Mixed objects are embedded into log pure objects

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Abstract

We prove that a variation of mixed Hodge structure is embedded in a logarithmic variation of pure Hodge structure, and a generalized version of this result. These results suggest some simple construction of the category of mixed motives by using log pure motives.

Introduction

0.1. In this paper, we develop our idea in [7] that a mixed object is embedded in a log pure object. We improve the result in [7] on this idea (Theorem 1.2) and propose a simple construction of the category of mixed motives over a field based on this idea without assuming any conjecture (Appendix in this paper).

0.2. The following is a standard example concerning this idea.

Let Δ be the unit disc $\{q \in \mathbb{C} \mid |q| < 1\}$, and let $\mathfrak{X}$ be a smooth complex manifold with a projective flat morphism $\mathfrak{X} \to \Delta$ which is smooth outside $0 \in \Delta$ and is of semistable reduction at $0 \in \Delta$. For $t \in \Delta$, let $\mathfrak{X}_t \subset \mathfrak{X}$ be the fiber over $t \in \Delta$. Then we have the mixed Hodge structure $H^1(\mathfrak{X}_0, \mathbb{Z})$. This mixed Hodge structure is embedded in the limit mixed Hodge structure "$\lim_{t \neq 0, t \to 0}\ H^1(\mathfrak{X}_t, \mathbb{Z})$" $\supset H^1(\mathfrak{X}_0, \mathbb{Z})$, and this limit mixed Hodge structure is associated with the log pure Hodge structure $H = H^1((\mathfrak{X}_0 \text{ with log})/(0 \text{ with log}), \mathbb{Z})$ of weight 1 on the standard log point $0 \in \Delta$. Thus the mixed object $H^1(\mathfrak{X}_0, \mathbb{Z})$ is embedded in the log pure object H.

0.3. In [7], we proved that a mixed Hodge structure is embedded in a log pure Hodge structure, which is the case $n = 0$ of the following more general result proved in [7]: A nilpotent orbit of mixed Hodge structures with n monodromy operators is embedded in a nilpotent orbit of pure Hodge structures with one more monodromy operators. This general result was successfully applied in [7] to deduce the SL(2)-orbit theorem for the degeneration of mixed Hodge structure from the SL(2)-orbit theorem of Cattani–Kaplan–Schmid ([3]) for degeneration of pure Hodge structure.

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In this paper, we prove the following further generalization (Theorem 1.2 in Section 1) of the result in [7]: A log mixed Hodge structure on an fs log analytic space X with polarizable graded quotients for the weight filtration is, locally on X, embedded into a log pure Hodge structure on $X \times S$, where S is the standard log point. The text of this paper (Sections 2–6) is devoted to the proof of this theorem.

0.4. In the theory of mixed motives over a field k, a big question is how to define the set of morphisms of mixed motives

$$(1) \quad h(Y)(r) \to h(Z)(s)$$

for schemes Y, Z of finite type over k and for $r, s \in \mathbb{Z}$. Here $h(Y)$ is the mixed motive associated with Y whose ℓ-adic realization for a prime number $\ell \neq \mathrm{char}(k)$ is $\bigoplus_m H^m_{\text{ét}}(Y \otimes_k \overline{k}, \mathbb{Q}_\ell)$ and (r) means the Tate twist. By the construction of the category (MM$**$) of mixed motives over k in A.20 in Appendix, we answer this question as follows. We define the category (LM$\flat**$) of limit mixed motives associated with log pure motives by using certain K-groups as the sets of morphisms, and define the mixed motive $h(Y)(r)$ as a functor from (LM$\flat**$) to the category of $\mathbb{Q}$-vector spaces, by using certain K-groups. Thus a morphism (1) is a morphism of functors.

We hope that this method is justified by its Hodge version (A.1): Our result on Hodge theory tells in particular that we can regard a mixed Hodge structure as a functor on the category of limit mixed Hodge structures associated with log pure Hodge structures.

The notion mixed motive is more difficult than the notion log pure motive (the latter is just the logarithmic version of the pure motive of Grothendieck) and our hope is that the difficult objects mixed motives are well-understood by using log pure motives which are simpler.

This Appendix (Section A), which discusses the motive theory, is independent of the text and one can read it first.

1 The results

1.1. As in [12], let $\mathcal{A}(\log)$ be the category of fs log analytic spaces (i.e., complex analytic spaces with fs log structures) and let $\mathcal{B}(\log) \supset \mathcal{A}(\log)$ be the category of locally ringed spaces over $\mathbb{C}$ with log structures which are locally subspaces of objects of $\mathcal{A}(\log)$ with the strong topologies ([12] 3.2).

Fix a subring R of $\mathbb{R}$.

Let X be an object of $\mathcal{B}(\log)$ and let H be as in one of the following (1) and (2).

(1) H is an R-log mixed Hodge structure (R-LMH) on X.
(2) H is an R-log variation of mixed Hodge structure (R-LVMH) on X.

For the definitions of R-LMH and R-LVMH, for the definitions of R-polarized log Hodge structure (R-PLH) and R-log variation of polarized Hodge structure (R-LVPH), and for the pre-versions (pre-R-LMH, etc.), cf. [12] 2.6 and [9] 1.3, where the cases $R = \mathbb{Z}$ are treated. The difference of R-LMH (resp. R-PLH) and R-LVMH (resp. R-LVPH) lies in that the latter must satisfy the big Griffiths transversality though the small Griffiths transversality is satisfied by the former ([12] 2.4.9).

In both situations (1) and (2), we assume that H satisfies the following conditions (i) and (ii).

(i) The local system H_R is locally free as a sheaf of R-modules on $X^{\log}$. Furthermore, $W_wH_R := H_R \cap W_w(H_R \otimes_{\mathbb{Z}} \mathbb{Q}) \subset H_R \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{gr}^W_w H_R := W_wH_R/W_{w-1}H_R$ for all w are locally free as sheaves of R-modules on $X^{\log}$.

(ii) For each w, there is an R-perfect $(-1)^w$-symmetric bilinear form $\mathrm{gr}^W_w H_R \times \mathrm{gr}^W_w H_R \to R \cdot (2\pi i)^{-w}$ which gives a polarization of $\mathrm{gr}^W_w H$.

If R is a field (as in the important cases $R = \mathbb{Q}$, $R = \mathbb{R}$), the condition (i) is empty and the condition (ii) simply says that H has polarizable gr^W.

The aim of this paper is to prove

Theorem 1.2. *Assume that we are in the situation* (1) *(resp.* (2)*) in* 1.1*. Let S be the standard log point. Then locally on X, there are an R-PLH (resp. R-LVPH) H' on $X \times S$ and an injective homomorphism $H_R \to H'_R$ of the local systems of R-modules on $(X \times S)^{\log}$ satisfying the following conditions* (i), (ii), (a), *and* (b) *below. If $W_wH = H$, there is such an H' of weight w.*

(i) *H'_R and H'_R/H_R are locally free as sheaves of R-modules on $(X \times S)^{\log}$.*

(ii) *The polarization of H' is given by an R-perfect $(-1)^{w'}$-symmetric bilinear form $H'_R \times H'_R \to R \cdot (2\pi i)^{-w'}$, where w' is the weight of H'.*

(These conditions (i) *and* (ii) *are automatically satisfied if R is a field.)*

(a) *The Hodge filtration of H is the restriction of that of H'. More precisely, the Hodge filtration of H on $H_{\mathcal{O}} = (\tau_X)_*(\mathcal{O}^{\log}_X \otimes_R H_R) = (\tau_{X\times S})_*(\mathcal{O}^{\log}_{X\times S} \otimes_R H_R)$ coincides with the restriction of the Hodge filtration of H' on $H'_{\mathcal{O}} = (\tau_{X\times S})_*(\mathcal{O}^{\log}_{X\times S} \otimes_R H'_R)$.*

(b) *The weight filtration of H is the restriction of the relative monodromy filtration of H'. More precisely, for every $t \in (X \times S)^{\log}$, the weight filtration of H on the stalk $H_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the restriction of the relative monodromy filtration on $H'_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ of the logarithm $H'_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q} \to H'_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ of the action of the standard generator of $\pi_1(S^{\log})$.*

Remark 1.3. (1) By duality, we have a result in which we replace the injection $H_R \to H'_R$ in Theorem 1.2 by a surjection $H'_R \to H_R$ and change the conditions (i), (a), and (b) accordingly.

(2) [7] Proposition 4.1 is a slightly weaker version of the case where X is an fs log point of this theorem. The structure of the proof of the above theorem given below is similar to that of the proof of [7] Proposition 4.1 given in Sections 6 and 7 of [7].

(3) On the other hand, the case of Theorem 1.2 where $X = (\mathrm{Spec}\,\mathbb{C}, \mathbb{C}^\times \oplus \mathbb{N}^n)$ implies that we can take all the a_{jk} to be 0 unless $j = k$ in [7] Proposition 4.1 (cf. the remark after ibid. Proposition 4.1). As explained in ibid. 5.9, this gives a characterization of $\mathbb{R}$-IMHM ([7] 5.2, [6]) without using relative monodromy filtrations. We state this below as Proposition 1.4.

(4) When X has the trivial log structure, this theorem implies the following. A variation of mixed Hodge structure with polarizable graded quotients on a complex analytic manifold X is, locally on X, embedded in a log variation of polarized Hodge structure.

Proposition 1.4. *Let* $(V, W, N_1, \dots, N_n, F)$ *be a pre-*$\mathbb{R}$*-IMHM* ([7] 5.2*). It is an* $\mathbb{R}$*-IMHM if and only if there is a pure nilpotent orbit* $(V', w, N'_0, \dots, N'_n, F')$ *and a surjective homomorphism* $(V', W(N'_0)[-w], N'_1, \dots, N'_n, F') \to (V, W, N_1, \dots, N_n, F)$ *of pre-*$\mathbb{R}$*-IMHMs.*

1.5. Inspired by Remark 1.3 (4), we expect that a motive theoretic version of the above theorem exists, that is, that a mixed motive can be embedded into a log pure motive. Based on this idea, we construct the category of mixed motives over a field in Appendix (Section A) by using log pure motives.

2 Preparation on log Hodge theory

We prove two propositions on log Hodge theory together. Proposition 2.1 will be used in the last part of the proof of Theorem 3.2. Proposition 2.2 will be used in the proof of Lemma 5.2.

Proposition 2.1. *Let* X *be an object of* $\mathcal{B}(\log)$*, let* R *be a subfield of* $\mathbb{R}$*, and let* H *be a pre-*R*-LMH on* X *satisfying the small Griffiths transversality. Assume that for each* $w \in \mathbb{Z}$*, we are given a* $(-1)^w$*-symmetric pairing* $\langle\cdot,\cdot\rangle_w : \mathrm{gr}^W_w H \otimes \mathrm{gr}^W_w H \to R(-w)$ *which induces an isomorphism* $\mathrm{gr}^W_w H \overset{\cong}{\to} (\mathrm{gr}^W_w H)^*(-w)$ *of pre-*R*-LMH, where* $(\cdot)^*$ *denotes the dual. Let* U *be the set of all* $x \in X$ *such that the pullback of* H *to the fs log point* x *is an* R*-LMH and such that* $\langle\cdot,\cdot\rangle_w$ *is a polarization for every* $w \in \mathbb{Z}$*. Then* U *is an open set of* X*.*

Proposition 2.2. *Let* X *be an object of* $\mathcal{B}(\log)$*, let* R *be a subring of* $\mathbb{R}$*, and let* H *be an* R*-LMH on* X *with polarized* gr^W *satisfying the condition* 1.1 (i)*. Then locally on* X*, there are a log manifold* Z *(for a log manifold, see* [12] *Definition* 3.5.7*) and a morphism* $X \to Z$ *of* $\mathcal{B}(\log)$ *such that* H *is the pullback of an* R*-LMH on* Z *with polarized* gr^W *satisfying the condition* 1.1 (i)*.*

Remark 2.3. In the case of $R = \mathbb{Z}$ or $\mathbb{Q}$, under the assumption that the local monodromy of H at each point of X is contained in a sharp cone, Proposition 2.2 is a consequence of the existence of the moduli space of LMH with polarized gr^W and of the fact that this moduli space is a log manifold as treated in [12] and [9]. Here we treat an R-LMH without such an assumption on local monodromy. The proof of Proposition 2.2 uses arguments in [12] 2.3.7 and [12] Section 8, and the space E (resp. $\check{E}$) which appears in 2.10 below is a variant of the space E_σ (resp. $\check{E}_\sigma$) in [12] and [9].

