A certain class of non-compact 4-symmetric spaces of exceptional type

Hiroyuki Kurihara and Koji Tojo

Abstract

Let $\mathfrak{g}$ be a non-compact simple exceptional Lie algebra over $\mathbb{R}$ with an automorphism σ of order four and $\mathfrak{h}$ the fixed point set of σ. Suppose that the dimension of the center of $\mathfrak{h}$ is at most one and $\mathfrak{h}_{\mathbb{C}}$ contains a Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$. In this paper we shall classify non-compact 4-symmetric pairs under the certain equivalence relation.

1 Introduction

It is known that k-symmetric spaces are generalizations of symmetric spaces. The definition is as follows:

Let G be a Lie group and H a closed subgroup of G. A homogeneous space $(G/H,\, \sigma)$ is called a *k-symmetric space* if there exists an automorphism σ of G such that

- $\sigma^k = \mathrm{Id}$ and $\sigma^l \neq \mathrm{Id}$ for any $l < k$,
- $G^\sigma_o \subset H \subset G^\sigma$, where G^σ and G^σ_o is the set of fixed points of σ in G and its identity component, respectively,

The classification of k-symmetric spaces is a fundamental problem for studying geometry of k-symmetric spaces. It is well known the classification of Riemannian symmetric spaces (cf. Helgason [3]). Gray [2] classified Riemannian 3-symmetric spaces (see also Wolf and Gray [8]). Moreover compact Riemannian 4-symmetric spaces is classified by Jeménez [4].

The classification of 3-symmetric spaces $(G/H, \sigma)$ was made by classifying involutions τ satisfying $\tau\sigma = \sigma\tau$. Similarly, involutions τ of a 4-symmetric space $(G/H, \sigma)$ satisfying $\tau\sigma = \sigma\tau$ are important for the classification of 4-symmetric spaces. Let $\mathfrak{g}$ (or $\mathrm{Lie}(G)$) denote the Lie algebra of G and $\mathfrak{g}^\sigma$ the fixed point set of σ in $\mathfrak{g}$. In two previous papers [5] and [6], we classified such involutions τ when $\mathfrak{g}$ is a compact simple Lie algebra of exceptional type and the dimension of the center of $\mathfrak{g}^\sigma$ is at most one. In this paper we classify the non-compact 4-symmetric spaces satisfying some certain conditions.

Let $(G/H, \sigma)$ be a 4-symmetric space such that G is a simple Lie group. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebra of G and H, respectively. The pair $(\mathfrak{g}, \mathfrak{h})$ (or $(\mathfrak{g}, \sigma)$) is called a *4-symmetric pair*. Note that, since the fixed point set $\mathfrak{g}^\sigma$ is equal to $\mathfrak{g}^{\sigma^{-1}}$, the 4-symmetric pair $(\mathfrak{g}, \sigma)$ is equal to $(\mathfrak{g}, \sigma^{-1})$.

Suppose that $\mathfrak{g}$ is of non-compact type. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\theta\sigma = \sigma\theta$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Let $\mathfrak{g}^*$ and $\mathfrak{h}^*$ denote the compact duals of $\mathfrak{g}$ and $\mathfrak{h} := \mathfrak{g}^\sigma$, respectively, that is, $\mathfrak{g}^* = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ and $\mathfrak{h}^* = \mathfrak{g}^\sigma \cap \mathfrak{k} \oplus \sqrt{-1}(\mathfrak{g}^\sigma \cap \mathfrak{p})$. Then σ induces an

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automorphism σ^* on $\mathfrak{g}^*$, and $(\mathfrak{g}^*, \sigma^*)$ becomes a compact 4-symmetric pair. Let $\mathfrak{z}((\mathfrak{g}^*)^{\sigma^*})$ denote the center of $(\mathfrak{g}^*)^{\sigma^*}$. Suppose that $\dim(\mathfrak{z}((\mathfrak{g}^*)^{\sigma^*}) \leq 1$. Then according to [5] and [6] if there exists an automorphism τ^* of $\mathfrak{g}^*$ such that $\tau^*((\mathfrak{g}^*)^{\sigma^*}) = (\mathfrak{g}^*)^{\sigma^*}$, then $\tau^*\sigma^* = \sigma^*\tau^*$ or $\tau^*\sigma^* = (\sigma^*)^{-1}\tau^*$. From the above, it is natural to define an isomorphism between the two triplets $(\mathfrak{g}_1, \sigma_1, \theta_1)$ and $(\mathfrak{g}_2, \sigma_2, \theta_2)$, where $\mathfrak{g}_i$ is a simple Lie algebra over $\mathbb{R}$, σ_i is an order four automorphism on $\mathfrak{g}_i$, and θ_i is an involution that commutes with $\sigma_i (i = 1, 2)$, as the existance of an isomorphism $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\mu(\mathfrak{g}_1^{\pm\theta_1}) = \mathfrak{g}_2^{\pm\theta_2}$ and $\mu(\mathfrak{g}_1^{\sigma_1}) = \mathfrak{g}_2^{\sigma_2}$. In this paper, we classify non-compact 4-symmetric pairs $(\mathfrak{g}, \sigma)$ of exceptional type when $\dim \mathfrak{z}(\mathfrak{g}^\sigma) \leq 1$ under this isomorphism using the involution on compact 4-symmetric pairs investigated in [5] and [6].

At first, we shall prove that there exists a Cartan involution θ of G satisfying $\theta\sigma = \sigma\theta$. This allows us to consider the compact dual $\mathfrak{g}^*$ of Lie(G) with respect to this Cartan involution θ. These automorphisms θ and σ of Lie(G) induce automorphisms θ^* and σ^* of $\mathfrak{g}^*$, respectively, which satisfy $(\theta^*)^2 = \mathrm{Id} = (\sigma^*)^4$ and $\theta^*\sigma^* = \sigma^*\theta^*$. The non-compact dual of a symmeric pair $(\mathfrak{g}^*, \theta^*)$ induce non-compact 4-symmetric pairs. This construction method runs out of non-compact 4-symmetric spaces. Therefore the existance of such a Cartan involution is most impotant. If $\mathfrak{g}^*$ is of an exceptional type, then these θ^* are classified by authors under the certain conditions ([5], [6]).

The organization of this paper is as follows:

In Section 2, we recall the notions of root systems needed for the remaining part of this paper and some results on inner automorphisms of order k ($k \leq 4$) of a semisimple Lie algebra.

In Section 3, let $\mathfrak{g}$ be a non-compact simple Lie algebra over $\mathbb{R}$ with an automorphism of order k. We prove that there is a maximal compact subgroup of Aut$(\mathfrak{g})$ containing a compact subgroup $\{\mathrm{Id}, \sigma, \sigma^2, \dots, \sigma^{k-1}\}$ of Aut$(\mathfrak{g})$ (Lemma 3.2). Using the Lie algebra of this maximal compact subgroup, we construct a Cartan decomposition and induced Cartan involution which is commute with σ. As mentioned above, this is key Proposition (Proposition 3.4).

In Section 4, we investigate some properties of automorphisms of the compact dual of non-compact 4-symmetric triple. These considerations shows that the compact dual constitutes a compact 4-symmetric triple.

In Section 5, we define the isomorphism between two triples $(\mathfrak{g}^*, \sigma^*, \theta^*)$ and describe the relationship between the isomorphic of two compact(resp. non-compact) triples and the isomorphic of their non-compact(resp. compact) duals. Moreover, if the dimension of the center of $(\mathfrak{g}^*)^{\sigma^*}$ is 0 or 1 and σ^* is an inner automorphism of $\mathfrak{g}^*$, then we prove that the non-compact triples are exhausted from all conjugate classes of involutions on $\mathfrak{g}^*$ under $\mathrm{Aut}_{(\mathfrak{g}^*)^{\sigma^*}}(\mathfrak{g}^*)$ which commutes with σ^*.

In Section 6, let $(G/H, \sigma)$ be a exceptional non-compact 4-symmetric space and suppose that the complexification of Lie(H) contains a Cartan subalgebra of the complexification of Lie(G) and the dimension of the center of Lie(H) is 0 or 1. Then using [5] and [6], we classify the exceptional non-compact 4-symmetric spaces $(G/H, \sigma)$ that satisfy these conditions under the isomorphism defined in Section 5.

2 Preliminaries.

Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be a compact semisimple Lie algebra and a maximal abelian subalgebra of $\mathfrak{g}^*$, respectively. Let $\mathfrak{g}^*_{\mathbb{C}}$ and $\mathfrak{t}^*_{\mathbb{C}}$ denote the complexifications of $\mathfrak{g}^*$ and $\mathfrak{t}^*$, respectively. Let $\Delta(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}})$ denote the root system of $\mathfrak{g}^*_{\mathbb{C}}$ with respect to $\mathfrak{t}^*_{\mathbb{C}}$, $\Pi(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}}) = \{\alpha_1, \dots, \alpha_n\}$ the set of fundamental roots of $\Delta(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}})$ with respect to a lexicographic order and

$$\mathfrak{g}^*_\alpha = \{X \in \mathfrak{g}^*_{\mathbb{C}} \ ; \ [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{t}^*_{\mathbb{C}}\}. \tag{2.1}$$

Since the Killing form B is non-degenerate, we can define $H_\alpha \in \mathfrak{t}^*_\mathbb{C}(\alpha \in \Delta(\mathfrak{g}^*_\mathbb{C}, \mathfrak{t}^*_\mathbb{C}))$ by $\alpha(H) = B(H_\alpha, H)$ for any $H \in \mathfrak{t}^*_\mathbb{C}$. As in Helgason [3], we take the Weyl basis $\{E_\alpha \in \mathfrak{g}^*_\alpha \ ; \ \alpha \in \Delta(\mathfrak{g}^*_\mathbb{C}, \mathfrak{t}^*_\mathbb{C})\}$ of $\mathfrak{g}^*_\mathbb{C}$ so that

$$\begin{aligned}
&[E_\alpha, E_{-\alpha}] = H_\alpha,\\
&[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad N_{\alpha,\beta} \in \mathbb{R},\\
&N_{\alpha,\beta} = -N_{-\alpha,-\beta},\\
&A_\alpha = E_\alpha - E_{-\alpha}, \quad B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{g}^*.
\end{aligned}$$

Let Δ^+ denote the set of positive roots of $\Delta(\mathfrak{g}^*_\mathbb{C}, \mathfrak{t}^*_\mathbb{C})$ with respect to the order.

