

AFFINE KILLING REEB VECTOR FIELD FOR A REAL HYPERSURFACE IN THE COMPLEX QUADRIC

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ABSTRACT. In this paper, first we introduce a general notion of affine Killing vector fields on the complex quadric Q^m , which is weaker than usual Killing vector field. Next, we give a complete classification of Hopf real hypersurfaces M with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$.

1. INTRODUCTION

It is well known that the importance of Killing vector fields on Riemannian manifolds and Mathematical Physics is highly evaluated (see [2], [22], [28], and [34]). Recall that a vector field V on a Riemannian manifold (\bar{M}, g) is said to be *Killing* if the Lie derivative of the metric tensor g along the direction of V is invariant, that is,

$$(1.1) \quad (\mathcal{L}_V g)(X, Y) = 0,$$

or equivalently,

$$(1.2) \quad g(\bar{\nabla}_X V, Y) = -g(\bar{\nabla}_Y V, X), \quad X, Y \in \mathfrak{X}(\bar{M}),$$

where $\bar{\nabla}$ denotes the Riemannian connection on (\bar{M}, g) and $\mathfrak{X}(\bar{M})$ the set of differentiable vector fields on \bar{M} . In terms of local components, (1.1) can be expressed as $\mathcal{L}_V g_{ij} = 0$.

When we consider a real hypersurface (M, g) in a Hermitian symmetric space (\bar{M}, g) , there exists a Reeb vector field ξ on M defined by $\xi = -JN$, where J denotes the Kähler structure on \bar{M} . Then, the Reeb vector field ξ is Killing (or *M has isometric Reeb flow*) if and only if the Lie derivative of the induced metric tensor g along the Reeb direction vanishes, $\mathcal{L}_\xi g = 0$. By using the Lie algebraic methods given in [1] and [3], Berndt and Suh [7] gave a complete classification of real hypersurfaces with isometric Reeb flow in Hermitian symmetric spaces.

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On a Riemannian manifold (\bar{M}, g) we say that a vector field V is *affine Killing* if it satisfies

$$\mathcal{L}_V \Gamma_{jk}^i = \bar{\nabla}_j \bar{\nabla}_k V^i + R^i_{kmj} V^m = 0$$

where Γ_{jk}^i and $\bar{\nabla}$ denote Christoffel symbols and the Riemannian connection defined on (\bar{M}, g) . From such a view point let us define an affine Killing vector field on a Riemannian manifold (\bar{M}, g) as follows:

Definition 1.1. Let (\bar{M}, g) be a Riemannian manifold with Riemannian connection $\bar{\nabla}$. A vector field V is said to be an *affine Killing vector field*, if it satisfies

$$(1.3) \quad (\mathcal{L}_V \bar{\nabla})(X, Y) = 0,$$

or equivalently,

$$(1.4) \quad [V, \bar{\nabla}_X Y] - \bar{\nabla}_{[V, X]} Y - \bar{\nabla}_X [V, Y] = 0$$

for all differentiable vector fields X and $Y \in \mathfrak{X}(\bar{M})$. In particular, the Reeb vector field ξ is said to be *affine Killing*, if $(\mathcal{L}_\xi \bar{\nabla})(X, Y) = 0$ (see also [11] and [13]).

In fact, the Riemannian connection $\bar{\nabla}$ of \bar{M} is the unique linear connection being torsion-free defined by $\bar{\nabla}(X, Y) = \bar{\nabla}_X Y$ for any $X, Y \in \mathfrak{X}(\bar{M})$. From this and the properties of Lie derivative, $\mathcal{L}_X Y = [X, Y]$ and $[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X$, we get

$$(1.5) \quad \begin{aligned} (\mathcal{L}_V \bar{\nabla})(X, Y) &= \mathcal{L}_V(\bar{\nabla}(X, Y)) - \bar{\nabla}(\mathcal{L}_V X, Y) - \bar{\nabla}(X, \mathcal{L}_V Y) \\ &= [V, \bar{\nabla}_X Y] - \bar{\nabla}_{[V, X]} Y - \bar{\nabla}_X [V, Y] \\ &= \bar{\nabla}_V \bar{\nabla}_X Y - \bar{\nabla}_{\bar{\nabla}_X V} Y - \bar{\nabla}_{[V, X]} Y - \bar{\nabla}_X \bar{\nabla}_V Y + \bar{\nabla}_X \bar{\nabla}_Y V \\ &= \bar{R}(V, X)Y - \bar{\nabla}_{\bar{\nabla}_X V} Y + \bar{\nabla}_X \bar{\nabla}_Y V, \end{aligned}$$

where the curvature tensor \bar{R} of M is a (1,3) tensor field defined by

$$(1.6) \quad \bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z$$

for any $X, Y, Z \in \mathfrak{X}(\bar{M})$.

In [35], Yano proved the existence of Killing vector fields on a compact orientable Riemannian manifold. Moreover, from the integrability condition of Killing vector fields we see that *every Killing vector field on a Riemannian manifold (\bar{M}, g) is affine Killing, but, the converse is not necessarily true* (see [11], [34], and [35]). Thus, in this paper we are interested in the study of real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m .

The complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [4], [6], [12], and [16]). Accordingly, Q^m admits two important geometric structures, so-called a real structure A and a complex structure J which anti-commute with each other, that is, $AJ = -JA$. By using the method of Lie algebra in [17], the triple (Q^m, J, g) is a Hermitian

symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see also [12], [27], and [29]).

