B}X \end{array}$$

with homeomorphic rows, where $\tilde{s} : B^Q_{\mathbb{Z}\times X} X \to \mathbb{R}$ and $s : B^Q_X X \to S^1 = \mathbb{R}/\mathbb{Z}$ are continuous functions, $p : B^Q_X X \to \hat{B}X$ is the quotient map, and $p_Q : B^Q_{\mathbb{Z}\times X} X \to B^Q_X X$ is the covering map.

As in [2], we can introduce a structure of a topological monoid on the quandle space $B^Q X$ and hence the action of the fundamental group on the homotopy groups is trivial. As seen above, the extended quandle space $\hat{B}X$ is homotopy equivalent to a covering space of $B^Q X$. Thus, each connected component of $\hat{B}X$ has the higher homotopy groups isomorphic to $\pi_n(B^Q X)$ and the action of fundamental group on them is trivial.

Proposition 4.2. *Let $\mathcal{S} = (\mathcal{C}, \mathcal{R})$ be a shadow X-coloring on any link diagram and let $r \in X$ be the color of the unbounded region. Then, identifying $\pi_2(\hat{B}X, r)$ and $\pi_2(B^Q X)$ as above, we have $\hat{\Xi}(\mathcal{S}) = \Xi(\mathcal{C})$.*

Proof. We take a continuous map $\xi : S^2 \to B^Q X$ representing $\Xi(\mathcal{C})$ and the lift $\tilde{\xi} : S^2 \to B^Q_X X$ as in the definitions of $\Xi(\mathcal{C})$ and $\Xi(\mathcal{S})$. By definition, we have $[p \circ \tilde{\xi}] = \hat{\Xi}(\mathcal{S}) \in \pi_2(\hat{B}X, r)$ and this concludes the proposition since the identification $\pi_2(\hat{B}X, r) \cong \pi_2(B^Q X)$ is given by the homomorphisms induced by $p : B^Q_X X \to \hat{B}X$ and the covering map $B^Q_X X \to B^Q X$. □

Quandle spaces $B^Q_Y X$ and $\hat{B}X$ are equipped with structures of CW-complexes and have (shifted) quandle homologies. In fact, their cellular complexes satisfy

$$(C_n(B^Q_Y X), \partial)_n \cong (C^Q_n(X, Y), \partial)_n, \quad (C_n(\hat{B}X), \partial)_n \cong (C^Q_{n+1}(X), -\partial)_n,$$

and hence $H_n(B^Q_Y X) \cong H^Q_n(X, Y)$, $H_n(\hat{B}X) \cong H^Q_{n+1}(X)$. Furthermore, the Hurewicz homomorphisms $h_Y : \pi_2(B^Q_Y X) \to H^Q_2(X, Y)$, $\hat{h} : \pi_2(\hat{B}X) \to H^Q_3(X)$ take the homotopy invariants $\Xi(\mathcal{S}), \hat{\Xi}(\mathcal{S})$ to the homology invariants $\Phi_{X,Y}(\mathcal{S}), \Phi_3(\mathcal{S})$.

Proof of Theorem 4.1. Let $q_0 : \hat{B}X \to B^Q_{\mathbb{Z}\times X} X$ be the composite

$$\hat{B}X \xrightarrow{\{0\}\times \mathrm{id}} \mathbb{R} \times \hat{B}X \xrightarrow{(\tilde{s}\times(p\circ p_Q))^{-1}} B^Q_{\mathbb{Z}\times X} X,$$

which is homotopy equivalent as recalled above, and define

$$q = (p_Q \circ q_0)_* : H^Q_3(X) \cong H_2(\hat{B}X) \to H_2(B^Q_X X) \cong H^Q_2(X, X).$$

