Liouville heat kernel upper bounds at large distances

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Abstract

We show that the Liouville heat kernel decays fast at large distances. In particular, the Liouville semigroup T_t is C_0-Feller, where C_0 is the space of real-valued continuous functions on $\mathbb{C}$ vanishing at infinity. This is a problem mentioned in the paper [2].

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1 Introduction

Liouville quantum gravity (LQG) was introduced by Polyakov in a seminal paper [19] and can be considered as the canonical 2-dimensional random Riemannian manifold. The Riemannian volume form can be formally written in the form

$$e^{\gamma X(z)}dz$$

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where X is a massive Gaussian free field (GFF) on $\mathbb{C}$; $\gamma \in (0,2)$ is a parameter; and dz is the Lebesgue measure on $\mathbb{C}$.

Of course the above form is not rigorous as the GFF is not a random function (but a distribution in the sense of Schwartz). Nonetheless, one can make sense of the volume form by the theory of Gaussian multiplicative chaos [15] or some other regularization procedure [6]. The rigorous construction of the random volume form is then referred as the Liouville measure M_γ.

The Liouville Brownian motion (LBM) is the canonical diffusion process for Liouville quantum gravity, which is constructed in [9, 3] as a time-changed Brownian motion of 2-dimension according to the Liouville measure (independent of the Brownian motion). More precisely, for the Liouville measure M_γ one can construct the associated positive continuous additive functional (PCAF) F of a Brownian motion W which can be formally written as

$$F_t = \int_0^t e^{\gamma X(W_s)}\, ds.$$

Then the LBM $\{Y_t\}_{t\geq 0}$ as a stochastic process is defined by $Y_t := W_{F_t^{-1}}$, where F^{-1} is the inverse of F (the inverse exists). For the rigorous discussion of LBM one can refer to [9, 3, 7] or see Section 2 of this paper.

The heat kernel of Liouville Brownian motion (LHK) is constructed in [8]. Further properties of LHK are studied in [2, 17, 5]. However none of them indicates large distance behavior of LHK.

In [9] it is shown that the semigroup T_t of Liouville Brownian motion is weak Feller, meaning that the semigroup operator T_t maps bounded continuous functions to bounded continuous functions. In [8] they show T_t is strong Feller, meaning that T_t maps bounded Borel measurable functions to continuous functions. But it is not clear whether it is C_0-Feller. That is, we don't know $T_t(C_0) \subseteq C_0$, where C_0 is the space of continuous functions vanishing at infinity (it is also mentioned in [2, Remark 2.3]). This is one of the motivations for this paper.

In this paper we show that the LHK decays fast at large distances (Theorem 3.9), which immediately implies C_0-Feller property. We also attach in Appendix a simple proof of Feller property without using estimates of LHK.

2 Background and preliminaries

2.1 The massive Gaussian free field and the Liouville measure

Given a real number $m > 0$, the whole-plane massive Gaussian free field (MGFF) (see [21] for more information about Gaussian free field) X is a centered Gaussian random distribution (in the sense of Schwartz) with covariance function given by the Green function of the operator $m^2 - \Delta$, that is,

$$\mathbb{E}[X(x)X(y)] = G_m(x,y) = \int_0^\infty e^{-\left(m^2/2\right)u - |x-y|^2/(2u)} \frac{du}{2u} \quad \text{for all } x, y \in \mathbb{C}.$$

Note that $G_m(x,y)$ can be written as

$$G_m(x,y) = \int_1^{+\infty} \frac{k_m(u(x-y))}{u} du$$

where $k_m(z) = \frac{1}{2}\int_0^\infty e^{-\frac{m^2}{2s}|z|^2 - \frac{s}{2}} ds$ is a continuous covariance kernel (see [1] for details about this expression). This expression helps us to decompose X into a sum of good Gaussian fields.

We then introduce the n-regularized field X_n. For this purpose, let $\{c_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of real numbers starting from $c_0 = 1$ and satisfying $\lim_{n\to\infty} c_n = \infty$. Let $(\eta_n)_{n\geq 1}$ be a family of independent continuous Gaussian fields on $\mathbb{C}$ with covariance

$$\mathbb{E}[\eta_n(x)\eta_n(y)] = \int_{c_{n-1}}^{c_n} \frac{k_m(u(x-y))}{u} du \quad \text{for all } x, y \in \mathbb{C}.$$

Note that for each n we can choose η_n to be continuous in space by applying Kolmogorov continuity theorem ([16, Theorem 2.23]). Define $X_n := \sum_{k=1}^n \eta_k$, and the associated random Radon measure $M_n = M_{\gamma,n}$ on $\mathbb{C}$ by

$$M_{n,\gamma}(dz) = \exp\left(\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(z)^2\right]\right) dz, \quad \gamma \in [0,\infty)$$

where dz is the Lebesgue measure on $\mathbb{C}$. By Kahane's theory of multiplicative chaos [15] almost surely M_n converges vaguely toward a limit Radon measure M, which is called the Liouville measure. The law of the limit does not depend on the choice of c_n and the limit measure is nontrivial if and only if $\gamma \in [0,2)$.

Recall (see [15, 20]) that the Liouville measure has an important property that for any bounded Borel set A and $p \in (-\infty, 4/\gamma^2)$ we have $\mathbb{E}[M(A)^p] < \infty$ and that

$$\sup_{r\in(0,1]} r^{-\xi_M(p)}\mathbb{E}\left[M(rA)^p\right] \leq C_p$$

for some constant C_p only depending on p, $\mathrm{diam}A (:= \sup_{x,y\in A}|x-y|)$, γ and m, where $\xi_M(q) = -\frac{\gamma^2}{2}q^2 + (2+\frac{\gamma^2}{2})q$ is the power law spectrum of M (see [1]).

2.2 Liouville Brownian motion

The Liouville Brownian motion is constructed in [9, 3] as the canonical diffusion process under the geometry induced by the measure M. More precisely, Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space that $(\eta_n)_{n\geq 1}$ live on. Let $\Omega' := C([0,\infty), \mathbb{C})$ and $W = (W_t)_{t\geq 0}$ be the coordinate procress on Ω'. Set $\mathcal{F} = \sigma(W_s, s < \infty)$ and $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Let $\{P_x\}_{x\in\mathbb{C}}$ be the family of probability measures on $(\Omega', \mathcal{F})$ such that W under P_x is a Brownian motion on $\mathbb{C}$ starting from $x \in \mathbb{C}$.

For each $n \in \mathbb{N}$ define $F^n(t) : \Omega \times \Omega' \to [0,\infty)$ to be

$$F^n(t) := \int_0^t \exp\left(\gamma X_n(W_s) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(W_s)^2\right]\right) ds, \quad t \geq 0.$$

Note that $F^n(t)$ is the positive continuous additive functional ([4], [7]) of W with Revuz measure $M_{\gamma,n}$. In [9, Theorem 2.7] (see also [3, Theorem 1.2]) they show that $\mathbb{P}$-a.s. there exists a unique positive continuous additive functional (PCAF) $F = (F(t))_{t\geq 0}$ of W such that the Revuz measure of F is M and

$$\lim_{n\to\infty} P_x[\sup_{t\leq T} |F^n(t) - F(t)| > \varepsilon] = 0 \quad \text{for all } \varepsilon > 0, T > 0, x \in \mathbb{C}.$$

And then the Liouville Brownian motion is defined to be

$$Y_t = W_{\bar{F}(t)}$$

where $\bar{F}(t) = F^{-1}(t) = \inf\{s \geq 0 : F(s) > t\}$. Note that it is proved in [9] (see also [3, Theorem 1.2]) that $\mathbb{P}$-a.s. for any $x \in \mathbb{C}$, P_x-a.s., F is continuous, strictly increasing and diverging to ∞.

2.3 Notation

Throughout this paper, we will fix $\gamma \in (0, 2)$. Define two constants in terms of γ which we will frequently use: $\alpha_1 = \frac{1}{2}(2+\gamma)^2$, $\alpha_2 = \frac{1}{2}(2-\gamma)^2$. Let X be a massive GFF on $\mathbb{C}$ and $M = M_\gamma$ be the Liouville measure constructed from X. We write

$$\tilde{\xi}(q) = -\xi_M(-q) = (2 + \frac{\gamma^2}{2})q + \frac{\gamma^2}{2}q^2$$

for $q > 0$. Let $\{Y_t\}_{t\geq 0}$ be a LBM and $p_t(x, y)$ be its heat kernel w.r.t. the Liouville measure M.

We denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Let $B_{x,r} = \{z \in \mathbb{C} : |z - x| \leq r\}$, in particular we write $B_R = \{z \in \mathbb{C} : |z| \leq R\}$. Let $\tau_{x,r} = \inf\{t \geq 0 : Y_t \notin B_{x,r}\}$ be the first exit time of LBM of the ball $B_{x,r}$.

