INTEGRAL MORSE COMPLEXES ON THE REAL GRASSMANNIANS

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Abstract. The purpose of this paper is to construct, following Morse homology theory, integral chain complexes of the canonical Morse-Smale functions on the Grassmannian $Gr(m,n)$ of m-dimensional subspaces and the Grassmannian $\widetilde{Gr}(m,n)$ of oriented m-dimensional subspaces in the real space $\mathbb{R}^{m+n}$. As an application, the homology groups of those chain complexes are computed straightforwardly, so as to recover the singular homology groups $H_*(Gr(m,n);\mathbb{Z})$ and $H_*(\widetilde{Gr}(2,n);\mathbb{Z})$ for any positive integers m and n.

1. Introduction

The real Grassmannian $Gr(m,n)$ is a space that parametrizes m-dimensional subspaces in the $(m+n)$-dimensional vector space $\mathbb{R}^{m+n}$, and the oriented Grassmannian $\widetilde{Gr}(m,n)$ parametrizes oriented m-dimensional subspaces. There exists a canonical two-sheeted covering map $\widetilde{Gr}(m,n) \to Gr(m,n)$. The purpose of this paper is to construct, by means of Morse theory, chain complexes whose homology groups are isomorphic to the singular homology groups with coefficients in $\mathbb{Z}$ of those Grassmannians.

A Morse function on a Riemannian manifold is said to satisfy *Morse-Smale condition* if the unstable and stable manifolds of any critical points intersect transversally. In Morse theory, a chain complex, called *Morse complex*, is defined as a module generated by the critical points of a Morse-Smale function with boundary operator obtained by counting the number of the gradient trajectories connecting two critical points differing in Morse index by one. If the coefficients are in $\mathbb{Z}$, the trajectories are counted with *sign* which is determined by fixing orientations of the unstable manifolds of critical points. On the other hand, a Morse-Smale function decomposes the manifold into the unstable manifolds of critical points, inducing a CW decomposition, which also defines a chain complex. Those two chain complexes are isomorphic. The computation of attaching maps in CW complexes is replaced by counting the number of gradient trajectories of the Morse function. The idea of Morse complex was renewed by Witten in [19], and took a new turn in infinite dimension after the

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works by Floer, for example in [9] and [10]. For a survey on this progress, we refer an expository paper by Guest [11], and comprehensive books by Audin and Damian [1] and by Banyaga and Hurtubise [2].

The (co)homology group of the real Grassmannian with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is similar to that of the complex Grassmannian with coefficients in $\mathbb{Z}$, whose Poincaré polynomial equals the q-binomial coefficient. Moreover, Schubert cells in the real Grassmannian generate independent (co)homology cycles of $\mathbb{Z}/2\mathbb{Z}$ coefficient, where the ring structure is expressed by using the Stiefel-Whitney characteristic classes. Ehresmann initiated the study on Schubert CW structure in [8], which was combined with the study of characteristic classes by Chern in [7], Borel in [5], Takeuchi in [18], and many others. For recent works on (co)homological structure of real Grassmannians, see [16] , [15] and references therein, and for more attempts, see also [4], [6], and [12]. On the other hand, the homology groups of the oriented Grassmannians are still unclear, even for coefficients in $\mathbb{Z}/2\mathbb{Z}$. Progresses on the theme are found, for example, in [3], [14] and [13].

There exists a Morse-Smale function on the real Grassmannian, which gives the same CW structure as the Schubert cell decomposition. As the main purpose of the paper, we investigate the Morse data of this function, namely solve the gradient flow equation, and compute the signs of the gradient trajectories between two critical points with successive Morse indices (Propositions 5.2 and 5.3), where the computations of the signs will be done along with general theory of Morse complex (Definition 2.1 and Definition 4.5). Proposition 5.2 gives immediately an explicit formula of the boundary operator of the Morse complex on $Gr(m,n)$ (Theorem 6.1). For oriented Grassmannians $\widetilde{Gr}(m,n)$, we use the covering map $\widetilde{Gr}(m,n) \to Gr(m,n)$, and lift the function to a Morse-Smale function on $\widetilde{Gr}(m,n)$, and the gradient trajectories to those in $\widetilde{Gr}(m,n)$. By those liftings, we obtain an explicit formula of boundary operators of the Morse complex on $\widetilde{Gr}(m,n)$ (Theorem 6.3). Those theorems are applied, in a straightforward manner, to compute the integral homology groups of $Gr(m,n)$ for all m and n in Section 7, and those of $\widetilde{Gr}(2,n)$ for all n in Section 8. Since Grassmannians play the role of classifying spaces, it is natural and significant to study cohomology rings from the viewpoint of characteristic classes. Nevertheless, the paper insists on calculating the homology groups of Morse complexes independently from the theory of characteristic classes.

2. Brief review of Morse homology theory

Let M be a closed Riemannian manifold, and $f : M \to \mathbb{R}$ be a Morse function. If the stable manifolds $W^s(x)$ and unstable manifolds $W^u(y)$ intersect transversally for any critical points x and y, the function f is called a *Morse-Smale* function. The Morse-Smale transversality condition and the closedness of the manifold guarantee that there exist finitely many gradient trajectories which connect two critical points with successive Morse indices. In order to define an integral Morse complex of a Morse-Smale function, we suppose that arbitrary orientations are fixed on the unstable manifolds of all critical points, which also fix co-orientations of stable manifolds. According to the orientations of unstable manifolds we define the sign of gradient trajectories connecting two critical points with successive Morse indices.

Definition 2.1. Let γ be a gradient trajectory between critical points x and y with Morse index $k+1$ and k respectively, and take a transverse slice $\mathcal{S}$ at the midpoint $\gamma(0)$ in the unstable manifold $W^u(x)$. Let $\{v_1, \cdots, v_k\}$ be a basis of the tangent space $T_{\gamma(0)}\mathcal{S}$ such that $\{\frac{d\gamma}{dt}(0), v_1, \cdots, v_k\}$ is an oriented basis of $T_{\gamma(0)}W^u(x)$. We define the *sign* $\mathrm{sgn}(\gamma) = 1$ or -1 according to whether the order of vectors $\{v_1, \cdots, v_k\}$ agrees with the co-orientation of the stable manifold $W^s(y)$ or not.

Denote by $\mathrm{crit}_k(f)$ the set of all critical points of f with Morse index k, and by $\mathcal{C}(f)_k$ the $\mathbb{Z}$-module formally generated over $\mathrm{crit}_k(f)$. Let $n(x, y)$ denote the number of gradient trajectories γ from x to y counted with $\mathrm{sgn}(\gamma)$;

$$n(x, y) = \sum \mathrm{sgn}(\gamma),$$

and define a homomorphism $\partial_{k+1} : \mathcal{C}(f)_{k+1} \to \mathcal{C}(f)_k$ by

$$\partial_{k+1} \langle x \rangle = \sum_{y \in \mathrm{crit}_k(f)} n(x, y) \langle y \rangle .$$

Theorem 2.2. *Let M be a closed Riemannian manifold, and $f : M \to \mathbb{R}$ be a Morse-Smale function. Then the complex $\mathcal{C}(f) = (\mathcal{C}(f)_*, \partial_*)$ defined above is a chain complex, and the homology group is independent of the choice of Morse-Smale functions nor orientations of unstable manifolds, which is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.*

For a proof, see [2, Chapter 7] for instance. The chain complex $\mathcal{C}(f)$ is called the *Morse complex* of a Morse-Smale function $f : M \to \mathbb{R}$.

3. Standard Embedding of Grassmannians

Let $Gr(m, n)$ denote the *Grassmannian* of m-dimensional subspaces in the $(m+n)$-dimensional real vector space $\mathbb{R}^{m+n}$. In the exterior product space $\bigwedge^m \mathbb{R}^{m+n}$, a non-zero decomposable element $v_1 \wedge \cdots \wedge v_m$ corresponds to a linear subspace of $\mathbb{R}^{m+n}$

spanned by $v_1, \cdots, v_m$, and hence a bijective correspondence between $Gr(m,n)$ and the set of all decomposable elements in the projective space $P\left(\bigwedge^m \mathbb{R}^{m+n}\right)$ is obtained;

$$Gr(m,n) \to P\left(\bigwedge^m \mathbb{R}^{m+n}\right) : \mathrm{span}\{v_1, \cdots, v_m\} \mapsto [v_1 \wedge \cdots \wedge v_m].$$

We adopt this identification, called the Plücker embedding, throughout the paper.