As a typical characterization, Berndt-Suh [5] considered a notion of isometric Reeb flow for real hypersurfaces in Q^m and gave a classification theorem as follows:

Theorem A ([5]). *Let M be a connected orientable real hypersurface in the complex quadric Q^m , $m \geq 3$. Then, the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^{2k} .*

It can be easily checked that the isometric Reeb flow is equivalent to the fact that the shape operator S of M commutes with the structure tensor ϕ , that is, $S\phi = \phi S$. From this, a real hypersurface M with isometric Reeb flow in Q^m is Hopf. The notion of *Hopf* means that the Reeb vector field ξ of M is principal for the shape operator S of M , that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. If the Reeb function $\alpha = g(S\xi, \xi)$ identically vanishes on M , we say that M has vanishing geodesic Reeb flow.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m . The complex quadric Q^m is a Hermitian symmetric space of rank 2. So, there exist two types of singular tangent vectors in Q^m : Let $V(A) = \{Z \in T_{[z]}Q^m \mid AZ = Z\}$ and $JV(A) = \{Z \in T_{[z]}Q^m \mid AZ = -Z\}$ be the (+1)-eigenspace and (-1)-eigenspace for the involution A on the tangent space $T_{[z]}Q^m$ of Q^m at any point $[z] \in Q^m$.

- If there exists a conjugation $A \in \mathfrak{A} = \{A_{\lambda\bar{z}} \mid \lambda \in S^1 \subset \mathbb{C}\}$ such that $W \in V(A)$, then W is singular, and it is called *\mathfrak{A} -principal*.
- If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $W/\|W\| = (Z_1 + JZ_2)/\sqrt{2}$, then W is singular, and it is called *\mathfrak{A} -isotropic*.

Related to the singularity of tangent vector fields of Q^m , Lee and Suh [20] gave a classification of Hopf real hypersurfaces in the complex quadric Q^m as follows:

Theorem B ([20]). *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 3$. Then the unit normal vector field N of M is \mathfrak{A} -principal if and only if M is locally congruent to an open part of a tube around the m -dimensional sphere S^m which is totally real and totally geodesic in Q^m .*

Remark 1.2. Usually, if the Reeb vector field ξ is Killing, that is, $\mathcal{L}_\xi g_{ij} = 0$, then it is affine Killing. So, the notion of affine Killing is more general and weaker than usual notion of Killing. In fact, we note that

$$\mathcal{L}_\xi \Gamma_{jk}^i = \frac{1}{2} g^{im} \left\{ \bar{\nabla}_j (\mathcal{L}_\xi g_{km}) + \bar{\nabla}_k (\mathcal{L}_\xi g_{jm}) - \bar{\nabla}_m (\mathcal{L}_\xi g_{jk}) \right\} = 0.$$

Therefore, the Reeb vector field ξ is affine Killing (see [13]). The detailed proof also will be given in Lemma 4.1 in section 4.

Motivated by Theorems A and B, and above Remark 1.2, we give a characterization of Hopf real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m as follows:

Theorem 1.3. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field ξ in the complex quadric Q^m , $m \geq 3$. Then, the unit normal vector field N of M is singular, that is, either \mathfrak{A} -principal or \mathfrak{A} -isotropic.*

Then, we can obtain a characterization for real hypersurfaces in Theorem A as follows:

Theorem 1.4. *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 3$. The Reeb vector field ξ is affine Killing if and only if M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^m , where $m = 2k$.*

Finally, as another generalization of Killing vector fields on a Riemannian manifold (\bar{M}, g) , we introduce the notion of conformal Killing vector field on (\bar{M}, g) as follows: A vector field V is said to be *conformal Killing*, if it satisfies

$$(1.7) \quad (\mathcal{L}_V g)(X, Y) = 2\delta g(X, Y)$$

for a differentiable function δ , or equivalently,

$$(1.8) \quad g(\bar{\nabla}_X V, Y) + g(\bar{\nabla}_Y V, X) = 2\delta g(X, Y)$$

for all differentiable vector fields $X, Y \in \mathfrak{X}(\bar{M})$. In this case, we say that the Reeb vector field ξ is *conformal Killing* if it satisfies (1.7).

From (1.7), a Killing vector field becomes a conformal Killing for a vanishing function, $\delta = 0$. As a converse problem, by Theorem 1.4, we obtain as a corollary that the Reeb vector field ξ should be Killing if ξ is conformal Killing as follows:

Corollary 1.5. *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 3$. The Reeb vector field ξ is conformal Killing if and only if M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^m , where $m = 2k$.*

2. THE COMPLEX QUADRIC

As mentioned in section 1, the complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where z_1, \cdots, z_{m+2} are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on (see [5]).

For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z , that is,

$$[z] = \mathbb{C}z = \{\lambda z \mid \lambda \in S^1 \subset \mathbb{C}\}.$$

Note that, by definition, $[z]$ is a point in $\mathbb{C}P^{m+1}$. For each $[z] \in Q^m \subset \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in \mathbb{C}^{m+2} . Then, the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in \mathbb{C}^{m+2} , where $\rho \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$ (see [17]).

For a unit normal vector ρ of Q^m at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ρ . The shape operator A is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the (+1)-eigenspace and $JV(A_\rho)$ is the (-1)-eigenspace of A_ρ . Geometrically this means that the shape operator A_ρ defines a real structure on the tangent space $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1 -subbundle \mathfrak{A} of the endomorphism bundle $\text{End}(TQ^m)$ consisting of real structures, that is, $\mathfrak{A} = \{A_{\lambda\rho} \mid \lambda \in S^1 \subset \mathbb{C}\}$. Then, the subbundle \mathfrak{A} is parallel, which means that there exists a certain 1-form q defined on TQ^m such that

$$(\bar{\nabla}_U A)W = q(U)JAW$$

for any vector fields U and W on Q^m (see [27] and [29]). Moreover, for every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$(2.1) \quad W = \cos(t)Z_1 + \sin(t)JZ_2$$

for some $t \in [0, \pi/4]$ (see Proposition 3 in [27]). The singular tangent vectors of Q^m correspond to the values $t = 0$ and $t = \pi/4$.

The Gauss equation for Q^m in $\mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor \bar{R} of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$(2.2) \quad \begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY \\ &\quad - 2g(JX, Y)JZ + g(AY, Z)AX \\ &\quad - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \end{aligned}$$

For more background to this section we refer to [5], [8], [14], [17], [19], [26], [27], [31], [32], and [33].

3. REAL HYPERSURFACES IN Q^m

Let M be a real hypersurface in Q^m and denote by N a unit normal vector field of M . Then, we obtain the induced almost contact metric structure (ϕ, ξ, η, g) on M . From this, for any vector field X tangent to M , we may put $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where ϕX is the tangential component of JX . The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of TM . The structure tensor field ϕ restricted to \mathcal{C} coincides with the complex structure J restricted to \mathcal{C} , and $\phi\xi = 0$.