As is well-known, a Lie algebra

$$\mathfrak{g}^* = \mathfrak{h}^* + \sum_{\alpha \in \Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$$

is a compact real form of $\mathfrak{g}^*_\mathbb{C}$. Here $\mathfrak{h}^* = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{-1}H_\alpha$. In particular let

$$\mathfrak{su}_\alpha(2) := \mathbb{R}\sqrt{-1}H_\alpha + \mathbb{R}A_\alpha + \mathbb{R}B_\alpha \cong \mathfrak{su}(2).$$

Let t_α denote the root reflections for $\alpha \in \Delta(\mathfrak{g}^*_\mathbb{C}, \mathfrak{t}^*_\mathbb{C})$ and $\tilde{t}_\alpha$ an inner automorphism of $\mathfrak{g}^*$ such that $\tilde{t}_\alpha|_{\mathfrak{t}^*} = t_\alpha$.

We define $K_j \in \mathfrak{t}^*_\mathbb{C}(j = 1, \dots, l)$ by

$$\alpha_i(K_j) = \delta_{ij}, \quad i, j = 1, \dots, l,$$

and denote the highest root δ by

$$\delta = \sum_{j=1}^{l} m_j \alpha_j, \quad m_j \in \mathbb{Z}.$$

3 The Cartan involution which commute to an automorphism of finite order.

Let $\mathfrak{g}$ be a n-dimentional non-compact simple Lie algebra over $\mathbb{R}$. Let θ be a Cartan involution of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \theta|_\mathfrak{k} = \mathrm{Id}|_\mathfrak{k}, \quad \theta|_\mathfrak{p} = -\mathrm{Id}|_\mathfrak{p}$$

the corresponding Cartan decomposition. Let σ be an automorphism of order k on $\mathfrak{g}$, i.e., $\sigma^k = \mathrm{Id}, \sigma^i \neq \mathrm{Id}(i = 1, 2, \dots, k-1)$. Then $\Gamma = \{\mathrm{Id}, \sigma, \sigma^2, \dots, \sigma^{k-1}\}$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Let $\tilde{\theta}$ be a Cartan involution of $\mathrm{GL}(n, \mathbb{R})$. From 1.1 of [1] any Cartan involutions of $\mathrm{Aut}(\mathfrak{g})$ are induced from $\tilde{\theta}$. For simplicity, these Cartan involutions of $\mathrm{Aut}(\mathfrak{g})$ are also represented by $\tilde{\theta}$. Since $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we have $\mathrm{ad}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{k}) \oplus \mathrm{ad}(\mathfrak{p})$.

Let $L = (\mathrm{Aut}(\mathfrak{g}))^{\tilde{\theta}} (= \{\varphi \in \mathrm{Aut}(\mathfrak{g}); \tilde{\theta}(\varphi) = \varphi\})$.

Lemma 3.1 ([1]). (i) *L is a maximal compact subgroup of* $\mathrm{Aut}(\mathfrak{g})$.

(ii) $\mathrm{Lie}(L) = \mathrm{ad}(\mathfrak{k})$.

(iii) *Every compact subgroup in* $\mathrm{Aut}(\mathfrak{g})$ *is conjugate to a subgroup in L.*

Lemma 3.2. *There exists a maximal compact subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *containing* Γ.

Proof. Since $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ and represented by an algebraic equation, $\mathrm{Aut}(\mathfrak{g})$ is a reductive algebraic subgroup of $\mathrm{GL}(n,\mathbb{R})$. Note that $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{ad}(\mathfrak{g})$. Since $\mathfrak{g}$ is simple, the center of $\mathfrak{g}$ is $\{0\}$, and hence we have $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$.

Now, from Lemma 3.1(iii) Γ is conjugate to some subgroup Γ' of L under $\tau \in \mathrm{Ad}(\exp(\mathrm{ad}(\mathfrak{p})))$, that is, $\Gamma' = \tau\Gamma\tau^{-1}$. From Lemma 3.1(i) $\tau^{-1}L\tau$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g})$ containing Γ. □

Let $\tilde{B}$ be a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g})$ containing Γ. Then there exists $\tau \in \mathrm{Aut}(\mathfrak{g})$ such that $\tilde{B} = \tau^{-1}L\tau$, which together with Lemma 3.1(ii) implies that $\mathrm{Lie}(\tilde{B}) \cong \mathrm{Lie}(L) = \mathrm{ad}(\mathfrak{k})$. Let $\tilde{\mathfrak{b}} = \mathrm{Lie}(\tilde{B})$, $\tilde{\mathfrak{b}} = \mathrm{ad}(\mathfrak{b})$ and $\sigma \in \Gamma$. Since $\Gamma \subset \tilde{B}$, we have

$$\sigma\tilde{B}\sigma^{-1} \subset \tilde{B}\cdot\tilde{B}\cdot\tilde{B} = \tilde{B}.$$

In particular, if $\tilde{B}_o$ is the identity component of $\tilde{B}$, then we have

$$\sigma\tilde{B}_o\sigma^{-1} = \tilde{B}_o.$$

$\tilde{B}_o$ is a closed set in a compact set $\tilde{B}$ so $\tilde{B}_o$ is a compact Lie group.

Lemma 3.3. $\sigma(\mathfrak{b}) = \mathfrak{b}$.

Proof. If $Y \in \mathfrak{b}$, then $\exp t(\mathrm{ad}\, Y) \in \tilde{B}_o$ and for any $t \in \mathbb{R}$

$$\exp(t\sigma(\mathrm{ad}\, Y)\sigma^{-1}) = \sigma(\exp t(\mathrm{ad}\, Y))\sigma^{-1} \in \sigma\tilde{B}_o\sigma^{-1} = \tilde{B}_o.$$

It follows that

$$\mathrm{ad}(\sigma(Y)) = \frac{d}{dt}\bigg|_{t=0} \exp t(\mathrm{ad}(\sigma(Y))) \in \mathrm{Lie}(\tilde{B}_o) = \tilde{\mathfrak{b}}.$$

Thus we have $\sigma(\mathfrak{b}) \subset \mathfrak{b}$. Therefore $\sigma(\mathfrak{b}) = \mathfrak{b}$. □

Proposition 3.4. *Let* $\mathfrak{g}$ *be an non-compact simple Lie algebra over* $\mathbb{R}$ *with an automorphism* σ *of order* k *and* $\mathfrak{g}^\sigma$ *the fixed point set of* σ *in* $\mathfrak{g}$. *Then*

(i) *There exists a Cartan involution that commutes with* σ.

(ii) *If* θ_1 *and* θ_2 *are Cartan involutions preserving* $\mathfrak{g}^\sigma$, *then there exists an automorphism* τ *of* $\mathfrak{g}$ *preserving* $\mathfrak{g}^\sigma$ *such that* $\tau\theta_1\tau^{-1} = \theta_2$.

Proof. (i) Let $\mathfrak{b}$ be constructed above. Since $\mathfrak{b}$ is a maximal compact subalgebra, there is a Cartan decomposition

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}^\perp$$

where $\mathfrak{b}^\perp$ denote the orthogonal complement with respect to the Killing form B. Let ω be the corresponding Cartan involution. From Lemma 3.3 for any $X \in \mathfrak{b}$ there exists $X' \in \mathfrak{b}$ such that $\sigma(X') = X$. So if $Y \in \mathfrak{b}^\perp$, then we have $B(X, \sigma(Y)) = 0$. It follows that $\sigma(\mathfrak{b}^\perp) \subset \mathfrak{b}^\perp$. If $X \in \mathfrak{b}^\perp$, then since $\mathfrak{b}^\perp \subset \mathfrak{g} = \sigma(\mathfrak{g})$, there exists $X' \in \mathfrak{g}$ such that $X = \sigma(X')$. For any $Y \in \mathfrak{b}$, from Lemma 3.3 we have $\sigma(Y) \in \mathfrak{b}$ so $B(Y, X') = B(\sigma(Y), X) = 0$. Thus $\mathfrak{b}^\perp \subset \sigma(\mathfrak{b}^\perp)$ and therefore $\mathfrak{b}^\perp = \sigma(\mathfrak{b}^\perp)$.

Now, if $X = X_{\mathfrak{b}} + X_{\mathfrak{b}^\perp} \in \mathfrak{g}$ ($X_{\mathfrak{b}} \in \mathfrak{b}, X_{\mathfrak{b}^\perp} \in \mathfrak{b}^\perp$), then

$$\sigma\omega(X_{\mathfrak{b}} + X_{\mathfrak{b}^\perp}) = \sigma(X_{\mathfrak{b}}) - \sigma(X_{\mathfrak{b}^\perp}).$$

Because of $\sigma(\mathfrak{b}) = \mathfrak{b}$ and $\sigma(\mathfrak{b}^\perp) = \mathfrak{b}^\perp$, we have

$$\omega\sigma(X_{\mathfrak{b}} + X_{\mathfrak{b}^\perp}) = \sigma(X_{\mathfrak{b}}) - \sigma(X_{\mathfrak{b}^\perp}).$$

Consequently, we have $\sigma\omega = \omega\sigma$.