3.1. **The standard embedding.** Let $\mathrm{Sym}(m+n;\mathbb{R})$ be the linear space of real symmetric matrices of size $m+n$, which is regarded as a Euclidean space equipped with inner product $\langle X, Y\rangle = \mathrm{trace}(XY)$. For each linear subspace $V \subset \mathbb{R}^{m+n}$, there exists a unique real symmetric matrix Y_V that has two eigenvalues ± 1 and the eigenspace V of eigenvalue 1. The embedding $\Phi : Gr(m,n) \to \mathrm{Sym}(m+n;\mathbb{R})$ defined by $V \mapsto \Phi(V) = Y_V$ is called the *standard embedding*. The image $\Phi(Gr(m,n))$ is characterized by two equations, namely $Y^2 = I$ (the unit matrix) and $\mathrm{trace}\, Y = m - n$;

$$\Phi : Gr(m,n) \to \{Y^2 = I, \mathrm{trace}\, Y = m-n\} \subset \mathrm{Sym}(m+n;\mathbb{R}).$$

The Riemannian structure on $Gr(m,n)$ induced by Φ will be fixed.

3.2. **A Morse function and the gradient flow.** Let Λ be a diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_{m+n}) \in \mathrm{Sym}(m+n;\mathbb{R}) \tag{3.1}$$

satisfying $\lambda_1 < \cdots < \lambda_{m+n}$, and let $h_\Lambda : \mathrm{Sym}(m+n;\mathbb{R}) \to \mathbb{R}$ be a linear function defined by $h_\Lambda(X) = 2\langle X, \Lambda\rangle = 2\,\mathrm{trace}(X\Lambda)$. We will show, in Section 4.3, that the composition of the standard embedding Φ with h_Λ is a Morse-Smale function;

$$Gr(m,n) \xrightarrow{\Phi} \mathrm{Sym}(m+n,\mathbb{R}) \xrightarrow{h_\Lambda} \mathbb{R}. \tag{3.2}$$

We denote by M the image of the standard embedding of $Gr(m,n)$. The tangent space $T_Y M$ equals $\{A \in \mathrm{Sym}(m+n,\mathbb{R})\,;\, 0 = AY + YA\}$, and the orthogonal projection from $\mathrm{Sym}(m+n,\mathbb{R})$ to the tangent space $T_Y M$ is given by $A \mapsto (A - YAY)/2$. Thus the gradient vector field of $h_\Lambda : M \to \mathbb{R}$ is $\nabla_{h_\Lambda}(Y) = -\Lambda + Y\Lambda Y$, and the *gradient trajectories* of h_Λ are the solutions of the ordinary differential equation

$$\frac{dY}{dt} = \nabla_{h_\Lambda}(Y) = -\Lambda + Y\Lambda Y. \tag{3.3}$$

3.3. **The gradient trajectory.** Let $\Lambda \in \mathrm{Sym}(m+n,\mathbb{R})$ be a diagonal matrix as above. Given a matrix $Y_0 \in M$, define a curve $t \mapsto Y(t) = -X(t)Z(t)^{-1}$ by setting

$$\begin{cases} X(t) = \big(\exp(t\Lambda)(I - Y_0) - \exp(-t\Lambda)(I + Y_0)\big)/2, \\ Z(t) = \big(\exp(t\Lambda)(I - Y_0) + \exp(-t\Lambda)(I + Y_0)\big)/2. \end{cases} \tag{3.4}$$

Lemma 3.1. *The curve Y defined above is the gradient trajectory of h_Λ with initial value $Y(0) = Y_0$.*

Proof. Since $X(0) = -Y_0$ and $Z(0) = I$, we have $Y(0) = Y_0$. The derivatives of X and Z by t equal $X' = \Lambda Z$ and $Z' = \Lambda X$ respectively, and $(Z^{-1})' = -Z^{-1}\Lambda X Z^{-1}$. Thus we have

$$Y' = -\Lambda Z Z^{-1} + X Z^{-1}\Lambda X Z^{-1} = -\Lambda + Y\Lambda Y = \nabla_{h_\Lambda}(Y).$$

□

We have the following geometric description of gradient trajectories:

Proposition 3.2. *If $Y(t)$ is a gradient trajectory in $M = \Phi(Gr(m,n))$, and if v is an eigenvector of $Y(0)$ with eigenvalue 1, then for any $t \in \mathbb{R}$, $\exp(-t\Lambda)v$ remains in the eigenspace of $Y(t)$ of eigenvalue 1. In other words, the gradient flow brings an element $V \in Gr(m,n)$ to $\exp(-t\Lambda)V$.*

Proof. Let X and Z be as above, and let v be an eigenvector of Y_0 with eigenvalue 1. Then it holds that $Zv = \exp(-t\Lambda)v$. The equality

$$(XZ^{-1} + I)Z = X + Z = \exp(t\Lambda)(I - Y_0)$$

implies that $(I - Y(t))Z = \exp(t\Lambda)(I - Y_0)$. Therefore, $(I - Y_0)v = 0$ implies

$$0 = (I - Y(t))Zv = (I - Y(t))\exp(-t\Lambda)v.$$

This completes the proof. □

4. Local charts and orientations of unstable manifolds

We will compute the signs of gradient trajectories in the next section. In preparation for this, we introduce local charts at critical points $V(\mu)$, and fix orientations of unstable manifolds in the real Grassmannian $Gr(m,n)$.

4.1. **Combinatorial view of the Morse complex.** We abbreviate as $[k]$ the set of positive integers $\{1, \cdots, k\}$. For positive integers m and n, denote by $S(m,n)$ the set of strictly increasing functions $\mu : [m] \to [m+n]$, which indexes coordinate m-dimensional planes $V(\mu) \in Gr(m,n)$;

$$V(\mu) = [e_{\mu(1)} \wedge \cdots \wedge e_{\mu(m)}],$$

where $\{e_1, \cdots, e_{m+n}\}$ is the standard basis of $\mathbb{R}^{m+n}$.

Lemma 4.1. *Let $h_\Lambda : Gr(m,n) \to \mathbb{R}$ be the function defined in (3.2). Then the set* $\mathrm{crit}(h_\Lambda)$ *of all critical points of h_Λ equals $\{V(\mu)\,;\, \mu \in S(m,n)\}$;*

$$\mathrm{crit}(h_\Lambda) = \{V(\mu)\,;\, \mu \in S(m,n)\}.$$

Proof. By Equation (3.3), $V \in Gr(m,n)$ is a critical point of h_Λ, if and only if $Y_V = \Phi(V)$ commutes with Λ, in which case, since Λ is a diagonal matrix with distinct entries, V is a coordinate m-plane. □

The Morse complex $\mathcal{C}(h_\Lambda)$ is a free $\mathbb{Z}$ module generated by the critical point set $\mathrm{crit}(h_\Lambda)$. The boundary operator of the Morse complex concerns certain pairs of critical points, which we describe below.