The symbols c, C stand for positive constants whose value may change from line to line, but they won't depend on any parameters in this article. By adding subscripts $X, \gamma, \alpha, ...$ to the symbols c, C we indicate their dependence on those subscripts, while some other symbols $\bar{C}_R, \hat{C}_R, C_*, ...$ are exclusively used in some propositions or theorems.

We use $\lesssim$ to indicate the inequality holds up to an absolute constant $C > 0$, i.e. $x \lesssim y$ if and only if $x \leq Cy$ for some $C > 0$. By adding subscripts $X, \gamma, \alpha, ...$ to the symbols $\lesssim$ we indicate dependence of the constant on those subscripts.

We use P_x, E_x to take the probability (expectation) w.r.t. the Brownian motion starting at $x \in \mathbb{C}$, and use $\mathbb{P}, \mathbb{E}$ to take the probabitlity (expectation) w.r.t. the massive GFF.

3 The estimates

In this section we will establish some estimates of the Liouville heat kernel (LHK).

3.1 Liouville measure at large distances

We first do some preparation for estimates of LHK. The following lemma will be used in Proposition 3.2, Proposition 3.3, and Lemma 3.5.

Lemma 3.1. *Let $\{Z_R\}_{R\geq 1}$ be a family of nonnegative random variables that are almost surely nondecreasing in R such that*

$$\mathbb{E}[Z_R^p] \leq CR^m \quad \text{for all } R \geq 1.$$

for some positive constants $p, m, C > 0$. Then for any $\theta > m/p$, almost surely there is a random constant $C_\theta > 0$ such that $Z_R \leq C_\theta R^\theta$ for all $R \geq 1$.

Proof. Let $R_n = 2^n$ and for any $\theta > 0$ define $A_n = \{Z_{R_n} \leq R_n^\theta\}$. Then

$$\mathbb{P}[A_n^c] \leq R_n^{-\theta p}\mathbb{E}Z_{R_n}^p \leq CR_n^{m-\theta p} \quad \text{for all } n \geq 0.$$

If $\theta > m/p$ then by Borel-Cantelli's lemma $\mathbb{P}[A_n^c \text{ i.o.}] = 0$ and thus almost surely there is a random constant $C_\theta > 0$ such that $Z_{R_n} \leq C_\theta R_n^\theta$ for any $n \geq 0$. By monotonicity we have

$$Z_R \leq Z_{R_{n+1}} \leq C_\theta R_{n+1}^\theta = C_\theta 2^\theta R_n^\theta \leq C_\theta 2^\theta R^\theta$$

provided $R_n \leq R \leq R_{n+1}$ for $n \geq 0$. Reassigning $C_\theta 2^\theta$ as C_θ finishes the proof. □

The following proposition gives the Liouville volume growth rate of Euclidian balls which will be used in Proposition 3.8.

Proposition 3.2. *For any $\varepsilon > 0$, $\mathbb{P}$-a.s. the Liouville measure satisfies*

$$M(B_R) \lesssim_{X,\gamma,\varepsilon} R^{2+\varepsilon} \qquad \text{for all } R \geq 1.$$

Proof. Notice that for any n-regularized Liouville measure M_n and bounded Borel set A we have

$$\mathbb{E}[M_n(A)] = \int_A \mathbb{E}\exp\left(\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(z)^2\right]\right) dz = \int_A 1\, dz.$$

Hence letting $n \to \infty$, by vague convergence (in fact we have $M_n(A) \to M(A)$) and Fatou's Lemma we have

$$\mathbb{E}[M(B_R)] \leq \mathbb{E}[\varliminf_{n\to\infty} M_n(B_R)] \leq \varliminf_{n\to\infty} \mathbb{E}[M_n(B_R)] = \pi R^2.$$

Then apply Lemma 3.1 to get the bound. □

Next we give the growth rate of the coefficients of Hölder continuity of the Liouville measure, which will be also used in Proposition 3.8.

Proposition 3.3. *For any $\gamma \in (0,2)$ and $\alpha \in (0, \alpha_2)$, set $m_0(\gamma, \alpha) = \frac{\gamma^2}{2} + \frac{4\alpha\gamma^2}{(\alpha_1-\alpha)(\alpha_2-\alpha)}$. Then there exists a random constant $\bar{C}_R$ depending on X, γ, α, R such that $\mathbb{P}$-a.s.*

$$\sup_{|x|\leq R} M(B_{x,r}) \leq \bar{C}_R r^\alpha \quad \text{for all } r \in (0,1]$$

and for any $\varepsilon > 0$ and any $R \geq 1$ we have

$$\bar{C}_R \lesssim_{X,\gamma,\alpha,\varepsilon} R^{m_0+\varepsilon}.$$

Proof. We prove it in a similar manner as in [9, Theorem 2.2], but give the coefficient estimates depending on R. The main idea is to improve Borel-Cantelli's lemma and use stationarity of the Liouville measure.

For $n \in \mathbb{N}$ we partition $[-8,8]^2$ into 2^{2n} dyadic squares $\{I_n^j : j = 1, 2, ..., 2^{2n}\}$ of equal size. Fix $\alpha > 0$, let A_n be the event that $M(I_n^j) \le 2^{-\alpha n}$ for all $1 \le j \le 2^{2n}$. Then for $p \in (0, 4/\gamma^2)$ we have using the stationarity of GFF and the power law of the Liouville measure

$$\begin{aligned}\mathbb{P}[A_n^c] &\le 2^{p\alpha n}\mathbb{E}\left[\sum_{1\le j\le 2^{2n}} M\left(I_n^j\right)^p\right]\\ &\le C_p 2^{-nK(\alpha,p)}\end{aligned}$$

where $K(\alpha,p) := \xi_M(p) - \alpha p - 2$. Set $E_n = \cap_{k=n}^\infty A_k$ and $\tilde{E}_0 = E_0$, $\tilde{E}_n = E_n \setminus E_{n-1}$ for $n \in \mathbb{N}^*$, then $\mathbb{P}[\tilde{E}_n] \le \mathbb{P}[A_{n-1}^c]$ for $n \in \mathbb{N}^*$, and $\tilde{E}_n$ are disjoint and $\mathbb{P}[\cup_{n=0}^\infty \tilde{E}_n] = \mathbb{P}[\cup_{n=0}^\infty E_n] = 1 - \mathbb{P}[A_n^c \text{ i.o.}] = 1$ by Borel-Cantelli's lemma.

Define

$$\bar{C}_0 := \begin{cases} 4 & \text{on } \tilde{E}_0 \\ 4 \vee \sup_{|x|\le 1, r\in(2^{-n},2)} \frac{M(B_{x,r})}{r^\alpha} & \text{on } \tilde{E}_n \text{ for } n \in \mathbb{N}^*. \end{cases}$$

Note that $\bar{C}_0$ is almost surely well-defined because $\tilde{E}_n$ are disjoint and $\mathbb{P}[\cup_{n=0}^\infty \tilde{E}_n] = 1$. Also $\bar{C}_0$ is $\mathcal{A}$-measurable as $\sup_{|x|\le 1, r\in(2^{-n},2)} \frac{M(B_{x,r})}{r^\alpha} = \sup_{x\in\mathbb{Q}^2, |x|\le 1, r\in(2^{-n},2)} \frac{M(B_{x,r})}{r^\alpha}$. This is because for $x \notin \mathbb{Q}^2$ with $|x| \le 1$ and $r \in (2^{-n}, 2)$ we can find $x_i \in \mathbb{Q}^2$ with $|x_i| \le 1$ and $r_i \in (2^{-n}, 2)$ such that $x_i \to x$, $r_i \downarrow r$, $B_{x,r} \subseteq B_{x_i,r_i}$ and hence $\frac{M(B_{x,r})}{r^\alpha} \le \limsup_{i\to\infty} \frac{M(B_{x_i,r_i})}{r_i^\alpha}$.

We claim $\mathbb{P}$-a.s. $M(B_{x,r}) \le \bar{C}_0 r^\alpha$ for any $|x| \le 1$ and $r \in (0,1]$. Indeed, on $\tilde{E}_n$, when $r \in (2^{-n}, 1]$ by the definition of $\bar{C}_0$ we have $M(B_{x,r}) \le \bar{C}_0 r^\alpha$; when $r \in (2^{-k-1}, 2^{-k}]$ for $k \ge n$, any ball $B_{x,r}$ is contained in at most 4 dyadic squares I_{k+1}^j and each square I_{k+1}^j (of size 2^{3-k}) has Liouville measure no greater than $2^{-(k+1)\alpha}$, hence $M(B_{x,r}) \le 4 \cdot 2^{-\alpha(k+1)} \le 4r^\alpha$.