2.4. Let X be an object of $\mathcal{B}(\log)$. Assume that we are given a pre-R-LMH H on X. For the proof of Proposition 2.1 (resp. 2.2), we assume that we are given $\langle\cdot,\cdot\rangle_w$ on $\mathrm{gr}^W_w H$ for each w which is as in the hypothesis of Proposition 2.1 (resp. which is a polarization).

2.5. Let $s \in X$ and let t be a point of $X^{\log}$ lying over s. We work around s. Let $(q_j)_{1 \le j \le n}$ be a finite family of local sections of M_X around s which forms a $\mathbb{Z}$-base of $(M^{\mathrm{gp}}_X/\mathcal{O}^\times_X)_s$. For $1 \le j \le n$, let γ_j be the element of $\pi_1(s^{\log}) = \mathrm{Hom}\,((M^{\mathrm{gp}}_X/\mathcal{O}^\times_X)_s, \mathbb{Z})$ which sends q_j to 1 and q_k to 0 for all $k \neq j$ (see [12] 2.2.9 for this identification). Then the action of γ_j on $H_{R,t}$ is unipotent ([12] Proposition 2.3.3 (ii)). Let $N_j = \log(\gamma_j) : H_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q} \to H_{R,t} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let z_j be a branch of $(2\pi i)^{-1}\log(q_j)$ around t. Then on an open neighborhood of t in $X^{\log}$, concerning $H_{\mathcal{O}} = \tau_*(\mathcal{O}_X^{\log} \otimes_R H_R)$, we have

$$H_{\mathcal{O}} = \exp\Big(\sum_{j=1}^{n} z_j N_j\Big)(\mathcal{O}_X \otimes_R H_R).$$

2.6. By replacing X by an open neighborhood of s in X if necessary, we may assume that there is a chart $\mathcal{S} \to M_X$ with an fs monoid $\mathcal{S}$ such that $\mathcal{S} \xrightarrow{\cong} (M_X/\mathcal{O}_X^\times)_s$. Let $\mathcal{T} = \mathrm{Spec}(\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ and let $X \to \mathcal{T}$ be the morphism induced by the composition $\mathcal{S} \to M_X \to \mathcal{O}_X$. This induces an isomorphism from s to the "origin" of $\mathcal{T}$. Since the induced map $s^{\log} \to \mathcal{T}^{\log}$ is a homotopy equivalence, the restriction of H_R to $s^{\log}$ extends uniquely to a local system $H_{R,\mathcal{T}}$ on $\mathcal{T}^{\log}$ which also has a weight filtration W and the family $(\langle\cdot,\cdot\rangle_w)_w$ of pairings. By the properness of $X^{\log} \to X$, for some open neighborhood V of s in X, we have an isomorphism between the pullbacks of $(H_R, W, (\langle\cdot,\cdot\rangle_w)_w)$ and $(H_{R,\mathcal{T}}, W, (\langle\cdot,\cdot\rangle_w)_w)$ to $V^{\log}$.

2.7. Let $H_0 := H_{R,t}$. We identify H_0 with the stalk of $H_{R,\mathcal{T}}$ at the image of t in $\mathcal{T}^{\log}$. Let $\Gamma := \pi_1(s^{\log}) = \pi_1(\mathcal{T}^{\log}) = \mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbb{Z})$. Then Γ acts on H_0, and the local system $H_{R,\mathcal{T}}$ has a canonical Γ-level structure with respect to the constant sheaf H_0 (that is, we have a canonical global section of the quotient sheaf $\mathcal{I}/\Gamma$ on $\mathcal{T}^{\log}$, where $\mathcal{I}$ is the sheaf of isomorphisms from $H_{R,\mathcal{T}}$ to H_0). Hence we have a canonical Γ-level structure on H_R with respect to H_0 on $V^{\log}$ for some open neighborhood V of s in X. We may assume that $X = V$.

2.8. The following happens on $\mathcal{T}^{\log}$ ([12] 2.3.7). We can regard H_0 as a constant subsheaf of $\mathcal{O}_{\mathcal{T}}^{\log} \otimes_R H_{R,\mathcal{T}}$ as follows.

Since $\Gamma = \pi_1(\mathcal{T}^{\log})$ is commutative, Γ acts on $H_{R,\mathcal{T}}$.

Define a local system H_0' on $\mathcal{T}^{\log}$ as follows. Taking $(q_j)_{1 \le j \le n}$ which is a $\mathbb{Z}$-base of $\mathcal{S}^{\mathrm{gp}}$, let

$$H_0' := \xi H_{R,\mathcal{T}} \subset \mathcal{O}_{\mathcal{T}}^{\log} \otimes_R H_{R,\mathcal{T}} \quad \text{with } \xi = \exp\Big(\sum_{j=1}^{n} z_j N_j\Big).$$

Here ξ depends on the choices of the branches z_j of $(2\pi i)^{-1}\log(q_j)$ $(1 \le j \le n)$, but H_0' is independent of the choice. Furthermore, $\xi \bmod \Gamma$ is independent of the choice of the $\mathbb{Z}$-base $(q_j)_j$ of $\mathcal{S}^{\mathrm{gp}}$.

Then by 2.5, H_0' descends to a local system on $\mathcal{T}$. Since $\mathcal{T}$ is contractible, H_0' is a constant sheaf. We have an isomorphism

$$H_0' \xrightarrow{\cong} H_0$$

by using a ring homomorphism $\mathcal{O}_{X,t}^{\log} \to \mathbb{C}$ which extends the evaluation $\mathcal{O}_{X,s} \to \mathbb{C}$ by $z_j \mapsto 0$.

We identify H_0' and H_0 via this isomorphism.

We regard H_0 as a constant sheaf on $\mathcal{T}$ via the above identification. We regard H_0 also as a constant sheaf on X. We have

$$\mathcal{O}_{\mathcal{T}}^{\log} \otimes_{\mathbb{Z}} H_{R,\mathcal{T}} = \mathcal{O}_{\mathcal{T}}^{\log} \otimes_R H_0, \quad \mathcal{O}_X^{\log} \otimes_R H_R = \mathcal{O}_X^{\log} \otimes_R H_0,$$

$$\tau_*(\mathcal{O}_{\mathcal{T}}^{\log} \otimes_{\mathbb{Z}} H_{R,\mathcal{T}}) = \mathcal{O}_{\mathcal{T}} \otimes_R H_0, \quad \tau_*(\mathcal{O}_X^{\log} \otimes_R H_R) = \mathcal{O}_X \otimes_R H_0.$$

Thus on X, $H_{\mathcal{O}} = \tau_*(\mathcal{O}_X^{\log} \otimes_R H_R)$ is identified with $\mathcal{O}_X \otimes_R H_0$.

Note that in the formula (1) in [12] 2.3.7, "with $\nu =$" should be replaced by "with $\xi =$".

2.9. Let h_w^p be the $\mathbb{C}$-dimension of the gr_F^p of gr_w^W of H at s. Note that we have W and $\langle \cdot, \cdot \rangle_w$ on $\mathrm{gr}_w^W H_{0,\mathbb{Q}}$.

Let $\check{D}$ be the space of all descending filtrations F on $H_{0,\mathbb{C}}$ such that the rank of gr_F^p of gr_w^W is the given h_w^p and such that the annihilator of $F^p \mathrm{gr}_{w,\mathbb{C}}^W$ in $\mathrm{gr}_{w,\mathbb{C}}^W$ under $\langle \cdot, \cdot \rangle_w$ is $F^{w+1-p} \mathrm{gr}_{w,\mathbb{C}}^W$. Then $\check{D}$ is a complex analytic manifold. See [19] for basic properties of $\check{D}$.

The Hodge filtration on $H_{\mathcal{O}} = \mathcal{O}_X \otimes_R H_0$ on X gives a morphism $X \to \check{D}$, and the Hodge filtration on $H_{\mathcal{O}}$ is the pullback of the universal Hodge filtration on $\mathcal{O}_{\check{D}} \otimes_R H_0$. Thus we have a morphism $X \to \check{E} := \mathcal{T} \times \check{D}$.

2.10. On $\check{E}$, we have the local system, the pullback $H_{R,\check{E}}$ of $H_{R,\mathcal{T}}$ with W and $\langle \cdot, \cdot \rangle_w$, and we have the Hodge filtration on $\mathcal{O}_{\check{E}} \otimes_R H_0$ which is the pullback of the universal Hodge filtration of $\mathcal{O}_{\check{D}} \otimes_R H_0$. We have also an isomorphism $\mathcal{O}_{\check{E}}^{\log} \otimes_R H_{R,\check{E}} \cong \mathcal{O}_{\check{E}}^{\log} \otimes_R H_0$. We denote this object by $H_{\check{E}}$. The H on X is the pullback of this $H_{\check{E}}$ under the canonical morphism (period map) $X \to \check{E}$.

2.11. Let $\tilde{E}$ (resp. E) be the set of all points z of $\check{E}$ such that the pullback of $H_{\check{E}}$ to z satisfies the Griffiths transversality (resp. is an R-LMH with polarized gr^W), and endow $\tilde{E}$ (resp. E) with the strong topology in $\check{E}$ in the sense of [12] Section 3.1 and with the inverse images of $\mathcal{O}_{\check{E}}$ and the log structure of $\check{E}$. Then E is an open set of $\tilde{E}$, and E and $\tilde{E}$ are log manifolds. This is seen by the arguments in [12] Section 7, [10] Appendix A.1, and [11] 4.5.

Let $H_{\tilde{E}}$ (resp. H_E) be the pullback of $H_{\check{E}}$ to $\tilde{E}$ (resp. E).

2.12. Since $X \to \check{E}$ is strict, for $x \in X$ with the image z in $\check{E}$, the pullback of H to the fs log point x satisfies the Griffiths transversality (resp. is an R-LMH with polarized gr^W) if and only if the pullback of $H_{\check{E}}$ to z has the same property.

2.13. We prove Proposition 2.1. Let $X \to \check{E}$ be as above. Assume that H satisfies the small Griffiths transversality. Then the morphism $X \to \check{E}$ factors through $X \to \tilde{E}$, and U is the inverse image of E. Since E is open in $\tilde{E}$, U is open in X.

2.14. We prove Proposition 2.2. Let $X \to \check{E}$ be as above. Assume that H is an R-LMH with polarized gr^W. Then the morphism $X \to \check{E}$ factors through $E \subset \check{E}$ and H is the pullback of H_E.