The statement (ii) is proved in the same way as in the proofs of Lemma 3 and Lemma 4 in [7] as follows. It is shown as in [3] that $\theta_2\theta_1$ is a self-adjoint transformation of $\mathfrak{g}$ with respect to positive definite inner product $B_{\theta_1}(B_{\theta_1}(X, Y) = -B(X, \theta_1(Y))$ for $X, Y \in \mathfrak{g}$). Since $\mathfrak{g}^\sigma$ is θ_1-stable and θ_2-stable, we can take an orthonormal basis $\{X_1, \cdots, X_n\}$ such that $\{X_1, \cdots, X_m\}$ is a basis of $\mathfrak{g}^\sigma$ and $\theta_2\theta_1$ is represented by a diagonal matrix with respect to this basis. Put $P = (\theta_2\theta_1)^2$ and define $P^t (t \in \mathbb{R})$ as in [3]. Then $P(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$ so $P^t(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$. Put $\tau = P^{1/4}$. Then it is easy to see that $\tau\theta_1\tau^{-1} = \theta_2$. Thus, τ is an automorphism of $\mathfrak{g}$ preserving $\mathfrak{g}^\sigma$. □

4 Compact dual.

Let $\mathfrak{g}$ be a non-compact simple Lie algebra over $\mathbb{R}$ with an automorphism σ of order four. By Proposition 3.4 there exists a Cartan involution θ of $\mathfrak{g}$ such that $\theta\sigma = \sigma\theta$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding Cartan decomposition of $\mathfrak{g}$.

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h} := \mathfrak{g}^\sigma$ in $\mathfrak{g}$ with respect to the Killing form, that is,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Then we have

$$\theta(\mathfrak{h}) = \mathfrak{h},\ \theta(\mathfrak{m}) = \mathfrak{m},\ \sigma(\mathfrak{k}) = \mathfrak{k},\ \sigma(\mathfrak{p}) = \mathfrak{p}. \tag{4.1}$$

In fact, Since $\theta\sigma = \sigma\theta$, it is obvious that $\theta(\mathfrak{h}) = \mathfrak{h}$, $\sigma(\mathfrak{k}) = \mathfrak{k}$ and $\sigma(\mathfrak{p}) = \mathfrak{p}$. Similarly as the proof of Lemma 3.3, we obtain $\theta(\mathfrak{m}) = \mathfrak{m}$.

Lemma 4.1. *The following decompositions are direct:*

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}),\ \mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p}),$$
$$\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{m}),\ \mathfrak{p} = (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{m}).$$

Proof. If $H_{\mathfrak{k}} + H_{\mathfrak{p}} \in \mathfrak{h}$ ($H_{\mathfrak{k}} \in \mathfrak{k}, H_{\mathfrak{p}} \in \mathfrak{p}$), then we have $H_{\mathfrak{k}} + H_{\mathfrak{p}} = \sigma(H_{\mathfrak{k}}) + \sigma(H_{\mathfrak{p}})$. It follows from (4.1) that $\sigma(H_{\mathfrak{k}}) \in \mathfrak{k}$ and $\sigma(H_{\mathfrak{p}}) \in \mathfrak{p}$ so $H_{\mathfrak{k}}, H_{\mathfrak{p}} \in \mathfrak{h}$. Thus we have a direct decomposition

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}).$$

If $X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{m}$ ($X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}} \in \mathfrak{p}$), then we have $\mathfrak{m} \ni \theta(X_{\mathfrak{k}} + X_{\mathfrak{p}}) = X_{\mathfrak{k}} - X_{\mathfrak{p}}$. Thus we have $2X_{\mathfrak{k}} = (X_{\mathfrak{k}} - X_{\mathfrak{p}}) + (X_{\mathfrak{k}} + X_{\mathfrak{p}}) \in \mathfrak{m}$ so $X_{\mathfrak{k}} \in \mathfrak{m}$. Therefore we obtain

$$\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p}).$$

If $X_{\mathfrak{h}} + X_{\mathfrak{m}} \in \mathfrak{k}$ $(X_{\mathfrak{h}} \in \mathfrak{h}, X_{\mathfrak{m}} \in \mathfrak{m})$, then we have $X_{\mathfrak{h}} + X_{\mathfrak{m}} = \theta(X_{\mathfrak{h}} + X_{\mathfrak{m}}) = \theta(X_{\mathfrak{h}}) + \theta(X_{\mathfrak{m}})$. It follows from (4.1) that $\theta(X_{\mathfrak{h}}) = X_{\mathfrak{h}}$ and $\theta(X_{\mathfrak{m}}) = X_{\mathfrak{m}}$ so $X_{\mathfrak{h}} \in \mathfrak{h} \cap \mathfrak{k}$ and $X_{\mathfrak{m}} \in \mathfrak{m} \cap \mathfrak{k}$. Therefore we get

$$\mathfrak{k} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{k}).$$

Similarly as above, we have $\mathfrak{p} = (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{m} \cap \mathfrak{p})$. □

Let $\mathfrak{g}^*$ be the compact dual of $\mathfrak{g}$, that is,

$$\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^* \ (\mathfrak{p}^* = \sqrt{-1}\mathfrak{p}),$$

and τ an automorphism of $\mathfrak{g}$ such that $\tau\theta = \theta\tau$. Then we have $\tau(\mathfrak{k}) = \mathfrak{k}$ and $\tau(\mathfrak{p}) = \mathfrak{p}$. Thus we can define the mapping $\tau^* : \mathfrak{g}^* \to \mathfrak{g}^*$ by $\tau^*(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}}) = \tau(X_{\mathfrak{k}}) + \sqrt{-1}\tau(X_{\mathfrak{p}})(X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}} \in \mathfrak{p})$.

Lemma 4.2. *τ^* is an automorphism of $\mathfrak{g}^*$. If the order of τ is k, then the order of τ^* is k. If $\tau^k \neq \mathrm{Id}$, then $(\tau^*)^k \neq \mathrm{Id}$.*

Proof. It follows from the definition of the mapping τ that τ^* preserve the bracket so $\tau^* \in \mathrm{Aut}(\mathfrak{g}^*)$. Since $\tau(\mathfrak{g}) = \mathfrak{g}$, we have $(\tau^*)^k(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}}) = \tau^k(X_{\mathfrak{k}}) + \sqrt{-1}\tau^k(X_{\mathfrak{p}})$. Thus the second assertion is trivial. □

We call the automorphism τ^* of $\mathfrak{g}^*$ defined in Lemma 4.2 *the automorphism of $\mathfrak{g}^*$ induced by τ.*

Lemma 4.3. *$\sigma^*\theta^* = \theta^*\sigma^*$.*

Proof. Since $\theta(\mathfrak{k}) = \mathfrak{k}$, $\theta(\mathfrak{p}) = \mathfrak{p}$ and (4.1), if $X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}} \in \mathfrak{g}^*$ $(X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}} \in \mathfrak{p})$, then we have $\sigma^*\theta^*(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}}) = \sigma(X_{\mathfrak{k}}) - \sqrt{-1}\sigma(X_{\mathfrak{p}}) = \theta^*\sigma^*(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}})$. □

Now, we have a direct decomposition

$$(\mathfrak{g}^*)^{\sigma^*} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p}). \tag{4.2}$$

Indeed, if $X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}} \in (\mathfrak{g}^*)^{\sigma^*} (X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}} \in \mathfrak{p})$, then we have $X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}} = \sigma^*(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}}) = \sigma(X_{\mathfrak{k}}) + \sqrt{-1}\sigma(X_{\mathfrak{p}})$. From (4.1) we have $\sigma(X_{\mathfrak{k}}) = X_{\mathfrak{k}}$ and $\sigma(X_{\mathfrak{p}}) = X_{\mathfrak{p}}$, and hence $X_{\mathfrak{k}}, X_{\mathfrak{p}} \in \mathfrak{h}$. It follows that $(\mathfrak{g}^*)^{\sigma^*} \subset (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})$. On the other hand, by the definition of the mapping σ^* it is clear that $(\mathfrak{g}^*)^{\sigma^*} \supset (\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})$.

Let $\mathfrak{g}_i (i = 1, 2)$ be a non-compact simple Lie algebra over $\mathbb{R}$ with an automorphism σ_i of order four and θ_i a Cartan involution of $\mathfrak{g}_i$ such that $\sigma_i\theta_i = \theta_i\sigma_i$. Let $\mathfrak{k}_i$ and $\mathfrak{p}_i$ denote eigenspaces of θ_i for the eigenvalues $+1$ and -1, respectively. Let $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ be the compact dual of $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$, respectively, that is, $\mathfrak{g}_1^* = \mathfrak{k}_1 \oplus \mathfrak{p}_1^*(\mathfrak{p}_1^* = \sqrt{-1}\mathfrak{p}_1)$, $\mathfrak{g}_2^* = \mathfrak{k}_2 \oplus \mathfrak{p}_2^*(\mathfrak{p}_2^* = \sqrt{-1}\mathfrak{p}_2)$. Suppose that there is an isomorphism $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying $\mu(\mathfrak{k}_1) = \mathfrak{k}_2$, $\mu(\mathfrak{p}_1) = \mathfrak{p}_2$ and $\mu((\mathfrak{g}_1)^{\sigma_1}) = (\mathfrak{g}_2)^{\sigma_2}$. Then, we can define the mapping $\mu^* : \mathfrak{g}_1^* \to \mathfrak{g}_2^*$ by $\mu^*(X_{\mathfrak{k}_1} + \sqrt{-1}X_{\mathfrak{p}_1}) = \mu(X_{\mathfrak{k}_1}) + \sqrt{-1}\mu(X_{\mathfrak{p}_1})$ $(X_{\mathfrak{k}_1} \in \mathfrak{k}_1, X_{\mathfrak{p}_1} \in \mathfrak{p}_1)$. It is obvious that $\mu^*(\mathfrak{k}_1) = \mathfrak{k}_2$ and $\mu^*(\mathfrak{p}_1^*) = \mathfrak{p}_2^*$. Moreover, it is easy to see that

