ON FIBRATION STABILITY AFTER DERVAN-SEKTNAN AND SINGULARITIES

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Abstract. We introduce $\mathfrak{f}$-stability, a modification of fibration stability of Dervan-Sektnan [11], and show that $\mathfrak{f}$-semistable fibrations have only semi log canonical singularities. Moreover, $\mathfrak{f}$-stability puts restrictions on semi log canonical centers on Fano fibrations.

Contents

1. Introduction

It is one of the most important problems in Kähler geometry that when a constant scalar curvature Kähler (cscK) metric exists on a polarized complex manifold (X, L). The Yau-Tian-Donaldson (YTD) conjecture predicts that the existence of cscK metrics is equivalent to a certain algebro-geometric condition called K-polystability (cf., [42], [39], [12]). Indeed, Berman-Darvas-Lu [4] proved that if (X, L) admits a cscK metric, then (X, L) is K-polystable (see also [37], [38]), and Chen-Donaldson-Sun [6] and Tian [40] proved the YTD conjecture in the Fano case independently. On the other hand, K-stability is the positivity of the leading term of the Chow weights and is also an important notion in terms of the geometric invariant theory (GIT [28]). Ross and Thomas [34] studied K-stability in an algebro-geometric way first, and Odaka [31] found out the relationship between K-stability and singularities by applying the minimal model program (MMP). He also proved that if a $\mathbb{Q}$-Gorenstein variety V is asymptotically Chow-semistable, then V has only slc singularities as a corollary.

On the other hand, the existence problem of cscK metrics on fibrations is well-studied in Kähler geometry. In this paper, fibrations mean algebraic fiber spaces $f : X \to B$, where f is a morphism of varieties with connected general fibers. For details, see Definition 2.17 below. We say that an algebraic fiber space $f : X \to B$ is

- a *Calabi-Yau fibration* if there exists a line bundle L_0 on B such that $K_X \sim_{\mathbb{Q}} f^*L_0$.
- a *Fano fibration* (resp., *a canonically polarized fibration*) if K_X is f-antiample (resp., f-ample).
- a *smooth* (resp., *flat*) *fibration* if f is smooth (resp., flat).

Fine obtained a sufficiency condition [14, Theorem 1.1] for existence of a cscK metric on a smooth fibration whose fibers have cscK metrics first. Dervan and Sektnan [10] introduced

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a differential geometric notion called optimal symplectic connection and proved the following generalization of the result of Fine:

Theorem ([10, Theorem 1.2])**.** *Let $f : (X, H) \to (B, L)$ be a polarized smooth fibration. If f admits an optimal symplectic connection and (B, L) admits a twisted cscK metric with respect to the Weil-Petersson metric, then $(X, \delta H + f^*L)$ has cscK metrics for sufficiently small $\delta > 0$.*

On the other hand, Jian-Shi-Song [23] proved that smooth good minimal models (i.e., K_X is semiample) have cscK metrics. Here, we emphasize that they treated not only smooth fibrations but Calabi-Yau fibrations admitting a singular fiber. Sjöström Dyrefelt [35] and Song [36] generalized independently the result of [23] to the case when X is a smooth minimal model (i.e., K_X is nef) later. On the other hand, the author proved K-stability of klt minimal models in [21].

Dervan and Sektnan [11] also conjectured that the existence of an optimal symplectic connection is equivalent to an algebro-geometric condition called fibration stability. Their theorem and conjecture predict that fibration stability and a certain stability of the base variety imply adiabatic K-stability (cf., Definition 2.18) of the total space as follows.

Conjecture ([11, 1.3])**.** *Let $f : (X, H) \to (B, L)$ be a smooth fibration. If f is fibration stable and (B, L) has a twisted cscK metric with respect to the Weil-Petersson metric, then (X, H) is adiabatically K-semistable i.e., $(X, \delta H + f^*L)$ is K-semistable for sufficiently small $\delta > 0$.*

If it was true, we could obtain a criterion for K-stability of fibrations. The condition on the base in the conjecture would be necessary. Indeed, it is known that twisted K-stability of the base is necessary for adiabatic K-stability of the total space by [8, Corollary 4.4].

In this paper, we introduce $\mathfrak{f}$-stability (cf., Definition 4.4) as a modification of fibration stability of Dervan-Sektnan [11] and prove fundamental results on this. This is a stronger condition than the fibration stability in numerical aspects. Dervan and Sektnan weakened the definition of fibration stability in [11] but the original one in [9] is the condition of positivity of W_0 and W_1 we will explain in the next page. Note that the original fibration stability coincides with $\mathfrak{f}$-stability for fibrations over curves such that each fiber is K-polystable. Since any polarized smooth curve is twisted K-stable, it is natural to conjecture that the original fibration stability implies existence of optimal symplectic connections on fibrations over smooth curves rather than the new one. More generally, for a smooth fibration $f : (X, H) \to (B, L)$, if B has a twisted cscK metric in the sense of the conjecture above and X has an optimal symplectic connection, it is easy to see that f is $\mathfrak{f}$-semistable by [10, Theorem 1.2]. Taking these facts into account, it would be worth studying $\mathfrak{f}$-stability for K-stability of fibrations.

First, we introduce invariants W_i as the Donaldson-Futaki invariant for $\mathfrak{f}$-stability. Similarly to results of [31] on K-stability, we establish the explicit formula to compute W_i by taking general hyperplane sections of the base, and show that $\mathfrak{f}$-stability puts some restrictions on singularities by applying MMP. We obtain the following.

Theorem A. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair (cf., Definition 2.17). If f is $\mathfrak{f}$-semistable, (X, Δ) has at most lc singularities.*

We remark that $\mathfrak{f}$-stability does not imply adiabatic K-stability in general. Indeed, the author showed in [22] that a rational elliptic surface with a section and a II^*, III^* or IV^*-fiber is $\mathfrak{f}$-stable but adiabatically K-unstable over $\mathbb{P}^1$. Thus, we can not apply [31] to Theorem A directly.

We also show that a $\mathfrak{f}$-semistable flat Fano fibration is Kawamata log terminal (klt) if the base variety has only klt singularities. Moreover, as an application of $\mathfrak{f}$-stability to K-stability, we show that adiabatically K-semistable flat Fano fibrations over klt varieties have at most klt singularities.

Theorem B. *Let $f : (X, H) \to (B, L)$ be a flat polarized algebraic fiber space such that all fibers are reduced. Suppose that there exist $\lambda \in \mathbb{Q}_{>0}$ and a line bundle L_0 on B such that*

$H+f^*L_0\equiv -\lambda K_X$ and B is klt. If f is $\mathfrak{f}$-semistable, then X is klt. In particular, if f is adiabatically K-semistable (Definition 2.18), then X is klt.

On the other hand, we prove that klt Calabi-Yau fibrations are $\mathfrak{f}$-stable similarly to [29, Theorem 2.10] in K-stability as follows.

Theorem C. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair with a line bundle L_0 on B. Suppose that (X,Δ) is klt and $K_X+\Delta\equiv f^*L_0$. Then f is $\mathfrak{f}$-stable.*

We explain the definition of $\mathfrak{f}$-stability briefly as follows. Suppose that $f:(X,H)\to(B,L)$ is a polarized algebraic fiber space. Let $m=\mathrm{rel.dim}\, f$ and $\dim B=n$. Roughly speaking, for any semiample test configuration $(\mathcal{X},\mathcal{H})$ for (X,H), we define constants $W_0(\mathcal{X},\mathcal{H}),\cdots,W_n(\mathcal{X},\mathcal{H})$ and a rational function $W_{n+1}(\mathcal{X},\mathcal{H})(j)$ in j so that

$$V(H+jL)M^{\mathrm{NA}}(\mathcal{X},\mathcal{H}+jL)=W_{n+1}(\mathcal{X},\mathcal{H})(j)+\sum_{i=0}^{n}j^iW_{n-i}(\mathcal{X},\mathcal{H})$$

where $\lim_{j\to\infty}W_{n+1}(\mathcal{X},\mathcal{H})(j)=0$ and M^{NA} is the non-Archimedean Mabuchi functional (cf., Definition 2.10 and Notation in §4). Then we say that $f:(X,H)\to(B,L)$ is $\mathfrak{f}$-*semistable* if

$$\sum_{i=0}^{n}j^iW_{n-i}(\mathcal{X},\mathcal{H})\geq 0$$

for sufficiently large $j>0$. We calculate W_i in Lemma 4.13 as follows,

$$W_i(\mathcal{X},\mathcal{H})=\binom{n+m}{n-i}\Bigg(K^{\log}_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-i}/\mathbb{P}^1}\cdot\mathcal{H}|^{m+i}_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-i}}$$
$$+\frac{S(X_b,H_b)}{m+i+1}\mathcal{H}|^{m+i+1}_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-i}}\Bigg)+\sum_{k=n-i+1}^{n}C_kJ^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_k}),$$

where C_k are constants and $D_k\in|L|$ are general elements. Then we prove Theorems A, B by applying MMP results (cf., [33, Theorem 1.1], [19] and the adjunction formula [25, §16, §17]) and Lemma 4.13. On the other hand, we also show these theorems hold for deminormal pairs with boundaries in §§5.2. More precisely,

Theorem D. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized deminormal algebraic fiber space pair (Definition 5.8).*

(1) If f is $\mathfrak{f}$-semistable, (X,Δ) has at most slc singularities.
(2) Suppose that there exist $\lambda\in\mathbb{Q}_{>0}$ and a line bundle L_0 on B such that $H+f^*L_0\equiv -\lambda(K_X+\Delta)$, and f is $\mathfrak{f}$-semistable. Then any slc-center C of (X,Δ) is of fiber type, i.e., $\mathrm{codim}_XC\leq\mathrm{codim}_Bf(C)$.

Note that if f is flat, C is of fiber type if and only if C contains an irreducible component of a fiber. Thus, Theorem B follows from Theorem D. On the other hand, $\mathfrak{f}$-semistability is a weaker condition than adiabatic K-semistability. Therefore, we also obtain the following.

Corollary E. *Let $f:(X,\Delta,H)\to(B,L)$ be an adiabatically K-semistable polarized deminormal algebraic fiber space pair such that $H+f^*L_0\equiv-\lambda(K_X+\Delta)$ where L_0 is a line bundle on B and $\lambda\in\mathbb{Q}_{>0}$. Then, (X,Δ) is slc and any slc-center of (X,Δ) is of fiber type.*

Outline of this paper. In §2, we prepare many terminology and facts on K-stability and algebraic fiber spaces. In §3, we recall results of Odaka [31] and Boucksom-Hisamoto-Jonsson [5] on the relationship between singularities and K-stability. From §4, we state our original results. In §4, we introduce $\mathfrak{f}_l$-(semi)stability, a generalization of $\mathfrak{f}$-(semi)stability, calculate W_i and deduce Theorem 4.20, a generalization of Theorem C for $\mathfrak{f}_l$-stability. In §§5.1, we apply results of MMP and the computations in §4 to obtain generalizations of main theorems for $\mathfrak{f}_l$-semistability. In §§5.2, we extend these theorems to the deminormal case.

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2. Notation

In this paper, we work over $\mathbb{C}$. If X is a *scheme*, we assume that X is of finite type over $\mathbb{C}$ in this paper. If X is a *variety*, we assume that X is an irreducible, reduced and separated scheme. We follow the definitions of $\mathbb{Q}$-line bundles, $\mathbb{Q}$-Weil divsiors and $\mathbb{Q}$-Cartier divisors, and the notations of the $\mathbb{Q}$-linearly equivalence $\sim_{\mathbb{Q}}$ and the numerical equivalence $\equiv$ from [20], [26] and [5]. A pair (X, L) is called a *polarized scheme* if X is a proper, reduced and equidimensional scheme over $\mathbb{C}$ and L is an ample $\mathbb{Q}$-line bundle over X. A $\mathbb{Q}$-Weil divisor Δ, such that there exist integral divisors F_i different from each other, $a_i \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $\Delta = \sum_{i=1}^{n} a_i F_i$, is called a *boundary* if $K_{(X,\Delta)} := K_X + \Delta$ is $\mathbb{Q}$-Cartier on a normal variety X. Then, we call the pair (X, Δ) a (normal) log pair. Here, we do not assume that $\lceil \Delta \rceil = \sum \lceil a_i \rceil F_i$ is a reduced divisor. We define $\Delta_{>1} = \sum_{a_i > 1} a_i F_i$. A $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{r} a_i D_i$ on a smooth variety X is *simple normal crossing* (snc) if each $\bigcap_{i \in J} D_i$ is smooth for any subset $J \subset \{1, 2, \cdots, r\}$.

Notation 2.1. If $\iota : Z \hookrightarrow X$ is the locally closed immersion, let $\overline{Z}$ be the scheme-theoretic image structure of ι and call $\overline{Z}$ the *Zariski closure* of Z.

First, recall the definitions of deminormal pairs and semi log canonical singularities,

Definition 2.2 (Deminormal pair [24, Chapter 5]). Let X be an equidimensional reduced scheme satisfying Serre's condition S_2. X is called a *deminormal scheme* if any codimension 1 point of X is smooth or nodal. A $\mathbb{Q}$-divisor Δ on X is called a boundary if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and any irreducible component of Δ is not contained in the singular locus $\mathrm{Sing}(X)$. Then we call (X, Δ) a *deminormal log pair*.

Let $\nu : \tilde{X} \to X$ be the normalization. Then, $\mathfrak{d} = \mathrm{Hom}_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$ is an ideal in both of $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$. We call the *conductors* of X and of $\tilde{X}$ the closed subschemes defined by $\mathfrak{d}$ and we will denote these by $\mathfrak{cond}_X \subset X$ and $\mathfrak{cond}_{\tilde{X}} \subset \tilde{X}$ respectively. Note that $\mathfrak{cond}_{\tilde{X}}$ is a reduced Weil divisor.

Furthermore, let L be a $\mathbb{Q}$-ample line bundle on X. Then, we call (X, Δ, L) a *polarized deminormal (log) pair*.

Notation 2.3. Let Y be a scheme and H be a $\mathbb{Q}$-line bundle over Y. If $\nu : \widetilde{Y} \to Y$ is the normalization, we denote $\nu^* H = \widetilde{H}$. Then, we call a pair $(\tilde{Y}, \tilde{H})$ the *normalization* of (Y, H).

Definition 2.4 (Log discrepancy). Let (X, Δ) be a normal log pair and v be a divisorial valuation on X. Suppose that $\sigma : Y \to X$ is a proper birational morphism from a normal variety such that there exist a positive constant $c > 0$ and a prime divisor F on Y such that $v = c\,\mathrm{ord}_F$. Then the *log discrepancy* of v with respect to (X, Δ) is

$$A_{(X,\Delta)}(v) = v(K_Y - \sigma^* K_{(X,\Delta)}) + c.$$

It is easy to see that the log discrepancy is independent of σ. We set $A_{(X,\Delta)}(v_{\mathrm{triv}}) = 0$ if v_{triv} is the trivial valuation.

We define (X, Δ) is

- *sub Kawamata log terminal (subklt)* if $A_{(X,\Delta)}(v) > 0$ for any non trivial divisorial valuation v.
- *sub log canonical (sublc)* if $A_{(X,\Delta)}(v) \geq 0$.

Let $c_X(v)$ be the center of v on X. $c_X(v)$ is called a non-lc (resp. lc) center of (X, Δ) if $A_{(X,\Delta)}(v) < 0$ (resp. $= 0$). (X, Δ) is klt (resp., lc) if (X, Δ) is subklt (resp., sublc) and Δ is effective. We also say that (X, Δ) has only klt (resp., lc) singularities.

Let (V, B) be a deminormal log pair and $\nu : V' \to V$ be the normalization. Then, (V, B) is *semi log canonical (slc)* if $(V', \nu_*^{-1}B + \mathfrak{cond}_{V'})$ is lc. We say that a point $v \in V$ is

- a *non-slc*, (resp. *slc*) center of (V, B) if there exists a non-lc, (resp. lc) center v' of $(V', \nu_*^{-1}B + \mathfrak{cond}_{V'})$ such that $\nu(v') = v$,
- a *non-klt* center if v is neither a non-slc nor slc center.

We also call an irreducible closed subset $F \subset V$ an lc (resp. slc, non-lc, non-klt) center if so is the generic point of F.

Recall the notion of the minimal model program (MMP). We follow the fundamental notations of MMP in [26] (e.g., dlt).

Definition 2.5 (Minimal model). Let S be a quasi projective normal variety. Let (X, Δ) be a normal log pair projective over S such that Δ is effective and $\lceil \Delta \rceil$ is reduced. Let Y be a normal variety projective over S. A birational map $\phi : X \dashrightarrow Y$ is called a *birational contraction* if there is no ϕ^{-1}-exceptional divisor. Suppose that $K_Y + \phi_*\Delta$ is also $\mathbb{Q}$-Cartier. Then, ϕ is called $(K_X + \Delta)$-non-positive (resp., $(K_X + \Delta)$-negative) if ϕ is a birational contraction and $p^*(K_X + \Delta) = q^*(K_Y + \phi_*\Delta) + E$, where E is effective (resp., E is effective and $\mathrm{Supp}\, E$ contains all ϕ-exceptional divisors). Here, Γ is the normalization of the graph of ϕ and $p : \Gamma \to X$ and $q : \Gamma \to Y$ are the canonical projections. Then Y is called

- a *minimal model* (resp., *good minimal model*) of (X, Δ) if $K_Y + \phi_*\Delta$ is nef (resp., semiample) over S and ϕ is $(K_X + \Delta)$-negative,
- the *log canonical model* (lc model) if $K_Y + \phi_*\Delta$ is ample over S and ϕ is $(K_X + \Delta)$-non-positive.