3 Polarized log mixed Hodge structure and PLH

3.1. Let X be an object of $\mathcal{B}(\log)$, let S be the standard log point with a fixed generator $q \in M_S$, let $w \in \mathbb{Z}$, and let R be a subfield of $\mathbb{R}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the following categories.

Let $\mathcal{C}_1$ be the category of pre-R-PLH (resp. pre-R-LVPH) P on $X \times S$ of weight w satisfying the following condition (a).

(a) Locally on X, for some morphism $X \times S \to X \times S$ over X, the pullback of P is an R-PLH on the left $X \times S$.

Let $\mathcal{C}_2$ be the category of R-LMH (resp. R-LVMH) H on X endowed with the following structures (i) and (ii) and satisfying the conditions (1) and (2).

(i) A homomorphism $H \otimes H \to R(-w)$ in the category of R-LMH such that the induced pairing $\langle \cdot, \cdot \rangle : H_R \times H_R \to R \cdot (2\pi i)^{-w}$ is non-degenerate and $(-1)^w$-symmetric.

(ii) A homomorphism $N : H \to H(-1)$ such that $N^n : H \to H(-n)$ is zero for some $n \geq 1$ and such that $\langle Nu, v\rangle + \langle u, Nv\rangle = 0$.

(1) The weight filtration on H_R coincides with $W(N)[-w]$, where $W(N)$ is the monodromy filtration of N.

(2) Let $k \geq w$ and let Prim_k be the primitive part of $\mathrm{gr}_k H_R$ for N. Then the pairing $\mathrm{Prim}_k \times \mathrm{Prim}_k \to R(-k)$; $(u, v) \mapsto \langle u, N^{k-w}v\rangle$ is a polarization of the pure PLH Prim_k of weight k.

Morphisms in $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined to be isomorphisms in the evident sense.

Note that for an object H of $\mathcal{C}_2$ and for $k \in \mathbb{Z}$, $\mathrm{gr}^W_k H$ is endowed with the polarization defined by the decomposition of $\mathrm{gr}^W_k H$ as the direct sum of various primitive parts which are endowed with the polarizations in the condition (2).

Theorem 3.2. *We have an equivalence $\mathcal{C}_1 \simeq \mathcal{C}_2$.*

$\mathcal{C}_1 \to \mathcal{C}_2$; $P \mapsto H$ is as follows. Let β be the canonical map $(X \times S)^{\log} = X^{\log} \times S^{\log} \to X^{\log}$. Then $H_R = \beta_*(\exp(\log(q)N)P_R)$, where $N = (2\pi i)^{-1}\log(\gamma)$ for the action γ of the canonical generator of $\pi_1(S^{\log})$. $H_{\mathcal{O}} = P_{\mathcal{O}}$ with the same Hodge filtration. The isomorphism $\mathcal{O}^{\log}_X \otimes_R H_R \cong \mathcal{O}^{\log}_X \otimes_{\mathcal{O}_X} H_{\mathcal{O}}$ is induced from the corresponding isomorphism for P.

$\mathcal{C}_2 \to \mathcal{C}_1$; $H \mapsto P$ is as follows. $P_R = H^{(N)}_R := \exp(-\log(q)N)H_R \subset \mathcal{O}^{\log}_{X\times S} \otimes_R H_R$. $P_{\mathcal{O}} = H_{\mathcal{O}}$ with the same Hodge filtration. The isomorphism $\mathcal{O}^{\log}_{X\times S} \otimes_R P_R \cong \mathcal{O}^{\log}_{X\times S} \otimes_{\mathcal{O}_X} P_{\mathcal{O}}$ is induced from the corresponding isomorphism for H.

3.3. Here, in the argument of $P \mapsto H$, the inverse image of H_R on $(X \times S)^{\log}$ is $\exp(\log(q)N)P_R$. In fact, γ does not change $\exp(\log(q)N)a$ for an element a of the stalk of P_R because $\gamma = (\gamma^*)^{-1}$ (γ^* is the pullback by γ) sends $\log(q)$ to $\log(q) - 2\pi i$ and hence $\gamma(\exp(\log(q)N)a) = \exp((\log(q) - 2\pi i)N)\exp(2\pi i N)a = \exp(\log(q)N)a$.

3.4. We first prove Theorem 3.2 in the case where X is an fs log point and $R = \mathbb{R}$. In this case, Theorem 3.2 is equivalent to the following Proposition 3.5. See 3.6. The case $n = 0$ of Proposition 3.5 is the well-known relation between nilpotent orbits and polarized mixed Hodge structures in [18] Theorem (6.16) and in [3] (3.13).

Proposition 3.5. *Let V be a finite dimensional $\mathbb{R}$-vector space. Let $w \in \mathbb{Z}$ and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be a non-degenerate $(-1)^w$-symmetric $\mathbb{R}$-bilinear form. Let $N_0, N_1, \ldots, N_n : V \to V$ be mutually commuting nilpotent linear operators such that $\langle N_j u, v\rangle + \langle u, N_j v\rangle = 0$ for all u, v in V and $0 \leq j \leq n$. Let $W = W(N_0)[-w]$, where $W(N_0)$ is the monodromy filtration of N_0. Let F be a descending filtration on $V_{\mathbb{C}}$ such that the annihilator of F^p for $\langle \cdot, \cdot \rangle$ is F^{w+1-p} for every p.*

Then, $(V, \langle \cdot, \cdot \rangle, N_0, aN_0+N_1, \ldots, aN_0+N_n, F)$ generates a pure nilpotent orbit of weight w for some, hence for any $a \gg 0$ if and only if the following two conditions (i) *and* (ii) *are satisfied.*

(i) $(V, W, N_1, \dots, N_n, F)$ *generates a mixed nilpotent orbit, where the polarizaitions on the graded pieces are determined by the following* (ii).

(ii) *Let* $k \geq w$*, let* $P_k \subset \mathrm{gr}_k^W$ *be the primitive part for* N_0*, and let* $\langle \cdot, \cdot \rangle_k : P_k \times P_k \to \mathbb{R}$ *be the bilinear form* $(u, v) \mapsto \langle u, N_0^{k-w} v \rangle$*. Then* $(P_k, \langle \cdot, \cdot \rangle_k, N_1, \dots, N_n, F(\mathrm{gr}_k^W)|_{P_k})$ *is a pure nilpotent orbit of weight* k*.*

3.6. The relation between Theorem 3.2 and Proposition 3.5 is that X in Theorem 3.2 provides $N_1, \dots, N_n$ of Proposition 3.5 and S in Theorem 3.2 provides N_0 of Proposition 3.5. For the equivalence of pure (resp. mixed) nilpotent orbits and PLH (resp. LMH), see [12] Section 2.5 (resp. [9] 2.2.2).

3.7. We prove the if part of Proposition 3.5. The argument below is similar to that in the proof of [7] Proposition 6.6. See also [8] Theorem 2.4.2 (ii).

By [7] 10.2, we have an action $\tau = (\tau_j)_{0 \leq j \leq n}$ of $\mathbb{G}_m^{\{0,\dots,n\}}$ on V associated with the mixed nilpotent orbit $H = (V, W, N_1, \dots, N_n, F)$ with polarized gr^W given by (i) and (ii) such that τ_0 splits W and τ_j for each $1 \leq j \leq n$ splits the relative monodromy filtration of $N_1 + \cdots + N_j$ with respect to W. For $y = (y_0, \dots, y_n)$, $y_j \in \mathbb{R}_{>0}$, let $t(y) = \prod_{j=0}^n \tau_j((y_{j+1}/y_j)^{1/2})$, where y_{n+1} denotes 1.

Fix $b : \{0, \dots, n\} \times \{0, \dots, n\} \to [0, \infty]$ such that $b_{j,k} b_{k,\ell} = b_{j,\ell}$ unless the set $\{b_{j,k}, b_{k,\ell}\}$ coincides with the set $\{0, \infty\}$, such that $b_{j,j} = 1$ for $0 \leq j \leq n$, and such that $b_{0,j} = \infty$ and $b_{j,0} = 0$ for $1 \leq j \leq n$. Then by the SL(2)-orbit theorem for mixed nilpotent orbit ([7] Theorem 0.5 and Section 10), we have the associated $\hat{F}$ and $\hat{N}_1, \dots, \hat{N}_n$ such that when $y_j \to \infty$ $(0 \leq j \leq n)$ and $(y_j/y_k)_{j,k}$ converges to b, then $t(y)^{-1} \exp(\sum_{j=0}^n i y_j N_j) F \in \check{D}$ converges to $\exp(\sum_{j=0}^n i \hat{N}_j) \hat{F} \in D$, where D is the classifying space of PH and $\check{D}$ is its compact dual (see [7] 0.1 for the precise definitions). Since D is open in $\check{D}$, we have $t(y)^{-1} \exp(\sum_{j=0}^n i y_j N_j) F \in D$ and hence $\exp(\sum_{j=0}^n i y_j N_j) F \in D$ if $(y_j/y_k)_{j,k}$ is sufficiently near to b.

This proves that $(V, \langle \cdot, \cdot \rangle, N_0, aN_0 + N_1, \dots, aN_0 + N_n, F)$ generates a pure nilpotent orbit of weight w for any $a \gg 0$.

3.8. We prove the only if part of Proposition 3.5. There is $c \in \mathbb{R}$ such that if $y_j \geq c$ for $0 \leq j \leq n$, $(V, \langle \cdot, \cdot \rangle, F_y)$ with $F_y := \exp(i y_0 N_0 + \sum_{j=1}^n i y_j (a N_0 + N_j)) F = \exp(i(y_0 + a \sum_{j=1}^n y_j) N_0 + \sum_{j=1}^n i y_j N_j) F$ is a polarized Hodge structure of weight w. Hence if $b_j \geq c$ for $1 \leq j \leq n$, $(V, \langle \cdot, \cdot \rangle, N_0, \exp(\sum_{j=1}^n i b_j N_j) F)$ generates a pure nilpotent orbit of weight w. Hence by the classical result of Schmid [18] Theorem (6.16) to which we referred to in 3.4, if $b_j \geq c$ for $1 \leq j \leq n$, $(V, W, \exp(\sum_{j=1}^n i b_j N_j) F)$ is a mixed Hodge structure and for $k \geq w$, $(P_k, \langle \cdot, \cdot \rangle_k, \exp(\sum_{j=1}^n i b_j N_j) F(\mathrm{gr}_k^W)|_{P_k})$ is a polarized Hodge structure. This proves that the conditions (i) and (ii) are satisfied.

3.9. Theorem 3.2 for general R is reduced to the case $R = \mathbb{R}$ because the conditions (a), (1), and (2) can be checked after tensoring $\mathbb{R}$. Theorem 3.2 for a general X is reduced to the case where X is an fs log point by Proposition 2.1. (Note that, by construction, the correspondence in Theorem 3.2 preserves the big Griffiths transversality.)

4 Study of Ext groups

4.1. Let X be an object of $\mathcal{B}(\log)$. We will consider the following six categories $\mathcal{H} \supset \mathcal{H}(*) \supset \mathcal{H}(**)$, $\mathcal{L}_0$, $\mathcal{L} \supset \mathcal{L}(*)$. Fix a subring R of $\mathbb{R}$.

Let $\mathcal{H}$ (resp. $\mathcal{H}(*)$, resp. $\mathcal{H}(**)$) be the category of pre-R-LMH (resp. R-LMH, resp. R-LVMH) on X satisfying the condition (i) in 1.1.