$$\mu^*([X_{\mathfrak{k}_1} + \sqrt{-1}X_{\mathfrak{p}_1}, Y_{\mathfrak{k}_1} + \sqrt{-1}Y_{\mathfrak{p}_1}]) = [\mu^*(X_{\mathfrak{k}_1} + \sqrt{-1}X_{\mathfrak{p}_1}), \mu^*(Y_{\mathfrak{k}_1} + \sqrt{-1}Y_{\mathfrak{p}_1})], \tag{4.3}$$

for $X_{\mathfrak{k}_1}, Y_{\mathfrak{k}_1} \in \mathfrak{k}_1, X_{\mathfrak{p}_1}, Y_{\mathfrak{p}_1} \in \mathfrak{p}_1$. Thus μ^* is an isomorphism of $\mathfrak{g}_1^*$ into $\mathfrak{g}_2^*$. Let $\sigma_i^*(i = 1, 2)$ be the automorphism of $\mathfrak{g}_i^*$ induced by σ_i. From Lemma 4.1 and (4.2) and the definition of the mapping μ we have

$$\begin{aligned} \mu^*((\mathfrak{g}_1^*)^{\sigma_1^*}) &= \mu((\mathfrak{g}_1)^{\sigma_1} \cap \mathfrak{k}_1) \oplus \sqrt{-1}\mu((\mathfrak{g}_1)^{\sigma_1} \cap \mathfrak{p}_1) \\ &= ((\mathfrak{g}_2)^{\sigma_2} \cap \mathfrak{k}_2) \oplus \sqrt{-1}((\mathfrak{g}_2)^{\sigma_2} \cap \mathfrak{p}_2) \\ &= (\mathfrak{g}_2^*)^{\sigma_2^*}. \end{aligned} \tag{4.4}$$

5 Isomorphism

We define the isomorphic between two compact simple Lie algebras with order two and four automorphisms. Let $\mathfrak{g}^*$ be a compact simple Lie algebra over $\mathbb{R}$ with an automorphism σ^* of order four and θ^* an involution of $\mathfrak{g}^*$ such that $\sigma^*\theta^* = \theta^*\sigma^*$. Let $\mathfrak{k}$ and $\mathfrak{p}^*$ denote eigenspaces of θ^* for the eigenvalues $+1$ and -1, respectively. In this section we denote it by $(\mathfrak{g}^*, \sigma^*, \theta^*)$ simply.

Two triples $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ and $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$ are called *isomorphic* if there exists an isomorphism $\mu^* : \mathfrak{g}_1^* \to \mathfrak{g}_2^*$ satisfying $\mu^*(\mathfrak{k}_1) = \mathfrak{k}_2$, $\mu^*(\mathfrak{p}_1^*) = \mathfrak{p}_2^*$ and $\mu^*((\mathfrak{g}_1^*)^{\sigma_1^*}) = (\mathfrak{g}_2^*)^{\sigma_2^*}$.

Remark 5.1. *As stated in Section 1, for any triple* $(\mathfrak{g}^*, \sigma^*, \theta^*)$ *the automorphism* $(\sigma^*)^{-1}$ *leaves* $(\mathfrak{g}^*)^{\sigma^*}$ *invariant. Thus* $(\mathfrak{g}^*, (\sigma^*)^{-1}, \theta^*)$ *will be identified with* $(\mathfrak{g}^*, \sigma^*, \theta^*)$.

Suppose that the triple $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$ and the corresponding isomorphism is μ^*.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the non-compact duals of $\mathfrak{g}_1^* = \mathfrak{k}_1 \oplus \mathfrak{p}_1^*$ and $\mathfrak{g}_2^* = \mathfrak{k}_2 \oplus \mathfrak{p}_2^*$, respectively, that is, $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1 (\mathfrak{p}_1 = \sqrt{-1}\mathfrak{p}_1^*)$, $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2 (\mathfrak{p}_2 = \sqrt{-1}\mathfrak{p}_2^*)$. Since $\mu^*(\mathfrak{k}_1) = \mathfrak{k}_2$ and $\mu^*(\mathfrak{p}_1^*) = \mathfrak{p}_2^*$, we can define the mapping $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$ by $\mu(X_{\mathfrak{k}_1} + \sqrt{-1}X_{\mathfrak{p}_1^*}) = \mu^*(X_{\mathfrak{k}_1}) + \sqrt{-1}\mu^*(X_{\mathfrak{p}_1^*})$ $(X_{\mathfrak{k}_1} \in \mathfrak{k}_1, X_{\mathfrak{p}_1^*} \in \mathfrak{p}_1^*)$. Let $\sigma_i (i = 1, 2)$ be the automorphism of $\mathfrak{g}_i$ induced by σ_i^*. Then by an argument similar to that in Section 4 μ is an isomorphism of $\mathfrak{g}_1$ into $\mathfrak{g}_2$ such that $\mu(\mathfrak{k}_1) = \mathfrak{k}_2$, $\mu(\mathfrak{p}_1) = \mathfrak{p}_2$ and $\mu((\mathfrak{g}_1)^{\sigma_1}) = (\mathfrak{g}_2)^{\sigma_2}$.

The following Lemma is a compact version of Lemma 4.2. The proof is also similar.

Lemma 5.2. *Let* $\mathfrak{g}^*$ *be a compact simple Lie algebra,* θ^* *an involution of* $\mathfrak{g}^*$ *and* $\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^*$ *the corresponding direct decomposition of* $\mathfrak{g}^*$. *Let* $\mathfrak{g}$ *denote the non-compact dual of* $\mathfrak{g}^*$. *Then for each automorphism* τ^* *of* $\mathfrak{g}^*$ *such that* $\tau^*\theta^* = \theta^*\tau^*$, *the mapping* $\tau : \mathfrak{g} \to \mathfrak{g}$ *by* $\tau(X_{\mathfrak{k}} + \sqrt{-1}X_{\mathfrak{p}^*}) = \tau^*(X_{\mathfrak{k}}) + \sqrt{-1}\tau^*(X_{\mathfrak{p}^*})(X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}^*} \in \mathfrak{p}^*)$ *is an automorphism of* $\mathfrak{g}$. *If the order of* τ^* *is* k, *then the order of* τ *is* k. *If* $(\tau^*)^k \neq \mathrm{Id}$, *then* $\tau^k \neq \mathrm{Id}$.

We also call the automorphism τ of $\mathfrak{g}$ defined in above Lemma *the automorphism of* $\mathfrak{g}$ *induced by* τ^*.

From Lemma 5.2 σ_i^* and θ_i^* $(i = 1, 2)$ induce $\sigma_i, \theta_i \in \mathrm{Aut}(\mathfrak{g}_i)$, which satisfy $(\sigma_i)^4 = \mathrm{Id} = (\theta_i)^2$.

Lemma 5.3. $\theta_2 = \mu\theta_1\mu^{-1}$, $\theta_i\sigma_i = \sigma_i\theta_i$ $(i = 1, 2)$.

Proof. If we take $X_{\mathfrak{k}_i} \in \mathfrak{k}_i$ and $X_{\mathfrak{p}_i^*} \in \mathfrak{p}_i^* (i = 1, 2)$, then from (4.1) we find $\sigma^*(X_{\mathfrak{k}_i}) \in \mathfrak{k}_i$ and $\sigma^*(X_{\mathfrak{p}_i^*}) \in \mathfrak{p}_i^*$. Thus it is easy to see that $\theta_i\sigma_i(X_{\mathfrak{k}_i} + \sqrt{-1}X_{\mathfrak{p}_i^*}) = \sigma_i\theta_i(X_{\mathfrak{k}_i} + \sqrt{-1}X_{\mathfrak{p}_i^*})$. Since $\mu(\mathfrak{k}_1) = \mathfrak{k}_2$ and $\mu(\mathfrak{p}_1^*) = \mathfrak{p}_2^*$, we see that $\mu\theta_1\mu^{-1}(X_{\mathfrak{k}_2} + \sqrt{-1}X_{\mathfrak{p}_2^*}) = X_{\mathfrak{k}_2} - \sqrt{-1}X_{\mathfrak{p}_2^*} = \theta_2(X_{\mathfrak{k}_2} + \sqrt{-1}X_{\mathfrak{p}_2^*})$. □

Let $\mathfrak{g}$ be a non-compact simple Lie algebra over $\mathbb{R}$ with an automorphism σ of order four and θ a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{k}$ and $\mathfrak{p}$ denote eigenspaces of θ for the eigenvalues $+1$ and -1, respectively. In this section we denote it by $(\mathfrak{g}, \sigma, \theta)$ simply.

From the above consideration, two triples $(\mathfrak{g}_1, \sigma_1, \theta_1)$ and $(\mathfrak{g}_2, \sigma_2, \theta_2)$ are called *isomorphic* if there exists an isomorphism $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying $\mu(\mathfrak{k}_1) = \mathfrak{k}_2$, $\mu(\mathfrak{p}_1) = \mathfrak{p}_2$ and $\mu((\mathfrak{g}_1)^{\sigma_1}) = (\mathfrak{g}_2)^{\sigma_2}$. The set of all isomorphisms $\mu : (\mathfrak{g}_1, \sigma_1, \theta_1) \to (\mathfrak{g}_2, \sigma_2, \theta_2)$ is denoted by $\mathrm{Isom}\{(\mathfrak{g}_1, \sigma_1, \theta_1), (\mathfrak{g}_2, \sigma_2, \theta_2)\}$.

Remark 5.4. *By an argument similar to Remark 5.1* $(\mathfrak{g}, \sigma^{-1}, \theta)$ *will be identified with* $(\mathfrak{g}, \sigma, \theta)$.

From Lemma 5.2, Lemma 5.3 and the argument as above we have following.