The set $S(m,n)$ has the following partial order $\rhd$: $\mu \rhd \nu$ if and only if $\mu(a) \geq \nu(a)$ for all $a \in [m]$. The relation $\rhd$ makes the set $S(m,n)$ a *partially ordered set* (*poset*). We say μ *covers* ν if $\mu \rhd \nu$ and there is no $\xi \in S(m,n) \setminus \{\mu,\nu\}$ such that $\mu \rhd \xi \rhd \nu$. If μ covers ν, there exists a unique $\alpha \in [m]$ such that

$$\nu = \mu_\alpha, \quad \text{where} \quad \mu_\alpha(a) = \begin{cases} \mu(a) - 1, & \text{if } a = \alpha, \\ \mu(a), & \text{otherwise.} \end{cases} \tag{4.1}$$

We denote by $\mathrm{cov}(\mu)$ the set of $\alpha \in [m]$ such that $\mu_\alpha : [m] \to [m+n]$ is strictly increasing;

$$\mathrm{cov}(\mu) = \{\alpha \in [m]\,;\, \mu_\alpha \in S(m,n)\}.$$

We define a *rank function* $\rho : S(m,n) \to \mathbb{Z}$ by

$$\rho(\mu) = \sum_{a=1}^{m} (\mu(a) - a), \tag{4.2}$$

which makes $S(m,n)$ a *ranked poset*. For each $\mu \in S(m,n)$, the Morse index $\mathrm{Ind}(V(\mu))$ of the critical point $V(\mu)$ of the function h_Λ equals $\rho(\mu)$;

$$\mathrm{Ind}(V(\mu)) = \rho(\mu).$$

Every gradient trajectory tends to a critical point as its parameter t tends to $\pm\infty$. Therefore Lemma 4.1 implies that, for each gradient trajectory $\gamma(t)$, there exist μ and $\nu \in S(m,n)$ such that

$$\lim_{t\to\infty} \gamma(t) = V(\nu), \quad \lim_{t\to-\infty} \gamma(t) = V(\mu), \tag{4.3}$$

where μ and $\nu \in S(m,n)$ are determined by the pivot positions of matrix representations of $\gamma(0)$; for μ, use a basis of $\gamma(0)$ in echelon form with zeroes in the upper-right corner, and for ν, with zeroes in the lower-left corner, respectively.

Lemma 4.2. *If* $\lim_{t\to\infty}\gamma(t) = V(\nu)$ *and* $\lim_{t\to-\infty}\gamma(t) = V(\mu)$*, then* $\mu \rhd \nu$*.*

Proof. For each $i \in [m+n]$, let $F_i \subset \mathbb{R}^{m+n}$ be the subspace spanned by $\{e_1, \cdots, e_i\}$, and $F_i^{\perp}$ be the orthogonal complement. Since the multiplication by the diagonal matrix $\exp(-t\Lambda)$ doesn't move F_i, the dimension $\dim(\gamma(t) \cap F_i)$ doesn't change as t varies. And also, since $V(\mu)$ and $V(\nu)$ are determined by the echelon matrices formed by properly chosen bases of $\gamma(t)$, it holds that $\lim_{t\to-\infty}\dim(\gamma(t)\cap F_i) = \dim(V(\mu)\cap F_i)$, and $\lim_{t\to\infty}\dim(\gamma(t)\cap {F_i}^{\perp}) = \dim(V(\nu)\cap {F_i}^{\perp})$. If $\mu(a) < \nu(a)$ for some index a, it holds that $\dim(\gamma(t)\cap F_{\mu(a)}) = \dim(V(\mu)\cap F_{\mu(a)}) = a$ and $\dim(\gamma(t)\cap {F_{\mu(a)}}^{\perp}) = \dim(V(\nu)\cap {F_{\mu(a)}}^{\perp}) \geq m-(a-1)$. This contradicts the fact $\dim\gamma(t) = m$. □

4.2. **Local charts in neighborhoods of critical points.** For each $\mu \in S(m,n)$, we use the following index sets:

$$[m+n]_\mu = [m+n]\setminus\mu([m]),\ \Xi_\mu = [m]\times[m+n]_\mu,\ \Xi_\mu^u = \{(a,b)\in\Xi_\mu\,;\, b<\mu(a)\},$$

and furthermore for each $a \in [m]$,

$$[m+n]_{\mu,a}^u = \{b \in [m+n]_\mu\,;\, b < \mu(a)\}.$$

The cardinalities $|\Xi_\mu|$ and $|\Xi_\mu^u|$ of Ξ_μ and Ξ_μ^u are equal to mn and $\sum_{a=1}^m(\mu(a)-a) = \rho(\mu)$, respectively. We will use the *lexicographic* order $\prec$ on the index sets Ξ_μ and Ξ_μ^u, namely, $(a,b)\prec(a',b')$ either if $a<a'$, or if $a=a'$ and $b<b'$. By this ordering, the vector space $\mathbb{R}^{|\Xi_\mu|}$, for instance, is regarded as oriented.

Given $\mu \in S(m,n)$, the map ψ_μ defined by

$$\psi_\mu : \left[\bigwedge_{a=1}^m\left(e_{\mu(a)} + \sum_{b\in[m+n]_\mu} x_{a,b}e_b\right)\right] \mapsto (x_{a,b}) \in \mathbb{R}^{mn} \tag{4.4}$$

is a local chart of $Gr(m,n)$ on a neighborhood Ω_μ of $V(\mu)$.

4.3. **Morse-Smale transversality.** By using Lemma 4.2, we verify the Morse-Smale transversality of the function $h_\Lambda : Gr(m,n)\to\mathbb{R}$.

Given a critical point $V(\mu)$, let Ω_μ be the local chart with coordinate functions $\{x_{a,b}\,;\,(a,b)\in\Xi_\mu\}$ defined in (4.4). For another critical point $V(\nu)$ with $\nu \lhd \mu$, let A be the index set consisting of $(a,b)\in\Xi_\mu$ satisfying either $\nu(a)\leq b<\mu(a)$ or $b=\nu(a')$ for some $a'<a$ with $\nu(a)\leq\mu(a')$. Then Proposition 3.2 implies that the variables $x_{a,b}$ with $(a,b)\in A$ parameterize the intersection $W^s(V(\nu))\cap W^u(V(\mu))$ in a neighborhood Ω_0 of a point in $W^s(V(\nu))\cap W^u(V(\mu))$. The cardinality of A equals $\rho(\mu)-\rho(\nu)$. We obtain a division of Ξ_μ into three subsets, namely

$$\Xi_\mu = (\Xi_\mu^u\setminus A)\cup A\cup(\Xi_\mu\setminus\Xi_\mu^u).$$

In the neighborhood Ω_0, the set of coordinate functions $x_{a,b}$ with $(a,b) \in \Xi_\mu^u$ corresponds to $W^u(V(\mu))$, and that with $(a,b) \in A \cup (\Xi_\mu \setminus \Xi_\mu^u)$ corresponds to the stable manifold $W^s(V(\nu))$, which implies that $W^u(V(\mu))$ transversely intersects with $W^s(V(\nu))$. Therefore we have proved the following:

Corollary 4.3. *The function $h_\Lambda : Gr(m,n) \to \mathbb{R}$ defined in (3.2) satisfies the Morse-Smale transversality condition.*

4.4. **Orientations of unstable manifolds.** In view of Proposition 3.2, we find that the local chart ψ_μ in (4.4) fits the gradient flow equation. Note that the gradient trajectories starting at a critical point $V(\mu)$ form the unstable manifold of $V(\mu)$, and that those ending at $V(\mu)$ form the stable manifold.

Remark 4.4. Let $\{x_{a,b}\}$ be the local coordinate functions defined in (4.4). Then the set of equations $\{x_{a,b} = 0\}$ with $(a,b) \notin \Xi_\mu^u$ defines the stable manifold $W^s(V(\mu))$, and a subset of functions $\{x_{a,b}\}$ with $(a,b) \in \Xi_\mu^u$ serves as a local coordinate system of the unstable manifold $W^u(V(\mu))$. We denote by ψ_μ^u the local chart on $W^u(V(\mu))$ with this set of coordinate functions.

Definition 4.5. Via the local chart ψ_μ^u, the unstable manifold $W^u(V(\mu))$ inherits *orientation* from $\mathbb{R}^{|\Xi_\mu^u|}$ with the lexicographic ordering of Ξ_μ^u. The stable manifold $W^s(V(\mu))$ is regarded as *co-oriented* by the ordering of Ξ_μ^u which indexes the set of defining equations $\{x_{a,b} = 0\}$ of $W^s(V(\mu))$.