Next, recall K-stability. One can find further details, for example, in [5].

Definition 2.6 (Test configuration). Let X be a proper scheme. Then, $\pi : \mathcal{X} \to \mathbb{A}^1$ is called a *test configuration* for X if $\pi : \mathcal{X} \to \mathbb{A}^1$ satisfies the following properties:

(1) $\mathcal{X}$ has a $\mathbb{G}_m$-action.
(2) π is a proper, flat and $\mathbb{G}_m$-equivariant morphism where $\mathbb{A}^1$ admits a canonical $\mathbb{G}_m$-action.
(3) $\mathcal{X}_1 := \pi^{-1}(1) \cong X$.

If there is no fear of confusion, we will simply denote $\pi : \mathcal{X} \to \mathbb{A}^1$ as $\mathcal{X}$. Let (X, L) be a polarized deminormal scheme. Then, $(\mathcal{X}, \mathcal{L})$ is called a test configuration for (X, L) if

(1) $\mathcal{X}$ is a test configuration for X.
(2) $\mathcal{L}$ is a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-line bundle.
(3) $\mathcal{L}|_{\mathcal{X}_1} = L$.

If $\mathcal{L}$ is $\mathbb{A}^1$-(semi)ample, then we call $(\mathcal{X}, \mathcal{L})$ (semi)ample. $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1}) = (X \times \mathbb{A}^1, L \times \mathbb{A}^1)$ with the trivial $\mathbb{G}_m$-action is an ample test configuration. We call this the *trivial test configuration*. Note that any test configuration is birational to $X_{\mathbb{A}^1}$. If $\mathcal{X}$ is $\mathbb{G}_m$-equivariantly isomorphic to $X_{\mathbb{A}^1}$ (in codimension 1), then we call $(\mathcal{X}, \mathcal{L})$ *(almost) trivial*.

We say $\mathcal{X}$ *dominates* $X_{\mathbb{A}^1}$ if there exists a birational morphism of test configurations $\rho_{\mathcal{X}} : \mathcal{X} \to X_{\mathbb{A}^1}$, and $(\mathcal{X}, \mathcal{L})$ is a *product* test configuration if $\mathcal{X}$ is isomorphic to $X \times \mathbb{A}^1$ as abstract varieties. The *central fiber* of $(\mathcal{X}, \mathcal{L})$ is the fiber of π over $0 \in \mathbb{A}^1$ and is denoted as $(\mathcal{X}_0, \mathcal{L}_0)$.

The following is an example of a test configuration and studied by [34].

Example 2.7 (Deformation to the normal cone). Let (X, B, L) be a polarized deminormal pair. For any non-void closed subscheme $Z \subset X$, *the deformation to the normal cone* $\mathcal{X}$ *of* Z is a test configuration for X such that $\mathcal{X}$ is the blow up along $Z \times \{0\}$ (see also [18, Chapter 5]). We usually consider an ample line bundle $\mathcal{L} = L_{\mathbb{A}^1} - t\mu^{-1}(\mathscr{I}_{Z \times \{0\}})$, where $\mathscr{I}_{Z \times \{0\}}$ is the ideal corresponding to $Z \times \{0\}$ and $t > 0$.

To define the Donaldson-Futaki invariant, we need the following definition of the weight of $\mathbb{G}_m$-representation. Let W be a finite dimensional $\mathbb{G}_m$-representation space over $\mathbb{C}$ and then

W has the unique weight decomposition

$$W = \bigoplus_{k=-\infty}^{\infty} W_k$$

where $\lambda \in \mathbb{G}_m$ acts on $v \in W_k$ in the way that $v \mapsto \lambda^k v$. Then, the *weight* of W is

$$-\sum_k k \dim W_k.$$

Definition 2.8 (Donaldson-Futaki invariant, [32] Definition 3.2). Let (X, Δ, L) be an n-dimensional polarized (demi)normal pair and $(\mathcal{X}, \mathcal{L})$ be an ample test configuration. Let $w(m)$ be the weight of $H^0(\mathcal{X}_0, m\mathcal{L}_0)$. It is well-known that $w(m) = \sum_{i=0}^{n+1} b_i m^{n+1-i}$ is a polynomial function for sufficiently large m (see [5, Theorem 3.1]). On the other hand, let $h^0(X, mL) = \sum_{i=0}^{n} a_i m^{n-i}$ be the Hilbert polynomial. Then, the *Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 2\frac{b_1 a_0 - a_1 b_0}{a_0^2}.$$

Moreover, let $\Delta_{\mathcal{X}}$ be the Zariski closure of $\Delta \times \mathbb{A}^1$ in $\mathcal{X}$. Let $\hat{w}(m) = \sum_{i=0}^{n} \hat{b}_i m^{n-i}$ be the weight polynomial of $H^0(\Delta_{\mathcal{X},0}, m\mathcal{L}|_{\Delta_{\mathcal{X},0}})$ and $h^0(\Delta, mL|_\Delta) = \sum_{i=0}^{n-1} \hat{a}_i m^{n-1-i}$ be the Hilbert polynomial. Here, we understand $h^0(\Delta, mL|_\Delta) = \sum c_i h^0(D_i, mL|_{D_i})$ where $\Delta = \sum c_i D_i$ and $\hat{w}(m)$ in the same way. Then, the *log Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}) + \frac{\hat{b}_0 a_0 - \hat{a}_0 b_0}{a_0^2}.$$

Notation 2.9. In this paper, we write line bundles and divisors interchangeably. For example,

$$L + mH = L \otimes H^{\otimes m}.$$

For simplicity, we will denote intersection products of line bundles or divisors as

$$L^m \cdot (H + D) = L^{\cdot m} \cdot (H \otimes \mathcal{O}_X(D)).$$

We define the non-Archimedean functionals as in [5].

Definition 2.10 ([5, §6, 7]). Let (X, Δ, L) be an n-dimensional polarized normal pair and $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ be a normal semiample test configuration. Let also $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ be the $\mathbb{G}_m$-*equivariant compactification over* $\mathbb{P}^1$ such that the ∞-fiber $(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}}_\infty)$ is $\mathbb{G}_m$-equivariantly isomorphic to (X, L) with the trivial $\mathbb{G}_m$-action (cf., [5, Definition 2.7]). Suppose that there exists a $\mathbb{G}_m$-equivariant morphism $\rho : \overline{\mathcal{X}} \to X_{\mathbb{P}^1}$ such that ρ is the identity on $X \times (\mathbb{P}^1 \setminus \{0\})$. Here, $(X_{\mathbb{P}^1}, L_{\mathbb{P}^1})$ is the $\mathbb{G}_m$-equivariant compactification of $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$. We call ρ the *canonical* map. We simply denote $\rho^* L_{\mathbb{P}^1}$ by $L_{\mathbb{P}^1}$. If

- $V = V(L) := L^n$,
- $S(X, \Delta, L) = -\frac{n(K_{(X,\Delta)} \cdot L^{n-1})}{(L^n)}$, where $K_{(X,\Delta)} = K_X + \Delta$,
- $K^{\log}_{(\overline{\mathcal{X}}, \Delta_{\overline{\mathcal{X}}})/\mathbb{P}^1} = K_{(\overline{\mathcal{X}}, \Delta_{\overline{\mathcal{X}}})/\mathbb{P}^1} + (\mathcal{X}_{0,\mathrm{red}} - \mathcal{X}_0)$, where $\mathcal{X}_{0,\mathrm{red}}$ is the reduced central fiber of $\pi : \mathcal{X} \to \mathbb{A}^1$ and $\Delta_{\overline{\mathcal{X}}}$ is the strict transform of $\Delta \times \mathbb{P}^1$ in $\overline{\mathcal{X}}$, and
- for any irreducible component of $\mathcal{X}_0$, the divisorial valuation v_E on X is the restriction of $b_E^{-1} \mathrm{ord}_E$ to X, where $b_E = \mathrm{ord}_E(\mathcal{X}_0)$ and $\mathrm{ord}_E|_X$ is non-trivial (cf., [5, §4]),

then

- the non-Archimedean Monge-Ampère energy of $(\mathcal{X}, \mathcal{L})$ is

$$E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V(L)},$$

- the non-Archimedean I-functional of $(\mathcal{X}, \mathcal{L})$ is

$$I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1}(\overline{\mathcal{L}} \cdot L^n_{\mathbb{P}^1}) - V^{-1}(\overline{\mathcal{L}} - L_{\mathbb{P}^1}) \cdot \overline{\mathcal{L}}^n,$$

- the non-Archimedean J-functional of $(\mathcal{X},\mathcal{L})$ is
$$J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=V^{-1}(\overline{\mathcal{L}}\cdot L^n_{\mathbb{P}^1})-E^{\mathrm{NA}}(\mathcal{X},\mathcal{L}),$$
- the $(\mathcal{J}^H)^{\mathrm{NA}}$-functional of $(\mathcal{X},\mathcal{L})$ is
$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=V^{-1}(H_{\mathbb{P}^1}\cdot\overline{\mathcal{L}}^n)-V^{-1}(nH\cdot L^{n-1})E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$
where H is a line bundle on X,
- the non-Archimedean Ricci energy of $(\mathcal{X},\mathcal{L})$ is
$$R^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L})=(\mathcal{J}^{K_{(X,\Delta)}})^{\mathrm{NA}}(\mathcal{X},\mathcal{L})-S(X,\Delta,L)E^{\mathrm{NA}}(\mathcal{X},\mathcal{L}),$$
- the non-Archimedean entropy of $(\mathcal{X},\mathcal{L})$ is (see [5, Corollary 7.18])
$$H^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L})=V^{-1}\sum_E b_E A_{(X,B)}(v_E)(E\cdot\overline{\mathcal{L}}^n)=V^{-1}(K^{\log}_{(\overline{\mathcal{X}},\Delta_{\overline{\mathcal{X}}})/\mathbb{P}^1}-\rho^*K_{(X_{\mathbb{P}^1},\Delta_{\mathbb{P}^1})/\mathbb{P}^1})\cdot\overline{\mathcal{L}}^n,$$
where E runs over the irreducible components of $\mathcal{X}_0$,
- the non-Archimedean Mabuchi functional of $(\mathcal{X},\mathcal{L})$ is
$$M^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L})=H^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L})+(\mathcal{J}^{K_{(X,\Delta)}})^{\mathrm{NA}}(\mathcal{X},\mathcal{L}).$$

If there is no fear of confusion, we denote $(\overline{\mathcal{X}},\overline{\mathcal{L}})$ as $(\mathcal{X},\mathcal{L})$. Note that these functionals are pullback invariant in the following sense. If there exists a $\mathbb{G}_m$-equivariant morphism $\mu:\tilde{\mathcal{X}}\to\mathcal{X}$ of normal test configurations for X such that μ is the identity on $X\times(\mathbb{A}^1\setminus\{0\})$ and F^{NA} is one of the functionals as above, then
$$F^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=F^{\mathrm{NA}}(\tilde{\mathcal{X}},\mu^*\mathcal{L}).$$

On the other hand, we say that a semiample test configuration $(\tilde{\mathcal{X}},\tilde{\mathcal{L}})$ *dominates* $(\mathcal{X},\mathcal{L})$ if there exists a $\mathbb{G}_m$-equivariant morphism $\mu:\tilde{\mathcal{X}}\to\mathcal{X}$ of normal test configurations for X such that μ is the identity on $X\times(\mathbb{A}^1\setminus\{0\})$ and $\tilde{\mathcal{L}}=\mu^*\mathcal{L}$. We define the set of all non-Archimedean positive metrics with respect to (X,L) as follows,
$$\mathcal{H}^{\mathrm{NA}}(L)=\{\text{All (semi)ample test configurations for }(X,L)\}/\sim$$
where $\sim$ is the equivalence relation generated by domination. See also [5, Proposition 2.17]. It is easy to see that $\mathcal{H}^{\mathrm{NA}}(L)$ is a set. Thus, the functionals F^{NA} are independent of the choice of $\phi\in\mathcal{H}^{\mathrm{NA}}(L)$. It is also easy to see that for any non-Archimedean positive metric $\phi\in\mathcal{H}^{\mathrm{NA}}(L)$, there exists a representative $(\mathcal{X},\mathcal{L})$ of ϕ such that $\mathcal{X}$ is normal and there exists a $\mathbb{G}_m$-equivariant morphism $\rho:\overline{\mathcal{X}}\to X_{\mathbb{P}^1}$ such that ρ is canonical.

Proposition 2.11 ([32, Theorem 3.7], [5, Proposition 3.12])**.** *Notations as above. Let (X,Δ,L) be a polarized normal pair and $(\mathcal{X},\mathcal{L})$ be a normal semiample test configuration. Then,*
$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})=V^{-1}(K_{(\overline{\mathcal{X}},\Delta_{\overline{\mathcal{X}}})/\mathbb{P}^1}-\rho^*K_{(X_{\mathbb{P}^1},\Delta_{\mathbb{P}^1})/\mathbb{P}^1})\cdot\overline{\mathcal{L}}^n+R^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L})+S(X,\Delta,L)E^{\mathrm{NA}}(\mathcal{X},\mathcal{L}).$$

Definition 2.12 (K-stability). Let (X,Δ,L) be a polarized deminormal pair. (X,L) is
- *K-semistable* if
$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})\ge 0$$
for any semiample test configuration,
- *K-polystable* if (X,Δ,L) is K-semistable and
$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})=0$$
if and only if $(\mathcal{X},\mathcal{L})$ is isomorphic to a product test configuration in codimension 1,
- *K-stable* if
$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})>0$$
for any non-almost-trivial ample test configuration,

- *uniformly K-stable* if there exists a positive constant $\epsilon > 0$ such that
$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L}) \geq \epsilon I^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$
for any semiample test configuration.

We remark that the non-Archimedean I and J-functionals are norms in the following sense. The non-Archimedean I and J-functionals are nonnegative and
$$J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = 0$$
if and only if the normalization of $(\mathcal{X},\mathcal{L})$ is trivial by [5, Theorem 7.9]. They satisfy that
$$\frac{1}{n} J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \leq I^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) - J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \leq n J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$
by [5, Proposition 7.8].

Remark 2.13. Dervan [7] introduced the minimum norm of test configurations defined by $I^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) - J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ and also proved that this is a norm as the non-Archimedean I and J-functionals.

It is well-known that M^{NA} detects K-stability of polarized normal pair. See [5, Proposition 8.2] or Proposition 2.25 below. Note that $M^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L}) = \mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})$ if $\mathcal{X}_0$ is reduced.

Definition 2.14. For $\phi \in \mathcal{H}^{\mathrm{NA}}(L)$, we define a positive non-Archimedean metric ϕ_d for $d \in \mathbb{Z}_{\geq 1}$ as follows. If $(\mathcal{X},\mathcal{L})$ is a representative of ϕ, then ϕ_d is represented by $(\mathcal{X}^{(d)},\mathcal{L}^{(d)})$ that is the normalization of the base change $\mathcal{X} \times_{\mathbb{A}^1} \mathbb{A}^1$ via the d-th power map $\mathbb{A}^1 \ni t \mapsto t^d \in \mathbb{A}^1$.

Note that $d\, M^{\mathrm{NA}}_\Delta(\phi) = M^{\mathrm{NA}}_\Delta(\phi_d)$ and $d\,\mathrm{DF}_\Delta(\phi) \geq \mathrm{DF}_\Delta(\phi_d)$ for any d. Moreover, for sufficiently divisible d, $M^{\mathrm{NA}}_\Delta(\phi_d) = \mathrm{DF}_\Delta(\phi_d)$ by [5, Proposition 7.16].

We will use the following notation in §4.

Definition 2.15. We say that a normal test configuration $(\mathcal{X},\mathcal{L})$ for (X, L) is *normalized with respect to the central fiber* if the following hold.

(1) There exists a $\mathbb{G}_m$-equivariant dominant morphism $\rho : (\mathcal{X},\mathcal{L}) \to (X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$.
(2) $D = \mathcal{L} - \rho^* L_{\mathbb{A}^1}$ has the support $\hat{X} \not\subset \mathrm{Supp}\, D \subset \mathcal{X}_0$, where $\hat{X}$ is the strict transform of $X \times \{0\}$ in $\mathcal{X}$.

Whenever there is no fear of confusion, we simply say that $(\mathcal{X},\mathcal{L})$ as above is a *normalized* test configuration. Note that $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ is normalized and if $(\mathcal{X},\mathcal{L}_1)$ and $(\mathcal{X},\mathcal{L}_2)$ are normalized as test configurations for (X, L_1) and (X, L_2) respectively, then $(\mathcal{X},\mathcal{L}_1+\mathcal{L}_2)$ is also normalized.