Lemma 5.5. *If $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$, then the non-compact duals of $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$, denoted as $\mathfrak{g}_1$ and $\mathfrak{g}_2$, generate non-compact triples $(\mathfrak{g}_1, \sigma_1, \theta_1)$ and $(\mathfrak{g}_2, \sigma_2, \theta_2)$ and these triples are isomorphic.*

Conversely, if two non-compact triples $(\mathfrak{g}_1, \sigma_1, \theta_1)$ and $(\mathfrak{g}_2, \sigma_2, \theta_2)$ are isomorphic, then using a similar argument to the one used to derive (4.3), (4.4), Lemma 5.2 and Lemma 5.3, we can construct two compact triples $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ and $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$ which are isomorphic, i.e., the following holds.

Lemma 5.6. *If $(\mathfrak{g}_1, \sigma_1, \theta_1)$ is isomorphic to $(\mathfrak{g}_2, \sigma_2, \theta_2)$, then the compact duals of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, denoted as $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$, generate compact triples $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ and $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$ and these triples are isomorphic.*

Remark 5.7. *If a compact triple $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ is not isomorphic to a compact triple $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$, then by Lemma 5.6 the non-compact triple $(\mathfrak{g}_1, \sigma_1, \theta_1)$ is not isomorphic to the non-compact triple $(\mathfrak{g}_2, \sigma_2, \theta_2)$. Suppose that the non-compact 4-symmetric pair $(\mathfrak{g}_1, \sigma_1)$ is isomorphic to the non-compact 4-symmetric pair $(\mathfrak{g}_2, \sigma_2)$. Put $\mathfrak{h}_1 = (\mathfrak{g}_1)^{\sigma_1}$ and $\mathfrak{h}_2 = (\mathfrak{g}_2)^{\sigma_2}$. Then there exists an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$. Since $(\mathfrak{g}_2)^{\varphi\sigma_1\varphi^{-1}} = \mathfrak{h}_2$, and Cartan involutions $\varphi\theta_1\varphi^{-1}$ and θ_2 preserve $\mathfrak{h}_2$, it follows from Proposition 3.4 (ii) that there exists $\tau \in \mathrm{Aut}_{\mathfrak{h}_2}(\mathfrak{g}_2)$ such that $\tau(\varphi\theta_1\varphi^{-1})\tau^{-1} = \theta_2$. It is obvious that*

$$(\tau\varphi)(\mathfrak{h}_1) = \mathfrak{h}_2, \quad (\tau\varphi)((\mathfrak{g}_1)^{\theta_1}) = (\mathfrak{g}_2)^{\varphi\theta_1\varphi^{-1}} = (\mathfrak{g}_2)^{\tau(\varphi\theta_1\varphi^{-1})\tau^{-1}} = (\mathfrak{g}_2)^{\theta_2}.$$

Thus $(\mathfrak{g}_1, \sigma_1, \theta_1)$ is isomorphic to $(\mathfrak{g}_2, \sigma_2, \theta_2)$, which is contradiction. Consequently if a compact triple $(\mathfrak{g}_1^*, \sigma_1^*, \theta_1^*)$ is not isomorphic to a compact triple $(\mathfrak{g}_2^*, \sigma_2^*, \theta_2^*)$, then the non-compact 4-symmetric pair $(\mathfrak{g}_1, \sigma_1)$ is not isomorphic to the non-compact 4-symmetric pair $(\mathfrak{g}_2, \sigma_2)$.

6 Classification

Let $\mathfrak{g}$ be a non-compact simple Lie algebra over $\mathbb{R}$ with an automorphism σ of order four. Let $\mathfrak{h}$ denote the fixed point set of σ in $\mathfrak{g}$. Suppose that $\mathfrak{h}_{\mathbb{C}}$ contains a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In this section we classify triples $(\mathfrak{g}, \sigma, \theta)$ under the isomorphism defined by Section 5 in the case where $\dim(\mathfrak{z}(\mathfrak{h})) \leq 1$.

Owing to Proposition 3.4, there exists a Cartan involution θ such that $\theta\sigma = \sigma\theta$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{g}^*$ denote the compact dual of $\mathfrak{g}$ and σ^* the automorphism of $\mathfrak{g}^*$ induced by σ. Let $\mathfrak{h}^*$ denote the compact dual of $\mathfrak{h}$. Then by (4.2) we have $\mathfrak{h}^* = (\mathfrak{g}^*)^{\sigma^*}$ and hence

$$\begin{aligned}
\mathfrak{h}^*_{\mathbb{C}} &= \mathfrak{h}^* \oplus \sqrt{-1}\mathfrak{h}^* \\
&= ((\mathfrak{h} \cap \mathfrak{k}) \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})) \oplus (\sqrt{-1}(\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})) \\
&= ((\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})) \oplus \sqrt{-1}((\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})) \\
&= \mathfrak{h} \oplus \sqrt{-1}\mathfrak{h} \\
&= \mathfrak{h}_{\mathbb{C}}.
\end{aligned}$$

Let $\mathfrak{t}^*$ be a maximal abelian subalgebra of $\mathfrak{h}^*$. Since $\mathfrak{h}_{\mathbb{C}}$ contains a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, the dimension of $\mathfrak{t}^*_{\mathbb{C}}$ is equal to the dimension of the maximal abelian subalgebra of $\mathfrak{g}^*$. Suppose that there exists a maximal abelian subalgebra $\tilde{\mathfrak{t}}$ of $\mathfrak{g}^*$ such that $\mathfrak{t}^* \subsetneq \tilde{\mathfrak{t}}$. If σ is an outer automorphism, then the dimension of $\mathfrak{t}^*_{\mathbb{C}}$ is less than the dimension of $\tilde{\mathfrak{t}}_{\mathbb{C}}$, which is a contradiction (cf. Theorem 5.15 of Chapter X of [3]). Thus $\mathfrak{t}^*$ is a maximal abelian subalgebra of $\mathfrak{g}^*$, so $\mathfrak{t}^* \subset \mathfrak{h}^* \subset \mathfrak{g}^*$. Therefore there exists $T \in \mathfrak{t}^*$ such that $\sigma^* = \mathrm{Ad}(\exp T)$ (cf. Proposition 5.3 of [3]).

Let $\mathrm{Int}(\mathfrak{g}^*)$ be the set of inner automorphisms of $\mathfrak{g}^*$. Since the dimension of the center of $\mathfrak{h}^*$ is 0 or 1, $\sigma^* \in \mathrm{Int}(\mathfrak{g}^*)$, Lemma 2.4 and Remark 2.2 of [5], σ^* is conjugate within $\mathrm{Int}(\mathfrak{g}^*)$ to one of the following σ_0^*:

$$\sigma_0^* = \mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}K_i)(m_i = 3, 4) \text{ or } \mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}(K_i + K_j))(m_i = m_j = 2),$$

where K_i and m_i is defined in Section 2, meaning that there exists $\tau_0^* \in \mathrm{Int}(\mathfrak{g}^*)$ such that $\tau_0^*\sigma^*(\tau_0^*)^{-1} = \sigma_0^*$. Thus the triple $(\mathfrak{g}^*, \sigma^*, \theta^*)$ is isomorphic to the triple $(\mathfrak{g}^*, \sigma_0^*, \tau_0^*\theta^*(\tau_0^*)^{-1})$. Let $\mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{g}^*)$ be the set of automorphisms of $\mathfrak{g}^*$ preserving $\mathfrak{h}^*$ and let $\mathfrak{h}_0 := (\mathfrak{g}^*)^{\sigma_0^*}$. According to [5] and [6], $\theta_1^* := \tau_0^*\theta^*(\tau_0^*)^{-1}$ is conjugate within $\mathrm{Aut}_{\mathfrak{h}_0^*}(\mathfrak{g}^*)$ to $\tilde{\theta}^*$, which is listed in [5] and [6], i.e, there exists $\tau_1^* \in \mathrm{Aut}_{\mathfrak{h}_0^*}(\mathfrak{g}^*)$ such that $\tilde{\theta}^* = \tau_1^*\theta_1^*(\tau_1^*)^{-1}$. By the definition of $\tilde{\theta}^*$ we have $\tau_1^*((\mathfrak{g}^*)^{\theta_1^*}) = (\mathfrak{g}^*)^{\tilde{\theta}^*}$. Thus the triple $(\mathfrak{g}^*, \sigma_0^*, \theta_1^*)$ is isomorphic to $(\mathfrak{g}^*, \sigma_0^*, \tilde{\theta}^*)$, so the triple $(\mathfrak{g}^*, \sigma^*, \theta^*)$ is isomorphic to the triple $(\mathfrak{g}^*, \sigma_0^*, \tilde{\theta}^*)$. Therefore, all non-compact triples $(\mathfrak{g}, \sigma, \theta)$ are isomorphic to one of the non-compact duals of the compact triples $(\mathfrak{g}^*, \sigma_0^*, \tilde{\theta}^*)$ classified in [5] and [6].

We suppose that $\mathfrak{g}^*$ is of type $\mathfrak{e}_7$. From what has been mentioned above, it suffices to consider the involution θ^* of $\mathfrak{e}_7$ that commute with each order four automorphism σ^* of $\mathfrak{e}_7$. Let $\mathfrak{t}^*$ be a maximal abelian subalgebra of the fixed point set $\mathfrak{h}^* := (\mathfrak{e}_7)^{\sigma^*}$ and let $\mathfrak{k}$ denote the fixed point set of θ^*.

First, we assume that $\theta^*|_{\mathfrak{t}^*} = \mathrm{Id}$. Then by Lemma 2.4 and Remark 2.2 of [5] we have $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$, where

$$K = K_4,\ K_3,\ K_5,\ K_1 + K_2,\ K_1 + K_6.$$

It follows from the lists of [5] and [6] that θ^* is conjugate within $\mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{e}_7)$ to one of automorphisms listed in Table I.