Since $p \circ p_Q \circ q_0 = \mathrm{id}_{\hat{B}X}$, the induced homomorphism $(q_0)_* : \pi_2(\hat{B}X) \to \pi_2(B^Q_{\mathbb{Z}\times X} X) \cong \pi_2(B^Q X)$ is isomorphic and gives the identification of Proposition 4.2. Then, the proposition shows that $(q_0)_* \hat{\Xi}(\mathcal{S}) = \Xi(\mathcal{S})$ for any shadow X-coloring $\mathcal{S}$. The universalities of homotopy invariants over homology invariants recalled above and the naturality of Hurewicz homomorphisms prove that

$$q(\Phi_3(\mathcal{S})) = (p_Q \circ q_0)_* \hat{h}(\hat{\Xi}(\mathcal{S})) = h_X((p_Q \circ q_0)_* \hat{\Xi}(\mathcal{S})) = h_X(\Xi(\mathcal{S})) = \Phi_X(\mathcal{S}),$$

as required. □

Remark 4.3. By using the cellular approximation of q_0 given in [8], we obtain a concrete description of $q : H^Q_3(X) \to H^Q_2(X, X)$ as follows:

$$q(x, y, z) = (x, y, z) - (x, x, z) + (x, x, y).$$

Then, for a shadow 2-cocycle $\phi : X^3 \to A$ valued on an abelian group A, the map $\phi' : X^3 \to A$ defined by $\phi'(x, y, z) = \phi(x, y, z) - \phi(x, x, z) + \phi(x, x, y)$ is a quandle 3-cocycle and $\phi(\Phi_X(\mathcal{S})) = \phi'(\Phi_3(\mathcal{S}))$ holds for any shadow X-coloring $\mathcal{S}$.

5. Efficiency of invariants

In this section, we see that invariants from a finitely generated connected quandles are efficient for knots. The efficiency of homotopy invariants and 3-(co)cycle invariants are shown in Section 5.1, and nonabelian cycle invariants are recalled and investigated in Section 5.2. Finally, Section 5.3 shows the independence of them, i.e., the efficiency of the product invariants.

5.1. **Efficiency of homotopy/3-cocycle invariants.** First, let us show the efficiency of homotopy invariants:

Proposition 5.1. *Let X be a quandle. Then, for any $\xi \in \pi_2(B^Q X)$, there exists an X-coloring $\mathcal{C}$ on a link diagram D such that $\Xi(\mathcal{C}) = \xi$. If X is finitely generated and connected, we may assume D to be a knot diagram and $\mathcal{C}$ surjective. Furthermore, the same claims hold for $\hat{\Xi}$.*

Proof. The general case is well known: We take a continuous map $f : S^2 \to B^Q X$ representing $\xi \in \pi_2(B^Q X)$ and, by a cellular approximation of f, find a link diagram D_0 with an X-coloring $\mathcal{C}_0$ which represents ξ, i.e., $\Xi(\mathcal{C}_0) = \xi$.

Let us assume X to be finitely generated and connected. We take a knot diagram D_1 with a surjective X-coloring $\mathcal{C}_1$ as in the proof of Proposition 3.1, and then we can easily check that $\Xi(\mathcal{C}_1) = 0$. Let D_2 be a disjoint union $D_0 \sqcup D_1$, where an X-coloring $\mathcal{C}_2$ is defined from $\mathcal{C}_0$ and $\mathcal{C}_1$. As in the proof of Proposition 3.1, we can take ribbons from D_1 to the components of D_0 so that the ribbon-sum D is a knot diagram which admits an X-coloring $\mathcal{C}$ which extends $\mathcal{C}_2$. Since a concordance with an X-coloring such as this ribbon-sum does not change the representing homotopy class, we have $\Xi(\mathcal{C}) = \xi$ as required.

The version of $\hat{\Xi}$ is now just a corollary of Proposition 4.2. □

Remark 5.2. The assumption of finite generation is necessary. For example, let X be the connected quandle $(\mathbb{Q}, *)$ defined by $x * y = 2y - x$. Since a link diagram admits a nontrivial X-coloring if and only if the determinant of the link equals zero (e.g., see [7]), every knot coloring is trivial and the homotopy invariant is zero. These are also true for X^2, but the homotopy group $\pi_2(B^Q X^2)$ is nontrivial: A map $\phi : X^2 \times X^2 \to \mathbb{Q}$ defined by $\phi((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$ is a quandle cocycle and we can easily find an X^2-coloring $\mathcal{C}$ of the $(4, 4)$-torus link such that $\phi(\Phi_2(\mathcal{C})) \neq 0$.