5. The signs of the gradient trajectories

Let μ and $\nu = \mu_\alpha$ with $\alpha \in \mathrm{cov}(\mu)$ be fixed (see (4.1)). Proposition 3.2 and Lemma 4.2 together with the local charts ψ_μ and ψ_{μ_α} imply the following :

Lemma 5.1. *Between $V(\mu)$ and $V(\mu_\alpha)$, there exist two gradient trajectories $\gamma_{\mu,\alpha}^\pm$, which are parametrized as follows:*

$$\gamma_{\mu,\alpha}^\pm(t) = [v_1(t) \wedge \cdots \wedge v_m(t)]$$

for $t \in \mathbb{R}$, where

$$v_\alpha(t) = \pm \exp(-t\lambda_{\nu(\alpha)}) e_{\nu(\alpha)} + \exp(-t\lambda_{\mu(\alpha)}) e_{\mu(\alpha)},$$

and $v_a(t) = e_{\mu(a)}$ *for* $a \neq \alpha$.

We still use notation $x_{a,b}$ for the coordinate functions of the local chart ψ_ν^u introduced in Remark 4.4, and on the other hand write $y_{a,b}$ instead of $x_{a,b}$ of the local chart ψ_μ^u. We define a transverse slice $\mathcal{S}^\pm$ of $\gamma^\pm_{\mu,\alpha}$ at the midpoint $\gamma^\pm_{\mu,\alpha}(0)$ by the equation $y_{\alpha,\nu(\alpha)} = \pm 1$. The local chart ψ_μ restricts to a local chart $\psi_\mu^u|_{\mathcal{S}^\pm}$ of the slice $\mathcal{S}^\pm$. More precisely, the local coordinate system $\psi_\mu^u|_{\mathcal{S}^\pm} : \mathcal{S}^\pm \to \mathbb{R}^{\rho(\mu)-1}$ is given by

$$\psi_\mu^u|_{\mathcal{S}^\pm}([v_1 \wedge \cdots \wedge v_m]) = (y_{a,b})$$

$((a,b) \in \Xi_\mu^u \setminus \{(\alpha, \nu(\alpha))\})$, where the vectors v_a are

$$v_a = \begin{cases} e_{\mu(a)} + \displaystyle\sum_{b \in [m+n]^u_{\mu,a}} y_{a,b} e_b, & \text{if } a \neq \alpha, \\ \pm e_{\nu(\alpha)} + e_{\mu(\alpha)} + \displaystyle\sum_{b \in [m+n]^u_{\mu,\alpha} \setminus \{\nu(\alpha)\}} y_{\alpha,b} e_b, & \text{if } a = \alpha. \end{cases}$$

The sign $\mathrm{sgn}(\gamma^\pm_{\mu,\alpha})$ concerns the following three terms:

(s1) permutation of Ξ_μ^u which moves $(\alpha, \nu(\alpha))$ to the first position,

(s2) $\left.\frac{dy_{\alpha,\nu(\alpha)}(\gamma^\pm_{\mu,\alpha}(t))}{dt}\right|_{t=0}$,

(s3) $\det\left(\frac{\partial x_{a,b}}{\partial y_{a',b'}}\right)$, where (a,b) runs through the index set Ξ_ν^u, and (a', b') through $\Xi_\mu^u \setminus \{(\alpha, \nu(\alpha))\}$.

The sign of the first term (s1) is common to $\gamma^+_{\mu,\alpha}$ and $\gamma^-_{\mu,\alpha}$, and determined by the parity of $-1 + \sum_{a=1}^{\alpha}(\mu(a) - a)$. The second (s2) is positive for $\gamma^+_{\mu,\alpha}$ and negative for $\gamma^-_{\mu,\alpha}$.

For the third (s3), note that the index sets Ξ_μ^u and Ξ_ν^u differ as follow:

$$\Xi_\nu^u \setminus \Xi_\mu^u = \{(a, \mu(\alpha))\,;\, a > \alpha\}, \quad \Xi_\mu^u \setminus \Xi_\nu^u = \{(a, \nu(\alpha))\,;\, a \geq \alpha\}. \tag{5.1}$$

We compute the sign of (s3) for $\gamma^+_{\mu,\alpha}$ and $\gamma^-_{\mu,\alpha}$, separately.

5.1. The sign of $\gamma^+_{\mu,\alpha}$.

Proposition 5.2. *The sign of the gradient trajectory* $\gamma^+_{\mu,\alpha}$ *is given by*

$$\mathrm{sgn}(\gamma^+_{\mu,\alpha}) = -(-1)^{m+\mu(\alpha)+\sum_{a=1}^{\alpha-1}(\mu(a)-a)}.$$

Proof. Since exterior product $\wedge$ is multilinear and alternative, we have, for $(a,b) \in \Xi^u_\nu$,

$$x_{a,b} = \begin{cases} -y_{a,\nu(a)}, & \text{if } \alpha+1 \le a \text{ and } b=\mu(a), \\ y_{a,b} - y_{\alpha,b}y_{a,\nu(\alpha)}, & \text{if } \alpha+1 \le a \text{ and } b \le \nu(\alpha)-1, \\ y_{a,b}, & \text{for the remaining } (a,b) \in \Xi^u_\nu. \end{cases}$$

With respect to lexicographic ordering of the index sets Ξ^u_ν and $\Xi^u_\mu \setminus \{\alpha, \nu(\alpha)\}$, the Jacobi matrix $\left(\frac{\partial(x_{a,b})}{\partial(y_{a',b'})}\right)$ at $\gamma^+_{\mu,\alpha}(0)$ is diagonal, and the number of -1 on the diagonal equals $m-\alpha$. □

5.2. **The sign of $\gamma^-_{\mu,\alpha}$.**

Proposition 5.3. *The sign of the gradient trajectory $\gamma^-_{\mu,\alpha}$ is given by*

$$\operatorname{sgn}(\gamma^-_{\mu,\alpha}) = -(-1)^{\sum_{a=1}^{\alpha-1}(\mu(a)-a)}.$$

Proof. At the midpoint $\gamma^-_{\mu,\alpha}(0)$, we have, for $(a,b) \in \Xi^u_\nu$,

$$x_{a,b} = \begin{cases} -y_{\alpha,b}, & \text{if } a=\alpha \text{ and } b \le \nu(\alpha)-1, \\ y_{a,b} + y_{\alpha,b}y_{a,\nu(\alpha)}, & \text{if } \alpha+1 \le a \text{ and } b \le \nu(\alpha)-1, \\ y_{a,\nu(\alpha)}, & \text{if } \alpha+1 \le a \text{ and } b=\mu(\alpha), \\ y_{a,b}, & \text{for the remaining } (a,b) \in \Xi^u_\nu. \end{cases}$$

The number of -1 on the diagonal of the Jacobi matrix at $\gamma^-(0)$ is equal to $(\mu(\alpha)-1)-1-(\alpha-1) = \mu(\alpha)-\alpha-1$. □

6. Boundary operators of the Morse complexes

For the real Grassmannians, the coefficients of the boundary operator of the Morse complex of $h_\Lambda : Gr(m,n) \to \mathbb{R}$ are obtained immediately from Propositions 5.2 and 5.3. The Morse complex for the oriented Grassmannians $\widetilde{Gr}(m,n)$ is constructed via the canonical double covering map $\widetilde{Gr}(m,n) \to Gr(m,n)$. The signs of gradient trajectories in $\widetilde{Gr}(m,n)$ are determined by lifting orientations of unstable manifolds in $Gr(m,n)$ to unstable manifolds in $\widetilde{Gr}(m,n)$.