Denote by ∇ and S the induced Riemannian connection and the shape operator on M , respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N, \quad \bar{\nabla}_X N = -SX,$$

where $\bar{\nabla}$ is the connection on Q^m . Also, we have

$$(3.1) \quad (\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi, \quad \nabla_X \xi = \phi SX.$$

Moreover, since the complex quadric Q^m has also a real structure A , we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$:

$$AX = BX + g(AX, N)N$$

where BX denotes the tangential component of AX .

On the other hand, since the normal vector field N belongs to $T_{[z]}Q^m$, $[z] \in M$, from (2.1), we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$(3.2) \quad N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. If $t = 0$, then $N = Z_1 \in V(A)$, therefore we see that the unit normal vector field N of M becomes an \mathfrak{A} -principal vector field. On the other hand, if $t = \frac{\pi}{4}$, then $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$. That is, N is \mathfrak{A} -isotropic.

It is known that the Reeb vector field ξ is defined by $\xi = -JN$. Then, it follows that

$$(3.3) \quad \begin{cases} \xi = \sin(t)Z_2 - \cos(t)JZ_1, \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi = \sin(t)Z_2 + \cos(t)JZ_1. \end{cases}$$

From this, we have $g(A\xi, N) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M . In particular, from the former equation we know that the unit vector field $A\xi$ is tangent to M . So, we get

$$(3.4) \quad AN = AJ\xi = -JA\xi = -\phi A\xi - \eta(A\xi)N,$$

where we have used the property of $JA = -AJ$. Hereafter, for our convenience sake, we denote the smooth function $g(A\xi, \xi)$ by κ , that is,

$$\kappa = g(A\xi, \xi) = -\cos(2t).$$

Moreover, by using the Gauss and Weingarten formulas, the left side of (2.2) becomes

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY \\ &\quad + \{g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z)\}N, \end{aligned}$$

where R and S denote the Riemannian curvature tensor and the shape operator of M in Q^m , respectively. From this formula, taking tangential and normal components of (2.2), we obtain respectively

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad - 2g(\phi X, Y)\phi Z + g(BY, Z)BX - g(BX, Z)BY \\ &\quad + g(\phi BY, Z)\phi BX + g(\phi BY, Z)g(X, \phi A\xi)\xi \\ (3.5) \quad &\quad + g(Y, \phi A\xi)\eta(Z)\phi BX - g(\phi BX, Z)\phi BY \\ &\quad - g(\phi BX, Z)g(Y, \phi A\xi)\xi - g(X, \phi A\xi)\eta(Z)\phi BY \\ &\quad + g(SY, Z)SX - g(SX, Z)SY \end{aligned}$$

and

$$\begin{aligned} (\nabla_X S)Y - (\nabla_Y S)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ (3.6) \quad &\quad - g(\phi A\xi, X)BY + g(\phi A\xi, Y)BX \\ &\quad + g(A\xi, X)\phi BY + g(A\xi, X)g(\phi A\xi, Y)\xi \\ &\quad - g(A\xi, Y)\phi BX - g(A\xi, Y)g(\phi A\xi, X)\xi, \end{aligned}$$

which are called the equations of Gauss and Codazzi.

Now let us assume that M is a Hopf real hypersurface in the complex quadric Q^m . That is, the shape operator S of M in Q^m satisfies $S\xi = \alpha\xi$ where the Reeb function α of M is given by $\alpha = g(S\xi, \xi)$. By Codazzi equation (3.6) and (3.4), we obtain:

Lemma 3.1 ([5]). *Let M be a Hopf hypersurface in Q^m , $m \geq 3$. Then, we obtain*

$$(3.7) \quad X\alpha = (\xi\alpha)\eta(X) - 2\kappa g(\phi A\xi, X)$$

and

$$(3.8) \quad \begin{aligned} 2S\phi SX - \alpha\phi SX - \alpha S\phi X - 2\phi X - 2g(\phi A\xi, X)A\xi \\ + 2g(A\xi, X)\phi A\xi + 2\kappa g(\phi A\xi, X)\xi - 2\kappa\eta(X)\phi A\xi = 0 \end{aligned}$$

for any tangent vector fields X and Y on M .

Remark 3.2 ([18]). By (3.7) we know that if M has vanishing geodesic Reeb flow, then the unit normal vector field N of M in Q^m becomes singular. Moreover, by the definition, the unit vector field N is \mathfrak{A} -isotropic if the smooth function $\kappa = g(A\xi, \xi)$ identically vanishes on M in Q^m .

From (3.8) we want to give some information about principal curvatures for Hopf real hypersurfaces in Q^m with \mathfrak{A} -principal normal vector field as follows:

Lemma 3.3 ([30]). *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is \mathfrak{A} -principal everywhere. Then, the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ , then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

In addition, when the unit normal vector field N of M is \mathfrak{A} -principal, we obtain that $A\xi = -\xi$ and $AN = N$ from (3.2) and (3.3). By using these formulas we get

Lemma 3.4 ([21]). *Let M be a real hypersurface with \mathfrak{A} -principal normal vector field N in the complex quadric Q^m , $m \geq 3$. Then we obtain:*

- (a) $AX = BX$,
 - (b) $A\phi X = -\phi AX$,
 - (c) $A\phi SX = -\phi SX$ and $q(X) = 2g(SX, \xi)$,
 - (d) $ASX = SX - 2g(SX, \xi)\xi$ and $SAX = SX - 2\eta(X)S\xi$
- for all $X \in TM$.