Let X be a quandle. Let $\mathcal{C}^r$ denote the shadow X-coloring determined by an X-coloring $\mathcal{C}$ of a diagram and the color $r \in X$ of the unbounded region. Since the action of $\pi_1(\hat{B}X)$ on $\pi_2(\hat{B}X)$ is trivial and $\hat{\Xi}$ is universal over Φ_3, the homology invariants $\Phi_3(\mathcal{C}^r)$ and $\Phi_3(\mathcal{C}^s)$ have the same value if r and s belong to the same connected component. In particular, if X is connected $\Phi_3(\mathcal{C}^r)$ does not depend on $r \in X$ and then we can denote it by $\Phi_3(\mathcal{C})$.

We shall assume X to be connected for the simplicity. Since $H_2(\hat{B}X) \cong H_3^Q(X)$, there exists a classifying homomorphism $c_* : H_3^Q(X) \to H_2(\pi_1(\hat{B}X))$, where $H_2(\pi_1(\hat{B}X))$ expresses the group homology with integer coefficient. A homology class $\phi \in H_3^Q(X)$ is said to be *realizable* if $c_*\phi = 0 \in H_2(\pi_1(\hat{B}X))$.

Proposition 5.3. *Let X be a finitely generated connected quandle and $\phi \in H_3^Q(X)$ a homology class. Then, there exists a pair $(D, \mathcal{C})$ of a knot diagram D and an X-coloring $\mathcal{C}$ of D such that $\Phi_3(\mathcal{C}) = \phi$ if and only if ϕ is realizable.*

Proof. By the five-term exact sequence of the Cartan-Leray spectral sequence for the universal covering of $\hat{B}X$, we find that

$$\pi_2(\hat{B}X) \xrightarrow{h_*} H_3^Q(X) \xrightarrow{c_*} H_2(\pi_1(\hat{B}X))$$

is exact, where h_* is the Hurewicz homomorphism. Since $\Phi_3(\mathcal{C}) = h_*\hat{\Xi}(\mathcal{C})$, we have $c_*\Phi_3(\mathcal{C}) = 0$, i.e., $\Phi_3(\mathcal{C})$ is realizable.

Conversely, if a homology class $\phi \in H_3^Q(X)$ is realizable, the exact sequence above shows that there exists $\xi \in \pi_2(\hat{B}X)$ such that $h_*\xi = \phi$. By Proposition 5.1 there exists an X-coloring $\mathcal{C}$ on a knot diagram such that $\hat{\Xi}(\mathcal{C}) = \xi$, which implies that $\Phi_3(\mathcal{C}) = h_*\hat{\Xi}(\mathcal{C}) = h_*\xi = \phi$, as required. □

5.2. **Non-abelian cocycle invariants.** Let X be a quandle and take a base point $x_0 \in X$. Following [3], we define the *fundamental group* $\pi_1(X, x_0)$ of (X, x_0) by

$$\pi_1(X, x_0) = \{g \in \mathrm{As}(X) \mid x_0 \cdot g = x_0,\ \varepsilon(g) = 0\},$$

where $\varepsilon : \mathrm{As}(X) \to \mathbb{Z}$ is the group homomorphism which maps the generators e_x to $1 \in \mathbb{Z}$.

It is known in [3] that the abelianization $\pi_1(X, x_0)_{\mathrm{ab}}$ is isomorphic to $H_2^Q(X)$ if the quandle X is connected. We can show this by checking $\pi_1(X, x_0) \cong \pi_1(\hat{B}X, x_0)$. In the following, we denote the image of $\lambda \in \pi_1(X, x_0)$ under the abelianization by $\lambda_{\mathrm{ab}} \in H_2^Q(X)$.