6.1. **Real Grassmannians.** We continue to use notations in Section 4.1. Denote by $S(m,n;k)$ the set of $\mu \in S(m,n)$ with $\rho(\mu)=k$. The k-dimensional chain group

$\mathcal{C}(h_\Lambda)_k$ of the Morse complex $\mathcal{C}(h_\Lambda)$ is generated by $\langle V(\mu)\rangle$ with $\mu \in S(m,n;k)$;

$$\mathcal{C}(h_\Lambda)_k = \bigoplus_{\mu\in S(m,n;k)} \mathbb{Z}\langle V(\mu)\rangle .$$

Theorem 6.1. *The boundary operator ∂_* of the Morse complex $\mathcal{C}(h_\Lambda)$ on the real Grassmannian $Gr(m,n)$ is given by*

$$\partial_k \langle V(\mu)\rangle = -\sum_{\alpha\in \mathrm{cov}(\mu)} (-1)^{\sum_{a=1}^{\alpha-1}(\mu(a)-a)} \left(1+(-1)^{m+\mu(\alpha)}\right)\langle V(\mu_\alpha)\rangle .$$

Proof. For a pair of critical points $V(\mu)$ and $V(\mu_\alpha)$, there exist two gradient trajectories $\gamma^{\pm}_{\mu,\alpha}$ between them (see Lemma 5.1). Their signs $\mathrm{sgn}(\gamma^{+}_{\mu,\alpha})$ and $\mathrm{sgn}(\gamma^{-}_{\mu,\alpha})$ are given in Propositions 5.2 and 5.3. The sum $\mathrm{sgn}(\gamma^{+}_{\mu,\alpha})+\mathrm{sgn}(\gamma^{-}_{\mu,\alpha})$ is the coefficient of $\langle V(\mu_\alpha)\rangle$ in $\partial_k\langle V(\mu)\rangle$. □

6.2. **Oriented Grassmannians.** Let $\mathbb{R}_+$ be the set of positive real numbers. The oriented Grassmannian $\widetilde{Gr}(m,n)$ is identified with the set of decomposable elements $[v_1\wedge\cdots\wedge v_m]_+$ in the sphere $\left(\bigwedge^m \mathbb{R}^{m+n}\setminus\{0\}\right)/\mathbb{R}_+$. Let $\pi : \widetilde{Gr}(m,n)\to Gr(m,n)$ be the canonical two-sheeted covering. Consider the Riemannian structure on $\widetilde{Gr}(m,n)$ induced by π from $Gr(m,n)$. Since h_Λ is a Morse-Smale function, so is $\widetilde{h}_\Lambda = h_\Lambda\circ\pi : \widetilde{Gr}(m,n)\to\mathbb{R}$.

For each $\mu\in S(m,n)$, let $V(\mu)^{\pm}$ be the oriented subspace of $\mathbb{R}^{m+n}$ defined by an ordered set of vectors $\{e_{\mu(1)},\cdots,e_{\mu(m-1)},\pm e_{\mu(m)}\}$. Then the critical point set of $\widetilde{h}_\Lambda$ is

$$\mathrm{crit}(\widetilde{h}_\Lambda) = \pi^{-1}(\mathrm{crit}(h_\Lambda)) = \{V(\mu)^{\varepsilon}\,;\,\mu\in S(m,n),\varepsilon\in\{+,-\}\}.$$

The unstable manifold of a critical point $V(\mu)^{\pm}$ is homeomorphically mapped through π to that of the critical point $V(\mu)=\pi(V(\mu)^{\pm})$. We adopt the *orientations* of the unstable manifolds of $V(\mu)^{\pm}$ so that the map π restricted to the unstable manifolds is orientation preserving. The Morse complex of $\widetilde{h}_\Lambda$ is denoted by $\mathcal{C}(\widetilde{h}_\Lambda)$.

6.2.1. *Lifts of the gradient trajectories.* For each $\mu\in S(m,n)$ and $\alpha\in\mathrm{cov}(\mu)$, we have the following four gradient trajectories $\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}$ in $\widetilde{Gr}(m,n)$ with $\varepsilon,\ \delta\in\{+,-\}$:

$$\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}(t) = [v_1(t)\wedge\cdots\wedge v_m(t)]_+\,, \tag{6.1}$$

where

$$v_\alpha(t) = \varepsilon\exp(-t\lambda_{\mu(\alpha)-1})e_{\mu(\alpha)-1} + \delta\exp(-t\lambda_{\mu(\alpha)})e_{\mu(\alpha)},$$

and $v_a(t) = e_{\mu(a)}$ for $a \neq \alpha$. It holds that $\pi\left(\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}\right) = \gamma^{\varepsilon\delta}_{\mu,\alpha}$ (see Lemma 5.1). The relation between the signs of $\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}$ and $\gamma^{\varepsilon\delta}_{\mu,\alpha}$ is as follows (c.f. Figure 1):

Lemma 6.2. $\mathrm{sgn}(\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}) = \mathrm{sgn}(\gamma^{\varepsilon\delta}_{\mu,\alpha})$ *for any* ε *and* $\delta \in \{+,-\}$.

Proof. The projection π restricts to a homeomorphism between neighborhoods of $\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}$ and $\gamma^{\varepsilon\delta}_{\mu,\alpha}$ which preserves the (co)orientations of the stable and unstable manifolds of their end points. Hence we have the equality $\mathrm{sgn}(\widetilde{\gamma}^{\varepsilon,\delta}_{\mu,\alpha}) = \mathrm{sgn}(\gamma^{\varepsilon\delta}_{\mu,\alpha})$. □

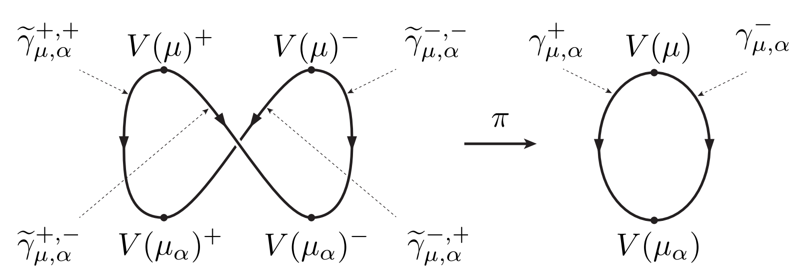

FIGURE 1. Projections of gradient trajectories.

6.2.2. *The Boundary operator.* The proof of the following theorem is immediate from Propositions 5.2, 5.3 and Lemma 6.2.

Theorem 6.3. *The boundary operator* $\widetilde{\partial}_*$ *of the Morse complex* $\mathcal{C}(\widetilde{h}_\Lambda)_*$ *on the oriented Grassmannian* $\widetilde{Gr}(m,n)$ *is given by*

$$\widetilde{\partial}_k \langle V(\mu)^\varepsilon\rangle = -\sum_{\alpha\in\mathrm{cov}(\mu)} (-1)^{\sum_{a=1}^{\alpha-1}(\mu(a)-a)} \left((-1)^{m+\mu(\alpha)}\langle V(\mu_\alpha)^\varepsilon\rangle + \langle V(\mu_\alpha)^{-\varepsilon}\rangle\right).$$

7. Homology groups of the real Grassmannians $Gr(m,n)$

We apply Theorem 6.1 to prove that the homology group $H_*(Gr(m,n);\mathbb{Z})$ has torsions only in $\mathbb{Z}/2\mathbb{Z}$, and thus is a direct sum of $(\mathbb{Z}/2\mathbb{Z})$'s and $\mathbb{Z}$'s; $H_k(Gr(m,n);\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{a(k)} \oplus \mathbb{Z}^{b(k)}$ (see Proposition 7.3). The generating functions $A(q) = \sum_k a(k)q^k$ and $B(q) = \sum_k b(k)q^k$ will be given explicitly in Proposition 7.4 and 7.6, respectively.

7.1. **q-binomial coefficient.** Since $\mu \in S(m,n)$ is strictly increasing, the sequence $\{\mu(m)-m,\cdots,\mu(1)-1\}$ is non-increasing and gives a partition of $\rho(\mu)$. The set $S(m,n)$ is in one-to-one correspondence with the set of partitions of numbers with largest part at most n and with at most m parts.

The subsets $S(m,n;k)=\rho^{-1}(k)\subset S(m,n)$ are enumerated via the *q-binomial coefficients* $\left[\begin{array}{c} m+n \\ m \end{array}\right]_q = \prod_{k=0}^{m-1}\frac{1-q^{m+n-k}}{1-q^{m-k}}$;

Lemma 7.1. $$\sum_{k=0}^{mn}|S(m,n;k)|q^k = \left[\begin{array}{c} m+n \\ m \end{array}\right]_q.$$

For a proof, see Stanley [17].