Lemma 2.16. *Let $(\mathcal{X},\mathcal{L})$ be a semiample test configuration for an n-dimensional polarized normal pair (X,Δ,L). Suppose that $(\mathcal{X},\mathcal{L})$ is normalized and there exists a canonical birational morphism $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$. Then the following hold,*

(1) $J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = -E^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$.
(2) $R^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{L}) = \frac{1}{V(L)} \rho^* K_{(X_{\mathbb{P}^1},\Delta_{\mathbb{P}^1})/\mathbb{P}^1} \cdot \mathcal{L}^n$.

Proof. If $(\mathcal{X},\mathcal{L})$ is normalized, $\mathcal{L} \cdot L^n_{\mathbb{P}^1} = 0$ by [5, Lemma 7.7]. Thus we have the assertions. □

We are interested in K-stability of the following object.

Definition 2.17. An *algebraic fiber space* $f : X \to B$ is a surjective morphism between proper normal varieties with geometrically connected fibers. If F is the generic geometric fiber, the *relative dimension* $\mathrm{rel.dim}\, f$ of f is $\dim F$. Note that F is normal and connected. If $f : X \to B$ is an algebraic fiber space and flat, we call f a *flat algebraic fiber space*. If $f : X \to B$ is an algebraic fiber space, H is an f-ample line bundle on X and L is an ample line bundle on B, then we call $f : (X,H) \to (B,L)$ a *polarized algebraic fiber space*. Moreover, if Δ is an effective boundary on X and f is as above, then we denote $f : (X,\Delta,H) \to (B,L)$ and call this a *polarized algebraic fiber space pair*.

Definition 2.18 (Adiabatic K-stability). Let $f : (X,\Delta,H) \to (B,L)$ be as above. Then (X,Δ,H) is called *adiabatically K-(poly, semi)stable over* (C,L) if $(X,\Delta,\epsilon H + f^*L)$ is K-(poly, semi)stable for sufficiently small $\epsilon > 0$.

Remark 2.19. In the above definition, we name the above stability after the adiabatic limit technique Fine [13], [14] and Dervan-Sektnan [11] used.

We need the following notation about algebraic fiber spaces.

Notation 2.20. Let $f : X \to B$ be an algebraic fiber space and H and M be line bundles on X. We say that H and M are $\mathbb{Q}$-linearly equivalent over B, $H \sim_{B,\mathbb{Q}} M$ if there exists a $\mathbb{Q}$-line bundle L on B such that $H \sim_{\mathbb{Q}} M + f^*L$. Furthermore, we will denote $H + L = H + f^*L$ if there is no fear of confusion.

To show our main theorems, we need the following concept.

Definition 2.21. Let $f : X \to B$ be a proper morphism of equidimensional schemes. We say that a closed subset Z of X is of *fiber type* if $\mathrm{codim}_X Z \le \mathrm{codim}_B f(Z)$. We say that a scheme-theoretic point $\xi \in X$ is of fiber type if so is $\overline{\{\xi\}}$.

For K-stability of deminormal varieties, we calculate the Donaldson-Futaki invariant by taking the normalization. We need the following,

Definition 2.22 (Partially normal test configuration, [30, Definition 3.7]). Let X be a deminormal scheme and $\mathcal{X}$ be a test configuration for X. Let also $\nu : \tilde{\mathcal{X}} \to \mathcal{X}$ be the normalization. The *partially normalization* $\mathcal{X}'$ of $\mathcal{X}$ is $\mathcal{S}pec_{\mathcal{X}}(\mathcal{O}_{X\times(\mathbb{A}^1\setminus\{0\})} \cap \nu_*\mathcal{O}_{\tilde{\mathcal{X}}})$. On the other hand, we say that $\mathcal{X}$ is *partially normal* if $\mathcal{X}$ coincides with its partially normalization. Partially normal test configurations for X are deminormal by [17, Proposition 3.3].

We recall the following fundamental result,

Proposition 2.23 ([34, Proposition 5.1], [30, Propositions 3.8 and 3.10])**.** *Let (X, Δ, L) be a polarized deminormal pair and $(\mathcal{X}, \mathcal{L})$ be a semiample test configuration for (X, L). Let also $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ be the normalization and $(\mathcal{X}', \mathcal{L}')$ be the partially normalization of $(\mathcal{X}, \mathcal{L})$, i.e., $\mathcal{L}'$ is the pullback of $\mathcal{L}$. Here, $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ is a semiample test configuration for the polarized normal pair $(\tilde{X}, \Delta_{\tilde{X}}, \nu^*L)$, where $\nu : \tilde{X} \to X$ is the normalization and $\Delta_{\tilde{X}} = \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}}$. Then,*

$$\mathrm{DF}_{\Delta_{\tilde{X}}}(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) = \mathrm{DF}_{\Delta}(\mathcal{X}', \mathcal{L}') \le \mathrm{DF}_{\Delta}(\mathcal{X}, \mathcal{L}).$$

We remark that an ample partially normal test configuration $(\mathcal{X}, \mathcal{L})$ is trivial if and only if so is the normalization by [17, Proposition 3.9].

Definition 2.24. Notations as in Proposition 2.23. If $(\mathcal{X}, \mathcal{L})$ is partially normal, we define the non-Archimedean Mabuchi functional of $(\mathcal{X}, \mathcal{L})$ to be

$$M_{\Delta}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = M_{\Delta_{\tilde{X}}}^{\mathrm{NA}}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}).$$

Combining these facts, we obtain the following criterion for K-stability of deminormal pairs.

Proposition 2.25. *Let (X, Δ, L) be a polarized deminormal pair. Then (X, L) is K-semistable (resp. uniformly K-stable) if and only if there exists $\epsilon \ge 0$ (resp. > 0) such that*

$$M_{\Delta}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \ge \epsilon I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

for any partially normal semiample test configuration.

Proof. The assertion when $\mathcal{X}$ is normal holds as [5, Proposition 8.2] even when X is reducible. For the general case, let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ be the normalization of $(\mathcal{X}, \mathcal{L})$. By what we stated in Definition 2.14,

$$\mathrm{DF}_{\Delta_{\tilde{X}}}(\widetilde{\mathcal{X}}^{(d)}, \widetilde{\mathcal{L}}^{(d)}) = M_{\Delta_{\tilde{X}}}^{\mathrm{NA}}(\widetilde{\mathcal{X}}^{(d)}, \widetilde{\mathcal{L}}^{(d)}) = dM_{\Delta_{\tilde{X}}}^{\mathrm{NA}}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$$

for sufficiently divisible $d \in \mathbb{Z}_{>0}$, where $(\widetilde{\mathcal{X}}^{(d)}, \widetilde{\mathcal{L}}^{(d)})$ is the normalization of the base change of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ via the d-th power map $g_d : \mathbb{A}^1 \ni t \mapsto t^d \in \mathbb{A}^1$. Let $(\mathcal{X}^{(d)}, \mathcal{L}^{(d)})$ be the partially normalization of the base change of $(\mathcal{X}, \mathcal{L})$ via g_d. Thus, we have

$$\mathrm{DF}_{\Delta}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) = M_{\Delta}^{\mathrm{NA}}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) = dM_{\Delta}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

by Proposition 2.23. ☐

3. Review of Basic Results on K-stability and singularities

Recall that Odaka proved that K-stability puts the restrictions on the singularities on varieties. In this section, recall those remarkable results of [31] and the logarithmic generalizations [32], [5].

Definition 3.1. Let Z be a closed subscheme of a normal variety X and $\mathfrak{a}_Z \subset \mathcal{O}_X$ be the coherent sheaf of ideals corresponding to Z. Let also $\pi : \tilde{X} \to X$ be the normalization of the blow up of X along Z and we call this the *normalized blow up* and call the Cartier divisor denoted by $D = \pi^{-1}(Z)$ corresponding to $\pi^{-1}(\mathfrak{a}_Z)$ the *inverse image* of Z. A divisorial valuation v on X is called a *Rees valuation* of Z if $v = \frac{\mathrm{ord}_E}{\mathrm{ord}_E(D)}$ where E is an irreducible component of the exceptional divisor of π. We will denote the set of Rees valuations of Z by $\mathrm{Rees}_X(Z)$.

The following is the main theorem of [31],

Theorem 3.2 ([31, Theorems 1.2, 1.3], [32, Theorem 6.1], [5, §9])**.** *Let (X, B, L) be a polarized deminormal pair such that B is effective. Then the following hold.*

(i) If (X, B, L) is K-semistable, then (X, B) has only slc singularities.
(ii) If (X, B, L) is K-semistable and $L = -K_X - B$, then (X, B) has only klt singularities.

Let us recall the proof of this theorem briefly. First, we need the following result in MMP, which is called the lc modification or slc modification,

Theorem 3.3 ([33], [16])**.** *(i) Let X be a normal variety and B be an effective $\mathbb{Q}$-divisor such that $\lceil B \rceil$ is reduced. If $f : (Y, \Delta_Y) \to X$ is a proper birational morphism such that $K_Y + \Delta_Y$ is f-ample and (Y, Δ_Y) is lc with $\Delta_Y = f_*^{-1}B + \sum E$ where E runs over the irreducible and reduced f-exceptional divisors, then we call $f : (Y, \Delta_Y) \to X$ the lc modification of (X, B) and the lc modification is unique up to isomorphism.*

(ii) Let (X, B) be a normal log pair such that B is effective and $\lceil B \rceil$ is reduced. Then, there exists the lc modification $f : (Y, \Delta_Y) \to X$ of (X, B).

(iii) Let (X, B) be a deminormal log pair such that B is effective and $\lceil B \rceil$ is reduced. Then, there exists a morphism $f : (Y, \Delta_Y) \to X$ such that

- *f is proper and isomorphic in codimension 1,*
- *(Y, Δ_Y) is slc for $\Delta_Y = f_*^{-1}B + E$ where E is the sum of all reduced f-exceptional divisors,*
- *$K_Y + \Delta_Y$ is f-ample.*

We call such $f : (Y, \Delta_Y) \to X$ the slc modification of (X, B) and the slc modification is unique up to isomorphism.

Let us recall the proof briefly. To prove (ii), we apply the results of [19] and [15]. (i) follows from the argument of [16, §4]. Note that we do not assume that $K_X + B$ is $\mathbb{Q}$-Cartier in (i). To prove (iii), we take the normalization $\nu : (X', B' + D) \to (X, B)$ where D is the conductor divisor on X'. Then, there exists the lc modification $f' : (Y', \Delta'_Y) \to (X', B' + D)$ by (ii). Let $n : D^{\mathrm{n}} \to D$ be the normalization and $D' = f'^{-1}_* D$. If $n' : D'^{\mathrm{n}} \to D'$ is the normalization,

$$(D'^{\mathrm{n}}, \mathrm{Diff}_{D'}(\Delta_Y - D')) \to (D^{\mathrm{n}}, \mathrm{Supp}(\mathrm{Diff}_D(B')_{>1}) + \mathrm{Diff}_D(B') - \mathrm{Diff}_D(B')_{>1})$$

is the lc modification in the sense of (i) (cf., [16, 4.4]). Here, $\mathrm{Diff}_D(B')$ is the different and see the definition in [25, §16]. Therefore, the involution of D^{n} lifts to D'^{n} and (iii) follows from [24, Theorem 5.13] (for details, see [33] and [16]).

By Theorem 3.3, we can conclude as follows.

Corollary 3.4 (cf. [5, Corollary 9.8])**.** *Let (X, B) be a non-slc pair such that B is effective and $\lceil B \rceil$ is reduced. Let also $\nu : (X', \nu_*^{-1}B + \mathfrak{cond}) \to (X, B)$ be the normalization. Then, there exists a closed subscheme $Z \subset X$ of $\mathrm{codim}_X Z \geq 2$ such that $A_{(X,B)}(v) := A_{(X', \nu_*^{-1}B + \mathfrak{cond})}(v) < 0$ for all $v \in \mathrm{Rees}_{X'}(\nu^{-1}Z)$.*

We also remark the following theorem,

Theorem 3.5 ([5, Theorem 4.8]). *Let $Z \subset X$ be a closed subscheme of a normal variety. Then, if $\mathcal{X}$ is the normalization of the deformation to the normal cone of Z, $\mathrm{Rees}_X(Z)$ coincides with the set of non-trivial valuations v_E, where E runs over the irreducible components of $\mathcal{X}_0$.*

Therefore, Theorem 3.2 follows from the same calculation in [5, Proposition 9.12]. We will prove in §5 that $\mathfrak{f}$-stability (we will define in §4 below) puts restrictions on the singularity.

4. Fibration stability after Dervan-Sektnan

In this section, we introduce the notion of $\mathfrak{f}$-stability that is a modified version of fibration stability, which is introduced by Dervan and Sektnan [11]. Next, we show the fundamental results on $\mathfrak{f}$-stability (e.g., Theorem C) and compare $\mathfrak{f}$-stability with fibration stability.

First, recall the notion of fibration degeneration introduced by Dervan and Sektnan [11] as test configurations in K-stability.

Definition 4.1. Let $\pi : (X, H) \to (B, L)$ be a polarized algebraic fiber space (cf., Definition 2.17). $\Pi : (\mathcal{X}, \mathcal{H}) \to B \times \mathbb{A}^1$ is called a *fibration degeneration* for π if the following hold.

- $(\mathcal{X}, \mathcal{H} + jL_{\mathbb{A}^1})$ is an ample test configuration for $(X, H + j\pi^*L)$ for sufficiently large $j > 0$,
- Π is $\mathbb{G}_m$-equivariant over $B_{\mathbb{A}^1}$,
- Let Π_1 be the restriction of Π to the fiber over $1 \in \mathbb{A}^1$. Then $\pi = \Pi_1$.

We will need a weaker concept which we call a *semi fibration degeneration* where $\mathcal{H}$ is relatively semiample over $B \times \mathbb{A}^1$.

When π is flat, this notion coincides with [11, Definition 2.16] due to [*loc.cit*, Lemma 2.15 and Corollary 2.24] and the next proposition.

Proposition 4.2. *Let $\pi : (X, H) \to (B, L)$ be a polarized algebraic fiber space. If $(\mathcal{X}, \mathcal{H})$ is a test configuration such that $(\mathcal{X}, \mathcal{H} + jL_{\mathbb{A}^1})$ is a semiample test configuration for $(X, H + jL)$ for sufficiently large j, where $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$ (hence also $B_{\mathbb{A}^1}$), then there exists a fibration degeneration for π dominated by $(\mathcal{X}, \mathcal{H})$.*

Proof. By assumption, $(\mathcal{X}, \mathcal{H} + jL_{\mathbb{A}^1})$ is semiample for some large j and hence $\mathcal{H}$ is semiample over $B_{\mathbb{A}^1}$. Therefore, if Π is a canonical morphism from $\mathcal{X}$ to $B_{\mathbb{A}^1}$, $\Pi^*\Pi_*(\mathcal{H}^{\otimes k}) \twoheadrightarrow \mathcal{H}^{\otimes k}$ induces a morphism $f_k : \mathcal{X} \to \mathcal{P}roj_{B\times\mathbb{A}^1}(\bigoplus_{l\geq 0} \mathcal{S}ym^l \Pi_*(\mathcal{H}^{\otimes k}))$ for sufficiently large k. If $(\mathcal{Y}_k, \frac{1}{k}\mathcal{O}_{\mathcal{P}roj}(1))$ is the image of f_k, this is a fibration degeneration for π dominated by $(\mathcal{X}, \mathcal{H})$. □

Then we define the almost triviality of fibration degenerations similarly to test configurations.

Definition 4.3. A semi fibration degeneration $(\mathcal{X}, \mathcal{H})$ is called *almost trivial* if $(\mathcal{X}, \mathcal{H})$ is almost trivial as a test configuration.

Now, we define $\mathfrak{f}$-stability as follows,

Definition 4.4. Suppose that $f : (X, \Delta, H) \to (B, L)$ is a polarized algebraic fiber space pair. Let $m = \mathrm{rel.dim}\, f$ and $\dim B = n$. For any normal fibration degeneration $(\mathcal{X}, \mathcal{H})$ for f, we define constants $W_0^\Delta(\mathcal{X}, \mathcal{H})$, $W_1^\Delta(\mathcal{X}, \mathcal{H}), \cdots, W_n^\Delta(\mathcal{X}, \mathcal{H})$ and a rational function $W_{n+1}^\Delta(\mathcal{X}, \mathcal{H})(j)$ so that the partial fraction decomposition of $M_\Delta^{\mathrm{NA}}(\mathcal{X}, \mathcal{H} + jL)$ in j is as follows:

$$V(H + jL) M_\Delta^{\mathrm{NA}}(\mathcal{X}, \mathcal{H} + jL) = W_{n+1}^\Delta(\mathcal{X}, \mathcal{H})(j) + \sum_{i=0}^{n} j^i W_{n-i}^\Delta(\mathcal{X}, \mathcal{H}).$$

See also [11, Lemma 2.25]. Let $0 \leq l \leq n$ be an integer. Then $f : (X, \Delta, H) \to (B, L)$ is called

- $\mathfrak{f}_l$-*semistable* if $W_0^\Delta \geq 0$ and
$$W_0^\Delta(\mathcal{X}, \mathcal{H}) = W_1^\Delta(\mathcal{X}, \mathcal{H}) = \cdots = W_i^\Delta(\mathcal{X}, \mathcal{H}) = 0 \Rightarrow W_{i+1}^\Delta(\mathcal{X}, \mathcal{H}) \geq 0$$
for $i = 0, 1, \cdots, l-1$ for any fibration degeneration. Equivalently, f is $\mathfrak{f}_l$-semistable if $\sum_{i=0}^{l} j^{n-i} W_i^\Delta(\mathcal{X}, \mathcal{H}) \geq 0$ for sufficiently large $j > 0$,
- $\mathfrak{f}$-*semistable* if f is $\mathfrak{f}_n$-semistable,

- $\mathfrak{f}$-*stable* if f is $\mathfrak{f}$-semistable and

$$W_i^\Delta(\mathcal{X},\mathcal{H})=0,\, i=0,1,\cdots,n-1 \Rightarrow W_n^\Delta(\mathcal{X},\mathcal{H})>0$$

  for any non-almost-trivial fibration degeneration for (X,H).