An X-coloring $\mathcal{C}$ on a diagram D of a knot K defines a group homomorphism from the knot group π_K to $\mathrm{As}(X)$ and then the image of a preferred longitude is contained in the fundamental group of X. Explicitly, we take a base point p_0 in an arc γ_0 of D and let $x_0 \in X$ be

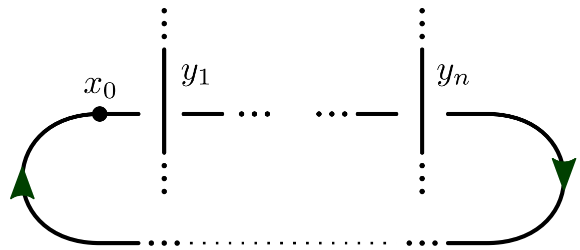


FIGURE 1. Definition of Λ

the color of γ_0. Starting at p_0, we go along K to come back to p_0; in this process, let y_i denote the color of the over-arc at the i-th crossing under which we pass, as illustrated in Figure 1. Then, we set

$$\Lambda(\mathcal{C}) = e_{x_0}^{-\sum_{i=1}^{n}\epsilon_i} e_{y_1}^{\epsilon_1} \cdots e_{y_n}^{\epsilon_n} \quad \in \mathrm{As}(X),$$

where n is the number of the crossings and $\epsilon_i = \pm 1$ is the sign of the i-th crossing. We can easily check that $\Lambda(\mathcal{C})$ belongs to the fundamental group $\pi_1(X, x_0)$ and call $\Lambda(\mathcal{C})$ the *non-abelian cycle invariant*. By the definitions, we can easily find $\Lambda(\mathcal{C})_{\mathrm{ab}} = \Phi_2(\mathcal{C}) \in H_2^Q(X)$.

Remark 5.4. Since $\Lambda(\mathcal{C})$ depends on the choice of the arc γ_0 of D, we should consider its conjugacy class in $\mathrm{As}(X)$ when we regard Λ as an invariant. However, for any $x_0 \in X$ belonging to the same connected component as $\mathcal{C}(\gamma_0)$, we can take $g \in \mathrm{As}(X)$ such that $\mathcal{C}(\gamma_0) \cdot g = x_0$ to find $g^{-1}\Lambda(\mathcal{C})g \in \pi_1(X, x_0)$. Furthermore, the conjugacy class of $g^{-1}\Lambda(\mathcal{C})g$ in $\pi_1(X, x_0)$, as well as in $\mathrm{As}(X)$, does not depend on the choice of g. Thus, if X is connected, we may consider $\Lambda(\mathcal{C})$ as in $\pi_1(X, x_0)$ for an arbitrarily fixed $x_0 \in X$ and identify it up to conjugation in $\pi_1(X, x_0)$ if necessary.

Since $H_2^Q(X) \cong H_2(B^Q X)$ and $\pi_1(B^Q X) \cong \mathrm{As}(X)$, there exists a classifying homomorphism $c_* : H_2^Q(X) \to H_2(\mathrm{As}(X))$. In the following, a homology class $\phi \in H_2^Q(X)$ is said to be *realizable* if $c_*\phi = 0$.

Proposition 5.5. *Let X be a finitely generated connected quandle and fix $x_0 \in X$. For $\lambda \in \pi_1(X, x_0)$, there exists a knot diagram D with an X-coloring $\mathcal{C}$ such that $\Lambda(\mathcal{C}) = \lambda$ if and only if $\lambda_{\mathrm{ab}} \in H_2^Q(X)$ is realizable.*

Proof. By the five-term exact sequence of the Cartan-Leray spectral sequence for the universal covering of $B^Q X$, we have an exact sequence

$$\pi_2(B^Q X) \xrightarrow{h_*} H_2^Q(X) \xrightarrow{c_*} H_2(\mathrm{As}(X)).$$

Since $h_*\Xi(\mathcal{C}) = \Phi_2(\mathcal{C})$, this sequence shows that $\Lambda(\mathcal{C})_{\mathrm{ab}} = \Phi_2(\mathcal{C})$ is realizable.