Let $\mathcal{L}_0$ be the category of locally constant sheaves of finite dimensional $\mathbb{R}$-vector spaces on $X^{\log}$ whose local monodromies are unipotent. Let $\mathcal{L}$ be the category of pairs (L, W), where L is an object of $\mathcal{L}_0$ and W is an increasing filtration (called weight filtration) on L such that each filter W_k is locally constant and that $W_k = L$ for some k and $W_k = 0$ for some k. Let $\mathcal{L}(*)$ be the full subcategory of $\mathcal{L}$ consisting of objects L such that the local monodromies of L are admissible (see [9] 1.2.4).

These categories are exact categories (in the sense of Quillen). For a contemporary treatment of exact categories, see [2]. A short exact sequence in $\mathcal{H}$, in $\mathcal{H}(*)$, or in $\mathcal{H}(**)$ is a sequence $0 \to H_1 \to H_2 \to H_3 \to 0$ such that $0 \to W_kH_{1,R} \to W_kH_{2,R} \to W_kH_{3,R} \to 0$ for all k and $0 \to F^pH_{1,\mathcal{O}} \to F^pH_{2,\mathcal{O}} \to F^pH_{3,\mathcal{O}} \to 0$ for all p are exact. A short exact sequence in $\mathcal{L}_0$ is an evident one, and that in $\mathcal{L}$ or in $\mathcal{L}(*)$ is a sequence $0 \to L_1 \to L_2 \to L_3 \to 0$ such that the sequences $0 \to W_kL_1 \to W_kL_2 \to W_kL_3 \to 0$ are exact for all k.

We have Yoneda's higher Ext group Ext^n for any exact category ([16] Appendix Proposition A.13). For example, Ext^1 is the set of isomorphism classes of extensions endowed with the group law given by Baer sums. A short exact sequence gives a long exact sequence of Ext^n.

4.2. If $\mathcal{C}$ is one of the above categories $\mathcal{H}$, $\mathcal{H}(*)$, $\mathcal{H}(**)$, $\mathcal{L}_0$, $\mathcal{L}$, and $\mathcal{L}(*)$, and if A, B are objects of $\mathcal{C}$, we have a sheaf $\mathcal{E}xt^n_{\mathcal{C}}(A, B)$ of abelian groups on X which is the sheafification of the presheaf $U \mapsto \mathrm{Ext}^n_{\mathcal{C}'}(A', B')$, where U is an open set of X and $\mathcal{C}', A', B'$ are the restrictions of $\mathcal{C}$, A, B over U, respectively.

The goal of this Section 4 is to prove the following proposition.

Proposition 4.3. *Let $\mathcal{C}$ be either $\mathcal{H}(*)$ or $\mathcal{H}(**)$. Let $0 \to R(1) \to P \to Q \to 0$ be an exact sequence in $\mathcal{C}$ and assume that the weights of $Q \leq -1$. Then the map $\mathcal{E}xt^1_{\mathcal{C}}(R, P) \to \mathcal{E}xt^1_{\mathcal{C}}(R, Q)$ is surjective.*

Lemma 4.4. *Let L be an object of $\mathcal{L}(*)$ such that the local monodromy actions on L are trivial and such that $W_{-2n}L = 0$.*

Then the canonical map $\mathcal{E}xt^n_{\mathcal{L}(*)}(\mathbb{R}, L) \to \mathcal{E}xt^n_{\mathcal{L}_0}(\mathbb{R}, L) = R^n\tau_*L$ is the zero map. Here $W_0\mathbb{R} = \mathbb{R}$, $W_{-1}\mathbb{R} = 0$, and τ is the canonical projection $X^{\log} \to X$. In other words (assume $n \geq 1$), if

$$0 \to L \to L_n \to \cdots \to L_2 \to L_1 \to \mathbb{R} \to 0$$

is an exact sequence in $\mathcal{L}(*)$, the induced section of $R^n\tau_*L$ is 0.

Proof. The case $n = 0$ is evident. Assume $n \geq 1$ and consider an exact sequence $0 \to L \to L_n \to \cdots \to L_2 \to L_1 \to \mathbb{R} \to 0$ in $\mathcal{L}(*)$. Let $t \in X^{\log}$ and let $x \in X$ be the image of t. Then $(R^n\tau_*L)_x$ is isomorphic to $(\bigwedge^n_{\mathbb{Z}} (M^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X)_x) \otimes_{\mathbb{Z}} L_t$ and is isomorphic to the Lie cohomology $H^n(\mathfrak{g}, L_t)$, where $\mathfrak{g}$ is the commutative Lie algebra $\mathrm{Hom}\,((M^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X)_x, \mathbb{Q})$

which acts on L_t trivially. Let σ be the monodromy cone $\mathrm{Hom}\,((M_X/\mathcal{O}_X^\times)_x, \mathbb{R}^{\mathrm{add}}_{\geq 0})$ of x. In the rest, we omit t in $(\cdot)_t$. By the admissibility, we have a relative monodromy filtration $W(\sigma)$ on L, L_j $(1 \leq j \leq n)$, $\mathbb{R}$ and the sequence $0 \to W(\sigma)_k L \to W(\sigma)_k L_n \to \cdots \to W(\sigma)_k L_1 \to W(\sigma)_k \mathbb{R} \to 0$ is exact for every k. For $0 \leq j \leq n$, let I_j be the image of $L_{j+1} \to L_j$, where L_0 denotes $\mathbb{R}$ and L_{n+1} denotes L. Hence I_0 is identified with $\mathbb{R}$ and I_n is identified with L. We compute the n-cocycle $f_n : \bigwedge^n_{\mathbb{Q}} \mathfrak{g} \to L$ corresponding to the element of $H^n(\mathfrak{g}, L)$ in problem, by the standard method: By induction on j, we get a j-cocycle $f_j : \bigwedge^j_{\mathbb{Q}} \mathfrak{g} \to I_j$ starting from $f_0 = 1 \in \mathbb{R} = I_0$. To get f_{j+1} from f_j, we lift f_j to a j-cochain $\tilde{f}_j : \bigwedge^j_{\mathbb{Q}} \mathfrak{g} \to L_{j+1}$ and obtain f_{j+1} from $\tilde{f}_j$. By induction on j, we get f_j whose image is in $W(\sigma)_{-2j} I_j$ and lift it to $\tilde{f}_j$ whose image is in $W(\sigma)_{-2j} L_{j+1}$, and get f_{j+1} whose image is contained in $(\sum_{N \in \mathfrak{g}} N W(\sigma)_{-2j} L_{j+1}) \cap I_{j+1} \subset W(\sigma)_{-2(j+1)} I_{j+1}$. Thus we get f_n whose image is in $W(\sigma)_{-2n} L = W_{-2n} L = 0$. □

Lemma 4.5. *Let $0 \to L \to P \to Q \to 0$ be an exact sequence in $\mathcal{L}(*)$. Assume that the local monodromy actions on L are trivial and $W_0 L = L$. For $a \in \mathcal{E}xt^1_{\mathcal{L}}(\mathbb{R}, P)$ and the image b of a in $\mathcal{E}xt^1_{\mathcal{L}}(\mathbb{R}, Q)$, a belongs to $\mathcal{E}xt^1_{\mathcal{L}(*)}(\mathbb{R}, P)$ if and only if b belongs to $\mathcal{E}xt^1_{\mathcal{L}(*)}(\mathbb{R}, Q)$.*

Proof. It is enough to prove the if part. We may assume that a is the class of an exact sequence $0 \to P \to \tilde{P} \to \mathbb{R} \to 0$ in $\mathcal{L}$ and b is the class of the exact sequence $0 \to Q \to \tilde{Q} \to \mathbb{R} \to 0$ in $\mathcal{L}(*)$, where $\tilde{Q} = \tilde{P}/L$. Let σ be as in the proof of Lemma 4.4. For each face σ' of σ, we have the relative monodromy filtration $W(\sigma')$ on $\tilde{P}$ defined as follows. If $k \leq -1$, $W(\sigma')_k \tilde{P} = W(\sigma')_k P$. If $k \geq 0$, $W(\sigma')_k \tilde{P}$ is the inverse image of $W(\sigma')_k \tilde{Q}$. Hence $\tilde{P}$ belongs to $\mathcal{L}(*)$. □

4.6. Note that for a topological space T and for a complex of sheaves of R-modules on T of the form $C = [C^0 \to C^1]$ (that is, a complex of sheaves of R-modules whose degree d-parts are zero unless $d = 0, 1$), the hyper-cohomology $H^1(T, C)$ is identified with the set of isomorphism classes of pairs of an exact sequence of the form $0 \to C^0 \to E \to R \to 0$ and a splitting $E \to C^1$.

4.7. For an object H of $\mathcal{H}$ satisfying the condition (i) in 1.1 on X, let

$$C(H) = [H_R \to (H_R \otimes \mathcal{O}_X^{\log})/F^0],$$

where H_R is of degree 0. Then we have identifications

$$\mathrm{Hom}_{\mathcal{H}}(R, H) = H^0(X^{\log}, C(H)) \quad \text{if the weights of } H \leq 0,$$

$$\mathrm{Ext}^1_{\mathcal{H}}(R, H) = H^1(X^{\log}, C(H)) \quad \text{if the weights of } H \leq -1.$$

This is shown by using 4.6.

Lemma 4.8. *Let $0 \to R(1) \to P \to Q \to 0$ be an exact sequence in $\mathcal{H}$ such that the weights of $Q \leq -1$. Then we have an exact sequence*

$$\mathcal{E}xt^1_{\mathcal{H}}(R, P) \to \mathcal{E}xt^1_{\mathcal{H}}(R, Q) \to R^2 \tau_* R(1).$$

Proof. By $\mathcal{E}xt^1_{\mathcal{H}}(R,P) \cong R^1\tau_*C(P)$ (4.7) and by the corresponding isomorphism for Q, we have an exact sequence $\mathcal{E}xt^1_{\mathcal{H}}(R,P) \to \mathcal{E}xt^1_{\mathcal{H}}(R,Q) \to R^2\tau_*C(R(1))$. Since $C(R(1)) = [R(1) \to \mathcal{O}^{\log}_X]$, we have an exact sequence $0 = R^1\tau_*\mathcal{O}^{\log}_X \to R^2\tau_*C(R(1)) \to R^2\tau_*R(1)$. □

Lemma 4.9. *Let $\mathcal{C}$ and $0 \to R(1) \to P \to Q \to 0$ be as in the hypothesis of Proposition 4.3. Then the map $\mathcal{E}xt^1_{\mathcal{H}}(R,Q) \to R^2\tau_*R(1)$ in Lemma 4.8 kills $\mathcal{E}xt^1_{\mathcal{C}}(R,Q)$.*

Proof. Since $R^2\tau_*R(1) \cong R \otimes_{\mathbb{Z}} \bigwedge^2_{\mathbb{Z}} (M^{\mathrm{gp}}_X/\mathcal{O}^\times_X)$, the map $R^2\tau_*R(1) \to R^2\tau_*\mathbb{R}(1)$ is injective. The map $\mathcal{E}xt^1_{\mathcal{C}}(R,Q) \to R^2\tau_*\mathbb{R}(1)$ factors as $\mathcal{E}xt^1_{\mathcal{C}}(R,Q) \to \mathcal{E}xt^1_{\mathcal{L}(*)}(\mathbb{R},Q_{\mathbb{R}}) \to \mathcal{E}xt^2_{\mathcal{L}(*)}(\mathbb{R},\mathbb{R}(1)) \to \mathcal{E}xt^2_{\mathcal{L}_0}(\mathbb{R},\mathbb{R}(1)) = R^2\tau_*\mathbb{R}(1)$. This is 0 by the case $n=2$ and $L=\mathbb{R}(1)$ of Lemma 4.4. □

By Lemmas 4.8 and 4.9, for the proof of Proposition 4.3, it is sufficient to prove the following.