Note that f is

- $\mathfrak{f}_l$-semistable if and only if $W_0^\Delta\geq 0$ and $W_0^\Delta(\mathcal{X},\mathcal{H})=W_1^\Delta(\mathcal{X},\mathcal{H})=\cdots=W_i^\Delta(\mathcal{X},\mathcal{H})=0\Rightarrow W_{i+1}^\Delta(\mathcal{X},\mathcal{H})\geq 0$ for $i\leq l-1$ for any normal test configuration $(\mathcal{X},\mathcal{H})$ dominating $X_{\mathbb{A}^1}$ such that $(\mathcal{X},\mathcal{H}+jL)$ is a semiample test configuration for sufficiently large j.
- $\mathfrak{f}$-stable if and only if f is $\mathfrak{f}$-semistable and $W_i^\Delta(\mathcal{X},\mathcal{H})=0$ for $i\leq n-1 \Rightarrow W_n^\Delta(\mathcal{X},\mathcal{H})>0$ for any non almost trivial test configuration $(\mathcal{X},\mathcal{H})$ for (X,H) dominating $X_{\mathbb{A}^1}$ such that $\mathcal{H}+jL$ is semiample for sufficiently large j.

This follows immediately from Proposition 4.2. If $\Delta=0$, we denote $W_i^\Delta=W_i$.

We remark the following trivial lemma.

Lemma 4.5. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair. Then $f:(X,\Delta,H)\to(B,L)$ is $\mathfrak{f}$-(semi)stable if and only if so is $f:(X,\Delta,H+jL)\to(B,L)$ for any $j\in\mathbb{Q}$.*

Remark 4.6. In the original definition of W_i in [9], Dervan-Sektnan used DF instead of M^{NA}. In other words, we can define $W_i'^\Delta$ of any semi fibration degeneration $(\mathcal{X},\mathcal{H})$ for f as

$$V(H+jL)\mathrm{DF}_\Delta(\mathcal{X},\mathcal{H}+jL)=W_{n+1}'^\Delta(\mathcal{X},\mathcal{H})(j)+\sum_{i=0}^n j^iW_{n-i}'^\Delta(\mathcal{X},\mathcal{H}).$$

However, $\mathfrak{f}$-stability defined by using DF coincides with the above one i.e., the positivity of $\sum_{i=0}^n j^iW_{n-i}^\Delta(\mathcal{X},\mathcal{H})$ coincides with that of $\sum_{i=0}^n j^iW_{n-i}'^\Delta(\mathcal{X},\mathcal{H})$. In fact, this follows from

$$V(H+jL)(\mathrm{DF}(\mathcal{X},(\mathcal{H}+jL))-M^{\mathrm{NA}}(\mathcal{X},(\mathcal{H}+jL)))=((\mathcal{X}_0-\mathcal{X}_{0,\mathrm{red}})\cdot(\mathcal{H}+jL)^{\dim X})\geq 0$$

and [5, Proposition 7.16]. On the other hand, if f is adiabatically K-semistable, then f is $\mathfrak{f}$-semistable.

We also remark that $\mathfrak{f}_1$-semistability coincides with the fibration semistability of [11]. On the other hand, strict $\mathfrak{f}$-stability coincides with the previous version [9] of fibration stability when B is a curve and all fibers are K-polystable rather than the one of [11]. See Remark 4.11 below.

Next, we want to calculate W_0 and W_1. For simplicity, we need the following notation,

Notation 4.7. If $f:X\to B$ is a morphism of varieties and D is a Cartier divisor on B such that $f(X)\not\subset D$, then we can define the Cartier divisor f^*D and we denote it by $X\cap D$. Otherwise, we understand $X\cap D$ be a Cartier divisor linearly equivalent to $f^*\mathcal{O}_B(D)$.

On the other hand, suppose that X is normal and let E be a Weil divisor. If E is Cartier at any point p of f^*D of $\mathrm{codim}_{f^*D}(\overline{\{p\}}\cap f^*D)\leq 1$, note that we can define the Weil divisor $E\cap D$ of $X\cap D$ (see [25, 16.3]).

Proposition 4.8. *Let $\pi:(X,H)\to(B,L)$ be a polarized algebraic fiber space. Suppose that $m=\mathrm{rel.dim}\,\pi$ and $n=\dim B\geq 2$. Let $(\mathcal{X},\mathcal{H})$ be a normal semi fibration degeneration for π. Then, for sufficiently large $k\in\mathbb{Z}$, there is $D\in|kL|$ such that $(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})$ is the test configuration of $(X\cap D,H|_{X\cap D})$ and $k\binom{n+m}{n-1}W_1(\mathcal{X},\mathcal{H})=\binom{n+m-1}{n-2}W_1(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})$.*

Proof. As in [11, p. 17], let

$$C_1(\mathcal{X},\mathcal{H}) = \frac{m}{m+2}\left(\frac{-K_X\cdot L^n\cdot H^{m-1}}{L^n\cdot H^m}\right)L^{n-1}\cdot\mathcal{H}^{m+2},$$
$$C_2(\mathcal{X},\mathcal{H}) = -\frac{m}{m+1}\left(\frac{(-K_X\cdot L^n\cdot H^{m-1})(L^{n-1}\cdot H^{m+1})}{(L^n\cdot H^m)^2}\right)L^n\cdot\mathcal{H}^{m+1},$$
$$C_3(\mathcal{X},\mathcal{H}) = \left(\frac{-K_X\cdot L^{n-1}\cdot H^m}{L^n\cdot H^m}\right)L^n\cdot\mathcal{H}^{m+1},$$
$$C_4(\mathcal{X},\mathcal{H}) = L^{n-1}\cdot\mathcal{H}^{m+1}\cdot K^{\log}_{\mathcal{X}/\mathbb{P}^1}.$$

Then, we obtain as [11]

$$W_1(\mathcal{X},\mathcal{H}) = \binom{n+m}{n-1}\sum_{j=1}^{4}C_j(\mathcal{X},\mathcal{H}).$$

The set of $D\in|kL|$ such that $X\cap D,\mathcal{X}\cap D$ and D are normal is a non-empty Zariski open subset of $|kL|$ for $k\gg 0$ by the Bertini theorem. We may assume that $k=1$ by replacing L by some multiple. Moreover, $X\cap D,\mathcal{X}\cap D,D$ are also connected by [20, III Exercise 11.3]. Furthermore, since $\mathcal{X}\cap D$ dominates $\mathbb{P}^1$, it is flat over $\mathbb{P}^1$. Thus, $(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})$ is a test configuration for $X\cap D$. We apply the adjunction formula (cf., [25, §16, §17]) to D and the pullback of this to X and obtain for example

$$(K_X+X\cap D)|_{X\cap D} = K_{X\cap D}.$$

Note that $K_{\mathcal{X}}$ and $\mathcal{X}_{0,\mathrm{red}}$ are Cartier in codimension 1 but not $\mathbb{Q}$-Cartier entirely in general. However, we can choose D so general that $K_{\mathcal{X}}$ is also Cartier on any point of codimension 1 in $\mathcal{X}\cap D$ and then the adjunction formula holds for such $\mathcal{X}\cap D$ (cf., [25, 16.4.3]). Hence, we obtain $C_1(\mathcal{X},\mathcal{H})+C_2(\mathcal{X},\mathcal{H}) = C_1(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})+C_2(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})$. Let $\Pi:\mathcal{X}\to B$ be the canonical projection. Since $\mathcal{X}\cap D=\Pi^*D$ and $X\cap D=\pi^*D$, we also have

$$C_3(\mathcal{X},\mathcal{H})+C_4(\mathcal{X},\mathcal{H}) = \left(\frac{-K_X\cdot L^{n-1}\cdot H^m}{L^n\cdot H^m}\right)L^n\cdot\mathcal{H}^{m+1}+L^{n-1}\cdot\mathcal{H}^{m+1}\cdot K^{\log}_{\mathcal{X}/\mathbb{P}^1}$$
$$= \left(\frac{-(K_{\pi^*D}-L|_{\pi^*D})\cdot L|^{n-2}_{\pi^*D}\cdot H|^m_{\pi^*D}}{L|^{n-1}_{\pi^*D}\cdot H|^m_{\pi^*D}}\right)L|^{n-1}_{\Pi^*D}\cdot\mathcal{H}|^{m+1}_{\Pi^*D}+L|^{n-2}_{\Pi^*D}\cdot\mathcal{H}|^{m+1}_{\Pi^*D}\cdot(K^{\log}_{\Pi^*D/\mathbb{P}^1}-L|_{\Pi^*D})$$
$$= \left(\frac{-K_{\pi^*D}\cdot L|^{n-2}_{\pi^*D}\cdot H|^m_{\pi^*D}}{L|^{n-1}_{\pi^*D}\cdot H|^m_{\pi^*D}}\right)L|^{n-1}_{\Pi^*D}\cdot\mathcal{H}|^{m+1}_{\Pi^*D}+L|^{n-2}_{\Pi^*D}\cdot\mathcal{H}|^{m+1}_{\Pi^*D}\cdot K^{\log}_{\Pi^*D/\mathbb{P}^1},$$

and hence we have $C_3(\mathcal{X},\mathcal{H})+C_4(\mathcal{X},\mathcal{H}) = C_3(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})+C_4(\mathcal{X}\cap D,\mathcal{H}|_{\mathcal{X}\cap D})$. □

Now, we can calculate W_0 and W_1 directly as follows. First, we need the following lemma,

Lemma 4.9 ([11, Lemma 2.33]). *Notations as in Proposition 4.8. Then*

$$W_0(\mathcal{X},\mathcal{H}) = \binom{n+m}{n}(H^m\cdot L^n)\cdot M^{\mathrm{NA}}(\mathcal{X}_b,\mathcal{H}_b)$$

holds for general $b\in B$.

Proof. For the readers' convenience, we prove this lemma here. First, we prove the following claim. Let $\mathcal{X}$ be a fibration degeneration for $\pi:X\to B$. Then a general fiber of $\mathcal{X}$ over B is a test configuration $\mathcal{X}_b$ for X_b.

It suffices to show that $\mathcal{X}_b$ is flat over $\mathbb{A}^1$. Since B is normal, $\mathcal{X}$ is flat over codimension 1 points of $B\times\mathbb{A}^1$. Therefore, $\mathcal{X}$ is also flat over $(b,0)\in B_{\mathbb{A}^1}$ for general points $b\in B$. On the other hand, by cutting by ample divisors, we may assume that $\dim B=1$ and that X is flat over B due to the same argument of the proof of Proposition 4.8 and the Bertini theorem. Therefore, $X\times(\mathbb{A}^1\setminus\{0\})$ is flat over $B\times(\mathbb{A}^1\setminus\{0\})$. Hence, for general point $b\in B$, $\mathcal{X}_b$ is flat over $\mathbb{A}^1$. Thus, the assertion follows as the proof of Proposition 4.8. □

To calculate W_1, it suffices to show the following by Proposition 4.8:

Lemma 4.10. *Let $\pi : (X, H) \to (B, L)$ be a polarized algebraic fiber space over a smooth curve B. Suppose that $\deg L = 1$. If $(\mathcal{X}, \mathcal{H})$ is a normal semi fibration degeneration for π, then*

$$W_1(\mathcal{X}, \mathcal{H}) = V(H)(M^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) + (S(X_b, H_b) - S(X, H))(E^{\mathrm{NA}}(\mathcal{H}) - E^{\mathrm{NA}}(\mathcal{H}_b))).$$

Proof. Notations as in the proof of Proposition 4.8. Then, it immediately follows that

$$\begin{aligned} C_1(\mathcal{X}, \mathcal{H}) &= V(H)E^{\mathrm{NA}}(\mathcal{H})S(X_b, H_b) \\ C_2(\mathcal{X}, \mathcal{H}) &= -V(H)E^{\mathrm{NA}}(\mathcal{H}_b)S(X_b, H_b) \\ C_3(\mathcal{X}, \mathcal{H}) &= V(H)E^{\mathrm{NA}}(\mathcal{H}_b)S(X, H) \\ C_4(\mathcal{X}, \mathcal{H}) &= V(H)(M^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) - E^{\mathrm{NA}}(\mathcal{H})S(X, H)) \end{aligned}$$

from assigning $n = 1$. □

For general cases, we calculate W_1 by cutting B by general divisors in $|L|_{\mathbb{Q}}$. Therefore, we conclude that the original fibration stability defined by Dervan and Sektnan [9] is the property of codimension 1 points of the base B. See also Example 4.18.

Remark 4.11. In [11, Definitions 2.26, 2.28], the fibration stability is defined as the positivity of $W_0(\mathcal{X}, \mathcal{H})$ and $W_1(\mathcal{X}, \mathcal{H})$ for fibration degeneration $(\mathcal{X}, \mathcal{H})$ whose minimum norm

$$\|(\mathcal{X}, \mathcal{H})\|_m = \frac{L^n \cdot \mathcal{H}^{m+1}}{m+1} + L^n \cdot \mathcal{H}^m \cdot (\mathcal{H} - H_{\mathbb{P}^1})$$

(cf., [11, Definition 2.27]) does not vanish. Here, the notations as in Proposition 4.8. It is easy to see that a normal fibration degeneration that has the minimum norm $\|(\mathcal{X}, \mathcal{H})\|_m = 0$ is not necessarily trivial but general fibers are trivial (see [11, Remark 2.30]). In this respect, $\mathfrak{f}$-stability seems to be different from fibration stability even when the base is a curve. Indeed, they are different at least in the logarithmic case as we see in Example 4.21 below.

On the other hand, we obtain the following when $\mathcal{X}$ is flat over $B \times \mathbb{A}^1$:

Proposition 4.12. *Let $\pi : (X, H) \to (B, L)$ be a flat polarized algebraic fiber space. Let $(\mathcal{X}, \mathcal{H})$ be a normal fibration degeneration for π such that $(\mathcal{X} \cap D, \mathcal{H}|_{\mathcal{X} \cap D})$ is trivial for general $D \in |mL|$ and for sufficiently large $m \gg 0$. If the canonical morphism $\Pi : \mathcal{X} \to B \times \mathbb{A}^1$ is flat, then $(\mathcal{X}, \mathcal{H})$ is the trivial test configuration.*

Proof. We may assume that $\mathcal{X} \cap D$ is trivial for general member D of $|mL|$ and for $m \gg 0$. This means that $\mathcal{X} \cap D \cong X_{\mathbb{A}^1|D} = D_{\mathbb{A}^1}$ via the canonical birational map $\phi : \mathcal{X} \dashrightarrow X \times \mathbb{A}^1$. By the assumption, $(\mathcal{X}, \mathcal{H} + jL)$ is an ample test configuration for all $j \gg 0$, and hence it suffices to show that $\mathcal{X}$ and $X \times \mathbb{A}^1$ are isomorphic in codimension 1 by the fact that if $(\mathcal{X}, \mathcal{H} + jL)$ is almost trivial and normal then it is trivial. Let $a_i \in \mathbb{N}$, D_i be the irreducible divisors and $\mathcal{X}_0 = \sum a_i D_i$ be the central fiber over $\mathbb{A}^1$. Note that $p_0 : \mathcal{X}_0 \to B$ is a flat proper morphism and hence any D_i dominates B via p_0. For general D, $\mathcal{X}_0 \cap D = \sum a_i (D_i \cap D)$ is the cycle decomposition and coincides with D as a cycle via the identification $\mathcal{X} \cap D \cong D_{\mathbb{A}^1}$. Therefore, $\mathcal{X}_0$ is integral.

By the valuative criterion for properness [20, II Theorem 4.7], we get a rational map $\psi : \mathcal{X}_0 \dashrightarrow X$ as the restriction of ϕ. Let Γ be the normalized graph of ϕ and $p : \Gamma \to \mathcal{X}$, $q : \Gamma \to X$ be the canonical projections respectively. Let Γ_0 be the central fiber of Γ over $0 \in \mathbb{A}^1$ and E_0 be the irreducible component of Γ_0 corresponding to $\mathcal{X}_0$. We have to show that $q|_{E_0}$ is birational. Take the largest open subset $U \subset \mathcal{X}$ such that U has a morphism $\iota : U \to \Gamma$ that is the local inverse of p. Let U_0 be the central fiber of U over $\mathbb{A}^1$. Note that $U_0 \neq \emptyset$ since $\mathrm{codim}(\mathcal{X} \setminus U) \geq 2$.