If a real hypersurface M in Q^m has an \mathfrak{A} -isotropic unit normal vector field N , it yields

$$g(A\xi, N) = 0 \quad \text{and} \quad g(AN, N) = g(A\xi, \xi) = 0.$$

From this and (3.4), we see that the two unit vector fields $A\xi$ and AN belong to the distribution $\mathcal{C} = [\xi]^\perp$, which is the orthogonal complement of the Reeb vector field ξ . Thus, it follows that $AN = -\phi A\xi$, so $A\xi \perp AN$. This implies that the tangent vector bundle TM of M can be decomposed as

$$TM = [\xi] \oplus \text{span}\{A\xi, AN\} \oplus \mathcal{Q},$$

where $[\xi] = \text{span}\{\xi\}$ and $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^\perp = \text{span}\{A\xi, AN\}$. Using such a decomposition and (3.8), we get the following:

Lemma 3.5 ([21]). *Let M be a Hopf hypersurface in Q^m , $m \geq 3$, such that the normal vector field N is \mathfrak{A} -isotropic everywhere. Then the following statements hold.*

- (a) *The Reeb function α is constant.*
- (b) *The unit tangent vector fields $A\xi$ and $AN = -\phi A\xi$ are principal for the shape operator and their principal curvature is zero, that is,*

$$SA\xi = SAN = S\phi A\xi = 0.$$

- (c) If $X \in \mathcal{Q}$ is a principal curvature vector of M with principal curvature λ , then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.

4. HOPF REAL HYPERSURFACES WITH AFFINE KILLING REEB VECTOR FIELD

In this section, first, we will give an important result concerned with Killing and affine Killing vector fields on Riemannian manifolds (\bar{M}, g) . In addition, by using this property, we will give a some basic formula about Hopf real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$.

We will prove the following.

Lemma 4.1. *Let (\bar{M}, g) be a Riemannian manifold with Riemannian connection $\bar{\nabla}$ and V be a differentiable vector field on \bar{M} . If V is a Killing vector field, then it is also affine Killing.*

Proof. The curvature tensor \bar{R} satisfies

$$g(\bar{R}(X, Y)U, W) = g(\bar{R}(U, W)X, Y) \quad (\text{Interchange Symmetry})$$

and

$$g(\bar{R}(X, Y)U, W) = -g(\bar{R}(X, Y)W, U) \quad (\text{Skew Symmetry})$$

for any $X, Y, U, W \in \mathfrak{X}(\bar{M})$. By using these properties and (1.6), we obtain

$$\begin{aligned} & g(\bar{R}(X, Y)V, W) + g(\bar{R}(Y, V)X, W) - g(\bar{R}(V, X)Y, W) \\ &= g(\bar{R}(X, Y)V, W) - g(\bar{R}(X, W)V, Y) - g(\bar{R}(Y, W)V, X) \\ (4.1) \quad &= g(\bar{\nabla}_X \bar{\nabla}_Y V, W) - g(\bar{\nabla}_Y \bar{\nabla}_X V, W) - g(\bar{\nabla}_{[X, Y]} V, W) \\ & \quad - g(\bar{\nabla}_X \bar{\nabla}_W V, Y) + g(\bar{\nabla}_W \bar{\nabla}_X V, Y) + g(\bar{\nabla}_{[X, W]} V, Y) \\ & \quad - g(\bar{\nabla}_Y \bar{\nabla}_W V, X) + g(\bar{\nabla}_W \bar{\nabla}_Y V, X) + g(\bar{\nabla}_{[Y, W]} V, X) \end{aligned}$$

for any vector fields X, Y, V and $W \in \mathfrak{X}(\bar{M})$. By the first Bianchi identity,

$$\bar{R}(X, Y)V + \bar{R}(Y, V)X + \bar{R}(V, X)Y = 0,$$

the left side of (4.1) becomes

$$\begin{aligned} (4.2) \quad & g(\bar{R}(X, Y)V, W) + g(\bar{R}(Y, V)X, W) \\ & - g(\bar{R}(V, X)Y, W) = -2g(\bar{R}(V, X)Y, W). \end{aligned}$$

On the other hand, from the assumption of V being Killing, (1.2) yields

$$(4.3) \quad g(\bar{\nabla}_X V, W) = -g(\bar{\nabla}_W V, X)$$

for all vector fields $X, W \in \mathfrak{X}(\bar{M})$. Since $[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X \in \mathfrak{X}(\bar{M})$, (4.3) gives us

$$(4.4) \quad \begin{aligned} g(\bar{\nabla}_{[X,Y]}V, W) &= -g(\bar{\nabla}_W V, [X, Y]) \\ &= -g(\bar{\nabla}_W V, \bar{\nabla}_X Y) + g(\bar{\nabla}_W V, \bar{\nabla}_Y X). \end{aligned}$$

Moreover, taking the covariant derivative of (4.3) along the Y -direction gives

$$(4.5) \quad g(\bar{\nabla}_Y \bar{\nabla}_X V, W) = -g(\bar{\nabla}_Y \bar{\nabla}_W V, X) - g(\bar{\nabla}_W V, \bar{\nabla}_Y X) - g(\bar{\nabla}_X V, \bar{\nabla}_Y W).$$

Then, from (4.4) and (4.5), we obtain

$$(4.6) \quad \begin{aligned} g(\bar{\nabla}_X \bar{\nabla}_Y V, W) - g(\bar{\nabla}_Y \bar{\nabla}_X V, W) - g(\bar{\nabla}_{[X,Y]}V, W) \\ = g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + g(\bar{\nabla}_Y \bar{\nabla}_W V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y). \end{aligned}$$

Similarly, we get

$$(4.7) \quad \begin{aligned} g(\bar{\nabla}_X \bar{\nabla}_W V, Y) - g(\bar{\nabla}_W \bar{\nabla}_X V, Y) - g(\bar{\nabla}_{[X,W]}V, Y) \\ = g(\bar{\nabla}_X \bar{\nabla}_W V, Y) + g(\bar{\nabla}_W \bar{\nabla}_Y V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_W Y) + g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} g(\bar{\nabla}_Y \bar{\nabla}_W V, X) - g(\bar{\nabla}_W \bar{\nabla}_Y V, X) - g(\bar{\nabla}_{[Y,W]}V, X) \\ = g(\bar{\nabla}_Y \bar{\nabla}_W V, X) + g(\bar{\nabla}_W \bar{\nabla}_X V, Y) + g(\bar{\nabla}_Y V, \bar{\nabla}_W X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W). \end{aligned}$$