Next, we assume that $\theta^*|_{\mathfrak{t}^*} \neq \mathrm{Id}$. Then by Theorem 10.1 of [5] and Theorem 8.1 of [6] we have $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$, where

$$K = K_4,\ K_1 + K_6.$$

Define $\varphi \in \mathrm{Aut}(\mathfrak{g}^*)$ by

$$\begin{array}{ll} (6.1) & \varphi(E_{\alpha_1}) = E_{\alpha_6},\quad \varphi(E_{\alpha_2}) = E_{\alpha_2},\quad \varphi(E_{\alpha_3}) = E_{\alpha_5},\quad \varphi(E_{\alpha_4}) = E_{\alpha_4}, \\ & \varphi(E_{\alpha_5}) = E_{\alpha_3},\quad \varphi(E_{\alpha_6}) = E_{\alpha_1},\quad \varphi(E_{\alpha_7}) = E_{\alpha_0}, \end{array}$$

where $\{E_{\alpha_0}, \ldots, E_{\alpha_7}\}$ is the Weyl basis of $\mathfrak{e}_7$. Then, θ^* is conjugate within $\mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{e}_7)$ to one of automorphisms listed in Table I (cf. [5], [6]).

Table I: $\mathfrak{g}^* = \mathfrak{e}_7$

$\mathfrak{h}^* = \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), \quad K = K_4$

$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
K_2	$\mathfrak{su}(8)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{so}(2)$
K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2)$
$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(2)$
$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$
$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(2)$
$K_2 + K_3 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{so}(2)$
$K_3 + K_4 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$

$\mathfrak{h}^* = \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, \quad K = K_3$

$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
K_2	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_1 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_3 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

$\mathfrak{h}^* = \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, \quad K = K_5$

$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
K_5	$\mathfrak{su}(8)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$K_1 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$K_3 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$

Table continued

Table I (continued)

$\mathfrak{h}^* = \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, \quad K = K_1 + K_2$

$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_2	$\mathfrak{su}(8)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$K_2 + K_5$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_2 + K_6$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

$\mathfrak{h}^* = \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, \quad K = K_1 + K_6$

$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
K_2	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}^2$
K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}^2$
$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(4) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}^2$
$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
$K_4 + K_7$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
$K_1 + K_3 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$

$\mathfrak{h}^* = \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), \quad K = K_4$

θ^*	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
φ	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(1)$
$\varphi \circ \tau_{K_2}$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(2)$
$\varphi \circ \tau_{K_4}$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(1)$

$\mathfrak{h}^* = \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, \quad K = K_1 + K_6$

θ^*	$\mathfrak{k}$	$\mathfrak{h}^* \cap \mathfrak{k}$
φ	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$
$\varphi \circ \tau_{K_2}$	$\mathfrak{su}(8)$	$(\mathfrak{so}(5) + \mathfrak{so}(3)) \oplus \mathfrak{su}(2)$

φ is the same involution as in (6.1) and $\tau_h = \mathrm{Ad}(\exp \pi\sqrt{-1}h)$.

In the case where $\theta^*|_{\mathfrak{t}^*} = \mathrm{Id}$, $K = K_4$ and $h = K_1$, since $\mathfrak{k} \cong \mathfrak{so}(12) \oplus \mathfrak{su}(2)$, the non-compact dual of $\mathfrak{e}_7$ is isomorphic to $\mathfrak{e}_{7(-5)}$ (cf. Table V of Chapter X of [3]). Let σ be the automorphism of $\mathfrak{e}_{7(-5)}$ induced by σ^* and $\mathfrak{h} := (\mathfrak{e}_{7(-5)})^\sigma$. Since

$$\mathfrak{h}^* \cong \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2),\ \ \mathfrak{h}^* \cap \mathfrak{k} \cong \mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2),$$

Lemma 4.1 and (4.2), $\mathfrak{h}$ is isomorphic to $\mathfrak{so}(6) \oplus \mathfrak{so}(4,2) \oplus \mathfrak{su}(2)$.

Similarly as above, from Section 5 we can determine $\mathfrak{g}$ and $\mathfrak{h} := \mathfrak{g}^\sigma$ up to isomorphism for all cases. We must be remark the case where there exists a center $\mathbb{R}\sqrt{-1}K$. In this case, we can check the center $\mathbb{R}\sqrt{-1}K$ is contained in $\mathfrak{h} \cap \mathfrak{k}$. Thus, similarly as above, for all h, we can determine $\mathfrak{g}$ and $\mathfrak{h}$ up to isomorphism listed in Table III and IV. For example, if $K = K_3$ and $h = K_1$, then we have

$$\mathfrak{g} \cong \mathfrak{e}_{7(-5)},\ \ \mathfrak{h} \cong \mathfrak{su}(6) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}.$$

In the case where $\theta^*|_{\mathfrak{t}^*} \neq \mathrm{Id}$, if $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_4)$ and $\theta^* \cong \varphi$, then similarly as in $\theta^*|_{\mathfrak{t}^*} = \mathrm{Id}$. If $K = K_1 + K_6$ and $\theta^* \cong \varphi$, then there exists the center $\mathbb{R}\sqrt{-1}(K_1 - K_6)$. Since

$$\begin{aligned} &\mathfrak{k} = \mathrm{span}\{K_1 + K_6 - 2K_7,\ K_2 - K_7,\ K_3 + K_5 - 3K_7,\ K_4 - 2K_7\}, \\ &\mathfrak{p}^* = \mathrm{span}\{K_1 - K_6,\ K_3 - K_5,\ K_7\}, \end{aligned} \tag{6.2}$$

the center $\mathbb{R}\sqrt{-1}(K_1 - K_6)$ is contained in $\mathfrak{h}^* \cap \mathfrak{p}^*$. Thus $\mathfrak{h} \cong \mathfrak{so}(7,1) \oplus \mathfrak{so}(3,1) \oplus \mathbb{R}$.

Remark 6.1. *In the case of* $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1 + K_2))$*, if* $\mathfrak{k}_1 = (\mathfrak{g}^*)^{\mathrm{Ad}(\exp\pi\sqrt{-1}K_6)}$ *and* $\mathfrak{k}_2 = (\mathfrak{g}^*)^{\mathrm{Ad}(\exp\pi\sqrt{-1}(K_1+K_6))}$*, then* $\mathfrak{k}_1^* \cong A_1 \oplus D_6 \cong \mathfrak{k}_2^*$*. However,* $\mathrm{Ad}(\exp\pi\sqrt{-1}K_6)$ *is not conjugate within* $\mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{g}^*)$ *to* $\mathrm{Ad}(\exp\pi\sqrt{-1}(K_1 + K_6))$*. In fact,* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *can be written as the direct decompositions*

$$\mathfrak{k}_1 = \mathfrak{t} \oplus \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(K_6)=0,2}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

$$\mathfrak{k}_2 = \mathfrak{t} \oplus \sum_{\substack{\alpha = \sum_{i=1}^7 n_i\alpha_i \in \Delta^+ \\ (n_1,n_6)=(0,0),(0,2),(1,1),(2,2)}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

respectively. Let $\Delta_{\mathfrak{k}_i} = \{\alpha \in \Delta^+(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}}); A_\alpha, B_\alpha \in \mathfrak{k}_i\} (i = 1,2)$*. If* $\alpha \in \Delta_{\mathfrak{k}_1^*}$ *and* $\alpha(K_6) = 0$*, then* $\alpha_7 \pm \alpha \notin \Delta$*. If* $\alpha \in \Delta_{\mathfrak{k}_1^*}$ *and* $\alpha(K_6) = 2$*, then the cofficients of* α_6 *and* α_7 *of* α *are* 2 *and* 1*, respectively, so* $\alpha_7 \pm \alpha \notin \Delta$*. Therefore for* $\mathfrak{k}_1$ *we have*

$$A_1 = \mathfrak{su}_{\alpha_7}(2) \subset \mathfrak{h}^*. \tag{6.3}$$

On the other hand, if $\alpha = \sum_{i=1}^7 n_i\alpha_i \in \Delta_{\mathfrak{k}_2}$ *and* $(n_1, n_6) = (0,2)$*, then* $\alpha = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$*. If* $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \in \Delta_{\mathfrak{k}_2^*}$*, then since there are no roots* $\alpha = \sum_{i=1}^7 m_i\alpha_i$ *such that* $(m_1, m_6) = (1,3), (2,4)$*, we have* $\beta \pm \gamma \notin \Delta$ *where* $\gamma = \sum_{i=1}^7 n_i\alpha_i$ $((n_1, n_6) = (1,1)$ *or* $(2,2))$*. Consequently, for any* $\alpha \in \Delta_{\mathfrak{k}_2}$*, we have* $\beta \pm \alpha \notin \Delta$*, so*

$$A_1 = \mathfrak{su}_\beta(2) \not\subset \mathfrak{h}^*. \tag{6.4}$$

Suppose that $\mathrm{Ad}(\exp\pi\sqrt{-1}K_6)$ *is conjugate to* $\mathrm{Ad}(\exp\pi\sqrt{-1}(K_1 + K_6))$*. Then there exists* $\mu^* \in \mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{g}^*)$ *such that*

$$\mu^*(\mathrm{Ad}(\exp\pi\sqrt{-1}K_6))(\mu^*)^{-1} = \mathrm{Ad}(\exp\pi\sqrt{-1}(K_1 + K_6))$$

This μ^* satisfies $\mu^*(\mathfrak{k}_1) = \mathfrak{k}_2$. Thus $\mu^*(\mathfrak{su}_{\alpha_7}(2)) = \mathfrak{su}_\beta(2)$, which is contradicts (6.3) and (6.4), that is, μ^* do not preserve $\mathfrak{h}^*$.