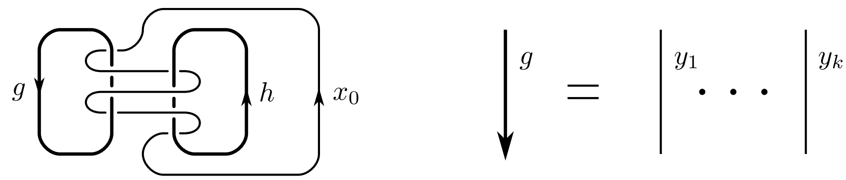


FIGURE 2. A diagram D_0 with an X-coloring $\mathcal{C}_0$

Assume $\lambda_{\mathrm{ab}} \in H_2^Q(X)$ to be realizable. As seen above, λ_{ab} comes from $\pi_2(B^Q X)$ and then Proposition 5.1 finds a surjective X-coloring $\mathcal{C}_0$ on a knot diagram D_0 such that $\Phi_2(\mathcal{C}_0) = h_* \Xi(\mathcal{C}_0) = \lambda_{\mathrm{ab}}$. Since X is connected, we may assume the color of an arc equal to x_0. By the universality of Λ over Φ_2, we find that $\Lambda(\mathcal{C}_0)^{-1}\lambda$ is contained in the commutator subgroup $[\pi_1(X), \pi_1(X)]$, and then Lemma 5.6 below finds an X-coloring $\mathcal{C}_1$ on a knot diagram D_1 such that $\Lambda(\mathcal{C}_1) = \Lambda(\mathcal{C}_0)^{-1}\lambda$. A connected sum D of D_0 and D_1 admits an X-coloring $\mathcal{C}$ obtained from $\mathcal{C}_0$ and $\mathcal{C}_1$, which satisfies the required condition $\Lambda(\mathcal{C}) = \lambda$. □

Lemma 5.6. *Let G denote the fundamental group $\pi_1(X, x_0)$. For any $\lambda \in [G, G]$, there exists an X-coloring $\mathcal{C}$ on a knot diagram such that $\Lambda(\mathcal{C}) = \lambda$ and $\Xi(\mathcal{C}) = 0$.*

We do not need the property $\Xi(\mathcal{C}) = 0$ in the proof of Proposition 5.5; this is used in the proof of Theorem 5.8.

Proof of Lemma 5.6. It is sufficient to consider the case where λ is a commutator $[g, h]$ of $g, h \in G$. In fact, any element of $[G, G]$ can be expressed as a product of commutators, and then we take a connected sum of colored knot diagrams corresponding to the commutators to obtain a required one.

Let D_0 be the link diagram with an X-coloring $\mathcal{C}_0$ described in Figure 2. Here, the component with label g represents a paralleled link whose colors express $g \in \pi_1(X, x_0) \subset \mathrm{As}(X)$: If $g = e_{y_1}^{\epsilon_1} \cdots e_{y_k}^{\epsilon_k}$, then we take a k-parallelization of that component and give colors and orientations according to y_i and ϵ_i; the component labeled h is similar. Then, we read the colors of over-arcs along the component labeled x_0 to find it equal to $[g, h]$.

In order to modify D_0 to be a knot diagram, we take a knot diagram D_1 with a surjective X-coloring $\mathcal{C}_1$ as in the proof of Proposition 3.1. We should remark that the proof gives a colored diagram with trivial non-abelian cycle invariant. We consider a disjoint union of D_0 and D_1, and connect D_1 and the components of D_0 with ribbons so that the colors of the arcs connected by each ribbon are same. Taking

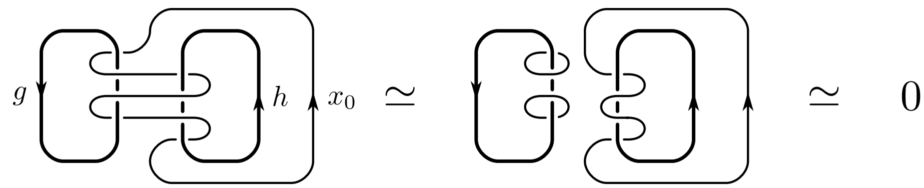


FIGURE 3. $(D_0, \mathcal{C}_0)$ is null-cobordant

the ribbon-sum, we obtain a knot diagram D with an X-coloring $\mathcal{C}$ satisfying $\Lambda(\mathcal{C}) = [g, h]$.