Moreover, for $\theta > 0$ by Hölder inequality for $q^{-1} + q'^{-1} = 1$

$$\begin{aligned}\mathbb{E}\bar{C}_0^\theta &\le 4^\theta + \sum_{n=1}^\infty 2^{n\alpha\theta}\mathbb{E}[M(B_3)^\theta; \tilde{E}_n]\\ &\le 4^\theta + \sum_{n=1}^\infty 2^{n\alpha\theta}\mathbb{E}[M(B_3)^{\theta q'}]^{1/q'}\mathbb{P}[\tilde{E}_n]^{1/q}\\ &\le 4^\theta + \sum_{n=1}^\infty 2^{n\alpha\theta}\mathbb{E}[M(B_3)^{\theta q'}]^{1/q'}(C_p 2^{-(n-1)K(\alpha,p)})^{1/q}\\ &= 4^\theta + C_p^{1/q}\mathbb{E}[M(B_3)^{\theta q'}]^{1/q'}\sum_{n=1}^\infty 2^{-n(K(\alpha,p)/q-\alpha\theta)+K(\alpha,p)/q}.\end{aligned}$$

The above is finite if $\theta q' < 4/\gamma^2$ and $K(\alpha,p)/q - \alpha\theta > 0$. So $\theta < \frac{K(\alpha,p)}{q\alpha} \wedge \frac{4}{\gamma^2 q'}$. Take $p = \frac{2+\gamma^2/2-\alpha}{\gamma^2}$ ($< \frac{4}{\gamma^2}$) (whence $K(\alpha,p) = \frac{(\alpha_1-\alpha)(\alpha_2-\alpha)}{2\gamma^2}$) and $q = \frac{(\alpha_1-\alpha)(\alpha_2-\alpha)}{8\alpha} + 1$ to maximize the right hand side to get $\mathbb{E}\bar{C}_0^\theta < \infty$ whenever $\theta < \frac{(\alpha_1-\alpha)(\alpha_2-\alpha)}{2\gamma^2((\alpha_1-\alpha)(\alpha_2-\alpha)/8+\alpha)}$.

Now do the same partition and reasoning for each region $z_k + [-8,8]^2$ where $z_k \in \mathbb{Z}^2$ and we get a sequence of $\bar{C}_{z_k}$ (defined similar to $\bar{C}_0$) with the same distribution as $\bar{C}_0$. Set $\bar{C}_R = \max_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \bar{C}_{z_k}$. Since any ball $B_{x,r}$ with $|x| \leq R$ and $r \in (0,1]$ is contained in one of the regions $\{z_k + [-8,8]^2\}_{z_k \in \mathbb{Z}^2 \cap B_{R+1}}$ (one can find $z_k \in \mathbb{Z}^2$ with $|x - z_k| \leq 1$ for each $x \in \mathbb{C}$ with $|x| \leq R$), thus $\sup_{|x|\leq R} M(B_{x,r}) \leq \bar{C}_R r^\alpha$ for any $r \in (0,1]$. Moreover, using union bound and the stationarity of GFF, we have for some absolute constant $C > 0$ that

$$\mathbb{E}\bar{C}_R^\theta \leq \sum_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \mathbb{E}\bar{C}_{z_k}^\theta \leq CR^2 \mathbb{E}\bar{C}_0^\theta.$$

By Lemma 3.1, we can show that $\bar{C}_R \lesssim_{X,\gamma,\alpha,m_1} R^{m_1}$ for $R \geq 1$ when $m_1 > 2/\theta$. Combining with the bound for θ, we get $m_1 > m_0(\gamma,\alpha) := \frac{\gamma^2}{2} + \frac{4\alpha\gamma^2}{(\alpha_1-\alpha)(\alpha_2-\alpha)}$. $\square$

It is natural to ask whether we can get similar estimates for the lower bound coefficients. Here we give the estimates but with some cost on the range of lower Hölder exponent α. We won't use the following proposition in the rest of this paper.

Proposition 3.4. *For any $\gamma \in (0,2)$ and $\alpha > \gamma^2/2 + 2\sqrt{2}\gamma + 2$ $(> \alpha_1)$, there is $m_{00}(\gamma,\alpha) > 0$ such that, there exists a random constant $\bar{c}_R$ depending on X, γ, α, R such that $\mathbb{P}$-a.s.*

$$\inf_{|x|\leq R} M(B_{x,r}) \geq \bar{c}_R r^\alpha \quad \text{for all } r \in (0,1]$$

and for any $\varepsilon > 0$ and any $R \geq 1$ we have

$$\bar{c}_R \gtrsim_{X,\gamma,\alpha,\varepsilon} R^{-m_{00}-\varepsilon}.$$

Proof. For each $n \in \mathbb{N}$ we partition $[-1,1]^2$ into 2^{2n} dyadic squares $\{I_n^j : j = 1,2,...,2^{2n}\}$ of equal size and define good events

$$A_n = \{\inf_{1\leq j\leq 2^{2n}} M(I_n^j) \geq 2^{-\alpha n}\}$$

and set $E_n = \cap_{k=n}^\infty A_k$ and $\tilde{E}_0 = E_0$, $\tilde{E}_n = E_n \setminus E_{n-1}$ for $n \in \mathbb{N}^*$. Note that $\tilde{E}_n \subseteq A_{n-1}^c$. Then for $p < 0$ by using Markov inequality, the stationarity of GFF and the power law of the Liouville measure we have

$$\begin{aligned}\mathbb{P}[\tilde{E}_{n+1}] \leq \mathbb{P}[A_n^c] &\leq 2^{p\alpha n}\mathbb{E}\left[\sum_{1\leq j\leq 2^{2n}} M\left(I_n^j\right)^p\right] \\ &\leq C_p 2^{-nK(\alpha,p)}\end{aligned}$$

where $K(\alpha,p) := \xi_M(p) - \alpha p - 2$. When $K(\alpha,p) > 0$ by Borel-Cantelli's lemma we have

$$\mathbb{P}[\cup_{n=0}^\infty \tilde{E}_n] = \mathbb{P}[\cup_{n=0}^\infty E_n] = 1 - \mathbb{P}[A_n^c \text{ i.o.}] = 1.$$

Define

$$\bar{c} := \begin{cases} 8^{-\alpha} & \text{on } \tilde{E}_0 \\ 8^{-\alpha} \wedge \inf_{|x|\le 1, r\in(2^{-n},1]} \frac{M(B_{x,r})}{r^\alpha} & \text{on } \tilde{E}_n \text{ for } n \in \mathbb{N}^*. \end{cases}$$

Note that $\bar{c}$ is almost surely well-defined because $\tilde{E}_n$ are disjoint and $\mathbb{P}[\cup_{n=0}^\infty \tilde{E}_n] = 1$. The $\mathcal{A}$-measurability of $\bar{c}$ can be shown in a similar way to the proof of the $\mathcal{A}$-measurability of $\bar{C}_0$ in Proposition 3.3. We claim $\mathbb{P}$-a.s. $M(B_{x,r}) \ge \bar{c} r^\alpha$ for any $|x| \le 1$ and $r \in (0,1]$. Indeed, on $\tilde{E}_n$, when $r \in (2^{-n}, 1]$ by the definition of $\bar{c}$ we have $M(B_{x,r}) \ge \bar{c} r^\alpha$; when $r \in (2^{-k-1}, 2^{-k}]$ for $k \ge n$, any ball $B_{x,r}$ contains at least 1 dyadic square I^j_{k+3} and each square I^j_{k+3} (of size 2^{-2-k}) has Liouville measure no less than $2^{-(k+3)\alpha}$, hence $M(B_{x,r}) \ge 2^{-\alpha(k+3)} \ge 8^{-\alpha} r^\alpha$.