Take $D \in |mL|$ such that $\mathcal{X} \cap D$, $\mathcal{X}_0 \cap D$, $X \cap D$, $E_0 \cap D$ and $\Gamma \cap D$ are normal by the theorem of Bertini. We may assume that $\mathcal{X} \cap D$, $\mathcal{X}_0 \cap D$, $X \cap D$, $E_0 \cap D$ and $\Gamma \cap D$ are also connected. Indeed, if $\dim B \geq 2$, they are connected. Otherwise, we may assume that $\deg D = 1$ and take a general fiber over B. We remark that $E_0 \cap D \neq \emptyset$ because E_0 dominates B. Since $\mathcal{X} \cap D$ and $D_{\mathbb{A}^1}$ are isomorphic, $U \cap D$ and $\Gamma \cap D$ are birational equivalent. Assume that $q|_{E_0} : E_0 \to X$ is not a birational morphism. Assume also that $q|_{E_0}$ is not dominant, then $q|_{E_0 \cap D}$ is not birational for general D. This contradicts to that $U_0 \cap D$ is embedded into X via $q \circ \iota$. Hence $q|_{E_0}$ is dominant. Since we assume that $q|_{E_0}$ is not birational, $\deg(q|_{E_0}) > 1$. This contradicts to $\deg(q|_{E_0}) = \deg(q|_{E_0 \cap D}) = 1$. Therefore, $q|_{E_0} : E_0 \to X \times \{0\}$ is a birational morphism. □

To prove Theorem 5.2 below, we need to calculate W_i^Δ of an arbitrary normalized semi fibration degeneration $(\mathcal{X}, \mathcal{H})$ for π and $i \geq 2$ as in Proposition 4.8. Indeed, we can prove the logarithmic version of Proposition 4.8 similarly. However, the coefficients of W_i^Δ are more complicated when $i \geq 2$ and we want to ignore the effects of points on B of higher codimension.

Lemma 4.13. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair with an ample line bundle H and $(\mathcal{X}, \mathcal{H})$ be a normal semiample test configuration for (X, H) dominating $X_{\mathbb{A}^1}$. Suppose that $m = \mathrm{rel.dim}\, f$, $n = \dim B$ and $(\mathcal{X}, \mathcal{H})$ is normalized with respect to the central fiber. Then, there exist constants $C_{n-i+1}^{(i)}, \cdots, C_n^{(i)}$ independent of $(\mathcal{X}, \mathcal{H})$ such that*

$$W_i^\Delta(\mathcal{X}, \mathcal{H}) = M^{-(n-i)} \binom{n+m}{n-i} \big(K^{\log}_{(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-i}, \Delta_{\mathcal{X}} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-i})/\mathbb{P}^1} \cdot \mathcal{H}|^{m+i}_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-i}}$$
$$- S(X_b, \Delta_b, H_b) J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-i}}) \big) + \sum_{k=n-i+1}^{n} C_k^{(i)} J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_k}),$$

where $D_1, D_2, \cdots, D_n \in |ML|$ are general ample divisors for sufficiently divisible integer $M > 0$ that ML is very ample. Here, $\Delta_{\mathcal{X}}$ is the strict transform of $\Delta \times \mathbb{P}^1$ on $\mathcal{X}$.

Notation 4.14. If there is no fear of confusion, we will denote

$$J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j}) = J^{\mathrm{NA}}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j, \mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j}).$$

If $j = \dim B$, we understand $J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j}) = V(ML) J^{\mathrm{NA}}(\mathcal{H}|_b)$ for general b.

For simplicity, we say that $(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j, \mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j})$ is trivial if one of the following holds.

- $j > \dim B$,
- $j = \dim B$ and any irreducible component of $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j$ is the trivial test configuration for the corresponding fiber of f, or
- $j < \dim B$ and $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j$ is trivial.

If $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-i+1}$ is trivial, then $J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_k}) = 0$ for $k \geq n-i+1$ and hence it is easy to calculate W_i^Δ by Lemma 4.13. To show our main results, it suffices to consider only such cases.

Proof of Lemma 4.13. Note that $(\mathcal{X}, \mathcal{H} + jL)$ is also normalized with respect to the central fiber (as a test configuration for $(X, H + jL)$) by what we explained after Definition 2.15, and so is $(\mathcal{X} \cap D_1, \mathcal{H}|_{\mathcal{X} \cap D_1})$. We have that

$$S(X, \Delta, H + jL) = -(n+m) \frac{K_{(X,\Delta)} \cdot (H + jL)^{n+m-1}}{(H+jL)^{n+m}}$$
$$= S(X_b, \Delta_b, H_b) + O(j^{-1}),$$

where the coefficients of $O(j^{-1})$ depend only on n, m, H and L. Note also that the j^{n-i}-term of $(\mathcal{H} + jL)^{n+m+1}$ is

$$\binom{n+m+1}{n-i} \mathcal{H}^{m+i+1} \cdot L^{n-i} = -M^{i-n}(n+m+1) \binom{n+m}{n-i} J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-i}}).$$

Here, we used Lemma 2.16. Thus, there exist constants $E_j^{(i)}$ independent of $(\mathcal{X},\mathcal{H})$ such that

$$S(X,\Delta,H+jL)E^{\mathrm{NA}}(\mathcal{H}+jL)=\sum_i j^{n-i}M^{i-n}\binom{n+m}{n-i}S(X_b,\Delta_b,H_b)J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-i}})$$
$$+\sum_i\sum_{k=n-i+1}^{n} j^{n-i}E_k^{(i)}J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_k}).$$

On the other hand, take $D_i\in|ML|$ so general that $K_{\mathcal{X}\cap D_1\cap\cdots\cap D_{i-1}}$, $\mathcal{X}_{0,\mathrm{red}}\cap D_1\cap\cdots\cap D_{i-1}$ and $\Delta_{\mathcal{X}}\cap D_1\cap\cdots\cap D_{i-1}$ are Cartier at any codimension 1 point of $\mathcal{X}\cap D_1\cap\cdots\cap D_i$. By the adjunction formula [25, §16], for example, we have

$$K^{\log}_{(\mathcal{X},\Delta_{\mathcal{X}})/\mathbb{P}^1}|_{\mathcal{X}\cap D_1}=K^{\log}_{(\mathcal{X}\cap D_1,\Delta_{\mathcal{X}}\cap D_1)/\mathbb{P}^1}-\mathcal{X}\cap D_1|_{\mathcal{X}\cap D_1},$$

as in the proof of Proposition 4.8 and hence we obtain the assertion. Indeed, if $k=n-i+1$, then $C_k^{(i)}=E_k^{(i)}+(n-i)M^{-(n-i)}$. Otherwise, $C_k^{(i)}=E_k^{(i)}$. □

On the other hand, we calculate W_i as Lemma 4.13 when there exists a line bundle L_0 on B such that $K_{(X,\Delta)}\equiv\lambda H+f^*L_0$ for $\lambda\in\mathbb{Q}$.

Proposition 4.15. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair with $m=\mathrm{rel.dim}\,f$ and $n=\dim B$. Furthermore, suppose that H is ample and L is very ample. Let $(\mathcal{X},\mathcal{H})$ be an arbitrary normal semiample test configuration for (X,H) normalized with respect to the central fiber. Suppose also that $\mathcal{X}$ dominates $X_{\mathbb{A}^1}$, that there exist $\lambda\in\mathbb{Q}$ and a line bundle M on B such that $\lambda H\equiv K_X+\Delta+f^*M$ and that $(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k+1},\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k+1}})$ is trivial for general $D_1,\cdots,D_{n-k+1}\in|L|$. Then*

$$\left(\binom{n+m}{n-k}(H^{m+k}\cdot L^{n-k})\right)^{-1}W_k^{\Delta}(\mathcal{X},\mathcal{H})=H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$$
$$+\lambda\big(I^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})-(k+1)J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})\big).$$

Proof. It follows from Lemma 2.16 that

$$J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=-E^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}).$$

Note that $J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k+s}})=0$ for $s\geq 1$ for general $D_j\in|L|$ by the assumption in the sense of Notation 4.14. Thus we have

$$\begin{aligned}(H^{m+k}\cdot L^{n-k})&R^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})\\
&=p^*K_{(X,\Delta)}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}\cdot(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})^{m+k}\\
&=p^*(\lambda H-f^*M)\cdot(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})^{m+k}\\
&=\lambda\,p^*H\cdot(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})^{m+k},\end{aligned}$$

where $p:\mathcal{X}\to X$ is the composition of the canonical morphisms $\mathcal{X}\to X_{\mathbb{A}^1}$ and $X_{\mathbb{A}^1}\to X$. Here, we apply Lemma 4.16 below to $(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$ and obtain the last equality. Then, we have as [5, Lemma 7.25],

$$p^*H\cdot(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})^{m+k}=(H^{m+k}\cdot L^{n-k})\Big(I^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$$
$$-(m+k+1)J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})\Big)$$

and $S(X_b,\Delta_b,H_b)=-m\lambda$. Therefore, the assertion follows from Lemma 4.13. □

Lemma 4.16. *Notations as in Proposition 4.15. If $(\mathcal{X}\cap D_1,\mathcal{H}|_{\mathcal{X}\cap D_1})$ is trivial for a general ample divisor $D_1\in|L|$, then for any line bundle M on B,*

$$\mathcal{H}^{n+m}\cdot p^*f^*M=0.$$

Proof. Let $F = H_{\mathbb{A}^1} - \mathcal{H}$. Since $\mathcal{H}$ is normalized, F is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and note that $\mathcal{H}|_F$ and $H_{\mathbb{P}^1}|_F$ are nef. By assumption, we have

$$0 = \mathcal{H}^{n+m} \cdot p^*f^*L = -\sum_{i=0}^{n+m-1} F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\mathbb{P}^1}^i \cdot p^*f^*L$$

since $H_{\mathbb{P}^1}^{n+m} \cdot p^*f^*L = 0$. On the other hand, $p^*f^*L|_F$ is also nef and hence $F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\mathbb{P}^1}^i \cdot p^*f^*L = 0$ for $0 \le i \le n+m-1$. Take $k \gg 0$ such that $kL \pm M$ are very ample on B. Then $\pm F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\mathbb{P}^1}^i \cdot p^*f^*M = F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\mathbb{P}^1}^i \cdot p^*f^*(kL \pm M) \ge 0$. Therefore,

$$\sum_{i=0}^{n+m-1} F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\mathbb{P}^1}^i \cdot p^*f^*M = -\mathcal{H}^{n+m} \cdot p^*f^*M = 0.$$

Hence, we obtain Lemma 4.16 and Proposition 4.15. □

Next, we discuss [11, Corollary 2.35]. Dervan and Sektnan conjectured that fibrations whose every fiber is K-stable are fibration stable in the original sense [9]. They also conjectured the following.

Conjecture 4.17 (Dervan-Sektnan [11])**.** *Let $f : (X, H) \to (B, L)$ be a polarized smooth fibration such that any fiber is K-stable. If $\mathrm{Aut}(X_b, H_b) = \{\mathrm{id}\}$ for any $b \in B$, f is fibration stable in the original sense* [9].

In fact, they proved such fibrations are fibration stable in the sense of [11]. However, there exists a following singular example for $\mathfrak{f}$-stability.

Example 4.18. Suppose that B is a projective cone of an elliptic curve and C is a proper smooth curve whose genus $g(C) > 2$ with no automorphism other than the identity (cf., [1]). Let $X = C \times B$ be an algebraic fiber space over B. Note that X has the relative canonical ample divisor over B. Let L be an ample line bundle on B and $H = K_{X/B} + rL$ for sufficiently large $r > 0$. Since B has an isolated singularity b_0 (cf., [24, Proposition 3.14]), X has a lc center of codimension 2. Take the deformation to the normal cone $(\mathcal{X}, \mathcal{H})$ of $C \times \{b_0\}$. Then $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$ by Proposition 4.8. We can compute $W_2(\mathcal{X}, \mathcal{H})$ as follows. Note that $H^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) = 0$ and $R^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) + S(X_b, K_{X_b})E^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) = I^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) - 3J^{\mathrm{NA}}(\mathcal{X}, \mathcal{H})$ by Proposition 4.15. Let $E = H_{\mathbb{P}^1} - \mathcal{H}$ be the exceptional divisor whose center has codimension 3 in $X \times \mathbb{P}^1$. We see that $H_{\mathbb{P}^1}^4 = H_{\mathbb{P}^1}^3 \cdot E = H_{\mathbb{P}^1}^2 \cdot E^2 = 0$. Then, we have

$$V(H)I^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) = E \cdot \mathcal{H}^3 = 3E^3 \cdot H_{\mathbb{P}^1} - E^4,$$
$$-V(H)J^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) = \frac{1}{4}\mathcal{H}^4 = -E^3 \cdot H_{\mathbb{P}^1} + \frac{1}{4}E^4.$$

Therefore,

$$V(H)(I^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}) - 3J^{\mathrm{NA}}(\mathcal{X}, \mathcal{H})) = -\frac{1}{4}E^4.$$

Since E is the pullback of the exceptional divisor on the blow-up of $B \times \mathbb{P}^1$, we have $E^4 = 0$. Hence, $(X, H) \to (B, L)$ is not $\mathfrak{f}$-stable. It also follows that Conjecture 4.17 does not hold in the original sense or for $\mathfrak{f}$-stability in the logarithmic case.

With this in mind, we define $\mathfrak{f}_l$-stability as follows.

Definition 4.19. Notations as Definition 4.4. Then $f : (X, \Delta, H) \to (B, L)$ is called *$\mathfrak{f}_k$-stable* if f is $\mathfrak{f}_k$-semistable and if $W_i^{\Delta}(\mathcal{X}, \mathcal{H}) = 0$ for $0 \le i \le k$, then $(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}})$ is trivial for any normal fibration degeneration $(\mathcal{X}, \mathcal{H})$ in the sense of Notation 4.14. Note that f is $\mathfrak{f}$-stable if and only if f is $\mathfrak{f}_i$-stable for $0 \le i \le n$.

Now we can show that klt Calabi-Yau fibrations are $\mathfrak{f}$-stable as the well-known theorem in K-stability [32, Theorem 4.1]. Indeed, we prove the following refinement of Theorem C.

Theorem 4.20. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair with a line bundle L_0 on B. Fix an integer $0\le k\le n=\dim B$ and let $Z\subset B$ be a closed subset of $\mathrm{codim}_B Z>k$ or $Z=\emptyset$. Suppose that $(X\setminus f^{-1}(Z),\Delta|_{X\setminus f^{-1}(Z)})$ is klt and $K_X+\Delta\equiv f^*L_0$. Then f is $\mathfrak{f}_l$-stable for $0\le l\le k$.*

Proof. We prove the assertion by induction on l. First, note that f is $\mathfrak{f}_0$-stable since general fibers of f are klt Calabi-Yau pairs (cf. [32, Theorem 4.1]). Assume that f is $\mathfrak{f}_l$-stable for $l<k$. Let $(\mathcal{X},\mathcal{H})$ be a normal semi fibration degeneration for f and $m=\mathrm{rel.dim}\,f$. Note that

$$M^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{H}+jL)=M^{\mathrm{NA}}_\Delta(\mathcal{X},\mathcal{H}+jL+d\mathcal{X}_0)$$

for any $d\in\mathbb{Q}$. By replacing H by $H+jL$ for $j\gg0$, we may assume that H is ample and $(\mathcal{X},\mathcal{H})$ is a normal semiample test configuration normalized with respect to the central fiber (cf. Lemma 4.5). Suppose that $(\mathcal{X}\cap D_1\cap\cdots\cap D_{n-k+l},\mathcal{H}|_{\mathcal{X}\cap D_1\cap\cdots\cap D_{n-k+l}})$ is trivial for general divisors $D_i\in|ML|$ for $l>0$ and $M\gg0$. By Proposition 4.15

$$\binom{n+m}{n-k}^{-1}M^{-(n-k)}W^\Delta_k(\mathcal{X},\mathcal{H})=(H^{m+k}\cdot L^{n-k})H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$$

for general $D_{n-k}\in|ML|$. It is easy to see that if $(X\setminus f^{-1}(Z),\Delta|_{X\setminus f^{-1}(Z)})$ is klt, then $(X\cap D_1\cap\cdots\cap D_{n-k},\Delta\cap D_1\cap\cdots\cap D_{n-k})$ is klt for general divisors $D_i\in|ML|$ and hence $H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})>0$ unless $J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=0$ by [5, Proposition 9.16]. □

Proof of Theorem C. It immediately follows from Theorem 4.20. □

On the other hand, we construct the following example as we stated in Remark 4.11.

Example 4.21. Let B be an elliptic curve and $f:X=B\times B\to B$ be the second projection. Fix a closed point $b\in B$ and let $\Delta=B\times\{b\}$. Then, it is easy to see that for any ample line bundles H on X and L on B, $f:(X,\Delta,H)\to(B,L)$ is strictly $\mathfrak{f}$-semistable as the proof of Theorem 4.20. Indeed, if $(\mathcal{X},\mathcal{H})$ is the deformation to the normal cone of Δ, then $W^\Delta_0(\mathcal{X},\mathcal{H})=W^\Delta_1(\mathcal{X},\mathcal{H})=0$. However, for any fibration degeneration $(\mathcal{X},\mathcal{H})$, $W^\Delta_0(\mathcal{X},\mathcal{H})=0$ if and only if $\|(\mathcal{X},\mathcal{H})\|_m=0$ in this case. Thus, f is fibration stable in the sense of [11]. Therefore, fibration stability and $\mathfrak{f}$-stability are different at least in the logarithmic setting.