Substituting (4.6), (4.7), and (4.8) into the right side of (4.1), it follows:

$$(4.9) \quad \begin{aligned} &g(\bar{R}(X, Y)V, W) + g(\bar{R}(Y, V)X, W) - g(\bar{R}(V, X)Y, W) \\ &= g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + g(\bar{\nabla}_Y \bar{\nabla}_W V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &\quad - g(\bar{\nabla}_X \bar{\nabla}_W V, Y) - g(\bar{\nabla}_W \bar{\nabla}_Y V, X) - g(\bar{\nabla}_X V, \bar{\nabla}_W Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \\ &= g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &\quad - g(\bar{\nabla}_X \bar{\nabla}_W V, Y) - g(\bar{\nabla}_W \bar{\nabla}_Y V, X) - g(\bar{\nabla}_X V, \bar{\nabla}_W Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \\ &\quad - g(\bar{\nabla}_W \bar{\nabla}_X V, Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_W X) - g(\bar{\nabla}_X V, \bar{\nabla}_Y W). \end{aligned}$$

Applying the formula (4.5) to the two terms $g(\bar{\nabla}_X \bar{\nabla}_W V, Y)$ and $g(\bar{\nabla}_W \bar{\nabla}_X V, Y)$ in the right side of (4.9), then it follows that

$$(4.10) \quad \begin{aligned} &g(\bar{R}(X, Y)V, W) + g(\bar{R}(Y, V)X, W) - g(\bar{R}(V, X)Y, W) \\ &= 2g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + 2g(\bar{\nabla}_W V, \bar{\nabla}_X Y). \end{aligned}$$

Then, by using (4.2) and (4.3), the formula (4.10) can be rewritten as follows:

$$\begin{aligned} -2g(\bar{R}(V, X)Y, W) &= g(\bar{R}(X, Y)V, W) + g(\bar{R}(Y, V)X, W) - g(\bar{R}(V, X)Y, W) \\ &= 2g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + 2g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &= 2g(\bar{\nabla}_X \bar{\nabla}_Y V, W) - 2g(\bar{\nabla}_{\bar{\nabla}_X Y} V, W), \end{aligned}$$

that is,

$$g(\bar{R}(V, X)Y, W) = -g(\bar{\nabla}_X \bar{\nabla}_Y V, W) + g(\bar{\nabla}_{\bar{\nabla}_X Y} V, W)$$

for any vector fields X, Y and $W \in \mathfrak{X}(\bar{M})$. From this, we get

$$\bar{R}(V, X)Y = -\bar{\nabla}_X \bar{\nabla}_Y V + \bar{\nabla}_{\bar{\nabla}_X Y} V.$$

By (1.5), it implies that for any differentiable vector fields $X, Y \in \mathfrak{X}(\bar{M})$

$$(\mathcal{L}_V \bar{\nabla})(X, Y) = 0,$$

which means that the Killing vector field V is affine Killing. \square

Bearing in mind this result, let us consider the case of the Reeb vector field ξ of M being affine Killing. Here, M denotes a Hopf real hypersurface in the complex quadric Q^m , $m \geq 3$. That is, the Reeb vector field ξ of M satisfies $\mathcal{L}_\xi \nabla = 0$, where ∇ is the Levi-Civita connection of M . Then, (1.5), together with $\nabla_X \xi = \phi S X$, yields

$$\begin{aligned} 0 &= (\mathcal{L}_\xi \nabla)(X, Y) = R(\xi, X)Y + \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi \\ (4.11) \quad &= R(\xi, X)Y + \nabla_X(\phi S Y) - \phi S(\nabla_X Y) \\ &= R(\xi, X)Y + (\nabla_X \phi)S Y + \phi(\nabla_X S)Y, \end{aligned}$$

where we used $\nabla_X(\phi S Y) - \phi S(\nabla_X Y) = (\nabla_X \phi)S Y + \phi(\nabla_X S)Y$ for any vector fields X and $Y \in TM$. Moreover, from (3.1) we get

$$(\nabla_X \phi)S Y = \eta(SY)S X - g(SX, SY)\xi.$$

By this formula and (3.5), the equation (4.11) becomes

$$\begin{aligned} 0 &= R(\xi, X)Y + \eta(SY)S X - g(SX, SY)\xi + \phi(\nabla_X S)Y \\ (4.12) \quad &= g(X, Y)\xi - \eta(Y)X + g(BX, Y)A\xi - g(A\xi, Y)B X + g(\phi B X, Y)\phi A\xi \\ &\quad + g(X, \phi A\xi)\eta(Y)\phi A\xi - g(\phi A\xi, Y)\phi B X - g(\phi A\xi, Y)g(X, \phi A\xi)\xi \\ &\quad + g(SX, Y)S\xi - g(SX, SY)\xi + \phi(\nabla_X S)Y. \end{aligned}$$

Moreover, from our assumption of M being Hopf, the Reeb vector field ξ satisfies $S\xi = \alpha\xi$. Then, (4.12) can be written as

$$\begin{aligned} (4.13) \quad &g(X, Y)\xi - \eta(Y)X + g(BX, Y)A\xi - g(A\xi, Y)B X + g(\phi B X, Y)\phi A\xi \\ &\quad + g(X, \phi A\xi)\eta(Y)\phi A\xi - g(\phi A\xi, Y)\phi B X - g(\phi A\xi, Y)g(X, \phi A\xi)\xi \\ &\quad + \alpha g(SX, Y)\xi - g(SX, SY)\xi + \phi(\nabla_X S)Y = 0 \end{aligned}$$

for any tangent vector fields X and Y on M . Consequently, (4.13) is equivalent to the assumption of the Reeb vector field ξ being affine Killing.