Moreover, for $\theta > 0$ by Hölder inequality for $q^{-1} + q'^{-1} = 1$

$$\begin{aligned} \mathbb{E}(\bar{c})^{-\theta} &\le 8^{\alpha\theta} + \sum_{n=1}^\infty \mathbb{E}[\sup_{|x|\le 1, r\in(2^{-n},1]} r^{\alpha\theta} M(B_{x,r})^{-\theta}; \tilde{E}_n] \\ &\le 8^{\alpha\theta} + \sum_{n=1}^\infty \mathbb{E}[\sup_{1\le j\le 2^{2n+4}} M(I^j_{n+2})^{-\theta}; \tilde{E}_n] \\ &\le 8^{\alpha\theta} + \sum_{n=1}^\infty 2^{2n+4} \mathbb{E}[M(I^1_{n+2})^{-\theta q'}]^{1/q'} \mathbb{P}[\tilde{E}_n]^{1/q} \\ &\le 8^{\alpha\theta} + C_{\gamma,p,q,\theta} \sum_{n=1}^\infty 2^{2n} \cdot 2^{-n\xi(-\theta q')/q'} \cdot 2^{-(n-1)K(\alpha,p)/q} \end{aligned}$$

The above is finite if

$$2 - K(\alpha,p)/q - \xi(-\theta q')/q' < 0.$$

Solving the above inequality, we have

$$\theta < -\frac{\xi^{-1}(q'(2 - K(\alpha,p)/q))}{q'}$$

where $\xi^{-1}(x) = \frac{1}{2} + \frac{2}{\gamma^2} - \frac{1}{\gamma^2}\sqrt{(2+\gamma^2/2)^2 - 2\gamma^2 x}$ by the quadratic formula. Since $\theta > 0$ we need $2 - K(\alpha,p)/q < 0$. Set $p = p(\alpha) = \frac{2+\gamma^2/2-\alpha}{\gamma^2} (< 0)$, and noting that q can be chosen arbitrarily close to 1, we have $\alpha > 2 + \frac{\gamma^2}{2} + 2\sqrt{2}\gamma$.

Now do the same partition and reasoning for each region $z_k + [-1,1]^2$ where $z_k \in \mathbb{Z}^2$ and we get a sequence of $\bar{c}_{z_k}$ (defined similar to $\bar{c}$) with the same distribution as $\bar{c}$. Set $\bar{c}_R = \min_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \bar{c}_{z_k}$. Since for any ball $B_{x,r}$ with $|x| \le R$ and $r \in (0,1]$, one can find $z_k \in \mathbb{Z}^2 \cap B_{R+1}$ with $|x - z_k| \le 1$, we have $\inf_{|x|\le R} M(B_{x,r}) \ge \bar{c}_R r^\alpha$ for any $r \in (0,1]$. Moreover, using union bound and the stationarity of GFF, we have for some absolute constant $C > 0$ that

$$\mathbb{E}\bar{c}_R^{-\theta} \le \sum_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \mathbb{E}\bar{c}_{z_k}^{-\theta} \le CR^2 \mathbb{E}\bar{c}^{-\theta}.$$

By Lemma 3.1, we can show that $\bar{c}_R \gtrsim_{X,\gamma,\alpha,m} R^{-m}$ for $R \ge 1$ when $m > 2/\theta$. Combining with the bound for θ we get

$$m > 2/\theta > 2q'\gamma^2 / \left(\sqrt{(2+\gamma^2/2)^2 - 4q'\gamma^2 + \frac{q'}{q}(\alpha_2 - \alpha)(\alpha_1 - \alpha)} - \frac{\gamma^2}{2} - 2 \right) =: m_{00}(\gamma, \alpha)$$

where we have chosen some $q = q(\alpha)$ such that $2 - K(\alpha, p(\alpha))/q < 0$. $\square$

3.2 Exit time estimates

Lemma 3.5. *For any $\gamma \in (0,2)$, $q > 0$, $p > 1, p' := p/(p-1)$, $\kappa > p(2 + \tilde{\xi}(q))$, and any $\varepsilon > 0$, there exists a random constant $\hat{C}_R$ depending on X, γ, q, κ, R such that $\mathbb{P}$-a.s.*

$$\sup_{|x| \le R} E_x[\tau_{x,r}^{-q}] \le \hat{C}_R r^{-\kappa} \quad \text{for all } r \in (0,1],$$

and for any $R \ge 1$,

$$\hat{C}_R \lesssim_{X,\gamma,q,p,\kappa,\varepsilon} R^{2p'+\varepsilon}.$$

Proof. We follow the proof in [2, Proposition 3.2], but give the coefficient estimates depending on R. The main idea is the same as Proposition 3.3, i.e., to improve Borel-Cantelli's lemma and use the stationarity of the Liouville measure.

Let $\mu_{y,r}^z$ be the harmonic measure of the circle $\partial B_{y,r}$ viewed at $z \in \mathbb{C}$. In particular when $z = y$, $\mu_{y,r}^z$ is the uniform distribution on $\partial B_{y,r}$ and we set $\mu_{y,r} = \mu_{y,r}^y$. When $|z - y| \le r/2$ we have $\mu_{y,r}^z \le C\mu_{y,r}$ for some absolute constant $C > 0$. For $n \in \mathbb{N}$, set $r_n := 2^{-n}$ and $\Xi_n := \{(i2^{-n}, j2^{-n}) : i, j \in [-2^n, 2^n] \cap \mathbb{Z}\}$. In the proof of [2, Proposition 3.2], they obtained that

$$\mathbb{E}E_{\mu_{x,r_n}}\left[\tau_{x,2r_n}^{-q}\right] \le C_{\gamma,q} r^{-\tilde{\xi}(q)}.$$

Define the event

$$A_n := \{\max_{x \in \Xi_{n+1}} E_{\mu_{x,r_n}}[\tau_{x,2r_n}^{-q}] \le r_n^{-\kappa}\}, \quad E_n := \cap_{k=n}^{\infty} A_k$$

and $\tilde{E}_0 := E_0$, $\tilde{E}_n := E_n \setminus E_{n-1}$ for $n \in \mathbb{N}^*$. For $n \in \mathbb{N}$ we have using the stationarity of GFF and the power law of the Liouville measure that

$$\begin{aligned}\mathbb{P}[\tilde{E}_{n+1}] \le \mathbb{P}[A_n^c] &\le r_n^{\kappa} \sum_{x \in \Xi_{n+1}} \mathbb{E}E_{\mu_{x,r_n}}[\tau_{x,2r_n}^{-q}] \\ &\le r_n^{\kappa} \cdot (2^{n+1} + 1)^2 C_{\gamma,q} r_n^{-\tilde{\xi}(q)} \\ &\le 9C_{\gamma,q} r_n^{\kappa - \tilde{\xi}(q) - 2}.\end{aligned}$$

By Borel-Cantelli's lemma $\mathbb{P}[\cup_{n=0}^{\infty} \tilde{E}_n] = \mathbb{P}[\cup_{n=0}^{\infty} E_n] = 1 - \mathbb{P}[A_n^c \text{ i.o.}] = 1$.

Now define

$$\hat{C}_0 := \begin{cases} C8^{\kappa} & \text{on } \tilde{E}_0, \\ C\left(8^{\kappa} \vee \max_{x \in \Xi_{n+3}} E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]\right) & \text{on } \tilde{E}_n \text{ for } n \in \mathbb{N}^*. \end{cases}$$

Note that $\hat{C}_0$ is well-defined because $\tilde{E}_n$ are disjoint and $\mathbb{P}[\cup_{n=0}^{\infty} \tilde{E}_n] = 1$. We claim

$$E_x[\tau_{x,r}^{-q}] \le \hat{C}_0 r^{-\kappa}$$

for all $x \in B_1$ and $r \in (0, 1]$. Indeed, fix $n_0 \in \mathbb{N}$. When $r \in (2^{-n+2}, 1]$ for $n \geq n_0 + 3$, we have for any $x \in B_1$, there is some $x_i \in \Xi_{n+1}$ such that $|x - x_i| \leq r_{n+1}$. By the strong Markov property

$$E_x \tau_{x,r}^{-q} \leq E_{\mu_{x_i,r_n}^x}[\tau_{x,r}^{-q}] \leq E_{\mu_{x_i,r_n}^x}[\tau_{x_i,2r_n}^{-q}] \leq C E_{\mu_{x_i,r_n}}[\tau_{x_i,2r_n}^{-q}],$$

and this is at most $\hat{C}_0 \leq \hat{C}_0 r^{-\kappa}$ on $\tilde{E}_{n-2}$. Thus the claim holds on $\tilde{E}_{n_0}$ for $r \in (2^{-n_0}, 1]$. Moreover on $\tilde{E}_{n_0} \subseteq E_n$, if $r \in (2^{-n+2}, 2^{-n+3}]$ we have

$$E_x \tau_{x,r}^{-q} \leq C E_{\mu_{x_i,r_n}}[\tau_{x_i,2r_n}^{-q}] \leq C r_n^{-\kappa} \leq C 8^\kappa r^{-\kappa} \leq \hat{C}_0 r^{-\kappa}.$$

Hence the claim is true.