5. Singularities of semistable algebraic fiber spaces

In this section, we see that $\mathfrak{f}$-semistability implies that X has only lc singularities similarly to [31]. Moreover, we obtain Theorem 5.3, which states that K-semistable Fano fibrations have no horizontal non-klt singularities. In the latter part of this section, we consider the case when X is non-normal.

5.1. **Normal case.** First, we prepare the following lemma.

Lemma 5.1. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair and Z be a closed subscheme of X with the corresponding ideal $\mathfrak{a}_Z$. Suppose that $\pi:\hat{X}\to X$ is the normalized blow up along Z, E is the inverse image of Z and $\mathrm{Rees}_X(Z)=\{\frac{\mathrm{ord}_{E_i}}{\mathrm{ord}_{E_i}(\mathfrak{a}_Z)}\}_{i=1}^r$.*

If $\dim B>1$, then the following hold for any general divisor $D\in|mL|$ and for sufficiently divisible $m\in\mathbb{Z}_{>0}$.

(1) $\pi|_{\hat{X}\cap D}:\hat{X}\cap D\to X\cap D$ is the normalized blow up along $Z\cap D$,
(2) $\pi|^{-1}_{\hat{X}\cap D}(\mathfrak{a}_Z\cdot\mathcal{O}_{X\cap D})=\mathcal{O}_{\hat{X}}(-E)|_{\hat{X}\cap D}$,
(3) $\mathrm{Rees}_D(Z\cap D)$ is the set of valuations v denoted as $\frac{\mathrm{ord}_F}{\mathrm{ord}_F(\mathfrak{a}_Z\cdot\mathcal{O}_{X\cap D})}$, where F is an irreducible component of $E_i\cap D$ for some i, and
(4) if v is as above and $A_{(X,\Delta)}(E_i)\le0$ (resp. <0), then $A_{(X\cap D,\Delta\cap D)}(v)\le0$ (resp. <0).

If $\dim B=1$, $m=1$ and L is a line bundle of $\deg L=1$, the above assertions also hold.

Proof. Note that D, $X \cap D$ and $\hat{X} \cap D$ are normal and irreducible for general D by the Bertini theorem. Since f is an algebraic fiber space, so is $f|_{X\cap D}$. Furthermore, it is easy to see that $\hat{X} \cap D$ is isomorphic to the normalized blow up of $X \cap D$ along $Z \cap D$ by [20, II, Corollary 7.15]. Furthermore, we may assume that any irreducible component of $E_i \cap D$ is a $\pi|_{\hat{X}\cap D}$-exceptional divisor by taking D so general. Here, it is easy to see that (1), (2) and (3) hold.

Next, we show the assertion (4). Since $X \cap D$ (resp. $\hat{X} \cap D$) is a normal Cartier divisor of X (resp. $\hat{X}$) and

$$K_{\hat{X}} + \hat{X} \cap D = \pi^*(K_X + \Delta + X \cap D) + \sum_i (A_{(X,\Delta)}(E_i) - 1)E_i,$$

we have by the adjunction formula (cf., [25, 16.3, 16.4, 17.2]),

$$K_{\hat{X}\cap D} = \pi|^*_{\hat{X}\cap D}(K_{X\cap D} + \Delta \cap D) + \sum_i (A_{(X,\Delta)}(E_i) - 1)(E_i \cap D).$$

Indeed, note that $E_i \cap D$'s are reduced divisors (maybe reducible) and any $\pi|_{\hat{X}\cap D}$-exceptional divisor coincides with one of irreducible components of $E_i \cap D$ by Bertini's theorem. We remark that the above adjunction formula indeed holds for sufficiently general D that each E_i or irreducible component of Δ is Cartier at codimension 1 points of $\hat{X} \cap D$. Therefore, we have

$$A_{(X\cap D,\Delta\cap D)}(F) = A_{(X,\Delta)}(E_i),$$

for any irreducible component F of $E_i \cap D$. Thus, we conclude that (4) holds. □

We prove the following generalization of Theorem A in terms of $\mathfrak{f}_k$-semistability.

Theorem 5.2. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair. Fix an integer $0 \le k \le n = \dim B$. If f is $\mathfrak{f}_k$-semistable, (X, Δ) has no non-lc center $\eta \in X$ such that $\mathrm{codim}_B f(\overline{\{\eta\}}) \le k$.*

Proof. Assume that $m = \mathrm{rel.dim}\, f$ and that (X, Δ) has a non-lc center η such that $\mathrm{codim}_B f(\overline{\{\eta\}}) \le k$. It suffices to show that then f is not $\mathfrak{f}_k$-semistable. Thus, we may further assume that $\mathrm{codim}_B f(\overline{\{\xi\}}) \ge k$ for any non-lc center ξ of (X, Δ).

First, we treat the case when $\eta \notin \mathrm{Supp}(\Delta_{>1})$. Then, by Theorem 3.3, there exists the lc modification $\hat{X}$ of $(X \setminus \mathrm{Supp}(\Delta_{>1}), \Delta \setminus \mathrm{Supp}(\Delta_{>1}))$. Thus, there exists a closed subscheme $Z \subset X \setminus \mathrm{Supp}(\Delta_{>1})$ such that every valuation $v \in \mathrm{Rees}_{X\setminus\mathrm{Supp}(\Delta_{>1})}(Z)$ satisfies $A_{(X,\Delta)}(v) < 0$ by Corollary 3.4 and $\pi : \hat{X} \to X \setminus \mathrm{Supp}(\Delta_{>1})$ is the blow up along Z. Let $\overline{Z}$ be the Zariski closure of Z in X in the sense of Notation 2.1. It is easy to see that the normalization of the deformation to the normal cone $\mathcal{X}$ of $\overline{Z}$ is a fibration degeneration for f. Let $\rho : \mathcal{X} \to X \times \mathbb{A}^1$ be the canonical projection, E be the inverse image of $\overline{Z} \times \{0\}$ and $\mathcal{H}_\epsilon = \rho^* H_{\mathbb{A}^1} - \epsilon E$ for sufficiently small $\epsilon > 0$. We may assume that H is ample and there exists a positive constant $\epsilon_0 > 0$ such that $(\mathcal{X}, \mathcal{H} = \mathcal{H}_{\epsilon_0})$ is a normal ample test configuration by replacing H by $H + jL$ for $j \gg 0$ (cf., Lemma 4.5). Hence, $(\mathcal{X}, \mathcal{H}_\epsilon)$ is an ample test configuration for $0 < \epsilon < \epsilon_0$. Note that $(\mathcal{X}, \mathcal{H}_\epsilon)$ is normalized with respect to the central fiber. Now, we prove that for sufficiently small $0 < \epsilon < \epsilon_0$, $W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) < 0$ but $W_i^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = 0$ for $i < k$. Take $D_1, D_2, \cdots, D_n \in |ML|$ general for sufficiently divisible integer $M > 0$. By replacing L by ML, we may assume that $M = 1$ as in Lemma 4.13. Then $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k+1}$ is trivial in the sense of Notation 4.14. Therefore, $W_i^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = 0$ for $i < k$. By Lemma 4.13,

$$\begin{aligned}
W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) &= \binom{n+m}{n-k} \left(K^{\log}_{(\mathcal{X},\rho_*^{-1}\Delta_{\mathbb{P}^1})/\mathbb{P}^1} \cdot L^{n-k} \cdot \mathcal{H}_\epsilon^{m+k} + \frac{S(X_b, \Delta_b, H_b)}{m+k+1} \mathcal{H}_\epsilon^{m+k+1} \cdot L^{n-k} \right) \\
&= \binom{n+m}{n-k} (L^{n-k} \cdot H^k) \big(H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) \\
&\quad + R^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) \\
&\quad + S(X_b, \Delta_b, H_b) E^{\mathrm{NA}}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) \big).
\end{aligned}$$

By the assumption, $E\cap D_1\cap D_2\cap\cdots\cap D_{n-k}\neq\emptyset$ and $(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$ is a non-trivial normal test configuration. Let $\{F_i\}_{i=1}^q$ be the set of prime exceptional divisors of $\hat{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}\to X\cap D_1\cap D_2\cap\cdots\cap D_{n-k}$ and suppose that

$$\min_{1\le i\le q}\mathrm{codim}_{X\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}\rho(F_i)=r.$$

Then, we see by [5, Proposition 9.12] that

$$R^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=O(\epsilon^{r+1}),\quad\text{and}$$
$$E^{\mathrm{NA}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=O(\epsilon^{r+1})$$

since $(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$ is also normalized with respect to the central fiber. On the other hand, there exists a positive constant $T>0$ such that

$$H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=-T\epsilon^r+O(\epsilon^{r+1})$$

since if $\mathrm{codim}_{X\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}\rho(F_i)=r$, then the center of F_i is not contained in $\mathrm{Supp}(\Delta_{>1})$ and therefore $A_{(X\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k})}(F_i)<0$ by Theorem 3.5 and Lemma 5.1. Thus, the assertion in this case holds.

Otherwise, replacing η by the generic point of some irreducible component of $\Delta_{>1}$ if necessary, we may assume that $\Delta=\sum a_iF_i$ for some $a_j>1$ and $\mathrm{codim}_Bf(F_j)\le k$, where F_i are distinct irreducible components. Fix such F_j. Let $(\mathcal{X},\mathcal{H}_\epsilon)$ be the deformation to the normal cone of F_j and $\mathcal{H}_\epsilon=H_{\mathbb{P}^1}-\epsilon E$ where E is the exceptional divisor for sufficiently small $\epsilon>0$. Then by the argument in the previous paragraph, $W^\Delta_i(\mathcal{X},\mathcal{H}_\epsilon)=0$ for $i<k$ and there exists a positive constant $T>0$ such that

$$H^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=-T\epsilon+O(\epsilon^2)$$

since for any $v\in\mathrm{Rees}_X(F_j)\setminus\{\mathrm{ord}_{F_j}\}$, $\mathrm{codim}_X\overline{\{c_X(v)\}}\ge 2$ even if F_j is not a $\mathbb{Q}$-Cartier divisor. On the other hand,

$$R^{\mathrm{NA}}_{\Delta\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=O(\epsilon^2)$$
$$E^{\mathrm{NA}}(\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k},\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})=O(\epsilon^2).$$

Thus, $W^\Delta_k(\mathcal{X},\mathcal{H}_\epsilon)<0$ for sufficiently small $\epsilon>0$. We complete the proof. □

Proof of Theorem A. It immediately follows from Theorem 5.2. □

Note that Theorem B holds by the following theorem.

Theorem 5.3. *Let $f:(X,\Delta,H)\to(B,L)$ be a polarized algebraic fiber space pair. Fix an integer $0\le k\le n=\dim B$. Suppose that there exist $\lambda\in\mathbb{Q}_{>0}$ and a line bundle L_0 on B such that $H+f^*L_0\equiv-\lambda(K_X+\Delta)$, and that f is $\mathfrak{f}_k$-semistable. Then any lc-center $C\subset X$ of (X,Δ) is of fiber type or $\mathrm{codim}_Bf(C)>k$.*

Proof of Theorem B. Note that an irreducible closed subset $Z\subset X$ is not of fiber type if and only if Z does not contain any irreducible component of any fiber of f since f is flat. Due to Theorem 5.3, it suffices to show that any non-klt center of X does not contain any irreducible component of $X_b=f^{-1}(b)$ for any $b\in B$ if B is klt. This fact immediately follows from Proposition 5.4 below. The last assertion of Theorem B follows from the fact that adiabatic K-semistability implies $\mathfrak{f}$-semistability (see Remark 4.6). □

Proposition 5.4. *Let $f:X\to B$ be a flat algebraic fiber space whose all fibers are reduced. Suppose that X is proper. If B has only klt singularities, there exists no lc center $C\subset X$ that contains any irreducible component of X_b for any $b\in B$.*

To show this, we need the following.

Lemma 5.5. *Let $f:X\to B$ be a smooth surjective morphism of normal varieties. If B is klt, so is X.*

Proof. Note first that $K_{X/B}$ is a Cartier divisor since f is smooth. Thus, it is easy to see that $K_X = K_{X/B} + f^*K_B$. Let $\pi : B' \to B$ be a resolution of singularities of B and $(X', \Delta') = (X \times_B B', X \times_B \Delta)$ where the exceptional set $\mathrm{Ex}(\pi) = \Delta$ is snc. Note that (X', Δ') is log smooth since $f' : X' \to B'$ induced by f is also a smooth morphism whose all fibers are connected. Note also that for any two prime divisors $D \neq D'$ on B', $f'^*(D) = \lfloor f'^*(D) \rfloor$ and $f'^*(D)$ and $f'^*(D')$ have no common component due to the property of f'. Let $\mu : X' \to X$ be the canonical projection. Since f' is the base change of f, $\mu^*\Omega_{X/B} = \Omega_{X'/B'}$ and hence $\mu^*K_{X/B} = K_{X'/B'}$. On the other hand, $K_{B'} + F = \pi^*K_B + E$ where E and F are effective divisors that have no common components and $\lfloor F \rfloor = 0$. Therefore,

$$K_{X'} + f'^*(F) = \mu^*K_X + f'^*(E)$$

where $f'^*(F)$ and $f'^*(E)$ are effective divisors that have no common components and $\lfloor f'^*(F) \rfloor = 0$. Thus, X has only klt singularities. □

Proof of Proposition 5.4. Note that X_b is generically smooth for any $b \in B$. Since f is faithfully flat, we conclude that there exists a closed subset Z such that f is smooth on $X \setminus Z$ and Z contains no component of X_b for any $b \in B$. Hence, the assertion follows from Lemma 5.5. □

In the proof of [31, Theorem 1.3], Odaka applied [3, 1.4.3]. To prove Theorem 5.3, it is necessary to blow only up lc centers of non-fiber type and we can not make use of the same argument directly. Hence, we need the slight modification of the technique developed for proving [31, Theorem 1.3] and [5, Proposition 9.9] as follows.

Proposition 5.6. *Let (X, Δ) be an lc pair and E_j be prime divisors over X such that $A_{(X,\Delta)}(E_j) = 0$ for $1 \leq j \leq r$. Then there exists a closed subscheme $Z \subset X$ such that*

$$\emptyset \neq \mathrm{Rees}_X(Z) \subset \left\{ \frac{\mathrm{ord}_{E_j}}{\mathrm{ord}_{E_j}(Z)} \,\middle|\, \mathrm{ord}_{E_j}(Z) \neq 0 \right\}_{j=1}^{r}.$$

Furthermore, suppose that one of the following holds:

(1) *$r = 1$ and E_1 is exceptional over X,*
(2) *Any E_j is a divisor on X.*

Then, we have

$$\mathrm{Rees}_X(Z) = \left\{ \frac{\mathrm{ord}_{E_j}}{\mathrm{ord}_{E_j}(Z)} \right\}_{j=1}^{r}.$$

Proof. Let $f : \tilde{X} \to (X, \Delta)$ be a log resolution such that E_j's are smooth divisors on $\tilde{X}$. Suppose that

$$K_{\tilde{X}} + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + \sum_i A_{(X,\Delta)}(F_i)F_i,$$

where F_i are irreducible components of f-exceptional divisors and $F = \sum_i F_i$. By assumption, $A_{(X,\Delta)}(F_i) \geq 0$. Then by the proof of [15, 4.1], we can conclude that the $K_{\tilde{X}} + f_*^{-1}\Delta + F$-MMP with scaling over X terminates with a $\mathbb{Q}$-factorial dlt minimal model $(X_1, (f_1^{-1})_*\Delta + D_1)$. Here, let $f_1 : X_1 \to X$ be the structure morphism and D_1 be the strict transform of F. Note that any F_i such that $A_{(X,\Delta)}(F_i) \neq 0$ is contracted but any E_j is not contracted on X_1. Then

$$K_{X_1} + (f_1)_*^{-1}\Delta + D_1 = f_1^*(K_X + \Delta)$$

Let D'_1 be the strict transform of $\sum_{j=1}^r E_j$ and $D''_1 = (f_1)_*^{-1}\Delta + D_1 - D'_1$. Since $K_{X_1} + (f_1)_*^{-1}\Delta + D_1 \sim_{X,\mathbb{Q}} 0$, the $K_{X_1} + D''_1 + (1-\delta)D'_1$-MMP with scaling over X terminates with a good minimal model for $0 < \delta < 1$ due to [2, Theorem 1.1] or [19, Theorem 1.6]. Therefore, there exists the lc model X_2 of $(X_1, D''_1 + (1-\delta)D'_1)$ over X. Let $f_2 : X_2 \to X$ be the structure morphism and the strict transforms of D'_1 and D''_1 on X_2 be D'_2 and D''_2 respectively. Then

$$K_{X_2} + D''_2 + (1-\delta)D'_2 = f_2^*(K_X + \Delta) - \delta D'_2$$

and hence $-D_2'$ is f_2-ample. Hence, the exceptional set $\mathrm{Ex}(f_2) \subset D_2'$ and any exceptional divisor other than $E_1, \cdots, E_r$ is contracted. If the condition (2) holds, $D_2' = (f_2)_*^{-1}(\sum_{j=1}^r E_j) \neq 0$ and hence we have the second assertion. On the other hand, it is easy to see that some E_j is not contracted on X_2 by the definition of the lc model. Therefore, the first assertion holds. Furthermore, we also have the second assertion by the first one if (1) holds. □

On the other hand, in Proposition 4.15, we need to discuss positivity of

$$I^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) - (k+1)J^{\mathrm{NA}}(\mathcal{H}|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}).$$

Lemma 5.7. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair and $Z \subset X$ be a non-empty closed subscheme of $\mathrm{codim}_X(Z) = r$. Suppose that L is very ample, $\mathcal{X}$ is the normalization of the deformation to the normal cone of Z and $\rho^{-1}(Z \times \{0\}) = E$, where $\rho : \mathcal{X} \to X_{\mathbb{A}^1}$ is the canonical morphism. Let $\mathcal{H}_\epsilon = \rho^* H_{\mathbb{A}^1} - \epsilon E$ for sufficiently small $\epsilon > 0$.*

If Z is of non-fiber type and $0 < \mathrm{codim}_B f(Z) = k \leq n = \dim B$, then $r > k$ and there exists a positive constant $C > 0$ such that

$$(1) \qquad I^{\mathrm{NA}}(\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) - (k+1)J^{\mathrm{NA}}(\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}}) = C\epsilon^{r+1} + O(\epsilon^{r+2})$$

for general $D_1, \cdots, D_{n-k} \in |L|$.