Lemma 4.10. *Let $\mathcal{C}$ and $0 \to R(1) \to P \to Q \to 0$ be as in the hypothesis of Proposition 4.3. Then for $a \in \mathcal{E}xt^1_{\mathcal{H}}(R,P)$ and the image b of a in $\mathcal{E}xt^1_{\mathcal{H}}(R,Q)$, a belongs to $\mathcal{E}xt^1_{\mathcal{C}}(R,P)$ if and only if b belongs to $\mathcal{E}xt^1_{\mathcal{C}}(R,Q)$.*

Proof. It is sufficient to prove the if part. We may assume that a is the class of an exact sequence $0 \to P \to \tilde{P} \to R \to 0$ in $\mathcal{H}$ and b is the class of the exact sequence $0 \to Q \to \tilde{Q} \to R \to 0$ in $\mathcal{H}(*)$, where $\tilde{Q} = \tilde{P}/R(1)$.

Then (the associated $\mathbb{R}$-local system of) $\tilde{P}$ satisfies the admissibility of local monodromy by Lemma 4.5.

If $\mathcal{C} = \mathcal{H}(*)$ (resp. $\mathcal{C} = \mathcal{H}(**)$), $\tilde{P}$ satisfies the small (resp. big) Griffiths transversality because

$$((\tilde{P}_R \otimes_R \mathcal{O}^{\log}_x)/F^{-1}) \otimes_{\mathbb{C}} \omega^1_x \xrightarrow{\cong} ((\tilde{Q}_R \otimes_R \mathcal{O}^{\log}_x)/F^{-1}) \otimes_{\mathbb{C}} \omega^1_x \quad \text{for } x \in X.$$

$$(\text{resp. } ((\tilde{P}_R \otimes_R \mathcal{O}^{\log}_X)/F^{-1}) \otimes_{\mathcal{O}_X} \omega^1_X \xrightarrow{\cong} ((\tilde{Q}_R \otimes_R \mathcal{O}^{\log}_X)/F^{-1}) \otimes_{\mathcal{O}_X} \omega^1_X.)$$

Finally, we prove that for $t \in X^{\log}$ lying over $x \in X$ and for $s \in \mathrm{sp}(t)$ ([12] 2.4.6), if $\exp(s(\log(\cdot))) : M^{\mathrm{gp}}_{X,x} \to \mathbb{C}^\times$ is sufficiently near to the structure homomorphism $\alpha_{X,x}$, then the associated specialization $\tilde{P}(s)$ is a mixed Hodge structure. If $k \neq 0$, $\mathrm{gr}^W_k\tilde{P} = \mathrm{gr}^W_k P$ and hence $\mathrm{gr}^W_k\tilde{P}(s)$ is a Hodge structure of weight k. We consider $\mathrm{gr}^W_0\tilde{P}(s)$. We have exact sequences $0 \to \mathrm{gr}^W_0 P_{\mathbb{C}} \to \mathrm{gr}^W_0\tilde{P}_{\mathbb{C}} \to \mathbb{C} \to 0$, $0 \to F^p\mathrm{gr}^W_0 P(s)_{\mathbb{C}} \to F^p\mathrm{gr}^W_0\tilde{P}(s)_{\mathbb{C}} \to \mathbb{C} \to 0$ for $p \leq 0$ and $F^p\mathrm{gr}^W_0 P(s)_{\mathbb{C}} \xrightarrow{\cong} F^p\mathrm{gr}^W_0\tilde{P}(s)_{\mathbb{C}}$ for $p \geq 1$. Hence we have the Hodge decomposition of $\mathrm{gr}^W_0\tilde{P}(s)$. □

5 Construction of an extension with three graded quotients

In this section, we prove Theorem 1.2 in the special case where $W_{-1}H = H$, $W_{-3}H = 0$, and $\mathrm{gr}^W_{-2}H = R(1)$. Let $Q := \mathrm{gr}^W_{-1}H$. Thus we have an exact sequence $0 \to R(1) \to H \to Q \to 0$.

5.1. Let $\tilde{Q} = H^*(1)$, where $(\cdot)^*$ denotes the dual. The intersection form $\langle\cdot,\cdot\rangle_{-1} : Q \times Q \to R(1)$ of a polarization gives an isomorphism $Q \xrightarrow{\cong} Q^*(1)$; $u \mapsto (v \mapsto \langle u, v\rangle)$. Hence $W_0\tilde{Q} = \tilde{Q}$, $\mathrm{gr}^W_0\tilde{Q} = R$, $\mathrm{gr}^W_{-1}\tilde{Q} = Q$, and $W_{-2}\tilde{Q} = 0$.

Let $\mathcal{C}$ be $\mathcal{H}(*)$ or $\mathcal{H}(**)$, and assume that H belongs to $\mathcal{C}$. By the surjectivity of $\mathcal{E}xt^1_{\mathcal{C}}(R, H) \to \mathcal{E}xt^1_{\mathcal{C}}(R, Q)$ (Proposition 4.3), locally on X, the class of $\tilde{Q}$ in $\mathrm{Ext}^1_{\mathcal{C}}(R, Q)$ lifts to the class in $\mathrm{Ext}^1_{\mathcal{C}}(R, H)$ of an exact sequence $0 \to H \to \tilde{H} \to R \to 0$ such that $\tilde{H}/R(1) = \tilde{Q}$.

Lemma 5.2. *There is a unique isomorphism $\tilde{H} \cong (\tilde{H})^*(1)$ whose gr^W_{-1} coincides with $Q \cong Q^*(1)$, whose gr^W_0 is the identity homomorphism of R, and whose gr^W_{-2} is the multiplication by -1 on $R(1)$.*

Proof. If we have two such isomorphisms $f, g : \tilde{H} \cong (\tilde{H})^*(1)$, $f - g : \tilde{H} \to (\tilde{H})^*(1)$ sends W_k to W_{k-1} and hence is zero. This proves the uniqueness.

We prove the existence.

First we consider the case $Q = 0$. We have an exact sequence $0 \to R(1) \to \tilde{H}_R \to R \to 0$. We have the wedge product $\langle\cdot,\cdot\rangle : \tilde{H}_R \times \tilde{H}_R \to \bigwedge^2_R \tilde{H}_R \cong R(1)$, where the last isomorphism is such that the induced map $\mathrm{gr}^W_0 \otimes \mathrm{gr}^W_{-2} \to R(1)$ is the canonical map $R \otimes R(1) \to R(1)$. This pairing induces a perfect pairing $\tilde{H}_{\mathcal{O}} \times \tilde{H}_{\mathcal{O}} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules. The Hodge filtration F of $H_{\mathcal{O}}$ satisfies $F^{-1} = H_{\mathcal{O}}$, F^0 is a line bundle, $F^1 = 0$, and the annihilator of F^0 under $\langle\cdot,\cdot\rangle$ is F^0. Hence $\langle\cdot,\cdot\rangle$ induces $\tilde{H} \cong (\tilde{H})^*(1)$; $u \mapsto (v \mapsto \langle u, v\rangle)$.

We consider the general case. A proof may be given by writing explicitly the involved Hodge filtrations, but we give a proof by some abstract argument. We may assume $\mathcal{C} = \mathcal{H}(*)$.

We have two sections $a, a^*(1)$ of $\mathcal{E}xt^1_{\mathcal{H}(*)}(R, H)$ given by $\tilde{H}$ and $(\tilde{H})^*(1)$, respectively. Here to define $a^*(1)$, we identify $W_{-1}((\tilde{H})^*(1))$ with H via $W_{-1}((\tilde{H})^*(1)) = (\tilde{Q})^*(1) = (H^*(1))^*(1) = H$.

For the proof of Lemma 5.2, it is sufficient to prove

(1) $a + a^*(1) = 0$.

In fact, then, there is a natural isomorphism from $\tilde{H}$ to the pushout of $(\tilde{H})^*(1)$ by $-1 : W_{-1}((\tilde{H})^*(1)) \to W_{-1}((\tilde{H})^*(1))$, whose gr^W_0 is the identity homomorphism of R such that the induced $H = W_{-1}(\tilde{H}) \to W_{-1}((\tilde{H})^*(1)) = H$ is the multiplication by -1. This is the desired isomorphism because $Q \cong Q^*(1) \cong (Q^*(1))^*(1) = Q$ is the multiplication by -1 on Q.

Note that we have already proved (1) in the case $Q = 0$, by the above consideration on the case $Q = 0$.

Concerning (1), we have the following (2) and (3).

(2) This element $a + a^*(1)$ belongs to $\mathcal{E}xt^1_{\mathcal{H}(*)}(R, R(1))$, since its image in $\mathcal{E}xt^1_{\mathcal{H}(*)}(R, Q)$ is zero.

(3) This sum $a + a^*(1)$ is determined by the image of a in $\mathcal{E}xt^1_{\mathcal{H}(*)}(R, Q)$. In fact, if the image of $b \in \mathcal{E}xt^1_{\mathcal{H}(*)}(R, H)$ in $\mathcal{E}xt^1_{\mathcal{H}(*)}(R, Q)$ coincides with that of a, we have $b = a + u$ for some $u \in \mathcal{E}xt^1_{\mathcal{H}(*)}(R, R(1))$. By the case $Q = 0$ considered above, $u + u^*(1) = 0$, and hence $b + b^*(1) = (a + u) + (a + u)^*(1) = (a + a^*(1)) + (u + u^*(1)) = a + a^*(1)$.

By (2) and (3), we obtain a homomorphism $\mathcal{E}xt^1_{\mathcal{H}(*)}(R,Q)\to\mathcal{E}xt^1_{\mathcal{H}(*)}(R,R(1))$. Note that $\mathcal{E}xt^1_{\mathcal{H}(*)}(R,R(1))\subset\mathcal{E}xt^1_{\mathcal{H}}(R,R(1))=\tau_*(\mathcal{O}_X^{\log}/R(1))$. We prove that this homomorphism is the zero map. By Proposition 2.2, this is reduced to the case where X is a log manifold. Let X_{triv} be the open set of X consisting of all points at which the log structure of X is trivial, which is a complex analytic manifold. In this case, for the inclusion map $j^{\log}: X_{\mathrm{triv}}\to X^{\log}$, $\mathcal{O}_X^{\log}\to j^{\log}_*(\mathcal{O}_{X_{\mathrm{triv}}})$ is injective and $R(1)\to j^{\log}_*(R(1))$ is an isomorphism, and hence the map $\mathcal{O}_X^{\log}/R(1)\to j^{\log}_*(\mathcal{O}_{X_{\mathrm{triv}}}/R(1))$ is injective. By this, replacing X by X_{triv}, we are reduced to the case where X is a complex analytic manifold with the trivial log structure. In this case, $\mathcal{H}(*)=\mathcal{H}$.

We first consider the case $X=\mathrm{Spec}(\mathbb{C})$ with the trivial log structure and $R=\mathbb{R}$. In this case, $\mathrm{Ext}\,^1_{\mathcal{H}}(\mathbb{R},Q)=Q_{\mathbb{R}}\setminus Q_{\mathbb{C}}/F^0$ (cf. 4.7) vanishes because Q is of weight -1, and hence the map $\mathrm{Ext}\,^1_{\mathcal{H}}(\mathbb{R},Q)\to\mathrm{Ext}\,^1_{\mathcal{H}}(\mathbb{R},\mathbb{R}(1))$ is the zero map.