Next we examine the moment of $\hat{C}_0$. Let $p > 1$ and $p' = \frac{p}{p-1}$. Then

$$\begin{aligned}
\mathbb{E}\hat{C}_0^{1/p'} &\leq (C8^\kappa)^{1/p'} + \sum_{n=1}^{\infty} \mathbb{E}\left[\left(C \sup_{x \in \Xi_{n+3}} E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]\right)^{1/p'}; \tilde{E}_n\right] \\
&\leq (C8^\kappa)^{1/p'} + \sum_{n=1}^{\infty} \left[C\mathbb{E} \sup_{x \in \Xi_{n+3}} E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]\right]^{1/p'} \mathbb{P}[\tilde{E}_n]^{1/p} \\
&\leq (C8^\kappa)^{1/p'} + \sum_{n=1}^{\infty} \left[C\mathbb{E} \sum_{x \in \Xi_{n+3}} E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]\right]^{1/p'} \mathbb{P}[\tilde{E}_n]^{1/p}.
\end{aligned}$$

Using the stationarity of GFF and the bounds for $\mathbb{E}E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]$ and $\mathbb{P}[\tilde{E}_n]$, we get

$$\begin{aligned}
\mathbb{E}\hat{C}_0^{1/p'} &\leq (C8^\kappa)^{1/p'} + \sum_{n=1}^{\infty} (C2^{2(n+5)})^{1/p'} \left[\mathbb{E}E_{\mu_{x,r_{n+2}}}[\tau_{x,r_{n+1}}^{-q}]\right]^{1/p'} \mathbb{P}[\tilde{E}_n]^{1/p} \\
&\leq (C8^\kappa)^{1/p'} + \sum_{n=1}^{\infty} (C2^{2(n+5)})^{1/p'} (C_{\gamma,q} r_{n+2}^{-\tilde{\xi}(q)})^{1/p'} (9C_{\gamma,q} r_n^{\kappa - \tilde{\xi}(q) - 2})^{1/p} \\
&= (C8^\kappa)^{1/p'} + C_{\gamma,q,p,\kappa} \sum_{n=1}^{\infty} r_n^{\frac{1}{p}(\kappa-2) - \frac{2}{p'} - \tilde{\xi}(q)}.
\end{aligned}$$

When $\frac{1}{p}(\kappa - 2) - \frac{2}{p'} - \tilde{\xi}(q) > 0$, i.e. $\kappa > p(2 + \tilde{\xi}(q))$, we have $\mathbb{E}\hat{C}_0^{1/p'} < \infty$.

Now do the same partition and reasoning for each region $z_k + [-1, 1]^2$ where $z_k \in \mathbb{Z}^2$ and we get a sequence of $\hat{C}_{z_k}$ (defined similar to $\hat{C}_0$) with the same distribution as $\hat{C}_0$. For $R \geq 1$ set $\hat{C}_R := \max_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \hat{C}_{z_k}$. Then

$$\sup_{|x| \leq R} E_x[\tau_{x,r}^{-q}] \leq \hat{C}_R r^{-\kappa} \quad \text{for all } r \in (0, 1].$$

Moreover

$$\mathbb{E}[\hat{C}_R^{1/p'}] \leq \sum_{z_k \in \mathbb{Z}^2 \cap B_{R+1}} \mathbb{E}[\hat{C}_{z_k}^{1/p'}] \leq C R^2 \mathbb{E}[\hat{C}_0^{1/p'}].$$

By Lemma 3.1, we can show that for any $\varepsilon > 0$ we have $\mathbb{P}$-a.s. $\hat{C}_R \lesssim_{X,\gamma,q,p,\kappa,\varepsilon} R^{2p'+\varepsilon}$ for $R \geq 1$. □

Corollary 3.6. *Let q, p, p', κ and $\hat{C}_R$ be as in Lemma 3.5. For any $\beta > \kappa/q$ and $\varepsilon \in (0,1)$, with $\delta_R := (\varepsilon/\hat{C}_R)^{1/q}$, $\mathbb{P}$-a.s.*

$$\sup_{r\in(0,1]}\sup_{|x|\le R} P_x[\tau_{x,r} \le \delta_R r^\beta] \le \varepsilon.$$

Proof. By Lemma 3.5 and Markov inequality we have for any $x \in B_R$ and $r \in (0,1]$

$$P_x\left[\tau_{x,r} \le \delta_R r^\beta\right] = P_x\left[\tau_{x,r}^{-q} \ge \left(\delta_R r^\beta\right)^{-q}\right] \le \hat{C}_R \delta_R^q r^{\beta q-\kappa} \le \varepsilon.$$

□

Now we come to the main result of this subsection, which gives the exit time estimate of large balls.

Proposition 3.7. *Let q, p, p', κ and $\hat{C}_R$ be as in Lemma 3.5. Then $\mathbb{P}$-a.s. for any $\varepsilon \in (0, 1/4]$ and any $\beta > \kappa/q$ the following holds. Let $R \ge 1$ and $\delta_{2R} := (\varepsilon/\hat{C}_{2R})^{1/q}$. Then for some $c_{\beta,\varepsilon} > 0$ we have for any $t \in (0, R\delta_{2R}/(2\beta))$ and $r \in [2\beta t/\delta_{2R}, R]$,*

$$\sup_{|x|\le R} P_x[\tau_{x,r} \le t] \le \frac{1}{1-2\varepsilon}\exp\left(-c_{\beta,\varepsilon}\left(\frac{\delta_{2R} r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right).$$

Proof. For any $x \in B_R$ and $r \in (0,R]$ set $\tilde{r} = r/K \le 1$ for some $K > 2$ to be determined. Let θ be the shift operator for $\{Y_t\}_{t\ge0}$. Define

$$\tau_0 := 0, \quad r_0 := 0, \quad \tau_n := \tau_{Y_{\tau_{n-1}},\tilde{r}} \circ \theta_{\tau_{n-1}} + \tau_{n-1}, \quad r_n := |Y_{\tau_n} - Y_0|, \qquad n \ge 1$$

and $N := \min\{n : r_n > r/2\}$. Note that $\{\tau_n\}_{n\ge0}$ are stopping times w.r.t. the right-continuous filtration generated by Y, because for $n \ge 1$

$$\tau_n = \tau_{n-1} + \inf\{s \ge 0 : |Y_{s+\tau_{n-1}} - Y_{\tau_{n-1}}| > \tilde{r}\} = \inf\{s \ge \tau_{n-1} : |Y_s - Y_{\tau_{n-1}}| > \tilde{r}\}$$

and hence by the continuity of sample paths of Y (and induction on that τ_{n-1} is a stopping time)

$$\{\tau_n < t\} = \cup_{s\in[0,t)\cap\mathbb{Q}}\{\tau_{n-1} \le s, |Y_s - Y_{\tau_{n-1}}| > \tilde{r}\} \in \sigma\{Y_s; s \le t\}.$$

By the strong Markov property we have

$$\begin{aligned} P_x\left[\tau_n \le t, N = n\right] &\le P_x\left[\max_{0\le i\le n-1}|Y_{\tau_i}| < 2R,\ \#\{i \in \{1,...,n\} : \tau_i - \tau_{i-1} \le \delta_{2R}\tilde{r}^\beta\} \ge n - t/(\delta_{2R}\tilde{r}^\beta)\right] \\ &\le 2^n \varepsilon^{n-t/(\delta_{2R}\tilde{r}^\beta)}. \end{aligned}$$

Note that $N \ge K/2$ since the path of Y needs to exit at least $\lceil K/2\rceil$ balls of radius $\tilde{r}$ before achieving $r_n > r/2$, and that $\tau_N \le \tau_{Y_0,r}$ by $\tilde{r} < r/2$. So by the estimates above we get

$$\begin{aligned}
P_x[\tau_{x,r} \le t] &\le \sum_{n=\lceil K/2\rceil}^{\infty} P_x\left[\tau_n \le t, N=n\right] \\
&\le \sum_{n=\lceil K/2\rceil}^{\infty} 2^n \varepsilon^{n-t/(\delta_{2R}\tilde{r}^\beta)} \\
&= \frac{\varepsilon^{-t/(\delta_{2R}\tilde{r}^\beta)}(2\varepsilon)^{\lceil K/2\rceil}}{1-2\varepsilon} \\
&\le \frac{1}{1-2\varepsilon}\exp\left(\frac{K}{2}\log(2\varepsilon) - \frac{K^\beta t}{\delta_{2R}r^\beta}\log\varepsilon\right).
\end{aligned}$$

Set $K = (\frac{\delta_{2R}r^\beta}{2\beta t})^{\frac{1}{\beta-1}}$. The assertion is obvious if $K \le 2$, by choosing $c_{\beta,\varepsilon} \le \frac{1}{2}(2\beta)^{\frac{-1}{\beta-1}}\log\frac{1}{1-2\varepsilon}$. When $K > 2$ and $r \ge 2\beta t/\delta_{2R}$ we have $K \ge r$ so that $\tilde{r} \le 1$, then we get

$$P_x[\tau_{x,r} \le t] \le \frac{1}{1-2\varepsilon}\exp\left(-c_{\beta,\varepsilon}(\delta_{2R}r^\beta/t)^{\frac{1}{\beta-1}}\right)$$

for $c_{\beta,\varepsilon} = \frac{1}{2}(\beta-2)(2\beta)^{-\frac{\beta}{\beta-1}}\log\frac{1}{\varepsilon} > 0$. □

3.3 Liouville heat kernel upper bounds

We first establish the on-diagonal bound of the Liouville heat kernel at large distances.