Proof. The first assertion is trivial. Since $J^{\mathrm{NA}}(\mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k+1}}), \cdots, J^{\mathrm{NA}}(\mathcal{H}_{\epsilon,b}) = 0$ for general $D_j \in |L|$ and $b \in B$, the value of (1) does not change when we replace H by $H+L$. Thus, we may assume that H and $\mathcal{H}_\epsilon$ are ample for sufficiently small $\epsilon > 0$. As we saw in Lemma 5.1, $(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_\epsilon|_{\mathcal{X}\cap D_1\cap D_2\cap\cdots\cap D_{n-k}})$ is the normalization of the deformation to the normal cone of $Z \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}$. Thus, we may replace X by $X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}$ and assume that $k = n$. Let $N = \dim X$ and decompose $E = E_1 + E_2 + \cdots + E_N$ where each irreducible component $E_i^{(s)}$ of E_i has the center $\eta_i^{(s)} \in X \times \{0\}$ such that $\mathrm{codim}_X \overline{\{\eta_i^{(s)}\}} = i$. Note that $r = \min\{i | E_i \neq \emptyset\} > n$. Let $\{E_j^{(s)}\}_{s=1}^{t_j}$ be the set of irreducible components of E_j. Let $F_{E_j^{(s)}}$ be the generic fiber of $\rho|_{E_j^{(s)}} : E_j^{(s)} \to \overline{\{\eta_j^{(s)}\}} \times \{0\}$. Since $E_j = \sum_{s=1}^{t_j} m_s E_j^{(s)}$ for some $m_s > 0$, we have as [5, Lemma 9.11] for sufficiently small $\epsilon > 0$ that

$$a_i^{(j)}(\epsilon) := \sum_{s=1}^{t_j} m_s E_j^{(s)} \cdot \mathcal{H}_\epsilon^i \cdot H_{\mathbb{P}^1}^{N-i} = \begin{cases} \epsilon^j \sum_{s=1}^{t_j} m_s b_j^{(s)} \binom{i}{j} (Z_{E_j^{(s)}} \cdot H^{n-j}) + O(\epsilon^{j+1}) & \text{for } i \geq j \\ 0 & \text{for } i < j, \end{cases}$$

where $b_j^{(s)} = (-1)^j (F_{E_j^{(s)}} \cdot E^j) > 0$ is independent of ϵ. Note also that $Z_{E_j^{(s)}} \cdot H^{n-j} > 0$. Thus, we have by [5, Lemma 7.4]

$$(H)^N\, I^{\mathrm{NA}}(\mathcal{H}_\epsilon) = \epsilon \sum_{j=r}^{N} a_N^{(j)}(\epsilon) = \epsilon a_N^{(r)}(\epsilon) + O(\epsilon^{r+2}),$$

$$(H)^N\, J^{\mathrm{NA}}(\mathcal{H}_\epsilon) = \frac{1}{N+1} \epsilon \sum_{j=r}^{N} \sum_{i=j}^{N} a_i^{(j)}(\epsilon) = \frac{1}{N+1} \epsilon \sum_{i=r}^{N} a_i^{(r)}(\epsilon) + O(\epsilon^{r+2}).$$

Therefore, we have

$$\begin{aligned}(H)^N (I^{\mathrm{NA}}(\mathcal{H}_\epsilon) - (n+1)J^{\mathrm{NA}}(\mathcal{H}_\epsilon)) &= \epsilon a_N^{(r)}(\epsilon) - \frac{n+1}{N+1} \epsilon \sum_{i=r}^{N} a_i^{(r)}(\epsilon) + O(\epsilon^{r+2}) \\ &= \epsilon^{r+1} \sum_s m_s b_r^{(s)} (Z_{E_r^{(s)}} \cdot H^{n-r}) \left(\binom{N}{r} - \frac{n+1}{N+1} \sum_{i=r}^{N} \binom{i}{r} \right) + O(\epsilon^{r+2}) \\ &= \epsilon^{r+1} \sum_s m_s b_r^{(s)} (Z_{E_r^{(s)}} \cdot H^{n-r}) \binom{N}{r} \left(1 - \frac{n+1}{r+1} \right) + O(\epsilon^{r+2}).\end{aligned}$$

Since $r > n$, $I^{\mathrm{NA}}(\mathcal{H}_\epsilon) - (n+1)J^{\mathrm{NA}}(\mathcal{H}_\epsilon) > 0$ for sufficiently small $\epsilon > 0$. $\square$

Proof of Theorem 5.3. First, we may assume that $\lambda = 1$. Suppose that there exists at least one non-klt center η of non-fiber type and $\mathrm{codim}_B f(\overline{\{\eta\}}) \leq k$. By Theorem 5.2, it follows that η is an lc center. If necessary, replacing k by a smaller integer, we may assume that $\mathrm{codim}_B f(\overline{\{\eta\}}) = k$ and that there exists a closed subset $W \subset B$ of $\mathrm{codim}_B(W) > k$ such that $(X \setminus f^{-1}(W), \Delta \setminus f^{-1}(W))$ is lc by Theorem 5.2. Then there exists a closed subscheme $Z \subset X \setminus f^{-1}(W)$ whose closure $\overline{Z}$ is of non-fiber type with $\mathrm{codim}_B f(\overline{Z}) = k$ such that there exists a unique valuation $v \in \mathrm{Rees}_{X \setminus f^{-1}(W)}(Z)$ such that $A_{(X,\Delta)}(v) = 0$ by applying Proposition 5.6 to $(X \setminus f^{-1}(W), \Delta \setminus f^{-1}(W))$. Thus, $f(c_X(w)) \in W$ for any $w \in \mathrm{Rees}_X(\overline{Z}) \setminus \{v\}$. Let $\mathcal{X}$ be the normalization of the deformation to the normal cone of $\overline{Z}$, E be the inverse image of $\overline{Z} \times \{0\}$ and $\mathcal{H}_\epsilon = H_{\mathbb{A}^1} - \epsilon E$. Here, we may assume that H and L are very ample.

We will show that $W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) < 0$ and $W_i^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = 0$ for $i < k$ for sufficiently small $\epsilon > 0$ as in the proof of Theorem 5.2. First, we take general D_i's of $|L|$. Note that $\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}$ is not trivial but $\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k+1}$ is trivial. Thus, we have $W_i^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = 0$ for $i < k$. On the other hand, $\overline{c_X(w)} \cap D_1 \cap \cdots \cap D_{n-k} = \emptyset$ for any $w \in \mathrm{Rees}_X(\overline{Z}) \setminus \{v\}$. Replace $\mathcal{X}$ by $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}$. Thus, we have

$$W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = H_\Delta^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}_\epsilon) - I^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}_\epsilon) + (k+1)J^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}_\epsilon)$$

and $H_\Delta^{\mathrm{NA}}(\mathcal{X}, \mathcal{H}_\epsilon) = 0$ by Proposition 4.15 and Lemma 5.1. It follows from Lemma 5.7 that $W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) < 0$. $\square$

5.2. **Non-normal case.** Next, we consider the deminormal case. We will define $\mathfrak{f}$-stability of deminormal algebraic fiber spaces and prove generalizations (Theorems 5.13 and 5.15) of Theorem 5.2 and Theorem 5.3 in this subsection.

Definition 5.8. Let X be a proper deminormal scheme and $\nu_i : X_i \to X$ be the normalization of irreducible components (compare this with Definition 2.17). A surjective proper morphism $f : (X, \Delta) \to B$ of equidimensional reduced schemes is called a *deminormal algebraic fiber space pair* if $f_i : X_i \to B_i$ is an algebraic fiber space for any X_i, where f_i is induced by $f \circ \nu_i$ and B_i is the normalization of $f \circ \nu_i(X_i)$, which is an irreducible component of B.

Definition 5.9 (cf., Definition 4.4). Suppose that $f : (X, \Delta, H) \to (B, L)$ is a deminormal polarized algebraic fiber space pair with a boundary Δ. Let $\mathrm{rel.dim}\, f = m$ and $\dim B = n$. We remark that we can define (semi) fibration degenerations for f similarly to Definition 4.1. For any fibration degeneration $(\mathcal{X}, \mathcal{H})$ for f, we set constants $W_0'^\Delta(\mathcal{X}, \mathcal{H})$, $W_1'^\Delta(\mathcal{X}, \mathcal{H}), \cdots, W_n'^\Delta(\mathcal{X}, \mathcal{H})$ and a rational function $W_{n+1}'^\Delta(\mathcal{X}, \mathcal{H})(j)$ so that the partial fraction decomposition of $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{H} + jL)$ in j is as follows:

$$V(H + jL)\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{H} + jL) = W_{n+1}'^\Delta(\mathcal{X}, \mathcal{H})(j) + \sum_{i=0}^{n} j^i W_{n-i}'^\Delta(\mathcal{X}, \mathcal{H}).$$

Then $f : (X, \Delta, H) \to (B, L)$ is called

- $\mathfrak{f}_l$-*semistable* if $W_0'^\Delta \geq 0$ and

$$W_0'^\Delta(\mathcal{X}, \mathcal{H}) = W_1'^\Delta(\mathcal{X}, \mathcal{H}) = \cdots = W_i'^\Delta(\mathcal{X}, \mathcal{H}) = 0 \Rightarrow W_{i+1}'^\Delta(\mathcal{X}, \mathcal{H}) \geq 0$$

  for $i = 0, 1, \cdots, l-1$ for any fibration degeneration. Equivalently, f is $\mathfrak{f}_l$-semistable if $\sum_{i=0}^{l} j^{n-i} W_i'^\Delta(\mathcal{X}, \mathcal{H}) \geq 0$ for sufficiently large $j > 0$,
- $\mathfrak{f}$-*semistable* if f is $\mathfrak{f}_n$-semistable,
- $\mathfrak{f}$-*stable* if f is $\mathfrak{f}$-semistable and

$$W_i'^\Delta(\mathcal{X}, \mathcal{H}) = 0,\ i = 0, 1, \cdots, n-1 \Rightarrow W_n'^\Delta(\mathcal{X}, \mathcal{H}) > 0$$

  for any non-almost-trivial fibration degeneration of (X, H).

Let $(\mathcal{X},\mathcal{H})$ be as above and be partially normal as a test configuration and $(\widetilde{\mathcal{X}},\widetilde{\mathcal{H}})$ be the normalization of $(\mathcal{X},\mathcal{H})$, where $\widetilde{\mathcal{H}}$ as Notation 2.3. Then we set constants $W_0^\Delta(\mathcal{X},\mathcal{H})$, $W_1^\Delta(\mathcal{X},\mathcal{H})$, $\cdots, W_n^\Delta(\mathcal{X},\mathcal{H})$ and a rational function $W_{n+1}^\Delta(\mathcal{X},\mathcal{H})(j)$ so that the partial fraction decomposition of $M_\Delta^{\mathrm{NA}}(\mathcal{X},\mathcal{H}+jL)$ in j is as follows:

$$V(H+jL)M_\Delta^{\mathrm{NA}}(\mathcal{X},\mathcal{H}+jL)=W_{n+1}^\Delta(\mathcal{X},\mathcal{H})(j)+\sum_{i=0}^{n} j^iW_{n-i}^\Delta(\mathcal{X},\mathcal{H}).$$

Note that f is $\mathfrak{f}_l$-semistable if and only if $\sum_{i=0}^{l} j^{n-i}W_i^\Delta(\mathcal{X},\mathcal{H})\geq 0$ for sufficiently large $j>0$. It is easy to check this fact as the proof of Proposition 2.25. See also Remark 4.6.

We prepare the following to calculate W_k^Δ by taking the normalization,

Definition 5.10. Let $X=\bigcup_{i=1}^r X_i$ be the irreducible decomposition. For $1\leq i\leq r$, let b_i be a general point of $f(X_i)$. Let also $(\widetilde{X},\widetilde{H})$ (resp. $\widetilde{X_i}$) be the normalization of (X,H) (resp. X_i) in the sense of Notation 2.3. We say that a polarized deminormal algebraic fiber space pair $f:(X,\Delta,H)\to(B,L)$ has *the same average scalar curvature with respect to the fiber of* f if for $1\leq i<j\leq r$,

$$S\left(\widetilde{(X_i)_{b_i}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_i)_{b_i}}},\widetilde{H}|_{\widetilde{(X_i)_{b_i}}}\right)=S\left(\widetilde{(X_j)_{b_j}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_j)_{b_j}}},\widetilde{H}|_{\widetilde{(X_j)_{b_j}}}\right).$$

Then we have the following:

Lemma 5.11 (cf. [21, Theorem 6.6])**.** *Notations as in Definition 5.10. If $f:(X,\Delta,H)\to(B,L)$ is $\mathfrak{f}_0$-semistable, then f has the same average scalar curvature with respect to the fiber of f.*

Proof. Suppose that $\dim B=n$ and $\mathrm{rel.dim}\, f=m$. Assume that

$$S\left(\widetilde{(X_1)_{b_1}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_1)_{b_1}}},\widetilde{H}|_{\widetilde{(X_1)_{b_1}}}\right)>S\left(\widetilde{(X_j)_{b_j}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_j)_{b_j}}},\widetilde{H}|_{\widetilde{(X_j)_{b_j}}}\right)$$

for $2\leq j\leq r$. Let $B=\bigcup B_k$ be the irreducible decomposition and $B_1=f(X_1)$. Then we obtain as [21, Theorem 6.6]

$$S\left(\widetilde{(X_1)_{b_1}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_1)_{b_1}}},\widetilde{H}|_{\widetilde{(X_1)_{b_1}}}\right)>\frac{\sum_k (L|_{B_k})^nS(X_{b_k},\Delta_{b_k},H_{b_k})}{L^n}.$$

Let $Z=X_1\cap\bigcup_{i\geq 2}X_i$ and $\mathcal{X}$ be the partially normalization of the blow up of $X_{\mathbb{A}^1}$ along $Z\times\{0\}$ with the exceptional divisor E. Let F be the strict transform of $X_1\times\{0\}$. By taking the finite base change via the d-th power map of $\mathbb{A}^1$, we may assume that the normalization $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ has the reduced central fiber as in the proof of [5, Proposition 7.16]. Choose $\eta>0$ such that $-E+\eta F$ is $X_{\mathbb{A}^1}$-ample and let

$$\mathcal{H}=H_{\mathbb{A}^1}-\epsilon(E-\eta F)$$

be an ample $\mathbb{Q}$-line bundle over $\mathcal{X}$ for sufficiently small $\epsilon>0$. Then

$$\begin{aligned}\binom{n+m}{n}^{-1}W_0^\Delta(\mathcal{X},\mathcal{H})&=\sum_k (L|_{B_k})^n\left(K_{(\mathcal{X}_{b_k},\Delta_{b_k})/\mathbb{P}^1}\cdot\mathcal{H}_{b_k}^m+S(X_{b_k},\Delta_{b_k},H_{b_k})\frac{\mathcal{H}_{b_k}^{m+1}}{m+1}\right)\\&=\epsilon\eta(H|_{X_1,b_1})^m\left(\sum_k (L|_{B_k})^nS(X_{b_k},\Delta_{b_k},H_{b_k})\right.\\&\left.-(L)^nS\left(\widetilde{(X_1)_{b_1}},(\pi_*^{-1}\Delta+\mathfrak{cond}_{\tilde{X}})|_{\widetilde{(X_1)_{b_1}}},H|_{\widetilde{(X_1)_{b_1}}}\right)\right)+O(\epsilon^2)\end{aligned}$$

for general $b_k\in B_k$ similarly to the proof of [21, Theorem 6.6]. Thus, $W_0^\Delta(\mathcal{X},\mathcal{H})<0$ for sufficiently small $\epsilon>0$. $\square$