This proves that in the case where X is a complex analytic manifold with the trivial log structure, the composite map $\mathcal{E}xt^1_{\mathcal{H}}(R,Q)\to\mathcal{E}xt^1_{\mathcal{H}}(R,R(1))=\mathcal{O}_X/R(1)\to\mathcal{O}_X/\mathbb{R}(1)$ is the zero map. Hence in this case, the map $\mathcal{E}xt^1_{\mathcal{H}}(R,Q)=Q_R\backslash(Q_R\otimes_R\mathcal{O}_X)/F^0\to\mathcal{E}xt^1_{\mathcal{H}}(R,R(1))=\mathcal{O}_X/R(1)$ has the image in $\mathbb{R}(1)/R(1)$. Since there is no non-zero homomorphism from $\mathcal{O}_X$ to the constant sheaf $\mathbb{R}(1)/R(1)$ which is functorial in X, we see that this map is zero. □

5.3. We return to the proof of Theorem 1.2. Let $N:\tilde{H}\to\tilde{H}(-1)$ be the homomorphism $\tilde{H}\to\mathrm{gr}^W_0\tilde{H}\cong R\cong(\mathrm{gr}^W_{-2}\tilde{H})(-1)\subset\tilde{H}(-1)$. Then the desired PLH (resp. LVPH) on $X\times S$ of weight -1 is the pullback H' of $\tilde{H}^{(N)}$ by $X\times S\to X\times S$ over X, q on the right hand side is sent to qf on the left-hand-side, where f is some global section of M_X. Here, $(\cdot)^{(N)}$ is as in Theorem 3.2. (Although R is a field there, the construction is the same. Note that since $2\pi iN\tilde{H}_R\subset\tilde{H}_R$ and $N^2=0$ here, we have $\exp(2\pi iN)\tilde{H}_R=\tilde{H}_R$ and hence $\tilde{H}^{(N)}$ has an R-structure.) We prove that this H' is a PLH (resp. LVPH). The small or big Griffiths transversality is easy to see. We have the isomorphism $H'\cong(H')^*(1)$ in Lemma 5.2 and it gives a polarization $\langle\cdot,\cdot\rangle: H'\times H'\to R(1)$ for which the isomorphism is $u\mapsto(v\mapsto\langle u,v\rangle)$. To prove that $\langle\cdot,\cdot\rangle$ is actually a polarization, we may assume that the base X is an fs log point and hence we reduce to the case of Theorem 3.2 where X is an fs log point and $R=\mathbb{R}$.

6 Proof of Theorem 1.2

We prove the general case of Theorem 1.2. We use the induction on n for the integer $n\geq 2$ such that there is $w\in\mathbb{Z}$ satisfying $W_wH=H$ and $W_{w-n}H=0$.

6.1. First assume $n=2$. Let $A=\mathrm{gr}^W_wH$, $B=\mathrm{gr}^W_{w-1}H=W_{w-1}H$. We consider $B^*(1)\otimes H$ and use $B^*(1)\otimes B\to R(1)$ to get P as the pushout. We have $W_{-1}P=P$, $W_{-3}P=0$, $\mathrm{gr}^W_{-1}P=B^*(1)\otimes A$, $\mathrm{gr}^W_{-2}P=R(1)$. We get $\tilde{P}$ as in the previous section. Let $\tilde{H}=B(-1)\otimes\tilde{P}$. Since $\tilde{P}$ is regarded as a PLH (resp. LVPH) of weight -1 by the previous section (after taking $(\cdot)^{(N)}$), $\tilde{H}$ is regarded as a PLH (resp. LVPH) of weight w. This $\tilde{H}$ regarded as a PLH (resp. LVPH) is the desired H'.

Here N is the evident isomorphism $\mathrm{gr}^W_{w+1}\to\mathrm{gr}^W_{w-1}(-1)$ induced by the identity map of B.

We embed H into $\tilde{H}$ by

$$H \to B \otimes B^* \otimes H \to B(-1) \otimes P \to B(-1) \otimes \tilde{P} = \tilde{H}.$$

The map $H_R \to \tilde{H}_R$ is injective (with locally free cokernel) because its gr^W_{w-1} is $B_R \to B_R \otimes B_R^* \otimes B_R \to B_R$, where the first arrow is $b \mapsto \sum_j e_j \otimes e_j^* \otimes b$ for a local base $(e_j)_j$ of B_R and for the dual base e_j^* of B_R^* and the second arrow is the map $u \otimes v \otimes b \mapsto uv(b)$, and the composite map $B_R \to B_R$ is the identity map because it gives $e_k \mapsto \sum_j e_j \otimes e_j^* \otimes e_k \mapsto e_k$.

6.2. Assume $n \geq 3$, $W_w H = H$, $W_{w-n}H = 0$. Then by the hypothesis of induction applied to $W_{w-1}H$, we find a PLH (resp. an LVPH) $I \supset W_{w-1}H$ on $X \times S$ of weight $w-1$. Let J be the pushout of $H \leftarrow W_{w-1}H \to I$. Then we have $W_w J = J$, $W_{w-2}J = 0$, $\mathrm{gr}^W_w J = \mathrm{gr}^W_w H$, $\mathrm{gr}^W_{w-1} J = I$. Hence we find a PLH (resp. an LVPH) $K \supset J$ on $X \times S \times S$ of weight w. We get the desired PLH (resp. LVPH) $H' \supset H$ on $X \times S$ of weight w as the pullback of K by the diagonal $S \to S \times S$.

It is clear that the condition (a) in Theorem 1.2 is satisfied. In 6.3 below, we show that the condition (b) in Theorem 1.2 is satisfied.

This will complete the proof of Theorem 1.2.

6.3. We prove that the condition (b) is satisfied. We may assume that X is an fs log point. We can use the following.

If $(W, N_1, \dots, N_n, F)$ is a mixed nilpotent orbit with polarizable gr^W, we have:

(1) We have a relative monodromy filtration $W^{(1)}$ of (W, N_1) and $(W^{(1)}, N_2, \dots, N_n, F)$ is a mixed nilpotent orbit.

(2) The relative monodromy filtration $W^{(2)}$ of $(W^{(1)}, N_2)$ coincides with the relative monodromy filtration of $(W, N_1 + N_2)$.

(3) $(W, N_1 + N_2, N_3, \dots, N_n, F)$ is a mixed nilpotent orbit.

In 6.2, we apply these to the pure weight filtration W of weight w and to the mixed nilpotent orbit $K = (W, N_{-1}, N_0, N_1, \dots, N_n, F)$, where $N_1, \dots, N_n$ comes from X, N_0 comes from S of $X \times S$, and N_{-1} comes from the second S of $X \times S \times S$. Then by (1) and (2), we have a mixed nilpotent orbit $(W^{(2)}, N_1, \dots, N_n, F)$, where $W^{(2)}$ is the relative monodromy filtration of $(W, N_{-1} + N_0)$. I (resp. J) is a sub mixed nilpotent orbit which is the weight $w-1$ (resp. $\leq w$) part of the mixed nilpotent orbit $(W^{(1)}, N_0, \dots, N_n, F)$ (consider (1)), where $W^{(1)}$ is the relative monodromy filtration of (W, N_{-1}). We have a mixed nilpotent orbit I' (resp. J') $= (W^{(2)}, N_1, \dots, N_n, F)$ whose underlying space is the same as I (resp. J).

The original $W_{w-1}H$ is a sub mixed nilpotent orbit of I'. Hence the weight filtration of $W_{w-1}H$ is the restriction of $W^{(2)}$. The original H satisfies that $H/W_{w-1}H$ is of weight w. J'/I' also has weight w. Hence the weight filtration of H is the restriction of $W^{(2)}$. Thus we complete the proof of that the condition (b) is satisfied.

A Appendix. Mixed motives and log pure motives

In this section, we will give some simple construction of the category of mixed motives over a field based on the idea that mixed motives should be embedded into log pure motives.

A.1. We present the Hodge analogue of our story on motives.

Here, Hodge or log Hodge structure means $\mathbb{Q}$-Hodge or $\mathbb{Q}$-log Hodge structure.

Let S be the standard log point over $\mathbb{C}$. Let (LH) be the category of polarizable log Hodge structures on S. For $H \in$ (LH), let $H^\flat$ be the associated mixed Hodge structure endowed with the monodromy operator $N : H^\flat \to H^\flat(-1)$.

Let (LH$\flat$) be the category of pairs of a mixed Hodge structure and N of the form $H^\flat$ with $H \in$(LH).

Let (MH) be the category of contra-variant functors from (LH$\flat$) to the category of $\mathbb{Q}$-vector spaces defined by a pair (H, V) as in (1) below, where $H \in$ (LH$\flat$) and V is a $\mathbb{Q}$-subspace of $H_\mathbb{Q}$ satisfying the following conditions (i) and (ii).

(i) For some $H' \in$ (LH$\flat$) and for some morphism $H' \to H$ in (LH$\flat$), V is the image of $H'_\mathbb{Q} \to H_\mathbb{Q}$.

(ii) N of H kills V.

(1) $H' \mapsto \{h \in \mathrm{Mor}_{(\mathrm{LH}\flat)}(H', H) \mid h(H'_\mathbb{Q}) \subset V\}$.

Then (MH) is equivalent to the category of mixed Hodge structures with polarizable gr^W by the case $X = \mathrm{Spec}(\mathbb{C})$ with the trivial log structure of Theorem 1.2 and Remark 1.3 (1).

A.2. Let k be a field, and let S be the standard log point over k.

We have the log absolute Galois group $\pi_1^{\log}(S)$. It is the automorphism group of the log separable closure (cf. [13] (2.5)) $\overline{S}$ of S over S. We have an exact sequence

$$0 \to \hat{\mathbb{Z}}(1)' \to \pi_1^{\log}(S) \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

where $\hat{\mathbb{Z}}(1)'$ is the product of $\mathbb{Z}_\ell(1)$ for all prime numbers ℓ which are not equal to the characteristic of k and $\overline{k}$ is the separable closure of k.

A.3. We fix a prime number ℓ which is different from the characteristic of k. Let $\mathcal{P}$ be the category of projective vertical log smooth saturated fs log schemes over S which have charts of the log structure Zariski locally. (The condition "saturated" here is not essential but we impose it because we would like to find the reason why our definitions are right in the analogy with the Hodge context presented in A.1. See 0.4.)

Let $X \in \mathcal{P}$. For $m \in \mathbb{Z}$, let

$$H^m(X)_\ell = H^m_{\text{logét}}(X \times_S \overline{S}, \mathbb{Q}_\ell).$$

It is a finite dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous action of $\pi_1^{\log}(S)$.

We make symbols $H^m(X)(r)$ and $H^m(X)(r)^\flat$ for $X \in \mathcal{P}$ and for $m, r \in \mathbb{Z}$ such that $m \geq 0$.

A.4. Let $K = K_n$ or $K = KH_n$, where K_n is Quillen's K-theory and KH_n is the homotopy K-theory of Weibel [21]. Following the method in [4] 2.4.6, for an fs log scheme X having charts Zariski locally, we define

$$K_{\lim}(X) = \varinjlim K(X'),$$

where X' ranges over all log modifications of X in the sense of [4] 2.3.6 and $K(X')$ means K of the underlying scheme of X'. The K-group $K_{0,\lim}$ is used in [4] and also in the first half of this Appendix, and the K-group $KH_{n,\lim}$ is used in the latter half of this Appendix.

A.5. Let $X, Y \in \mathcal{P}$.