7.2. **Torsions in $H_k(Gr(m,n);\mathbb{Z})$.** The following was shown by Ehresmann in [8]. We will give a more precise description of the torsion group of $H_k(Gr(m,n);\mathbb{Z})$ in Proposition 7.3 :

Proposition 7.2. *The torsion coefficients of $H_k(Gr(m,n);\mathbb{Z})$ equal 2.*

We draw the Hasse diagram of the poset $S(m,n)$ in the Euclidean space $\mathbb{R}^m$ as follows: let $\Delta(m,n)$ be the subset of the integral point set $\mathbb{Z}^m\subset\mathbb{R}^m$ defined by

$$\Delta(m,n)=\{(\mu_1,\cdots,\mu_m)\in\mathbb{Z}^m\,;\,1\le\mu_1<\cdots<\mu_m\le m+n\}.$$

Through the canonical identification between $S(m,n)$ and $\Delta(m,n)$, the arrow running from $\mu\in S(m,n)$ to μ_α is a segment which is of length 1, and parallel to the α-th coordinate axis of $\mathbb{R}^m$.

By Theorem 6.1, the coefficient $n(\mu,\alpha)=(-1)^{\sum_{a=1}^{\alpha-1}(\mu(a)-a)}\left(1+(-1)^{m+\mu(\alpha)}\right)$ of $\langle V(\mu_\alpha)\rangle$ in the boundary $\partial\langle V(\mu)\rangle$ is equal to 0, or ±2. The vanishing of $n(\mu,\alpha)$ is equivalent to the condition that there exists an integer j such that

$$\mu_\alpha(\alpha)+m<2j+\frac{1}{2}<\mu(\alpha)+m.$$

The inequalities are equivalent to

$$\mu_\alpha(\alpha)<2\lfloor\frac{m}{2}\rfloor+\frac{(-1)^m}{2}<\mu(\alpha).$$

For each $\mu \in \Delta(m,n)$, denote by $X(\mu)$ the set of pairs $(a,j) \in [m] \times \mathbb{Z}$ satisfying the inequalities

$$\mu(a-1) < 2(j-1) + \frac{(-1)^m}{2} < \mu(a) < 2j + \frac{(-1)^m}{2} < \mu(a+1).$$

Define the collection $\mathcal{H}_m$ of affine hyperplanes in $\mathbb{R}^m$ by

$$\left\{ (x_1, \cdots, x_m) \in \mathbb{R}^m \,;\, x_i = 2j + \frac{(-1)^m}{2} \right\} \tag{7.1}$$

for $i \in [m]$, $j \in \mathbb{Z}$, so that the arrow from μ to μ_α in $\Delta(m,n)$ crosses a hyperplane in $\mathcal{H}_m$, if and only if $\mu(\alpha)$ and m have opposite parities, namely if and only if $n(\mu,\alpha) = 0$ (see Theorem 6.1).

We obtain sub-diagrams Δ_i $(i \in I)$ of $\Delta(m,n)$ by removing the arrows that cross a hyperplane in $\mathcal{H}_m$; $\bigcup_{i \in I} \Delta_i = \Delta(m,n)$. Since each Δ_i is surrounded by hyperplanes in $\mathcal{H}_m$, any two vertices μ and $\mu' \in \Delta_i$ have the same index set $X(\mu) = X(\mu')$. Thus a set $X_i \subset [m]$ is determined by

$$X_i = \{a\,;\,(a,j) \in X(\mu) \text{ for some } j \text{ and } \mu \in \Delta_i\}.$$

The sub-diagram Δ_i can be identified with the Hasse diagram of the Boolean poset defined over the set X_i, and has a unique element $\mu^i_{\max}$ such that $\rho(\mu^i_{\max}) \geq \rho(\mu)$ for any $\mu \in \Delta_i$.

Each sub-diagram Δ_i corresponds to a sub-poset S_i of $S(m,n)$. The decomposition of the poset $S(m,n)$ into S_i's induces a direct sum decomposition of the chain complex $\mathcal{C}(h_\Lambda) = \bigoplus_i \mathcal{C}_i$, and a decomposition of the homology group $H_*(\mathcal{C}(h_\Lambda);\mathbb{Z}) = \bigoplus_i H_*(\mathcal{C}_i;\mathbb{Z})$.

Proposition 7.3. *If $X_i = \emptyset$, then $H_k(\mathcal{C}_i;\mathbb{Z}) \cong \mathbb{Z}$. If $X_i \neq \emptyset$, then $H_*(\mathcal{C}_i;\mathbb{Z})$ equals*

$$H_k(\mathcal{C}_i;\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\binom{|X_i|-1}{\rho(\mu^i_{\max})-k-1}}$$

for k with $\rho(\mu^i_{\max}) - |X_i| \leq k \leq \rho(\mu^i_{\max}) - 1$, and $H_k(\mathcal{C}_i;\mathbb{Z}) \cong \{0\}$ for k not in this range. Thus we have

$$H_k(Gr(m,n);\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{a(k)} \oplus \mathbb{Z}^{b(k)}. \tag{7.2}$$

Proof. If $X_i = \emptyset$, then $S_i = \{\mu^i_{\max}\}$, and hence $H_*(\mathcal{C}_i;\mathbb{Z}) = \mathbb{Z}$.

If $|X_i| = 1$, the complex $\mathcal{C}_i$ is, up to a shift of dimension, isomorphic to

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\times(\pm 2)} \mathbb{Z} \longrightarrow \{0\} \;,$$

and the proposition holds.

Suppose that $|X_i| \geq 2$. Let $\alpha = \max X_i$. Define the sub-posets $S_i^{\pm}$ by

$$S_i^+ = S_i \cap \{\mu \in S_i\,;\, \mu(\alpha) = \mu^i_{\max}(\alpha)\},$$
$$S_i^- = S_i \cap \{\mu \in S_i\,;\, \mu(\alpha) = \mu^i_{\max}(\alpha) - 1\},$$

and restrict the boundary operators ∂ to them to obtain chain complexes $\mathcal{C}(S_i^{\pm}, \partial)$. Let $f_* : \mathcal{C}(S_i^+, \partial) \to \mathcal{C}(S_i^-, \partial)$ be the chain map defined by $f_k(x) = 2\epsilon(-1)^k x$. Then for a suitable choice of $\epsilon = \pm 1$ depending on $\mathrm{cov}(\mu^i_{\max})$ the (homological) mapping cone of f_* is isomorphic to $\mathcal{C}(S_i, \partial)$. In other words, we have a commutative diagram

$$\begin{array}{lccccccccc}
\mathcal{C}(S_i^+, \partial): & \cdots & \longrightarrow & \mathcal{C}_k & \xrightarrow{\partial_k} & \mathcal{C}_{k-1} & \longrightarrow & \cdots & & \\
 & & \searrow & & \searrow^{f_k} & & \searrow & & & \\
\mathcal{C}(S_i^-, \partial): & \cdots & \longrightarrow & \mathcal{C}_{k+1} & \xrightarrow[\partial_{k+1}]{} & \mathcal{C}_k & \longrightarrow & \cdots & &
\end{array}$$

where $\mathcal{C}(S_i^+, \partial)$ and $\mathcal{C}(S_i^-, \partial)$ are isomorphic up to a shift of degree 1, and $\mathcal{C}(f_*)_k = \mathcal{C}(S_i^+, \partial)_{k-1} \oplus \mathcal{C}(S_i^-, \partial)_k = \mathcal{C}(S_i, \partial)_k$. We have an exact sequence

$$\cdots \to \ker(f_{k-1}) \to H_k(\mathcal{C}(f_*)) \to \mathrm{coker}(f_k) \to \cdots .$$

Since f_k are homotheties, we have $\ker(f_k) = \{0\}$, and the coefficients of ∂ are ± 2, $\mathrm{coker}(f_*) \cong \mathcal{C}(S_i^-, \partial) \otimes (\mathbb{Z}/2\mathbb{Z})$ has trivial boundary operators. This completes the proof. $\square$

7.3. **Rank of $H_k(Gr(m,n);\mathbb{Z})$.** By Proposition 7.3, we have a decomposition of the integral homology group $H_k(Gr(m,n);\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{a(k)} \oplus \mathbb{Z}^{b(k)}$ with some nonnegative integers $a(k)$ and $b(k)$ for $k = 0, \cdots, mn$. The generating function $B(q) = \sum_{k=0}^{mn} b(k)q^k$ is the $\mathbb{Z}$-Poincaré polynomial of $Gr(m,n)$.