Finally, we shall check that $\Xi(\mathcal{C}) = 0$. Since the ribbon-sum operation does not change the homotopy invariant, it is sufficient to show $\Xi(\mathcal{C}_0) = \Xi(\mathcal{C}_1) = 0$. We can easily find $\Xi(\mathcal{C}_1) = 0$ from the definition of $\mathcal{C}_1$ (the proof of Proposition 3.1). The other equation $\Xi(\mathcal{C}) = 0$ can be seen as illustrated in Figure 3. □

Remark 5.7. For a diagram D of a knot K and an arc m, there is a one-to-one correspondence between the set of X-colorings $\mathcal{C}$ such that $\mathcal{C}(m) = x_0$ and $\Lambda(\mathcal{C}) = \lambda$ and the set of group homomorphisms $\pi_K \to \mathrm{As}(X)$ which sends the meridian-longitude pair to (e_{x_0}, λ). Then, we can use [10] to obtain an alternative proof of Proposition 5.5; for example, we can check the vanishing of the Pontryagin product $\langle e_{x_0}, \lambda \rangle \in H_2(\mathrm{As}(X))$ by showing a formula $\langle e_{x_0}, \lambda \rangle = c_*(\lambda_{\mathrm{ab}})$. We omit the details, for this proof does not deduce Theorem 5.8 in the next section.

5.3. **Non-abelian invariants vs. 3-cocycle invariants.** As seen above, homotopy invariants and non-abelian invariants are efficient under some conditions. As the goal of this paper, we see the independence of these invariants:

Theorem 5.8. *Let X be a finitely generated connected quandle and take $x_0 \in X$. For $\xi \in \pi_2(B^Q X)$ and $\lambda \in \pi_1(X, x_0)$, there exists a pair $(D, \mathcal{C})$ of a knot diagram D and an X-coloring $\mathcal{C}$ of D such that $\Xi(\mathcal{C}) = \xi$ and $\Lambda(\mathcal{C}) = \lambda$ if and only if $h_*\xi = \lambda_{\mathrm{ab}} \in H_2^Q(X)$, where $h_* : \pi_2(B^Q X) \to H_2^Q(X)$ is the Hurewicz homomorphism.*

Proof. For an X-coloring $\mathcal{C}$ on a knot diagram, the universalities of Ξ and Λ over Φ_2 shows that $h_*\Xi(\mathcal{C}) = \Phi_2(\mathcal{C}) = \Lambda(\mathcal{C})_{\mathrm{ab}}$. Conversely, if $\xi \in \pi_2(B^Q X)$ and $\lambda \in \pi_1(X, x_0)$ satisfy $h_*\xi = \lambda_{\mathrm{ab}}$, we can construct a required pair $(D, \mathcal{C})$ just as in the proof of Proposition 5.5: We take a pair $(D_0, \mathcal{C}_0)$ with $\Xi(\mathcal{C}_0) = \xi$ by Proposition 5.1 and then correct Λ by taking a connected sum with a pair $(D_1, \mathcal{C}_1)$ obtained by Lemma 5.6; we should note that the connected-sum operation does not change the homotopy invariant because of the property $\Xi(\mathcal{C}_1) = 0$ in Lemma 5.6. □

Corollary 5.9. *For $\phi \in H_3^Q(X)$ and $\lambda \in \pi_1(X, x_0)$, there exists a pair $(D, \mathcal{C})$ of a knot diagram D and an X-coloring $\mathcal{C}$ of D such that $\Phi_3(\mathcal{C}) = \phi$ and $\Lambda(\mathcal{C}) = \lambda$ if and only if ϕ is realizable and $p_* \circ q(\phi) = \lambda_{\mathrm{ab}} \in H_2^Q(X)$, where $p : B_X^Q X \to B^Q X$ is the covering map.*

Remark 5.10. In Corollary 5.9, $p_* \circ q : H_3^Q(X) \to H_2^Q(X)$ is equal to the shifting homomorphism introduced in [6].

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