By a morphism $H^m(X)(r)^\flat \to H^n(Y)(s)^\flat$, we mean a $\mathbb{Q}_\ell$-linear map $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ which is obtained as below from an element of

$$\mathrm{gr}^u K_{0,\lim}(X \times_S Y \times \mathbb{G}_m^t) \otimes \mathbb{Q},$$

where $t = (n-2s) - (m-2r)$ and $u = d + n - m + r - s$ with d being the dimension of X (the dimension is defined as a locally constant function on X), and gr^u is the graded quotient for the γ-filtration. (If X is not equi-dimensional, this K-group is defined as the direct sum of the K-group of connected components of X by using the dimension of each connected component.)

If $m - 2r > n - 2s$, there is no non-zero morphism. We assume $m - 2r \leq n - 2s$. We have homomorphisms

$$\mathrm{gr}^u K_{0,\lim}(X \times_S Y \times \mathbb{G}_m^t) \otimes \mathbb{Q} \to H^{2u}(X \times_S Y \times \mathbb{G}_m^t)_\ell(u)$$
$$\to H^{2d-m}(X)_\ell \otimes H^n(Y)_\ell(d+s-r) \otimes H^t(\mathbb{G}_m^t)_\ell(t) \to \mathrm{Hom}\,(H^m(X)_\ell(r), H^n(Y)_\ell(s)).$$

Here to have the first homomorphism, we use the fact that the log blowing-up along the log structure does not change the log étale cohomology. The second homomorphism is by Künneth formula, and the third one is by Poincaré duality and by the canonical map $H^t(\mathbb{G}_m^t)_\ell(t) \stackrel{\cong}{\to} \mathbb{Q}_\ell$ induced by $H^1(\mathbb{G}_m)_\ell \cong \mathbb{Q}_\ell(-1)$. (For basic properties of log étale cohomology, see [14].)

As is easily seen, a linear map $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ commutes with the action of $\pi_1^{\log}(S)$ if it is a morphism $H^m(X)(r)^\flat \to H^n(Y)(s)^\flat$.

By a morphism $H^m(X)(r) \to H^n(Y)(s)$, we mean a morphism $H^m(X)(r)^\flat \to H^n(Y)(s)^\flat$ such that we can take $t = 0$ in the above.

If $m - 2r \neq n - 2s$, there is no morphism $H^m(X)(r) \to H^n(Y)(s)$.

Proposition A.6. (1) *The identity map of $H^m(X)_\ell(r)$ is a morphism $H^m(X)(r) \to H^m(X)(r)$ and hence a morphism $H^m(X)(r)^\flat \to H^m(X)(r)^\flat$.*

(2) *For morphisms $H^{m(1)}(X_1)(r(1))^\flat \to H^{m(2)}(X_2)(r(2))^\flat$ and $H^{m(2)}(X_2)(r(2))^\flat \to H^{m(3)}(X_3)(r(3))^\flat$, the composition is a morphism $H^{m(1)}(X_1)(r(1))^\flat \to H^{m(3)}(X_3)(r(3))^\flat$. The non-$\flat$ version is also true.*

Proof. The proof for the non-$\flat$ version is given in [4] Propositions 3.1.4 and 3.1.6. The $\flat$ version is proved in the same way. □

Thus we have the category (LM$\flat$) of $H^m(X)(r)^\flat$ and the category (LM) of $H^m(X)(r)$. A variant of the latter was considered in [4].

A.7. We define the category (MM) as the category of contra-variant functors from (LM$\flat$) to the category of $\mathbb{Q}$-vector spaces which are obtained as in (1) below, from an object $H^n(Y)(s)^\flat$ of (LM$\flat$) and a $\mathbb{Q}_\ell$-subspace V of $H^n(Y)_\ell(s)$ satisfying the following conditions (i) and (ii).

(i) There is a morphism $H^m(X)(r)^\flat \to H^n(Y)(s)^\flat$ for some X, m, r such that V is the image of $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$.

(ii) The action of $\pi_1^{\log}(S)$ on V factors through $\mathrm{Gal}(\bar{k}/k)$.

(1) $H^m(X)(r)^\flat \mapsto$ the set of all morphisms $H^m(X)(r)^\flat \to H^n(Y)(s)^\flat$ such that the image of $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ is contained in V.

A.8. We expect that this category (MM) is the category of mixed motives over k.

A.9. The above may be one of the simplest constructions of the category of mixed motives, and by the comparison with the Hodge version in A.1, we expect that the obtained category is the right one.

However, it is not clear whether the above (MM) contains the "usual" mixed motives $H^m(T)(r)$ associated with schemes T of finite type over k. We give below another construction of the category of mixed motives over k containing these "usual" objects, again by using log pure motives, and will conjecture that these two constructions give the same category.

A.10. For this, we use the homotopy K-theory KH_n ($n \in \mathbb{Z}$, it is important for us that n can be negative here) defined by Weibel [21]. There is a canonical homomorphism $K_n \to KH_n$ from Quillen's K-theory K_n which is an isomorphism for regular Noetherian schemes. The reason why we use KH_n, not Quillen's K-theory, is that we use the Riemann–Roch theorem for KH_n proved in [15].

A.11. For a scheme T of finite type over k, let $H^m(T)_\ell = H^m_{\text{ét}}(T \otimes_k \overline{k}, \mathbb{Q}_\ell)$.

Let $X \in \mathcal{P}$. Let Y be an object of $\mathcal{P}$ (resp. a scheme of finite type over k). By a morphism $h : H^m(X)(r)^{\flat *} \to H^n(Y)(s)^{\flat *}$ (resp. $H^m(X)(r)^{\flat *} \to H^n(Y)(s)$) of symbols, we mean a $\mathbb{Q}_\ell$-homomorphism $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ obtained from some element of

$$\mathrm{gr}^{d-r+s} KH_{(m-2r)-(n-2s),\lim}(Z) \otimes \mathbb{Q}, \quad \text{where } \ Z = X \times_S Y \ \ (\text{resp. } Z = X \times Y).$$

Here d is the dimension of X. Note that an element of this K-group goes by the Chern class map to $H^{2d-m+n}(Z)_\ell(d-r+s)$, and by Künneth formula and by Poincaré duality of X, to $\mathrm{Hom}_{\mathbb{Q}_\ell}(H^m(X)_\ell(r), H^n(Y)_\ell(s))$.

For such a morphism h and for a morphism $g : H^{m(1)}(X_1)(r(1))^{\flat *} \to H^m(X)(r)^{\flat *}$ with $X_1 \in \mathcal{P}$, the composition $h \circ g : H^{m(1)}(X_1)_\ell(r(1)) \to H^n(Y)_\ell(s)$ is a morphism $H^{m(1)}(X_1)(r(1))^{\flat *} \to H^n(Y)(s)^{\flat *}$ (resp. $H^{m(1)}(X_1)(r(1))^{\flat *} \to H^n(Y)(s)$). The identity map $H^m(X)(r)^{\flat *} \to H^m(X)(r)^{\flat *}$ is a morphism. These are proved in the same way as the non-$\flat$ case in Proposition A.6, by replacing the Riemann–Roch theorem for K_0 by the Riemann–Roch theorem for KH_n in [15] which works for projective morphisms locally of complete intersection.

Thus we have a category (LM$\flat *$), and for a scheme Y of finite type over k, we have a contra-variant functor

$$H^n(Y)(s) : H^m(X)(r)^{\flat *} \mapsto \{\text{morphisms } H^m(X)(r)^{\flat *} \to H^n(Y)(s)\}$$

from (LM$\flat *$) to the category of $\mathbb{Q}$-vector spaces.

Let (MM$*$) be the smallest full subcategory $\mathcal{C}$ of the category of contra-variant functors from (LM$\flat *$) to the category of $\mathbb{Q}$-vector spaces satisfying the following conditions (i) and (ii).

(i) $\mathcal{C}$ contains the functors $H^n(T)(s)$ for schemes T of finite type over k and for $n, s \in \mathbb{Z}$.

(ii) The kernel of every morphism of $\mathcal{C}$ belongs to $\mathcal{C}$.

That is, if $\mathcal{C}_0$ denotes the category of the functors $H^n(T)(s)$ for schemes T of finite type over k and for $n, s \in \mathbb{Z}$ and if $\mathcal{C}_{i+1}$ is the category of functors which are kernels of some morphisms of $\mathcal{C}_i$, then (MM$*$)$= \bigcup_{i \geq 0} \mathcal{C}_i$.

Thus (MM$*$) is an additive category with kernels of morphisms. The authors expect that it is an abelian category, but have not yet proved it. The authors have not yet proved that the category (MM) is stable under taking kernels.

A.12. For any scheme T and for an integer $t \geq 0$, we have a canonical homomorphism $K_0(T \times \mathbb{G}_m^t) \to KH_{-t}(T)$, and the Chern class map on the former K-group factors through the Chern class map on the latter K-group. Hence we have a functor

$$(\mathrm{LM}\flat) \to (\mathrm{LM}\flat *)$$

(the objects are the same but the set of morphisms might be enlarged in the latter category).

Conjecture A.13. (LM$\flat$) $=$ (LM$\flat *$) *and* (MM) $=$ (MM$*$).

To check that our definitions of the category of mixed motives are reasonable, we show an example A.18 with our definitions for which the problems on Tate conjecture and Hodge conjecture (A.14) and the monodromy conjecture (Conjecture A.17) on mixed motives have affirmative answers (Proposition A.19).

A.14. Let Y and Z be schemes of finite type over k (resp. objects of $\mathcal{P}$) and let $m, n, r, s \in \mathbb{Z}$. We ask whether the following (1) and (2) are true.

(1) (Tate conjecture.) Assume that k is finitely generated over the prime field. Then

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathrm{Mor}_{(\mathrm{MM}*)}(H^m(Y)(r), H^n(Z)(s)) \overset{\cong}{\to} \mathrm{Hom}_{\,\mathrm{Gal}(\bar{k}/k)}(H^m(Y)_\ell(r), H^n(Z)_\ell(s))$$

$$(\text{resp. } \mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathrm{Mor}_{(\mathrm{LM}\flat)}(H^m(Y)(r)^\flat, H^n(Z)(s)^\flat) \overset{\cong}{\to} \mathrm{Hom}_{\,\pi_1^{\log}(S)}(H^m(Y)_\ell(r), H^n(Z)_\ell(s))).$$

(2) (Hodge conjecture.) Assume that $k = \mathbb{C}$. Then

$$\mathrm{Mor}_{(\mathrm{MM}*)}(H^m(Y)(r), H^n(Z)(s)) \overset{\cong}{\to} \mathrm{Hom}_{\,(\mathrm{MH})}(H^m(Y)(r)_H, H^n(Z)(s)_H)$$

$$(\text{resp. } \mathrm{Mor}_{(\mathrm{LM}\flat)}(H^m(Y)(r)^\flat, H^n(Z)(s)^\flat) \overset{\cong}{\to} \mathrm{Hom}_{\,(\mathrm{LH}\flat)}(H^m(Y)(r)^\flat_H, H^n(Z)(s)^\flat_H))).$$

Here $(\cdot)_H$ is the associated mixed Hodge structure (resp. mixed Hodge structure with N).

A.15. The conjectures in A.14 for the first isomorphisms in (1), (2) (i.e., for (MM$*$)) are in general false. The example in Appendix of [5] written by S. Bloch is a counter-example for the first isomorphism in (2) in which $Y = \mathrm{Spec}(\mathbb{C})$, $m = 0$, $r = 0$, Z is the W there which is three dimensional and singular, $n = 4$, $s = 2$. A counter-example for the first isomorphism in (1) is obtained from by defining this W over a number field.