We put $\beta := \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Then it is easy to see that $\tau := t_\beta \circ t_{\alpha_6} \in \mathrm{Aut}_{\mathfrak{h}^*}(\mathfrak{g}^*)$,

$$\tau(\mathrm{Ad}(\exp \pi\sqrt{-1}(K_1 + K_6)))\tau^{-1} = \mathrm{Ad}(\exp \pi\sqrt{-1}K_6)$$

and

$$\tau(\mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}(K_1 + K_2)))\tau^{-1} = \mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}(2K_1 + K_2 + 3K_5 + K_7)).$$

Therefore two triples

$$(\mathfrak{e}_7, \mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}(K_1 + K_2)), \mathrm{Ad}(\exp \pi\sqrt{-1}(K_1 + K_6)))$$

and

$$(\mathfrak{e}_7, \mathrm{Ad}(\exp \frac{\pi}{2}\sqrt{-1}(2K_1 + K_2 + 3K_5 + K_7)), \mathrm{Ad}(\exp \pi\sqrt{-1}K_6))$$

are isomorphic.

In the same way as above, for which $\mathfrak{g}^*$ is of all types we can determine $\mathfrak{g}$ and $\mathfrak{h}$ up to isomorphism , which are listed in Table II–IV.

Remark 6.2. *There are three additional sets of non-isomorphic triples $(\mathfrak{g}^*, \sigma^*, \theta^*)$ where $\mathfrak{g}^*, (\mathfrak{g}^*)^{\sigma^*}$ and $(\mathfrak{g}^*)^{\theta^*}$ are equal, similar to those shown in Remark 6.1 (see Lemma 7.1 of [6]). These have the following isomorphism.*

The triple $(\mathfrak{e}_7, \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1+K_6)), \mathrm{Ad}(\exp \pi\sqrt{-1}(K_1+K_6)))$ becomes isomorphic to the triple $(\mathfrak{e}_7, \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(3K_1 + 2K_6)), \mathrm{Ad}(\exp \pi\sqrt{-1}K_1))$ using the root reflection $t_\beta t_{\alpha_1}$ where $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$.

The triple $(\mathfrak{e}_8, \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1 + K_8)), \mathrm{Ad}(\exp \pi\sqrt{-1}(K_1 + K_8)))$ becomes isomorphic to the triple $(\mathfrak{e}_8, \mathrm{Ad}(\exp(3\pi/2)\sqrt{-1}K_8), \mathrm{Ad}(\exp \pi\sqrt{-1}K_8))$ using the root reflection $t_\beta t_{\alpha_8}$ where $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

The triple $(\mathfrak{f}_4, \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1+K_4)), \mathrm{Ad}(\exp \pi\sqrt{-1}(K_1+K_4)))$ becomes isomorphic to the triple $(\mathfrak{f}_4, \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1+2K_4)), \mathrm{Ad}(\exp \pi\sqrt{-1}K_1))$ using the root reflection $t_{\alpha_1+\alpha_2+\alpha_3}$.

Consequently, noting Remark 5.7, we obtain the following classification theorem.

Theorem 6.3. *Let $(G/H, \sigma)$ be a 4-symmetric space such that G is a non-compact simple Lie group of the exceptional type with the Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ denote the Lie algebra of H with the center $\mathfrak{z}$. Suppose that $\dim \mathfrak{z} = 0$ or 1 and $\mathfrak{h}_\mathbb{C}$ contains a Cartan subalgebra in $\mathfrak{g}_\mathbb{C}$. Then the following Table II–IV gives the complete list of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}(= \mathfrak{g}^\theta)$ and $\mathfrak{h} \cap \mathfrak{k}$ of the possibilities up to isomorphism.*

Table II: $\dim \mathfrak{z} = 0,\ \sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$ and $\mathfrak{k} = \mathfrak{g}^\theta$)

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(6) \oplus \mathfrak{so}(4,2) \oplus \mathfrak{su}(2), K_4)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{sl}(2,\mathbb{R}), K_4)$	K_2	$\mathfrak{su}(8)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(6) \oplus \mathfrak{so}(4,2) \oplus \mathfrak{sl}(2,\mathbb{R}), K_4)$	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(4,2) \oplus \mathfrak{so}(4,2) \oplus \mathfrak{su}(2), K_4)$	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(6) \oplus \mathfrak{so}^*(6) \oplus \mathfrak{su}(2), K_4)$	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(4,2) \oplus \mathfrak{so}(4,2) \oplus \mathfrak{sl}(2,\mathbb{R}), K_4)$	$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(6) \oplus \mathfrak{so}^*(6) \oplus \mathfrak{sl}(2,\mathbb{R}), K_4)$	$K_2 + K_3 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(6) \oplus \mathfrak{so}^*(6) \oplus \mathfrak{su}(2), K_4)$	$K_3 + K_4 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(8) \oplus \mathfrak{sl}(2,\mathbb{R}), K_3)$	K_1	$\mathfrak{so}(16)$	$\mathfrak{su}(8) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(8) \oplus \mathfrak{su}(2), K_3)$	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(8) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}^*(8) \oplus \mathfrak{su}(2), K_3)$	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(4) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(4,4) \oplus \mathfrak{su}(2), K_3)$	K_6	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}^*(8) \oplus \mathfrak{su}(2), K_3)$	$K_3 + K_4$	$\mathfrak{so}(16)$	$\mathfrak{sp}(4) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(4,4) \oplus \mathfrak{su}(2), K_3)$	$K_3 + K_6$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathfrak{su}(2)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(6,2) \oplus \mathfrak{su}(2), K_3)$	$K_1 + K_4$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(6,2) \oplus \mathfrak{su}(2), K_3)$	$K_1 + K_6$	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(6), K_6)$	K_1	$\mathfrak{so}(16)$	$(\mathfrak{so}(8) + \mathfrak{so}(2)) \oplus \mathfrak{so}(6)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(6), K_6)$	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(6) + \mathfrak{so}(4)) \oplus \mathfrak{so}(6)$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(10) \oplus \mathfrak{so}(6), K_6)$	K_6	$\mathfrak{so}(16)$	$\mathfrak{so}(10) \oplus \mathfrak{so}(6)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(10) \oplus \mathfrak{so}(4,2), K_6)$	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(10) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(4,2), K_6)$	$K_1 + K_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(8) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}^*(6), K_6)$	$K_2 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{u}(5) \oplus \mathfrak{u}(3)$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(4,2), K_6)$	$K_3 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{so}(6) + \mathfrak{so}(4)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}^*(6), K_6)$	$K_2 + K_6 + K_7$	$\mathfrak{so}(16)$	$\mathfrak{u}(5) \oplus \mathfrak{u}(3)$

Table continued

Table II (continued)

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{f}_{4(4)}, \mathfrak{so}(4,2) \oplus \mathfrak{so}(3), K_3)$	K_1	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(3)$
$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(6) \oplus \mathfrak{so}(3), K_3)$	K_3	$\mathfrak{so}(9)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(3)$
$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(6) \oplus \mathfrak{so}(2,1), K_3)$	K_4	$\mathfrak{so}(9)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
$(\mathfrak{f}_{4(4)}, \mathfrak{so}(6) \oplus \mathfrak{so}(2,1), K_3)$	$K_1 + K_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$

$(\mathfrak{g}, \mathfrak{h}, K)$	θ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(4) \oplus \mathfrak{su}^*(2), K_4)$	φ	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(1)$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4) \oplus \mathfrak{sl}(2,\mathbb{R}), K_4)$	$\varphi \circ \tau_{K_2}$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4) \oplus \mathfrak{su}^*(2), K_4)$	$\varphi \circ \tau_{K_4}$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{sp}(1)$

$\varphi \,:\, E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4},\ E_{\alpha_7} \mapsto E_{\alpha_0},\ \tau_h = \mathrm{Ad}(\exp \pi\sqrt{-1}h).$

Table III: $\dim \mathfrak{z} = 1$, $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$ and $\mathfrak{k} = \mathfrak{g}^\theta$

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	K_1	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	K_4	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	K_5	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	$K_1 + K_2$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	$K_1 + K_5$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_4)$	$K_2 + K_4$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_4)$	$K_1 + K_2 + K_5$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	$K_1 + K_4 + K_5$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_3)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(5,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_2	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(3,3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(5,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_3)$	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,2) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_3)$	$K_1 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_3)$	$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	$K_3 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(3,3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

Table continued

Table III (continued)

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,1) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(3,2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	K_5	$\mathfrak{su}(8)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(5) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(5) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4,1) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,1) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(4,1) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	$K_1 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(3,2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(3,2) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	$K_3 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(3,2) \oplus \mathfrak{su}(2,1) \oplus \mathbb{R}, K_5)$	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(7,1) \oplus \mathbb{R}, K_2)$	K_1	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(7) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(8) \oplus \mathbb{R}, K_2)$	K_2	$\mathfrak{so}(16)$	$\mathfrak{su}(8) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(6,2) \oplus \mathbb{R}, K_2)$	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(5,3) \oplus \mathbb{R}, K_2)$	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(3)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(4,4) \oplus \mathbb{R}, K_2)$	K_5	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(7,1) \oplus \mathbb{R}, K_2)$	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(7) + \mathfrak{u}(1)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(6,2) \oplus \mathbb{R}, K_2)$	$K_2 + K_3$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{su}(5,3) \oplus \mathbb{R}, K_2)$	$K_2 + K_4$	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(3)) \oplus \mathbb{R}$

Table continued

Table III (continued)

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	K_1	$\mathfrak{so}(16)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	K_2	$\mathfrak{so}(16)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	K_7	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_6 \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_7)$	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	$K_1 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_7)$	$K_1 + K_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	$K_2 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_7)$	$K_2 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_{4(4)}, \mathfrak{su}(3) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_2)$	K_1	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_{4(4)}, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_2)$	K_2	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_{4(-20)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_2)$	K_4	$\mathfrak{so}(9)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_{4(4)}, \mathfrak{su}(2,1) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_2)$	$K_1 + K_3$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_{4(4)}, \mathfrak{su}(2,1) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_2)$	$K_2 + K_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) \oplus \mathbb{R}, K_1)$	K_1	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_1)$	K_2	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(2) \oplus \mathbb{R}$