Proposition 3.8. *For any $\gamma \in (0,2)$ and $\alpha \in (0,\alpha_2)$ we have $\mathbb{P}$-a.s. for any $R > 2$ and $t \in (0,1/2]$,*

$$\sup_{|x|,|y|<R} p_t^R(x,y) \lesssim_{X,\gamma,\alpha} (\log R)t^{-1}\log t^{-1}$$

where p_t^R is the Liouville heat kernel killed upon exiting B_R, and

$$\sup_{|x|,|y|<R} p_t(x,y) \lesssim_{X,\gamma,\alpha,q,\kappa} (\log R)t^{-1}\log t^{-1}$$

where q,κ are from Lemma 3.5 and $q > 2$.

Proof. To get the bound for $p_t^R(x,y)$, we first show a Faber-Krahn-type inequality (an estimate for the smallest eigenvalue of the generator). For a fixed non-empty bounded open set $U \subseteq B_R$, let $\lambda_1(U)$ be the smallest eigenvalue of the generator $-\mathcal{L}_U$ of the LBM killed upon leaving U and $G_U f(x) = (-\mathcal{L}_U)^{-1}f(x) = \int g_U(x,y)f(y)M(dy)$ where g_U is the Green kernel of the standard Brownian motion killed upon leaving U. For g_U we have (see e.g. [18, Lemma 3.37]) for any $x,y \in U \subseteq B_R$

$$\begin{aligned}
g_U(x,y) &\le g_{B_{2R}}(x,y) \\
&= \frac{1}{\pi}\log\frac{1}{|x-y|} + \frac{1}{\pi}\mathbb{E}_x\left[\log|W_{T_{2R}} - y|\right] \\
&\le \frac{1}{\pi}\log\frac{1}{|x-y|} + \frac{1}{\pi}\log(3R)
\end{aligned}$$

where $T_{2R} = \inf\{t \geq 0 : W_t \notin B_{2R}\}$. We have for $\beta > 0$

$$\begin{aligned}
\|G_U 1\|_\infty &= \sup_{x\in U} \int g_U(x,y) M(dy) \\
&\lesssim \sup_{x\in U} \int_U \left(\log(3R) + \log \frac{1}{|x-y|} \right) M(dy) \\
&\lesssim M(U) \left[\log(3R) + \sup_{x\in U} \beta^{-1} \int_U \log \frac{1}{|x-y|^\beta} \frac{M(dy)}{M(U)} \right] \\
&\lesssim M(U) \left[\log(3R) + \sup_{x\in U} \beta^{-1} \log \int_U \frac{1}{|x-y|^\beta} \frac{M(dy)}{M(U)} \right],
\end{aligned}$$

where the last inequality follows from Jensen's inequality. By Proposition 3.3, one can get for any $x \in B_R$

$$\begin{aligned}
\int_U \frac{1}{|x-y|^\beta} \frac{M(dy)}{M(U)} &\leq 1 + \sum_{n=1}^\infty \int_{U\cap\{2^{-n}<|x-y|\leq 2^{-n+1}\}} \frac{1}{|x-y|^\beta} \frac{M(dy)}{M(U)} \\
&\leq 1 + \sum_{n=1}^\infty \frac{2^{\beta n} M(\{|x-y| \leq 2^{-n+1}\})}{M(U)} \\
&\leq 1 + \frac{2^\alpha \bar{C}_R}{M(U)} \sum_{n=1}^\infty 2^{(\beta-\alpha)n},
\end{aligned}$$

where $\alpha > 0$ is from Proposition 3.3. Choose $\beta = \alpha/2$, and the sum $C_\alpha = 2^\alpha \sum_{n=1}^\infty 2^{-\alpha n/2}$ (> 1) is finite. Hence

$$\begin{aligned}
\|G_U 1\|_\infty &\lesssim M(U) \left[\log(3R) + \log(1 + \frac{C_\alpha \bar{C}_R}{M(U)}) \right] \\
&\lesssim M(U) \left[\log(3R) + \log(C_\alpha \bar{C}_R) + \log(\frac{1}{C_\alpha \bar{C}_R} + \frac{1}{M(U)}) \right] \\
&\lesssim \left(\log(3R) + \log(C_\alpha \bar{C}_R) \right) M(U) \log\left(2 + \frac{1}{M(U)} \right).
\end{aligned}$$

By [13, Lemma 3.2] we know $\lambda_1(U)^{-1} \leq \|G_U 1\|_\infty$ and hence

$$\lambda_1(U) \gtrsim \frac{C_9}{M(U) \log\left(2 + \frac{1}{M(U)}\right)}$$

where $C_9^{-1} = \log(3R) + \log(C_\alpha \bar{C}_R) \lesssim_{X,\gamma,\alpha} \log(R)$ provided $R > 2$.

Now we apply the proof of [2, Proposition 5.3]. Let $T_t^{B_R}$ be the semigroup operator associated to the heat kernel p_t^R. They obtained that

$$\|T_t^{B_R}\|_{L^1(B_R)\to L^\infty(B_R)} \leq m(t)$$

for some function $m(t)$. By following their proof, it is straightforward to check that for $t \in (0, 1/2]$

$$m(t) \leq 4C_9^{-1} t^{-1} \log t^{-1} \lesssim_{X,\gamma,\alpha} (\log R) t^{-1} \log t^{-1}.$$

Hence the bound for $p_t^R(x, y)$ follows.

To extend the bound to $p_t(x, y)$, we can use Kigami's iteration argument [14, Lemma 5.6]. Let $Q_t(R) := C_Q (\log R) t^{-1} \log t^{-1}$ where $C_Q = C_Q(X, \gamma, \alpha)$ is the constant so that

$$\sup_{|x|,|y| \leq R} p_t^R(x, y) \leq Q_t(R).$$

Notice that for any $s \in [t/2, t]$, $\lambda \in [1, 4]$ we have

$$Q_s(\lambda R) \leq 12 Q_t(R).$$

Let $L := 12$ and $\varepsilon := \frac{1}{2L}$. Now choose $R_0 = R_0(X, \gamma, q, \kappa) > 0$ large enough so that for any $R \geq R_0$,

$$\delta_{2R} R > \beta \quad \text{and} \quad 2 \exp\left(-c_{\beta,1/4} (2\delta_{2R} R^\beta)^{\frac{1}{\beta-1}}\right) \leq \varepsilon.$$

This can be done because when $1 - 2p'/q > 0$ we have $\delta_{2R} R \to \infty$ as $R \uparrow \infty$. Then from Proposition 3.7 we get

$$\sup_{|x| \leq R} P_x[\tau_{0,4R} \leq t] \leq \sup_{|x| \leq R} P_x[\tau_{x,R} \leq t] \leq \varepsilon.$$

Define for $k = 0, 1, 2, \ldots$ the sequences

$$t_k = \frac{1}{2}(1 + 2^{-k}) t, \ R_k = 4^k R_0, \ B_k = B_{R_k}.$$

Let

$$\sup_U p_t^R := \sup_{x,y \in U} p_t^R(x, y)$$

for any set $U \subseteq \mathbb{C}$. We apply the inequality [14, Theorem 4.6] to get

$$\begin{aligned}
\sup_{B_k} p_{t_k}^R &\leq \sup_{B_{k+1}} p_{2^{-(k+2)}t}^{R_{k+1}} + \varepsilon \sup_{B_{k+1}} p_{t_{k+1}}^R \\
&\leq Q_{2^{-(k+2)}t}(R_{k+1}) + \varepsilon \sup_{B_{k+1}} p_{t_{k+1}}^R \\
&\leq L^{k+2} Q_t(R_0) + \varepsilon \sup_{B_{k+1}} p_{t_{k+1}}^R
\end{aligned}$$

as long as $R_{k+1} \leq R$. Let $n \geq 1$, set $R = R_n$ and by iteration we get

$$\begin{aligned}
\sup_{B_0} p_t^{R_n} &\leq L^2 \left(1 + L\varepsilon + (L\varepsilon)^2 + \ldots\right) Q_t(R_0) + \varepsilon^n \sup_{B_n} p_{t_n}^{R_n} \\
&\leq 2L^2 Q_t(R_0) + (L\varepsilon)^n Q_t(R_0).
\end{aligned}$$

Since $\lim_{n\to\infty} p_t^{R_n}(x,y) = p_t(x,y)$ for any $x,y \in B_0$ by [2, Proof of Theorem 5.1 for unbounded U], let $n \to \infty$ and we get

$$\sup_{B_0} p_t \le 2L^2 Q_t(R_0) = 288 C_Q (\log R_0) t^{-1} \log t^{-1}.$$

Since $R_0 = R_0(X,\gamma,q,\kappa)$ can be chosen to be any larger value, it follows that

$$\sup_{|x|,|y|\le R} p_t(x,y) \lesssim_{X,\gamma,\alpha,q,\kappa} (\log R) t^{-1} \log t^{-1}$$

for any $R > 2$. □

Now comes the main result of this paper.