We expect that the conjectures for the second isomorphisms in (1), (2) (i.e., for (LM$\flat$)) are true in general. We expect that the above conjectures for the first isomorphisms in (1), (2) are true for smooth Y, Z, and more generally, for the underlying schemes of log smooth saturated fs log schemes over the standard log point.

Remark A.16. (1) For singular varieties, the Hodge conjecture [a Hodge class in homology = an algebraic cycle class] and the Tate conjecture [a Tate class (a Galois invariant element) in homology = an algebraic cycle class] are formulated in Part II of Jannsen [5] and are shown to be equivalent to the classical Hodge conjecture and Tate conjecture for projective smooth varieties (and hence are believed to be true), but the Hodge conjecture [a Hodge class in cohomology = an algebraic cycle class] and the Tate conjecture [a Tate class in cohomology = an algebraic cycle class] are false by Appendix of [5] written by Bloch. The counter-examples in A.15 appear because our theory considers cohomology $H^m(X)(r)$, not homology $H_m(X)(r)$.

(2) In Part II of [5], for smooth varieties, conjectures [Hodge classes in cohomology come from Quillen's K-theory] and [Tate classes in cohomology come from Quillen's K-theory] (for various Tate twists of the cohomology) are formulated. These are essentially the first isomorphisms in (1), (2) of A.14 for $Y = \mathrm{Spec}(k)$ and $m = 0$, $r = 0$, and Z smooth, though we use the homotopy K-theory KH, not Quillen's K-theory.

Conjecture A.17. *(Monodromy conjecture which tells that the monodromy operator comes from geometry, not only from Galois theory.)*

For $X \in \mathcal{P}$, *the monodromy operator* $N : H^m(X)_\ell \to H^m(X)_\ell(-1)$ *is a morphism* $H^m(X)^\flat \to H^m(X)(-1)^\flat$, *and hence is a morphism* $H^m(X)^{\flat *} \to H^m(X)(-1)^{\flat *}$.

A.18. Example. Let Λ be a discrete valuation ring with residue field k, and let $\mathfrak{X}$ be a projective regular flat scheme over Λ of relative dimension one with smooth generic fiber and with semistable reduction. We assume that the special fiber of $\mathfrak{X}$ is a simple normal crossing divisor. Endow $\mathrm{Spec}(\Lambda)$ and $\mathfrak{X}$ with the canonical log structures. We regard S as the closed point of $\mathrm{Spec}(\Lambda)$ with the induced log structure. Let X be the fs log scheme $\mathfrak{X} \times_{\mathrm{Spec}(\Lambda)} S$ over S. Then $X \in \mathcal{P}$. Let T be the underlying scheme $\mathfrak{X} \otimes_\Lambda k$ over k of X. We have a canonical injective homomorphism $H^1(T)_\ell \to H^1(X)_\ell$.

Remark. $H^1(X)^\flat$ (or $H^1(X)^{\flat *}$) is regarded as the limit mixed motive, an analogue of limit mixed Hodge structure.

Proposition A.19. *Let the notation be as in* A.18.

(1) *The Tate conjecture and the Hodge conjecture* A.14 *for* (MM$*$) *are true in the case* $Y = \mathrm{Spec}(k)$, $Z = T$, $m = 0$, $n = 1$, $r = s = 0$.

(2) *The Tate conjecture and the Hodge conjecture* A.14 *for* (LM$\flat$) *are true in the case* $Y = S$, $Z = X$, $m = 0$, $n = 1$, $r = s = 0$, *and also in the case* $Y = X$, $Z = S$, $m = 1$, $n = 0$, $r = 0$, $s = -1$.

(3) *The monodromy operator* $N : H^1(X)_\ell \to H^1(X)_\ell(-1)$ *is a morphism of* (LM$\flat$) *(and hence a morphism of* (LM$\flat *$)*).*

Proof. In (1) and (2), we only discuss the Tate conjecture. The proof for the Hodge conjecture is similar.

In the discussion about (MM$*$) (resp. (LM$\flat$)), we denote $H^0(\mathrm{Spec}(k))$ (resp. $H^0(S)^\flat$) by $\mathbb{Q}$. With this notation, the Tate conjecture in (1) is written as

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathrm{Mor}_{(\mathrm{MM}*)}(\mathbb{Q}, H^1(T)) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{Gal}(\bar{k}/k)}(\mathbb{Q}_\ell, H^1(T)_\ell),$$

and the statements on the Tate conjecture in (2) are written as

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathrm{Mor}_{(\mathrm{LM}\flat)}(\mathbb{Q}, H^1(X)^\flat) \xrightarrow{\cong} \mathrm{Hom}_{\pi_1^{\log}(S)}(\mathbb{Q}_\ell, H^1(X)_\ell),$$

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathrm{Mor}_{(\mathrm{LM}\flat)}(H^1(X)^\flat, \mathbb{Q}(-1)) \xrightarrow{\cong} \mathrm{Hom}_{\pi_1^{\log}(S)}(H^1(X)_\ell, \mathbb{Q}_\ell(-1)).$$

By Galois descent, we may and do assume that all singular points of T are k-rational. Let A be the set of all singular points of T and let B be the set of all generic points of T. The following (i) and (ii) are well-known. (See, for example, [17].)

(i) We have a canonical isomorphism $Q \xrightarrow{\cong} H^1_{\text{ét}}(T, \mathbb{Z})$, where Q is the cokernel of a natural homomorphism $\mathbb{Z}^B \to \mathbb{Z}^A$, and it induces an isomorphism from $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} Q$ to the G-invariant part of $H^1(X)_\ell$, where $G = \pi_1^{\log}(S)$. Hence by the Poincaré duality, we have an isomorphism from the G-coinvariant of $H^1(X)_\ell(1)$ to $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} P$, where $P = \mathrm{Hom}\,(Q, \mathbb{Z})$.

(ii) The monodromy logarithm $N : H^1(X)_\ell \to H^1(X)_\ell(-1)$ is the composition

$$(*) \quad H^1(X)_\ell \to \mathbb{Q}_\ell(-1) \otimes_{\mathbb{Z}} P \to \mathbb{Q}_\ell(-1)^A \to \mathbb{Q}_\ell(-1) \otimes_{\mathbb{Z}} Q \to H^1(X)_\ell(-1).$$

By Theorems 3.3 and 5.1 of [21] and by Lemma 2.3 of [22], we have an isomorphism $KH_{-1}(T) \cong H^1_{\text{ét}}(T, \mathbb{Z})$ and the Chern class map $KH_{-1}(T) \otimes \mathbb{Q} \to \mathrm{gr}^0 KH_{-1}(T) \otimes \mathbb{Q} \to H^1(T)_\ell$ corresponds to the canonical map $H^1_{\text{ét}}(T, \mathbb{Z}) \to H^1(T)_\ell$. By definition, $\mathrm{Mor}_{(\mathrm{MM}*)}(\mathbb{Q}, H^1(T))$ is the image of $KH_{-1}(T) \otimes \mathbb{Q} \to H^1(T)_\ell$ and hence we have (1).

Next we consider (2). We prove first the version of the Tate conjecture in (2) in which we replace (LM♭) by (LM♭∗). By definition, $\mathrm{Mor}_{(\mathrm{LM}\flat*)}(\mathbb{Q}, H^1(X)^{\flat*})$ is the image of $KH_{-1}(T) \otimes \mathbb{Q} \to H^1(X)_\ell$ and $\mathrm{Mor}_{(\mathrm{LM}\flat*)}(H^1(X)^{\flat*}, \mathbb{Q}(-1))$ is the image of $KH_{-1}(T) \otimes \mathbb{Q} \to H^1(X)_\ell \cong \mathrm{Hom}\,(H^1(X)_\ell, \mathbb{Q}_\ell(-1))$ and hence we have the (LM♭∗) version of (2).

Now we consider (LM♭). By definition, $\mathrm{Mor}_{(\mathrm{LM}\flat)}(\mathbb{Q}, H^1(X)^\flat)$ is the image of $\mathrm{gr}^1 K_0(T \times \mathbb{G}_m) \otimes \mathbb{Q} \to H^1(X)_\ell$ and $\mathrm{Mor}_{(\mathrm{LM}\flat)}(H^1(X), \mathbb{Q}(-1))$ is the image of $\mathrm{gr}^1 K_0(T \times \mathbb{G}_m) \otimes \mathbb{Q} \to H^1(X)_\ell \cong \mathrm{Hom}\,(H^1(X)_\ell, \mathbb{Q}_\ell(-1))$. These Chern class maps factor through the above Chern class maps on $\mathrm{gr}^1 KH_{-1}(T) \otimes \mathbb{Q}$. Since the map $\mathrm{gr}^1 K_0(T \times \mathbb{G}_m) \otimes \mathbb{Q} \to \mathrm{gr}^1 KH_{-1}(T) \otimes \mathbb{Q} \cong H^1_{\text{ét}}(T, \mathbb{Z}) \otimes \mathbb{Q}$ has a right inverse defined by $H^1_{\text{ét}}(T, \mathbb{Z}) \to H^1_{\text{ét}}(T \times \mathbb{G}_m, \mathbb{G}_m) = \mathrm{Pic}\,(T \times \mathbb{G}_m) \to \mathrm{gr}^1 K_0(T \times \mathbb{G}_m)$ in which the first arrow is the product with the coordinate function of $\mathbb{G}_m$, we have (2).

(3) follows from the above (ii) because every arrow in $(*)$ in (ii) is a morphism in (MM). □

A.20. We have considered the category of mixed motives modulo homological equivalence. The above method of the construction of (MM∗) works without homological equivalence as follows, by using the K-groups as the sets of morphisms. We define the modified version (LM♭∗∗) of (LM♭∗) as the category of symbols $h(X)(r)^{\flat**}$, where $X \in \mathcal{P}$ and $r \in \mathbb{Z}$. We define the set of morphisms from $h(X)(r)^{\flat**}$ to $h(Y)(s)^{\flat**}$ to be $\bigoplus_{n \in \mathbb{Z}} \mathrm{gr}^{d-r+s} KH_{n,\lim}(X \times_S Y) \otimes \mathbb{Q}$, where d is the dimension of X. We define the modified version (MM∗∗) of (MM∗) as the category $\bigcup_{i \geq 0} \mathcal{C}_i$ of contra-variant functors from (LM♭∗∗) to the category of $\mathbb{Q}$-vector spaces, where $\mathcal{C}_0$ is the category of the functors

$$h(T)(s) : h(X)(r)^{\flat**} \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{gr}^{d-r+s} KH_{n,\lim}(X \times T) \otimes \mathbb{Q}$$

for schemes T of finite type over k and for $s \in \mathbb{Z}$, and $\mathcal{C}_{i+1}$ is the category of functors which are kernels of some morphisms of $\mathcal{C}_i$.

Thus (MM$**$) is an additive category with kernels of morphisms. We expect that it is an abelian category.

A.21. In the case where the characteristic of k is 0, a definition of the category of mixed motives is given in Part I of Jannsen [5] by considering smooth (not necessarily proper) schemes. In his definition, a morphism of mixed motives is a compatible family of homomorphisms of various realizations (including the $\mathbb{Q}$-Betti realization; K-theory is not used in this definition). His definition and our definition are connected by the Tate conjecture for the first isomorphism in (1) of A.14 for smooth schemes.

Our definition works also in positive characteristic in which we do not have the Betti realization.

The authors do not see how our definition is related to the work [20] on mixed motives and [1] on log motives.

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