$(\mathfrak{g}, \mathfrak{h}, K)$	θ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{6(-26)}, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	ψ	$\mathfrak{f}_4$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(6)}, \mathfrak{su}(3) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}, K_4)$	$\psi \circ \tau_{K_2}$	$\mathfrak{sp}(4)$	$\mathfrak{su}(3) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(6)}, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	$\psi \circ \tau_{K_4}$	$\mathfrak{sp}(4)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

$\psi \ : \ E_{\alpha_1} \mapsto E_{\alpha_6}, \ E_{\alpha_2} \mapsto E_{\alpha_2}, \ E_{\alpha_3} \mapsto E_{\alpha_5}, \ E_{\alpha_4} \mapsto E_{\alpha_4}$

Table IV: $\dim \mathfrak{z} = 1$, $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$ and $\mathfrak{k} = \mathfrak{g}^\theta$

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(6) \oplus \mathfrak{so}^*(4) \oplus \mathbb{R}, K_3 + K_6)$	K_1	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus \mathfrak{u}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(4,2) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_3 + K_6)$	K_2	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_3 + K_6)$	K_3	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(5,1) \oplus \mathfrak{so}(3,1) \oplus \mathbb{R}, K_3 + K_6)$	$K_1 + K_2$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(6) \oplus \mathfrak{so}(2,2) \oplus \mathbb{R}, K_3 + K_6)$	$K_1 + K_6$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(4,2) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_3 + K_6)$	$K_2 + K_3$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_3 + K_6)$	$K_3 + K_5$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(5,1) \oplus \mathfrak{so}(3,1) \oplus \mathbb{R}, K_3 + K_6)$	$K_1 + K_2 + K_5$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	K_2	$\mathfrak{su}(8)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_2)$	K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_2)$	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, 2K_1 + K_2 + 3K_5 + K_7)$	K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_2)$	$K_2 + K_5$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	$K_2 + K_6$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_2)$	$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4,2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_2)$	$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

Table continued

Table IV (continued)

$(\mathfrak{g}, \mathfrak{h}, K)$	$h(\theta^* = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_1 + K_6)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_1 + K_6)$	K_2	$\mathfrak{su}(8)$	$\mathfrak{u}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}^*(4) \oplus \mathbb{R}, K_1 + K_6)$	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(4) \oplus \mathfrak{u}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(4,4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_1 + K_6)$	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(8) \oplus \mathfrak{so}(2,2) \oplus \mathbb{R}, K_1 + K_6)$	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(8) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, K_1 + K_6)$	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{u}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}, 3K_1 + 2K_6)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}(2,2) \oplus \mathbb{R}, K_1 + K_6)$	$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(4) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}^*(4) \oplus \mathbb{R}, K_1 + K_6)$	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{u}(4) \oplus \mathfrak{u}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(4,4) \oplus \mathfrak{so}(2,2) \oplus \mathbb{R}, K_1 + K_6)$	$K_4 + K_7$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(8) \oplus \mathfrak{so}^*(4) \oplus \mathbb{R}, K_1 + K_6)$	$K_1 + K_3 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{u}(4) \oplus \mathfrak{u}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	K_1	$\mathfrak{so}(16)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(12) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_8)$	K_2	$\mathfrak{so}(16)$	$\mathfrak{u}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{u}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(8,4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(8) + \mathfrak{so}(4)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_8)$	K_5	$\mathfrak{so}(16)$	$(\mathfrak{so}(6) + \mathfrak{so}(6)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(12) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}, K_1 + K_8)$	$K_1 + K_2$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{u}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	$K_1 + K_3$	$\mathfrak{so}(16)$	$\mathfrak{u}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, 3K_8)$	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(8,4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_1 + K_8)$	$K_4 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{so}(8) + \mathfrak{so}(4)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$

Table continued

Table IV (continued)

$(\mathfrak{g},\mathfrak{h},K)$	$h(\theta^*=\tau_h)$	$\mathfrak{k}$	$\mathfrak{h}\cap\mathfrak{k}$
$(\mathfrak{f}_{4(4)},\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R},K_1+K_4)$	K_1	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$(\mathfrak{f}_{4(4)},\mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{sp}(1,\mathbb{R})\oplus\mathbb{R},K_1+K_4)$	K_2	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{u}(2)\oplus\mathfrak{u}(1)\oplus\mathbb{R}$
$(\mathfrak{f}_{4(-20)},\mathfrak{sp}(1,1)\oplus\mathfrak{sp}(1)\oplus\mathbb{R},K_1+K_4)$	K_3	$\mathfrak{so}(9)$	$(\mathfrak{sp}(1)+\mathfrak{sp}(1))\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$(\mathfrak{f}_{4(4)},\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R},K_1+K_4)$	K_4	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$(\mathfrak{f}_{4(4)},\mathfrak{sp}(1,1)\oplus\mathfrak{sp}(1)\oplus\mathbb{R},K_1+K_4)$	K_1+K_3	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$(\mathfrak{sp}(1)+\mathfrak{sp}(1))\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$(\mathfrak{f}_{4(4)},\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R},K_1+2K_4)$	K_1	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
$(\mathfrak{g},\mathfrak{h},K)$	θ	$\mathfrak{k}$	$\mathfrak{h}\cap\mathfrak{k}$
$(\mathfrak{e}_{6(-26)},\mathfrak{so}(6)\oplus\mathfrak{so}(3,1),K_3+K_5)$	ψ	$\mathfrak{f}_4$	$\mathfrak{so}(6)\oplus\mathfrak{so}(3)\oplus\mathbb{R}$
$(\mathfrak{e}_{6(6)},\mathfrak{so}(3,3)\oplus\mathfrak{so}^*(4),K_3+K_5)$	$\psi\circ\tau_{K_2}$	$\mathfrak{sp}(4)$	$(\mathfrak{so}(3)+\mathfrak{so}(3))\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
$(\mathfrak{e}_{7(-25)},\mathfrak{so}(7,1)\oplus\mathfrak{so}(3,1)\oplus\mathbb{R},K_1+K_6)$	φ	$\mathfrak{e}_6\oplus\mathbb{R}$	$\mathfrak{so}(7)\oplus\mathfrak{so}(3)$
$(\mathfrak{e}_{7(7)},\mathfrak{so}(5,3)\oplus\mathfrak{so}(3,1)\oplus\mathbb{R},K_1+K_6)$	$\varphi\circ\tau_{K_2}$	$\mathfrak{su}(8)$	$\mathfrak{so}(5)\oplus\mathfrak{so}(3)$

φ and ψ are same involutions as in Table I and Table III.

Remark 6.4. *$K \in \mathfrak{g}^*$ can be uniquely written as*

$$K = K_{\mathfrak{k}} + K_{\mathfrak{p}^*}, \quad K_{\mathfrak{k}} \in \mathfrak{k}, K_{\mathfrak{p}^*} \in \mathfrak{p}^*.$$

Thus $K_{\mathfrak{p}^*} = 0$ if and only if $K \in \mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}^*$. Therefore, if $K \in \mathfrak{h}^* \cap \mathfrak{k}$, then $\sigma^* \in \mathrm{Int}(\mathfrak{g}^*)$. In the case of $\mathfrak{g}^* = \mathfrak{e}_7$ and $\theta^* \cong \varphi$, $\mathfrak{k}$ and $\mathfrak{p}^*$ is given by (6.2). Thus $K_4 = (K_4 - 2K_7) + 2K_7$, $K_1 + K_6 = (K_1 + K_6 - 2K_7) + 2K_7$ are not elements in $\mathfrak{h}^* \cap \mathfrak{k}$, so the automorophisms which induced by $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_4)$ and $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_1 + K_6))$ are not inner. Similarly as above we can check that all the other $\sqrt{-1}K$ in Table II–V are the elements in $\mathfrak{h} \cap \mathfrak{k}$.

Consequently, except for above examples in the case where $\mathfrak{g}^* = \mathfrak{e}_7$ and $\theta|_{\mathfrak{t}^*} \neq \mathrm{Id}$, all automorphisms σ^* of order four of $\mathfrak{g}^*$ can be written as $\sigma^* = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K)$ for some $K \in \mathfrak{k}$. Therefore σ^* is an inner automorphism of $\mathfrak{g}$.

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References

[1] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. **75** (1962), 485-535.

[2] A. Gray, *Riemannian manifolds with geodesic symmetries of order 3,* J. Differential Geom. **7** (1972), 343-369.

[3] S. Helgason, Differential Geometry, Lie groups, and Symmetric Spaces, Academic Press, New York London, 1978.

[4] J. A. Jiménez, *Riemannian 4-symmetric spaces*, Trans. Amer. Math. Soc. **306** (1988), 715-734.

[5] H. Kurihara and K. Tojo, *Involutions of compact Riemannian* 4*-symmmetric spaces*, Osaka J. Math. **45** (2008), 643-689.

[6] H. Kurihara and K. Tojo, *Involutions of a compact* 4*-symmmetric space of exceptional type*, Osaka J. Math. **52** (2015), 1101-1124.

[7] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331-357.

[8] J. A. Wolf and A. Gray, *Homogeneous spaces defined by Lie group automorphisms I,II*, J. Differential Geom. **2** (1968), 77-114, 115-159.

Department of Mathematics,
Ibaraki University
Bunkyo, Mito, 310-8512
Japan
E-mail address : hiroyuki.kurihara.math@vc.ibaraki.ac.jp

Department of Mathematics
Chiba Institute of Technology
Shibazono, Narashino, Chiba 275-0023
Japan
E-mail address : tojo.koji@it-chiba.ac.jp