5. PROOF OF THEOREM 1.3

In this section, we want to show that the unit normal vector field N is singular for Hopf real hypersurfaces M in Q^m with affine Killing Reeb vector field. Then, we can use (4.13). So, if we put $Y = \xi$ in (4.13), it follows that

$$\begin{aligned} 0 &= \eta(X)\xi - X + g(A\xi, X)A\xi - \kappa BX + g(\phi A\xi, X)\phi A\xi + \phi(\nabla_X S)\xi \\ &= \eta(X)\xi - X + g(A\xi, X)A\xi - \kappa BX + g(\phi A\xi, X)\phi A\xi \\ &\quad - \alpha SX + \alpha^2 \eta(X)\xi - \phi S\phi SX, \end{aligned}$$

where we have used $(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX$ and $\phi^2 SX = -SX + g(SX, \xi)\xi$. Applying the structure tensor ϕ of M to this equation becomes

$$\begin{aligned} &-\phi X + g(A\xi, X)\phi A\xi - \kappa\phi BX - g(\phi A\xi, X)A\xi \\ &+ \kappa g(\phi A\xi, X)\xi - \alpha\phi SX + S\phi SX = 0, \end{aligned}$$

that is,

$$(5.1) \quad \begin{aligned} &S\phi SX - \phi X + g(A\xi, X)\phi A\xi - g(\phi A\xi, X)A\xi \\ &+ \kappa g(\phi A\xi, X)\xi = \alpha\phi SX + \kappa\phi BX. \end{aligned}$$

Moreover, from (3.8) we obtain

$$(5.2) \quad \begin{aligned} &2S\phi SX - 2\phi X + 2g(A\xi, X)\phi A\xi - 2g(\phi A\xi, X)A\xi + 2\kappa g(\phi A\xi, X)\xi \\ &= \alpha\phi SX + \alpha S\phi X + 2\kappa\eta(X)\phi A\xi. \end{aligned}$$

Hence, both equations (5.1) and (5.2) give

$$(5.3) \quad \alpha(\phi S - S\phi)X = 2\kappa\{\eta(X)\phi A\xi - \phi BX\}$$

for any tangent vector field X on M .

As mentioned in Remark 3.2, we know that *if either the Reeb function $\alpha = g(S\xi, \xi)$ or the smooth function $\kappa = g(A\xi, \xi)$ identically vanishes on M , then the unit normal vector field N becomes singular*. Thus, in the remaining part of this section we only consider the case of both α and κ being non-vanishing.

Proposition 5.1. *Let M be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m , $m \geq 3$. If the Reeb vector field ξ of M is affine Killing and $\kappa \neq 0$, then the unit normal vector field N of M is \mathfrak{A} -principal.*

Proof. Putting $X = A\xi$ in (5.3) and using $BA\xi = A^2\xi - g(A^2\xi, N)N = \xi$, together with $\alpha \neq 0$, we get

$$(5.4) \quad S\phi A\xi = \phi SA\xi + \sigma\phi A\xi,$$

where $\sigma = -\frac{2\kappa^2}{\alpha}$. Taking $X = \phi A\xi$ in (3.8) and using (5.4), together with $g(\phi A\xi, \phi A\xi) = 1 - \kappa^2$, we obtain

$$(5.5) \quad S^2 A\xi = -2\kappa^3 \xi + (\alpha - \sigma)SA\xi.$$

On the other hand, taking the inner product of (4.13) with ξ and using $B\xi = A\xi$, we obtain

$$(5.6) \quad \begin{aligned} &g(X, Y) - \eta(X)\eta(Y) + \kappa g(BX, Y) - g(A\xi, X)g(A\xi, Y) \\ &\quad - g(\phi A\xi, X)g(\phi A\xi, Y) + \alpha g(SX, Y) - g(SX, SY) = 0 \end{aligned}$$

for any tangent vector fields X and Y on M . Taking $X = A\xi$ in (5.6) we obtain

$$\alpha g(SA\xi, Y) - g(S^2A\xi, Y) = 0 \quad \forall Y \in TM,$$

which implies that

$$(5.7) \quad S^2A\xi = \alpha SA\xi.$$

From this and (5.5), we see that

$$(5.8) \quad SA\xi = \alpha \kappa \xi.$$

Moreover, from (5.8) equation (5.4) becomes

$$(5.9) \quad S\phi A\xi = \sigma \phi A\xi.$$

Putting $X = \phi A\xi$ in (5.6), and using (5.9), we get

$$(5.10) \quad \begin{aligned} 0 &= \phi A\xi + \kappa B\phi A\xi - (1 - \kappa^2)\phi A\xi + \alpha\sigma\phi A\xi - \sigma^2\phi A\xi \\ &= (2\kappa^2 + \alpha\sigma - \sigma^2)\phi A\xi \\ &= -\sigma^2\phi A\xi, \end{aligned}$$

where we used

$$\begin{aligned} B\phi A\xi &= A\phi A\xi - g(A\phi A\xi, N)N \\ &= A(-AN - \kappa N) - g(AN, -AN - \kappa N)N \\ &= -N - \kappa AN + g(AN, AN)N + \kappa g(AN, N)N \\ &= -\kappa AN + \kappa g(AN, N)N \\ &= \kappa\phi A\xi + \kappa^2 N + \kappa g(AN, N)N \\ &= \kappa\phi A\xi, \end{aligned}$$

together with $AN = -\phi A\xi - \kappa N$, $A^2X = X$ and $\kappa = g(A\xi, \xi) = -g(AN, N)$.

Since the smooth functions α and κ are non-vanishing on M , we get

$$\sigma = -\frac{2\kappa^2}{\alpha} \neq 0.$$

Thus, (5.10) tells that the vector field $\phi A\xi$ vanishes on M , that is, $\phi A\xi = 0$. Since $g(\phi A\xi, \phi A\xi) = 1 - \kappa^2$, it means $\kappa = \pm 1$. Meanwhile, from (3.3) we see that the smooth function $\kappa = g(A\xi, \xi)$ is given by $\kappa(t) = -\cos(2t)$ where $t \in [0, \frac{\pi}{4}]$. By such two facts related to κ , we consequently have $t = 0$. This means that the unit normal vector field N satisfies $N = Z_1 \in V(A)$. Therefore, we claim that the unit normal vector field N is \mathfrak{A} -principal. \square

Summing up Remark 3.2 and Proposition 5.1 we give a complete proof of our Theorem 1.3.