Proposition 7.4. *The $\mathbb{Z}$-Poincaré polynomial $B(q)$ of $Gr(m,n)$ is given by*

$$B(q) = \begin{cases} \begin{bmatrix} \lfloor (m+n)/2 \rfloor \\ \lfloor m/2 \rfloor \end{bmatrix}_{q^4}, & \text{if } mn \text{ is even,} \\[2ex] (1+q^{m+n-1}) \begin{bmatrix} (m+n)/2 - 1 \\ (m-1)/2 \end{bmatrix}_{q^4}, & \text{if } mn \text{ is odd,} \end{cases}$$

Proof. In view of Proposition 7.3, it suffices to find all the sub-poset S_i with $X_i = \emptyset$, namely those S_i with a single vertex $\mu^i_{\max}$ and no arrows. Such $\mu = \mu^i_{\max}$ is characterized according to the parities of m and n as follows:

(1) if m is even, then $\mu(2i-1) + 1 = \mu(2i)$, and $\mu(2i)$ is even for each $i = 1, \cdots, \frac{m}{2}$,

(2) if m is odd, and n is even, then $\mu(1) = 1$, and $\mu(2i)+1 = \mu(2i+1)$, and $\mu(2i+1)$ is odd for each $i = 1, \cdots, \frac{m-1}{2}$,
(3) if m and n are odd, there are two cases:
  (i) $\mu(1) = 1$, and $\mu(2i) + 1 = \mu(2i + 1)$, and $\mu(2i + 1)$ is odd for each $i = 1, \cdots, \frac{m-1}{2}$,
  (ii) $\mu(m) = m + n$, and $\mu(2i - 1) + 1 = \mu(2i)$, and $\mu(2i)$ is odd for each $i = 1, \cdots, \frac{m-1}{2}$,

Let an elements $\mu \in S(m, n)$ be depicted by an $m \times (m + n)$-grid with bullets in the entries $(a, \mu(a))$ for all $a \in [m]$. In order to count the number of μ of the above types, we reduce the grid size by treating 2×2-blocks as a unit square, and apply Lemma 7.1.

Figure 2 shows an example of $\mu \in S_{4,4}$ of type (1) in the above list, where a $4 \times (4+4)$-grid is used and bullets are in entries $(a, \mu(a)) = (1, 1), (2, 2), (3, 5), (4, 6)$, from the upper left to the lower right. In this example, since m is even, vertical heavy lines, which correspond to the hyperplane correction $\mathcal{H}_m$ defined in (7.1), are drawn on the right of even-numbered columns, so that a 2×2-block that contains a pair of successive bullets at $(2i - 1, \mu(2i - 1))$ and $(2i, \mu(2i))$ lies between heavy lines, and reduces to a unit of the 2×4 grid in the right of Figure 2. Lemma 7.1 is applied to the reduced grid to count the number of S_i's with $X_i = \emptyset$. The term q^4 appeared along with this reduction. □

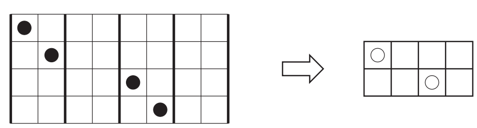

FIGURE 2. Reduction of grid size.

7.4. **Integral homology group.** In order to calculate $a(k)$'s in (7.2), we mention the following:

Proposition 7.5. *For each integer k, let $c(k)$ be the $\mathbb{Z}/2\mathbb{Z}$-dimension of the homology group $H_k(Gr(m, n); \mathbb{Z}/2\mathbb{Z})$. Then the generating function $C(q) = \sum_{k=0}^{mn} c(k)q^k$, namely the $\mathbb{Z}/2\mathbb{Z}$-Poincaré polynomial of $Gr(m, n)$ is given by the q-binomial coefficient;*

$$C(q) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$$

Proof. By Theorem 6.1, the coefficients of the boundary operators of the Morse complex are ± 2 or 0. Hence the Morse complex $\mathcal{C}(S(m,n),\partial)\otimes(\mathbb{Z}/2\mathbb{Z})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ has trivial boundary operators. The degree of each chain $\langle V(\mu)\rangle$ with $\mu\in S(m,n)$ equals the rank $\rho(\mu)$. Therefore the $\mathbb{Z}/2\mathbb{Z}$-Poincaré polynomial of $Gr(m,n)$ is coincide with the q-binomial coefficient. □

Proposition 7.6. *The generating function $A(q)$ of the ranks $a(k)$ of the torsion subgroups of $H_*(Gr(m,n);\mathbb{Z})$ is given by*

$$A(q)=\begin{cases}\frac{1}{1+q}\left(\begin{bmatrix} m+n\\ m\end{bmatrix}_q-\begin{bmatrix}\lfloor (m+n)/2\rfloor\\ \lfloor m/2\rfloor\end{bmatrix}_{q^4}\right), & \text{if } mn \text{ is even,}\\[2ex] \frac{1}{1+q}\left(\begin{bmatrix} m+n\\ m\end{bmatrix}_q-(1+q^{m+n-1})\begin{bmatrix}(m+n)/2-1\\ (m-1)/2\end{bmatrix}_{q^4}\right), & \text{if } mn \text{ is odd}\end{cases}$$

Proof. Let $\{b(k)\}$ and $\{c(k)\}$ be the sequences given in Propositions 7.4 and 7.5. Since the homology group $H_*(Gr(m,n);\mathbb{Z})$ satisfies (7.2), the universal coefficient theorem asserts that

$$c(k)=a(k-1)+a(k)+b(k)$$

holds for each k. In other words, the generating functions $A(q)=\sum_{k=0}^{mn}a(k)q^k$, $B(q)=\sum_{k=0}^{mn}b(k)q^k$, $C(q)=\sum_{k=0}^{mn}c(k)q^k$ satisfy

$$C(q)=(1+q)A(q)+B(q).$$

The formulas in Propositions 7.4 and 7.5 are combined to complete the proof. □

8. The homology group of the oriented 2-plane Grassmannian

We compute the homology group $H_*(\widetilde{Gr}(2,n);\mathbb{Z})$ via the Morse complex $\mathcal{C}(\widetilde{h}_\Lambda)$ obtained in Theorem 6.3.

For each $\mu\in S(2,n)$ and $\varepsilon\in\{+,-\}$, we write $V(\mu(1),\mu(2))^\varepsilon$ in place of $V(\mu)^\varepsilon$. The k-dimensional chain group $\mathcal{C}(\widetilde{h}_\Lambda)_k$ is generated by $\langle V(i,j)^\varepsilon\rangle$ with $\varepsilon=\pm$ and

$$(i,j)=\begin{cases}(1,k+2),\cdots,(\lfloor\frac{k}{2}\rfloor+1,\lceil\frac{k}{2}\rceil+2) & \text{if } k\le n,\\[1ex] (k-n+1,n+2),\cdots,(\lfloor\frac{k}{2}\rfloor+1,\lceil\frac{k}{2}\rceil+2) & \text{if } n<k.\end{cases}$$

Theorem 6.3 asserts that the boundary of a chain $\langle V(i,j)^\varepsilon\rangle$ is

$$\begin{aligned}\widetilde{\partial}\langle V(i,j)^\varepsilon\rangle=&-(-1)^i\langle V(i-1,j)^\varepsilon\rangle-\left\langle V(i-1,j)^{-\varepsilon}\right\rangle\\ &+(-1)^{i+j}\langle V(i,j-1)^\varepsilon\rangle+(-1)^i\left\langle V(i,j-1)^{-\varepsilon}\right\rangle,\end{aligned}\tag{8.1}$$

where the terms with $i-1=0$ or $i=j-1$ should be omitted.