Theorem 3.9. *For any $\gamma \in (0,2)$, $p > 1$, $p' = \frac{p}{p-1}$, $q > 2p'$, $\alpha \in (0,\alpha_2)$ and $\beta > \frac{p}{q}(2+\tilde{\xi}(q))$, there exist $c_*, C_* > 0$ depending on X,γ,q,p,β, such that $\mathbb{P}$-a.s. for any $t \in (0,1/2]$, $R > 2$, and $x,y \in B_R$ with $|x-y| > c_* R^{2p'/q} t$, we have*

$$p_t(x,y) \lesssim_{X,\gamma,\alpha,q,p,\beta} (\log R) t^{-1} \log t^{-1} \exp\left(-C_* \left(\frac{|x-y|^\beta}{tR^{2p'/q}}\right)^{\frac{1}{\beta-1}}\right).$$

Proof. We apply a result in [10, Theorem 10.4] (see also [11, Theorem 5.1]) that, if U,V are non-empty open subsets of $\mathbb{C}$ with $U \cap V = \varnothing$, then for any $(x,y) \in V \times U$,

$$p_t(x,y) \le \psi^V\left(x,\frac{t}{2}\right) \sup_{t/2\le s\le t} \sup_{v\in\partial V} p_s(v,y) + \psi^U\left(y,\frac{t}{2}\right) \sup_{t/2\le s\le t} \sup_{u\in\partial U} p_s(u,x)$$

where $\psi^V(z,s) = P_z[\tau_V \le s]$, and τ_V is the first exit time of Liouville Brownian motion from V. Note that we can change "esup" to "sup" because $p_t(x,y)$ has been proved to have a (t,x,y)-jointly continuous version (see [2, Theorem 1.1]) and $\psi^V(x,t/2)$ is continuous in $x \in V$ by [2, Theorem 5.1(ii)].

Set $r = |x-y|/2$, $V = B_{x,r}$, $U = B_{y,r}$. Applying Proposition 3.7 with $\varepsilon = 1/4$ and $\kappa = \frac{1}{2}(q\beta + p(2+\tilde{\xi}(q)))$ (so that $\beta > \kappa/q$) leads to

$$\psi^V\left(x,\frac{t}{2}\right) \vee \psi^U\left(y,\frac{t}{2}\right) \le 2\exp\left(-c_{\beta,1/4}(2\delta_{2R} r^\beta/t)^{\frac{1}{\beta-1}}\right)$$

provided $\delta_{2R} r > \beta t$. In particular by Lemma 3.5 it is true if $|x-y| > c_* R^{2p'/q} t$ for some $c_* = c_*(X,\gamma,q,p,\beta) > 0$.

Furthermore, by Proposition 3.8 we have

$$\sup_{t/2\le s\le t} \sup_{v\in\partial V} p_s(v,y) \vee \sup_{t/2\le s\le t} \sup_{u\in\partial U} p_s(u,x) \lesssim_{X,\gamma,\alpha,q,\kappa} (\log R) t^{-1} \log t^{-1}.$$

Hence

$$\begin{aligned} p_t(x,y) &\lesssim_{X,\gamma,\alpha,q,p,\beta} (\log R) t^{-1} \log t^{-1} \exp\left(-\frac{1}{2} c_{\beta,1/4}(\delta_{2R}|x-y|^\beta/t)^{\frac{1}{\beta-1}}\right) \\ &\lesssim_{X,\gamma,\alpha,q,p,\beta} (\log R) t^{-1} \log t^{-1} \exp\left(-C_* \left(\frac{|x-y|^\beta}{tR^{2p'/q}}\right)^{\frac{1}{\beta-1}}\right) \end{aligned}$$

for some $C_* = C_*(X,\gamma,q,p,\beta) > 0$. □

Corollary 3.10. *Under the setting of Theorem 3.9, set $R = |x| \vee |y| \vee 2$. If in addition $R \le c|x-y|$ for some $c > 0$, then*

$$p_t(x,y) \lesssim_{X,\gamma,\alpha,q,p,\beta} (\log R)t^{-1}\log t^{-1} \exp\left(-\tilde{C}_* \left(\frac{|x-y|^{\beta-2p'/q}}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for some $\tilde{C}_* = \tilde{C}_*(X,\gamma,q,p,\beta,c) > 0$.

Corollary 3.11. *Under the setting of Theorem 3.9, for any $t \in (0,1/2]$ and $R_0 > 1$, there exists $R_1 = R_1(c_*, R_0, p, q) > R_0$ such that for any $y \in B_{R_0}$, $x \notin B_{R_1}$, we have*

$$p_t(x,y) \lesssim_{X,\gamma,\alpha,q,p,\beta} \exp\left(-0.5\tilde{C}_* \left(\frac{|x-y|^{\beta-2p'/q}}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for some $\tilde{C}_* = \tilde{C}_*(X,\gamma,q,p,\beta) > 0$.

Proof. Choose R_1 such that $R_1 > (c_*/2+1)^{\frac{q}{q-2p'}} \vee (R_0+1)^{\frac{q}{2p'}} \vee (2R_0)$ (recall that $q-2p' > 0$). Then for any $y \in B_{R_0}$, $x \notin B_{R_1}$ we have $2|x-y| \ge 2(|x|-R_0) \ge R$ and $|x-y| > c_* R^{2p'/q}t+1$, where $R = |x| \vee |y|$. The result follows from Corollary 3.10 with $c = 2$ by absorbing $(\log R)t^{-1}\log t^{-1}$ into the exponential. □

Corollary 3.12. *Liouville Brownian motion is C_0-Feller in the sense that T_t is a positive contraction strongly continuous semigroup on C_0, where C_0 is the space of continuous functions on $\mathbb{C}$ vanishing at infinity.*

4 Appendix: A simple proof of the Feller property

In this section, we give a simple proof of the C_0-Feller property without using heat kernel estimates.

We slightly change the notation. Let X be a whole plane (massive) Gaussian free field defined on some probability space Ω with the law denoted by $\mathbb{P}^X$, and B be a Brownian motion defined on another probability space. Let $\mathbb{P}_x^B$ be the law of Brownian motion starting from $x \in \mathbb{C}$. By making the product space and setting $\mathbb{P}_x = \mathbb{P}^X \otimes \mathbb{P}_x^B$, then X and B are independent under $\mathbb{P}_x$. We use $\mathbb{E}_x$, $\mathbb{E}_x^B$ and $\mathbb{E}^X$ to mean taking expectation under $\mathbb{P}_x, \mathbb{P}_x^B$ and $\mathbb{P}^X$ respectively. When $x = 0$, we drop the subscript.

Let F denote the PCAF of B whose Revuz measure is the Liouville measure and $\bar{F}$ be the inverse of F, i.e., $\bar{F}(t) = F^{-1}(t) = \inf\{s \ge 0 : F(s) > t\}$. We denote the Liouville Brownian motion by $Y_t = B_{\bar{F}(t)}$, and define the running supremum $Y_t^* := \max_{s\le t}|Y_s - Y_0|$.

Let $\mathbb{D}_R$ be the open disk with center 0 and radius $R > 0$ and $\bar{\sigma}_R(dx)$ be the uniform probability measure on the circle $\partial\mathbb{D}_R$. For a finite set S, we use $|S|$ to denote the number of elements in S.

The following discussion is for $\mathbb{P}^X$-a.e. element of Ω. Recall that it has already been proved that T_t maps C_b to C_b, where C_b is the set of bounded continuous functions on $\mathbb{C}$. Fix $t > 0$. To show T_t maps C_0 to C_0, it is enough to show that for any $R > 0$,

$$\lim_{x\to\infty} \mathbb{P}_x^B[Y_t \in \mathbb{D}_R] = 0.$$

Indeed, for any $f \in C_0$ and $\varepsilon > 0$ there is a continuous function $f_\varepsilon \in C_K$ with compact support such that $\|f - f_\varepsilon\|_\infty < \varepsilon$. Choose R large enough so that the support of f_ε is contained in $\mathbb{D}_R$, then $|T_t f(x)| \le \varepsilon + \|f_\varepsilon\|_\infty \mathbb{P}_x^B[Y_t \in \mathbb{D}_R]$. Let $x \to \infty$ and then $\varepsilon \to 0$, and we get $\lim_{x\to\infty} T_t f(x) = 0$.