6. PROOF OF THEOREM 1.4

In this section, unless otherwise stated, let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$.

First, by virtue of our Theorem 1.3, we consider the case where M has an \mathfrak{A} -principal normal vector field N in Q^m . Then, Theorem B tells that M with \mathfrak{A} -principal normal vector field N is locally congruent to an open part of a tube around the m -dimensional sphere S^m . Hereafter, we call such a model space (resp. a model space in Theorem A) a real hypersurface of type (B) (resp. a real hypersurface of type (A)) and denote such a model space by (\mathcal{T}_B) (resp. (\mathcal{T}_A)).

As a converse of this statement, naturally, the following question arises.

Has the real hypersurface (\mathcal{T}_B) of type (B) in Q^m an affine Killing Reeb vector field ξ ?

To solve this problem, we introduce the following proposition given in [30].

Proposition A. *Let (\mathcal{T}_B) be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m -dimensional sphere S^m in Q^m . Then the following statements hold:*

- (i) (\mathcal{T}_B) is a Hopf hypersurface.
- (ii) The normal bundle of (\mathcal{T}_B) consists of \mathfrak{A} -principal vector fields.
- (iii) (\mathcal{T}_B) has three distinct constant principal curvatures. The principal curvatures and corresponding principal curvature spaces of (\mathcal{T}_B) are as follows:

principal curvature	eigenspace	multiplicity
$\alpha = -\sqrt{2} \cot(\sqrt{2}r)$	$T_\alpha = \mathbb{R}JN$	1
$\lambda = \sqrt{2} \tan(\sqrt{2}r)$	$T_\lambda = V(A) \cap \mathcal{C} = \{X \in \mathcal{C} \mid AX = X\}$	$m - 1$
$\mu = 0$	$T_\mu = JV(A) \cap \mathcal{C} = \{X \in \mathcal{C} \mid AX = -X\}$	$m - 1$

- (iv) $S\phi + \phi S = 2\tau\phi$, $\tau = -\frac{1}{\alpha} \neq 0$ (contact hypersurface).

By (i) and (ii) in Proposition A, we know that (\mathcal{T}_B) is a Hopf real hypersurface with \mathfrak{A} -principal normal vector field N in Q^m , $m \geq 3$.

Assume that the Reeb vector field ξ of (\mathcal{T}_B) satisfies the affine Killing property (4.11). It implies (4.13) for all tangent vector fields X and Y on $T(\mathcal{T}_B) = T_\alpha \oplus T_\lambda \oplus T_\mu$. Then, by Lemma 3.4 and (4.13) we obtain for $Y = \xi$

$$(6.1) \quad 2\eta(X)\xi - X + AX - \alpha SX + \alpha^2\eta(X)\xi - \phi S\phi SX = 0,$$

where we have used $A\xi = -\xi$ and $\phi(\nabla_X S)\xi = \alpha^2\eta(X)\xi - \alpha SX - \phi S\phi SX$. Then, by Proposition A, the left side of (6.1) becomes

$$(6.2) \quad 2\eta(X)\xi - X + AX - \alpha SX + \alpha^2\eta(X)\xi - \phi S\phi SX = \begin{cases} 0 & \text{if } X \in T_\alpha \\ 2X & \text{if } X \in T_\lambda \\ -2X & \text{if } X \in T_\mu. \end{cases}$$

Then, comparing (6.1) with (6.2), we get a contradiction. Summing up all the facts, we get the following

Lemma 6.1. *There does not exist any Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$, whose unit normal vector field N is \mathfrak{A} -principal.*

From this lemma, together with Theorem 1.3, it follows that

Proposition 6.2. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$. Then, the unit normal vector field N in Q^m is \mathfrak{A} -isotropic.*

By virtue of Proposition 6.2, the unit normal vector field N of M is \mathfrak{A} -isotropic. Thus, N is expressed as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vector fields $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the (+1)-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Since $AJ = -JA$ and $J^2X = -X$, it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2), \quad \text{and} \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2).$$

Then it gives that

$$\kappa = g(\xi, A\xi) = g(JN, AJN) = 0 = -g(AN, N)$$

and

$$g(\xi, AN) = -g(JN, AN) = 0,$$

which means that the vector fields $AN = -\phi A\xi$ and $A\xi$ are tangent to M . From $\kappa = 0$ and (5.3), it follows that

$$(6.3) \quad \alpha(\phi S - S\phi)X = 0$$

for any tangent vector field X on M . If M has non-vanishing geodesic Reeb flow, that is, $\alpha \neq 0$, (6.3) gives that the shape operator S commutes with the structure tensor ϕ on M , that is, $S\phi = \phi S$. So, the Reeb vector field ξ becomes Killing if the Reeb function α is non-vanishing on M .

In the following proposition let us consider the case that M has vanishing geodesic Reeb flow, that is, $\alpha = 0$. In fact, from Lemma 3.5 we obtain:

Proposition 6.3. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m , $m \geq 3$. If M has vanishing geodesic Reeb flow, the Reeb vector field ξ is Killing. That is, the shape operator S of M commutes with the structure tensor ϕ on M .*

Proof. Since M is a Hopf real hypersurface with affine Killing Reeb vector field ξ in Q^m , we see that the unit normal vector field N is \mathfrak{A} -isotropic. That is, it implies $\kappa = g(A\xi, \xi) = 0$. By using this fact and our assumption $\alpha = 0$, if we take the inner product of (4.13) with ξ , it follows that

$$\begin{aligned} g(X, Y) - \eta(X)\eta(Y) - g(A\xi, X)g(A\xi, Y) \\ - g(\phi A\xi, X)g(\phi A\xi, Y) - g(SX, SY) = 0 \end{aligned}$$

for any tangent vector fields X and Y on M . Then, we get

$$(6.4) \quad X - \eta(X)\xi - g(A\xi, X)A\xi - g(\phi A\xi, X)\phi A\xi - S^2X = 0$$

for any $X \in TM$.