We first compute the homology group $H_n(\widetilde{Gr}(2,n);\mathbb{Z})$ of middle dimension in the cases n is even or odd, separately, and then compare $H_n(\widetilde{Gr}(2,n+1);\mathbb{Z})$ with $H_n(\widetilde{Gr}(2,n);\mathbb{Z})$ as an inductive step. Finally Poincaré duality is applied to obtain the ranks of $H_k(\widetilde{Gr}(2,n);\mathbb{Z})$ with $n<k\leq 2n$.

8.1. $H_n(\widetilde{Gr}(2,n);\mathbb{Z})$ **with** $n=2k$ **even.** Applying Formula (8.1) to compute the boundary of a chain

$$c=\sum_{i=1}^{k+1}\left(x_i\left\langle V(i,2k+3-i)^+\right\rangle+y_i\left\langle V(i,2k+3-i)^-\right\rangle\right)$$

in $\mathcal{C}(\widetilde{h}_\Lambda)_{2k}$, we get a list of independent equations of $\ker\widetilde{\partial}_{2k}$;

$$0=-x_i+(-1)^i y_i+(-1)^i x_{i+1}-y_{i+1}\quad (i=1,\cdots,k). \tag{8.2}$$

Let $\xi:\mathcal{C}(\widetilde{h}_\Lambda)_{2k}\to\mathbb{Z}^{k+2}$ be a homomorphism defined by

$$\sum_{i=1}^{k+1}\left(x_i\left\langle V(i,2k+3-i)^+\right\rangle+y_i\left\langle V(i,2k+3-i)^-\right\rangle\right)\mapsto(y_1,\cdots,y_k,x_{k+1},y_{k+1}).$$

Then the restriction of ξ to $\ker\widetilde{\partial}$ is bijective.

On the other hand, it holds that

$$\widetilde{\partial}\left\langle V(i+1,2k+3-i)^+\right\rangle=-(-1)^i\widetilde{\partial}\left\langle V(i+1,2k+3-i)^-\right\rangle.$$

Let $c_i\in\mathcal{C}(\widetilde{h}_\Lambda)_{2k}$ $(i=1,\cdots,k+2)$ be chains defined by

$$\begin{aligned} c_i &= \widetilde{\partial}\left\langle V(i+1,2k+3-i)^+\right\rangle,\quad (i=1,\cdots,k)\\ c_{k+1} &= (1,-1,-1,1\cdots)\in\mathcal{C}(\widetilde{h}_\Lambda)_{2k}\cong\mathbb{Z}^{2k+2},\\ c_{k+2} &= (1,-1,0,\cdots,0)\in\mathcal{C}(\widetilde{h}_\Lambda)_{2k}\cong\mathbb{Z}^{2k+2}.\end{aligned} \tag{8.3}$$

From Equation (8.2), it follows that c_{k+1} and c_{k+2} are independent cycles, and that the images $\xi(c_i)\in\mathbb{Z}^{k+2}$ of those cycles c_i with $i=1,\cdots,k+2$ form a unimodular matrix. Thus the homology group $H_{2k}(\widetilde{Gr}(2,2k);\mathbb{Z})$ is generated by the cycles c_{k+1} and c_{k+2}, and isomorphic to $\mathbb{Z}^2$.

8.2. $H_n(\widetilde{Gr}(2,n);\mathbb{Z})$ **with** $n=2k+1$ **odd.** Now we consider the case $n=2k+1$. By using (8.1), we compute the boundary of a chain

$$c=\sum_{i=1}^{k+1}\left(x_i\left\langle V(i,2k+4-i)^+\right\rangle+y_i\left\langle V(i,2k+4-i)^-\right\rangle\right),$$

and get a list of equations of the kernel $\ker \widetilde{\partial}_{2k+1} \subset \mathcal{C}(\widetilde{h}_\Lambda)_{2k+1} \cong \mathbb{Z}^{2k+2}$, that is,

$$0 = x_i + (-1)^i y_i, \quad (i = 1, \cdots, k+1). \tag{8.4}$$

Let $\xi : \mathcal{C}(\widetilde{h}_\Lambda)_{2k+1} \to \mathbb{Z}^{k+1}$ be a homomorphism defined by $\xi(c) = (y_1, \cdots, y_{k+1})$. The restriction of ξ to the kernel $\ker \widetilde{\partial}_{2k+1}$ is an isomorphism. By a simple computation, we find that the images $\xi\left(\widetilde{\partial}_{2k+2}\left(\langle V(i, 2k+5-i)^+\rangle\right)\right)$ in $\mathbb{Z}^{k+1}$ with $i = 1, \cdots, k+1$ form a unimodular matrix, which implies that $H_{2k+1}\left(\widetilde{Gr}(2, 2k+1); \mathbb{Z}\right) = \{0\}$.

8.3. **Inductive step.** Let $\mathcal{C}_*(n)$ denote the Morse complexes of $\widetilde{h}_\Lambda : \widetilde{Gr}(2, n) \to \mathbb{R}$. By comparing the Hasse diagrams of the posets $S(2, n)$ and $S(2, n+1)$, it is clear that the chain complexes $\mathcal{C}_q(n)$ and $\mathcal{C}_q(n+1)$ in dimension $q \leq n$ are isomorphic;

$$\begin{array}{ccc} \mathcal{C}_q(n+1) & \xrightarrow{\widetilde{\partial}_q(n+1)} & \mathcal{C}_{q-1}(n+1) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{C}_q(n) & \xrightarrow{\widetilde{\partial}_q(n)} & \mathcal{C}_{q-1}(n) \end{array}$$

$(q = 1, \cdots, n)$, and hence that the homology groups $H_q(\widetilde{Gr}(2, n))$ and $H_q(\widetilde{Gr}(2, n+1))$ are isomorphic for $q = 0, \cdots, n-1$.

In the case n is odd, we have shown that $H_n(\widetilde{Gr}(2, n); \mathbb{Z}) = \{0\}$. Since $\mathcal{C}_{n+1}(n+1) \supset \mathcal{C}_{n+1}(n)$, it holds that

$$\operatorname{Im} \widetilde{\partial}_{n+1}(n+1) \supset \operatorname{Im} \widetilde{\partial}_n(n+1) = \ker \widetilde{\partial}_n(n+1) = \ker \widetilde{\partial}_{n+1}(n+1),$$

where $\mathcal{C}_n(n+1)$ and $\mathcal{C}_n(n)$ are naturally identified. This implies that $H_n(\widetilde{Gr}(2, n+1); \mathbb{Z})$ also vanishes.

If n is even, the homology group $H_n(\widetilde{Gr}(2, n); \mathbb{Z}) \cong \mathbb{Z}^2$ is generated by two cycles c_{k+1} and c_{k+2} of $\mathcal{C}_n(n)$ as described in (8.3). In the chain complex $\mathcal{C}_n(n+1)$, the boundary $\widetilde{\partial}(\mathcal{C}_{n+1}(n+1))$ is the same as $\widetilde{\partial}(\mathcal{C}_n(n+1))$ except the images $\widetilde{\partial}_{n+1}\langle V(1, n+2)^\pm\rangle$, which equals $-c_{k+2}$. This implies that $H_n(\widetilde{Gr}(2, n+1); \mathbb{Z}) \cong \mathbb{Z}$. We complete the induction step.

We obtain the following:

Proposition 8.1. *Let n be a positive integer, and let q be nonnegative and less than or equal to $2n = \dim \widetilde{Gr}(2, n)$. Then it holds that*

$$H_q(\widetilde{Gr}(2, n); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } q \text{ is even and } q \neq n \\ \mathbb{Z}^2, & \text{if } n \text{ is even and } q = n \\ \{0\}, & \text{otherwise.} \end{cases} \tag{8.5}$$

Proof. In the case $n = 1$, the formula is trivial, since $\widetilde{Gr}(2, 1) \cong S^2$. In the case $n > 1$, we have shown the formula holds for $q \leq n$ by induction. We complete the proof by the Poincaré duality. □

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