Now fix $R > 0$ and $t > 0$. Define $g(x) = g(x, X) := \mathbb{P}_x^B[Y_t^* \ge |x| - R]$.

Lemma 4.1. *Let $\theta \in \mathbb{C}$ and $|\theta| = 1$. Then $g(\theta x)$ and $g(x)$ have the same law under $\mathbb{P}^X$. In particular we have $\mathbb{E}^X g(x) = \mathbb{E}^X g(|x|)$.*

Proof. Let $X^\theta = X(\cdot/\theta)$ and $B^\theta = \theta B$. First we show that $\mathbb{P}^X$-a.s., $F_t(X, B) = F_t(X^\theta, B^\theta)$ P_x^B-a.s. for any $x \in \mathbb{C}$. Indeed, we have

$$F^n(t) = \int_0^t \exp\left(\gamma X_n(B_s) - \frac{\gamma^2}{2}\mathbb{E}^X\left[X_n(B_s)^2\right]\right) ds = \int_0^t \exp\left(\gamma X_n^\theta(B_s^\theta) - \frac{\gamma^2}{2}\mathbb{E}^X\left[X_n^\theta(B_s^\theta)^2\right]\right) ds.$$

Let $n \to \infty$ and by the uniqueness of the limit we have $F_t(X, B) = F_t(X^\theta, B^\theta)$, and consequently $\bar{F}_t(X, B) = \bar{F}_t(X^\theta, B^\theta)$.

Notice that

$$\begin{aligned}
g(x, X) &= \mathbb{P}_x^B[Y_t^* \ge |x| - R] \\
&= \mathbb{P}_x^B[\max_{s\le t} |B_{\bar{F}_s(X,B)} - B_0| \ge |x| - R] \\
&= \mathbb{P}_x^B[\max_{s\le t} |\theta B_{\bar{F}_s(X^\theta,B^\theta)} - \theta B_0| \ge |x| - R] \\
&= \mathbb{P}_x^B[\max_{s\le t} |B^\theta_{\bar{F}_s(X^\theta,B^\theta)} - B^\theta_0| \ge |x| - R] \\
&= \mathbb{P}_{\theta x}^B[\max_{s\le t} |B_{\bar{F}_s(X^\theta,B)} - B_0| \ge |\theta x| - R] \\
&= g(\theta x, X^\theta).
\end{aligned}$$

Since X and X^θ have the same law under $\mathbb{P}^X$ by the rotation invariance of the covariance function G_m, we see that $g(\theta x)$ and $g(x)$ also have the same law under $\mathbb{P}^X$. Now fix $x \in \mathbb{C}$, choose θ such that $\theta x = |x|$, take the expectation, and we get $\mathbb{E}^X g(x) = \mathbb{E}^X g(|x|)$. □

Lemma 4.2. *Let $x_n = n \in \mathbb{C}$. Then $\lim_{n\to\infty} \mathbb{E}^X g(x_n) = 0$.*

Proof. For $x \in \mathbb{C} \setminus \{0\}$ and $\varepsilon > 0$, set $s = \varepsilon |x|^2$, and we get

$$\begin{aligned}
g(x) &\le \mathbb{P}_x^B\left[Y_t^* \ge |x| - R, \bar{F}(t) \le s\right] + \mathbb{P}_x^B\left[\bar{F}(t) > s\right] \\
&\le \mathbb{P}^B\left[\max_{l\le s} |B_l| \ge |x| - R\right] + \mathbb{P}_x^B\left[\frac{\bar{F}(t)}{|x|^2} > \varepsilon\right] \\
&= \mathbb{P}^B\left[\max_{l\le 1} |B_l| \ge \frac{|x| - R}{\sqrt{\varepsilon}|x|}\right] + \mathbb{P}_x^B\left[\frac{\bar{F}(t)}{|x|^2} > \varepsilon\right].
\end{aligned}$$

Thus by the translation invariance of the law of X we have

$$\mathbb{E}^X g(x_n) \le \mathbb{P}^B \left[\max_{l \le 1} |B_l| \ge \frac{1}{2\sqrt{\varepsilon}} \right] + \mathbb{P}[\bar{F}(t)/n^2 > \varepsilon]$$

provided n is large enough so that $R/n \le 1/2$. Now let $n \to \infty$, then $\varepsilon \to 0$ and we get the desired result. □

Now we are ready to prove that T_t is Feller.

Theorem 4.3. *$\mathbb{P}^X$-a.s., T_t maps C_0 to C_0.*

Proof. By Lemma 4.1 we have

$$\mathbb{E}^X \int g(x)\bar{\sigma}_n(dx) = \int \mathbb{E}^X g(n)\bar{\sigma}_n(dx) = \mathbb{E}^X g(n).$$

By Lemma 4.2 we get

$$\lim_{n\to\infty} \mathbb{E}^X \int g(x)\bar{\sigma}_n(dx) = \lim_{n\to\infty} \mathbb{E}^X g(n) = 0.$$

Thus there is a subsequence of n along which $\int g(x)\bar{\sigma}_n(dx) \to 0$ $\mathbb{P}^X$-a.s.. Then for any $\varepsilon > 0$ and $\delta > 0$, there is some $n > R$ sufficiently large such that

$$\bar{\sigma}_n(\{x \in \partial\mathbb{D}_n : g(x) > \varepsilon\}) < \delta.$$

Set $S_n = \{x \in \partial\mathbb{D}_n : g(x) \le \varepsilon\}$, then we have $\bar{\sigma}_n(S_n^c) \le \delta$.

Let $\tau_n = \inf\{s > 0 : Y_s \in \partial\mathbb{D}_n\}$, then $|Y_{\tau_n}| = n$. When $|x| > n > R$, using the strong Markov property (see, e.g., [12, Proposition 3.4]), we have

$$\begin{aligned}
\mathbb{P}_x^B[Y_t \in \mathbb{D}_R] &= \mathbb{P}_x^B[Y_t \in \mathbb{D}_R, \tau_n < t] \\
&= \int_{\{\tau_n < t\}} \mathbb{P}^B_{Y_{\tau_n}(\omega)}[Y_{t-\tau_n(\omega)} \in \mathbb{D}_R]\mathbb{P}_x^B(d\omega) \\
&\le \mathbb{E}_x^B \left[P^B_{Y_{\tau_n}}[Y_t^* \ge n - R] \right] \\
&\le \mathbb{E}_x^B \left[g(Y_{\tau_n}), Y_{\tau_n} \in S_n \right] + \mathbb{P}_x^B[Y_{\tau_n} \in S_n^c] \\
&= \int_{S_n} g(z)\mu_{x,n}(dz) + \mu_{x,n}(S_n^c)
\end{aligned}$$

where $\mu_{x,n}(dz) = \mathbb{P}_x^B[Y_{\tau_n} \in dz]$, which is the harmonic measure of the Brownian motion viewed at x.

We claim $\mu_{x,n} \to \bar{\sigma}_n$ in total variation as $x \to \infty$. Indeed. Let $\phi(z) = n^2 z/|z|^2$. Notice that ϕ is analytic on $\mathbb{C} \setminus \{0\}$ and $\phi|_{\partial\mathbb{D}_n}$ is the identity map. We have

$$\mu_{x,n}(dz) = \mathbb{P}_x^B[B_{\bar{F}(\tau_n)} \in dz] = \mathbb{P}_{x'}^{B'}[{B'}_{{\tau_n}'} \in dz]$$

where $x' = \phi(x)$, $B' = \phi(B)$ (which is a time-change of a Brownian motion) and ${\tau_n}' = \inf\{s > 0 : B'_s \in \partial\mathbb{D}_n\}$. Thus

$$\mu_{x,n}(dz) = \mu_{x',n}(dz) = p_n(x', z)\bar{\sigma}_n(dz)$$

where $p_n(x', z)$ is the Poisson kernel on $\partial\mathbb{D}_n$. Hence

$$\|\mu_{x,n} - \bar{\sigma}_n\|_{\text{total variation}} = \int |p_n(x', z) - 1|\bar{\sigma}_n(dz) \to 0$$

as $x' \to 0$ ($x \to \infty$). So we have $\mathbb{P}^X$-a.s.

$$\begin{aligned}\limsup_{x\to\infty} \mathbb{P}_x^B[Y_t \in \mathbb{D}_R] &\le \int_{S_n} g(z)\bar{\sigma}_n(dz) + \bar{\sigma}_n(S_n^c) \\ &\le \varepsilon + \delta.\end{aligned}$$

Let $\varepsilon \to 0$ and $\delta \to 0$, and combining the discussion at the very beginning of this section, we complete the proof. □

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