Therefore, if a reducible algebraic fiber space $f : (X, \Delta, H) \to (B, L)$ is $\mathfrak{f}$-semistable, then we can decompose W_k^{Δ} as follows,

Lemma 5.12. *Let $f : (X, \Delta, H) \to (B, L)$ be a deminormal polarized algebraic fiber space pair that has the same average scalar curvature with respect to the fiber of f, $0 \le k \le \dim B = n$ and $(\mathcal{X}, \mathcal{H})$ be a partially normal semiample test configuration for (X, H) dominating $X_{\mathbb{A}^1}$. Suppose that H is ample, L is very ample and $\mathcal{X}$ has the reduced central fiber. Let $\nu : \tilde{X} \to X$ be the normalization and $\tilde{X} = \bigcup_{i=1}^r \widetilde{X_i}$ be the irreducible decomposition. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}})$ be the normalization of $(\mathcal{X}, \mathcal{H})$ and $\widetilde{\mathcal{X}} = \bigcup_{i=1}^r \widetilde{\mathcal{X}_i}$ be the irreducible decomposition where the indices corresponding to those of $\tilde{X} = \bigcup_{i=1}^r \widetilde{X_i}$.*

If $(\widetilde{\mathcal{X}_i}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i}})$ is normalized and $(\widetilde{\mathcal{X}_i} \cap D_1 \cap \cdots \cap D_{n-k+1}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i} \cap D_1 \cap \cdots \cap D_{n-k+1}})$ is trivial for general ample divisors $D_1, \cdots, D_{n-k+1} \in |L|$, then

$$W_k^{\Delta}(\mathcal{X}, \mathcal{H}) = \sum_{i=1}^r W_k^{(\nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})|_{\tilde{X}_i}}(\widetilde{\mathcal{X}_i}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i}}).$$

Proof. By the assumption, we have

$$\mathrm{DF}_{\Delta}(\mathcal{X}, \mathcal{H} + jL) = M^{\mathrm{NA}}_{(\nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}} + jL)$$

for any j. Moreover, since each $(\widetilde{\mathcal{X}_i}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i}})$ is normalized, we have

$$W_k^{(\nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}}) = \sum_{i=1}^r W_k^{(\nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})|_{\tilde{X}_i}}(\widetilde{\mathcal{X}_i}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i}})$$

by Lemma 4.13. ☐

Note that if $(\mathcal{X}, \mathcal{H})$ is a deformation to the normal cone of a closed subscheme Z of X with the inverse image E of $Z \times \{0\}$ such that $\dim Z < \dim X$ and $\mathcal{H} = H_{\mathbb{A}^1} - \epsilon E$ for sufficiently small $\epsilon > 0$, then $(\widetilde{\mathcal{X}_i}, \widetilde{\mathcal{H}}|_{\widetilde{\mathcal{X}_i}})$ is normalized with respect to the central fiber in Lemma 5.12.

Theorem 5.13. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized deminormal algebraic fiber space pair. Fix an integer $0 \le k \le n = \dim B$. If f is $\mathfrak{f}_k$-semistable, (X, Δ) has no non-slc center η such that $\mathrm{codim}_B f(\overline{\{\eta\}}) \le k$.*

Proof. By Lemma 5.11, we may assume that f has the same average scalar curvature with respect to the fiber of f. Thus, f satisfies the assumption of Lemma 5.12. Assume that (X, Δ) has a non-slc center η such that $\mathrm{codim}_B f(\overline{\{\eta\}}) = k$ and $\mathrm{codim}_B f(\overline{\{\xi\}}) \ge k$ for any non-slc center $\xi \in X$ as in the proof of Theorem 5.2.

First, we treat the case when $\{\eta\} \not\subset \mathrm{Supp}(\Delta_{>1})$. By Theorem 3.3, there exists the slc modification $\pi : Y \to X \setminus \mathrm{Supp}(\Delta_{>1})$ of $(X \setminus \mathrm{Supp}(\Delta_{>1}), \Delta \setminus \mathrm{Supp}(\Delta_{>1}))$. It is easy to see that there exists a closed subscheme $Z \subset X \setminus \mathrm{Supp}(\Delta_{>1})$ such that π is the blow up along Z. Let $\nu : \tilde{X} \to X$ be the normalization and $D = \mathfrak{cond}_{\tilde{X}}$ be the conductor. If $\nu_Y : \tilde{Y} \to Y$ is the normalization of Y and $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is the induced morphism, then $(\tilde{Y}, \Delta_{\tilde{Y}} + \tilde{\pi}_*^{-1}D)$ is the lc modification of $(\tilde{X} \setminus \nu^{-1}(\mathrm{Supp}(\Delta_{>1})), (\Delta_{\tilde{X}} + D) \setminus \nu^{-1}(\mathrm{Supp}(\Delta_{>1})))$ where $\Delta_{\tilde{Y}} = (\nu_Y)_*^{-1}\Delta_Y$ and $\Delta_{\tilde{X}} = \nu_*^{-1}\Delta$. Let $(\mathcal{X}, \mathcal{H}_\epsilon = H_{\mathbb{A}^1} - \epsilon E)$ be the partially normalization of the deformation to the normal cone of $\overline{Z}$ the Zariski closure of Z, where E is the inverse image of $\overline{Z} \times \{0\}$ and $\epsilon > 0$. Let also $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}_\epsilon})$ be the normalization of $(\mathcal{X}, \mathcal{H}_\epsilon)$. Then it is easy to see that $\widetilde{\mathcal{X}}$ is the normalization of the deformation to the normal cone of $\nu^{-1}\overline{Z}$. For any irreducible component $\widetilde{\mathcal{X}_i}$ of $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}_i}$ is the normalization of the deformation to the normal cone of $\nu^{-1}\overline{Z} \cap \widetilde{X_i}$, where $\widetilde{X_i}$ is the irreducible component corresponding to $\widetilde{\mathcal{X}_i}$. Hence, we have $W_k^{\Delta_{\tilde{X}} + D}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}_\epsilon}) < 0$ and $W_i^{\Delta_{\tilde{X}} + D}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}_\epsilon}) = 0$ for $i < k$ and for sufficiently small $\epsilon > 0$ by the proof of Theorem 5.2 and by applying Lemma 5.12. By

$$W_i^{(\Delta_{\tilde{X}} + D)}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{H}_\epsilon}) = W_i^{\Delta}(\mathcal{X}, \mathcal{H}_\epsilon)$$

for $0 \leq i \leq \dim B$, we have the assertion in this case.

Next, we treat the case when $\overline{\{\eta\}}$ is an irreducible component of $\Delta_{>1}$. Then, as in the proof of Theorem 5.2, the partially normalization of the deformation to the normal cone $(\mathcal{X}, \mathcal{H})$ of $\overline{\{\eta\}}$ with some polarization $\mathcal{H}$ satisfies that $W_k^{\Delta}(\mathcal{X}, \mathcal{H}) < 0$ and $W_i^{\Delta}(\mathcal{X}, \mathcal{H}) = 0$ for $i < k$. $\square$

As in Proposition 5.6, we need the following partial resolution.

Lemma 5.14. *Let (X, Δ) be a quasi projective slc pair and fix an slc center $\eta \in X$ such that* $\mathrm{codim}_X \overline{\{\eta\}} \geq 2$. *Let $\nu : \tilde{X} \to X$ be the normalization. Then there exists an irreducible closed subscheme Z that satisfies the following.*

(1) The reduced structure $\mathrm{red}\,(Z) = \overline{\{\eta\}}$, *and*

(2) $A_{(\tilde{X}, \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})}(v) = 0$ *for any* $v \in \mathrm{Rees}_{\tilde{X}}(\nu^{-1}Z)$.

Proof. Fix an ample line bundle H and let $\mathscr{I}$ be the ideal sheaf corresponding to the reduced structure of $\overline{\{\eta\}}$. Then the linear system $\mathfrak{d} = H^0(X, \mathscr{I} \otimes \mathcal{O}(mH))$ is base point free outside from $\overline{\{\eta\}}$ for sufficiently large $m > 0$. We can choose $D \in \mathfrak{d}$ such that D contains no slc center other than those contained in $\overline{\{\eta\}}$ and $X_i \not\subset \mathrm{Supp}(D)$ for any irreducible component X_i of X. Let $f : Y \to \tilde{X}$ be a log resolution of $(\tilde{X}, \nu_*^{-1}\Delta + \nu^*D + \mathfrak{cond}_{\tilde{X}})$ and a resolution of the base locus of $\mathfrak{d}$. By replacing D by general one, we may assume that $f^*\nu^*D$ does not contain any prime divisor E on Y such that $A_{(\tilde{X}, \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})}(E) = 0$ and $\nu \circ f(E) \not\subset \overline{\{\eta\}}$ due to the theorem of Bertini. Let Δ_Y be the $\mathbb{Q}$-divisor satisfying that $f_*\Delta_Y = \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}}$ and

$$K_Y + \Delta_Y = f^*(K_{\tilde{X}} + \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}}).$$

Then, since (Y, Δ_Y) is log smooth and sublc, for any non-lc center η' of $(\tilde{X}, \nu_*^{-1}\Delta + \epsilon\nu^*D + \mathfrak{cond}_{\tilde{X}})$, $\nu(\eta') \in \overline{\{\eta\}}$ for sufficiently small rational $\epsilon > 0$. Note that $(X, \Delta + \epsilon\nu^*D)$ is not slc along $\overline{\{\eta\}}$ but $\lceil \epsilon\nu^*D + \Delta \rceil$ is reduced. Thanks to Theorem 3.3, we take the slc modification $g : W \to X$ of $(X, \Delta + \epsilon\nu^*D)$ and there exists a closed subscheme Z such that g is the blow up along Z. It is easy to see that $\mathrm{red}\,(Z) = \overline{\{\eta\}}$ and $A_{(\tilde{X}, \nu_*^{-1}\Delta + \mathfrak{cond}_{\tilde{X}})}(v) = 0$ for any $v \in \mathrm{Rees}_{\tilde{X}}(\nu^{-1}Z)$. $\square$

Theorem 5.15. *Let $f : (X, \Delta, H) \to (B, L)$ be a polarized deminormal algebraic fiber space. Suppose that there exist $\lambda \in \mathbb{Q}_{>0}$ and a line bundle L_0 on B such that $H + f^*L_0 \equiv -\lambda(K_X + \Delta)$, and f is $\mathfrak{f}_k$-semistable for $0 \leq k \leq n = \dim B$. Then, (X, Δ) has no slc-center η of non-fiber type such that* $\mathrm{codim}_B f(\overline{\{\eta\}}) \leq k$.

Proof. We may assume that H is ample and L is very ample. It follows from Theorem 5.13 that (X, Δ) is slc. Assume that $\mathrm{codim}_B f(\overline{\{\xi\}}) \geq k$ for any slc center ξ of non-fiber type and there exists at least one slc center η of non-fiber type such that $\mathrm{codim}_B f(\overline{\{\eta\}}) = k$. Then we will show that f is $\mathfrak{f}_k$-unstable as Theorem 5.3. By Theorem 5.13, we may further assume that there exists a closed subset $W \subset B$ of $\mathrm{codim}_B W > k$ such that $(X \setminus f^{-1}(W), \Delta \setminus f^{-1}(W))$ is slc. Throughout the proof, let $\nu : \tilde{X} \to X$ be the normalization and $\tilde{D} = \mathfrak{cond}_{\tilde{X}}$ be the conductor divisor. Note that we may assume that f satisfies the assumption of Lemma 5.12 as Theorem 5.13.

First, we treat the case when V is an irreducible component of $D = \mathfrak{cond}_X$. Note that $\tilde{D}$ need not to be $\mathbb{Q}$-Cartier in general. Due to Proposition 5.6, there exists a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_{\tilde{X}}(-\tilde{D}) = \mathcal{O}_X(-D)$ satisfying the following.

- Let Z be the closed subscheme of X corresponding to $\mathfrak{a}$. Then it holds that $f \circ \nu(c_{\tilde{X}}(v)) \in W$ or there exists an irreducible component E of $\tilde{D}$ such that $v = \frac{\mathrm{ord}_E}{\mathrm{ord}_E(\mathfrak{a})}$ for any $v \in \mathrm{Rees}_{\tilde{X}}(\nu^{-1}(Z))$, and
- $\mathrm{ord}_E(\mathfrak{a}) \neq 0$ for any irreducible component E of $\tilde{D}$.

Then consider the partially normalization of the deformation to the normal cone $(\mathcal{X}, \mathcal{H}_\epsilon = H_{\mathbb{A}^1} - \epsilon E)$ of Z where E is the exceptional divisor for $\epsilon > 0$. We may assume that the central fiber $\mathcal{X}_0$ of $\mathcal{X}$ is reduced by the same argument of the proof of Theorem 5.13. If $(\tilde{\mathcal{X}}, \tilde{\mathcal{H}}_\epsilon)$ is the normalization of $(\mathcal{X},\mathcal{H}_\epsilon)$, then $H^{\mathrm{NA}}_\Delta(\tilde{\mathcal{X}}_b,\widetilde{\mathcal{H}_\epsilon}|_{\tilde{\mathcal{X}}_b})=0$ for general $b\in B$ and hence $W^\Delta_0(\mathcal{X},\mathcal{H}_\epsilon)<0$. Therefore, we may assume that any singular locus of codimension 1 of (X,Δ) is of fiber type. Furthermore, we may assume that any irreducible component of a general fiber is lc.

Next, we treat the case when η is the generic point of an irreducible component V of $\lfloor\Delta\rfloor$. Let X_1 be the irreducible component of X containing η. There exists a closed subscheme Z of X_1 such that for any valuation $v\in\mathrm{Rees}_{\tilde{X}}(\nu^{-1}Z)\setminus\{\mathrm{ord}_{\nu_*^{-1}V}\}$, $f\circ\nu(c_{\tilde{X}}(v))\in W$ by applying Proposition 5.6 to $X_1\setminus\overline{(X\setminus X_1)}$ similarly to the proof of Theorem 5.3. For any general point $b\in f(X_1)\subset B$, any irreducible component of (X_b,Δ_b) is lc and $\mathrm{Rees}_{(X_1)_b}(Z_b)=\{\mathrm{ord}_{V_b}\}$ by what we assumed in the previous paragraph. Therefore, it is easy to see that if $(\mathcal{X},\mathcal{H}_\epsilon)$ is the partially normalization of the deformation of X to the normal cone of Z where E is the inverse image of $Z\times\{0\}$ and $\mathcal{H}_\epsilon=H_{\mathbb{A}^1}-\epsilon E$, then we have $W^\Delta_0(\mathcal{X},\mathcal{H}_\epsilon)<0$ for sufficiently small $\epsilon>0$. Thus, we may assume that $\lfloor\Delta\rfloor$ is of fiber type.

Finally, we treat the case when $\mathrm{codim}_X\overline{\{\eta\}}\geq 2$. By applying Lemma 5.14 to $(X\setminus f^{-1}(W),\Delta\setminus f^{-1}(W))$, there exists a closed subscheme $Z\subset X\setminus f^{-1}(W)$ such that $\overline{\{\eta\}}=\overline{\mathrm{red}(Z)}$ and it holds for any $v\in\mathrm{Rees}_{\tilde{X}\setminus\nu^{-1}f^{-1}(W)}(\nu^{-1}Z)$ that $A_{(\tilde{X},\nu_*^{-1}\Delta+\tilde{D})}(v)=0$. Hence, if $\overline{Z}$ is the Zariski closure of Z, we have $f\circ\nu(c_{\tilde{X}}(w))\in W$ if $A_{(\tilde{X},\nu_*^{-1}\Delta+\tilde{D})}(v)\neq 0$ for any $w\in\mathrm{Rees}_{\tilde{X}}(\nu^{-1}\overline{Z})$. Now, let $(\mathcal{X},\mathcal{H}_\epsilon)$ be the partially normalization of the deformation of X to the normal cone of $\overline{Z}$, E be the inverse image of $\overline{Z}\times\{0\}$ and $\mathcal{H}_\epsilon=H_{\mathbb{A}^1}-\epsilon E$. Here, we may assume that $\dim B=k$ by replacing X by $X\cap D_1\cap\cdots\cap D_{n-k}$ for general divisors $D_1,\cdots,D_{n-k}\in|L|$. Since $\dim f(\overline{Z})=0$, any center of $v'\in\mathrm{Rees}_{\tilde{X}}(\nu^{-1}\overline{Z})$ is of non-fiber type. Hence, we see that $W^\Delta_k(\mathcal{X},\mathcal{H}_\epsilon)<0$ and $W^\Delta_i(\mathcal{X},\mathcal{H}_\epsilon)=0$ for $i<k$ and sufficiently small $\epsilon>0$ similarly to Theorems 5.13 and 5.3 by applying Lemma 5.7. □

Proof of Theorem D. It immediately follows from Theorem 5.13 and Theorem 5.15. □

Proof of Corollary E. It immediately follows from Theorem D and the fact that adiabatic K-semistability implies $\mathfrak{f}$-semistability. □

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