On the other hand, by the fact (b) in Lemma 3.5, the tangent bundle TM of M is given by

$$TM = [\xi] \oplus \mathcal{C} = [\xi] \oplus \mathcal{Q}^\perp \oplus \mathcal{Q},$$

where $\mathcal{C} = [\xi]^\perp$ and $\mathcal{Q}^\perp = \text{span}\{A\xi, AN\} \subset \mathcal{C}$. So, (6.4) provides

$$(6.5) \quad \begin{aligned} S^2X &= X - \eta(X)\xi - g(A\xi, X)A\xi - g(\phi A\xi, X)\phi A\xi \\ &= \begin{cases} 0 & \text{if } X \in [\xi] \oplus \mathcal{Q}^\perp \\ X & \text{if } X \in \mathcal{Q}. \end{cases} \end{aligned}$$

Now, choose some unit tangent vector field X_0 of \mathcal{Q} such that $SX_0 = \lambda X_0$. Substituting $X = X_0$ in (6.5) yields $\lambda^2 = 1$, that is, $\lambda = \pm 1$. Moreover, from Lemma 3.5 we see that the corresponding unit vector field ϕX_0 is also a principal curvature vector field of M with principal curvature $\mu := \frac{1}{\lambda}$. That is, we obtain either $S\phi X_0 = \phi X_0$ if $SX_0 = X_0$ or $S\phi X_0 = -\phi X_0$ if $SX_0 = -X_0$. From this, we know that the eigenspace $T_\lambda = \{X \in \mathcal{Q} \subset \mathcal{C} \mid SX = \lambda X\}$ is ϕ -invariant, that is, $\phi T_\lambda = T_\lambda$.

On the other hand, it is well-known that the shape operator S is diagonalizable, that is, S has a basis of eigenvectors. From such a view point, we take

$$\mathfrak{B} = \left\{ \begin{array}{l} \xi, A\xi, AN, e_1, e_2 = \phi e_1, \dots, e_{p-1}, e_p = \phi e_{p-1}, \\ e_{p+1}, e_{p+2} = \phi e_{p+1}, \dots, e_{2m-5}, e_{2m-4} := \phi e_{2m-5} \end{array} \right\}$$

as the basis of M satisfying

$$S\xi = SA\xi = SAN = 0$$

and

$$Se_k = \begin{cases} e_k & \text{for } k = 1, 2, \dots, p, \\ -e_k & \text{for } k = p+1, \dots, 2m-4. \end{cases}$$

Then, any tangent vector field $X \in \mathcal{Q}$ is expressed by the basis \mathfrak{B} as

$$\begin{aligned}
 (6.6) \quad X &= g(X, \xi)\xi + g(X, A\xi)A\xi + g(X, AN)AN + \sum_{k=1}^{2m-4} g(X, e_k)e_k \\
 &= \sum_{k=1}^{2m-4} g(X, e_k)e_k.
 \end{aligned}$$

Applying the operator ϕS of TM to (6.6) yields

$$\begin{aligned}
 (6.7) \quad \phi SX &= \sum_{k=1}^{2m-4} g(X, e_k)\phi Se_k = \sum_{k=1}^p g(X, e_k)\phi Se_k + \sum_{k=p+1}^{2m-4} g(X, e_k)\phi Se_k \\
 &= \sum_{k=1}^p g(X, e_k)\phi e_k - \sum_{k=p+1}^{2m-4} g(X, e_k)\phi e_k \\
 &= g(X, e_1)\phi e_1 + g(X, e_2)\phi e_2 + \cdots + g(X, e_{p-1})\phi e_{p-1} \\
 &\quad + g(X, e_p)\phi e_p - g(X, e_{p+1})\phi e_{p+1} - g(X, e_{p+2})\phi e_{p+2} \\
 &\quad - \cdots - g(X, e_{2m-5})\phi e_{2m-5} - g(X, e_{2m-4})\phi e_{em-4} \\
 &= g(X, e_1)e_2 + g(X, e_2)\phi^2 e_1 + \cdots + g(X, e_{p-1})e_p \\
 &\quad + g(X, e_p)\phi^2 e_{p-1} - g(X, e_{p+1})e_{p+2} - g(X, e_{p+2})\phi^2 e_{p+1} \\
 &\quad - \cdots - \cdots - g(X, e_{2m-5})e_{2m-4} - g(X, e_{2m-4})\phi^2 e_{em-5} \\
 &= g(X, e_1)e_2 - g(X, e_2)e_1 + \cdots + g(X, e_{p-1})e_p \\
 &\quad - g(X, e_p)e_{p-1} - g(X, e_{p+1})e_{p+2} + g(X, e_{p+2})e_{p+1} \\
 &\quad - \cdots - g(X, e_{2m-5})e_{2m-4} + g(X, e_{2m-4})e_{em-5} \\
 &= g(X, e_1)e_2 + \cdots + g(X, e_{p-1})e_p + g(X, e_{p+2})e_{p+1} \cdots \\
 &\quad + g(X, e_{2m-4})e_{2m-5} - g(X, e_2)e_1 - \cdots - g(X, e_p)e_{p-1} \\
 &\quad - g(X, e_{p+1})e_{p+2} - \cdots - g(X, e_{2m-5})e_{2m-4},
 \end{aligned}$$

where we have used $\phi^2 e_k = -e_k + \eta(e_k)\xi = -e_k$ for any $k = 1, 2, \dots, 2m - 4$.

On the other hand, by using linear combination with our basis and the symmetric property of S , the tangent vector field $S\phi X$ is given as follows:

$$\begin{aligned}
 (6.8) \quad S\phi X &= g(S\phi X, \xi)\xi + g(S\phi X, A\xi)A\xi + g(S\phi X, AN)AN \\
 &\quad + g(S\phi X, e_1)e_1 + g(S\phi X, \phi e_1)\phi e_1 + \cdots + g(S\phi X, e_{p-1})e_{p-1} \\
 &\quad + g(S\phi X, \phi e_{p-1})\phi e_{p-1} + g(S\phi X, e_{p+1})e_{p+1} + g(S\phi X, \phi e_{p+1})\phi e_{p+1} \\
 &\quad + \cdots + g(S\phi X, e_{2m-5})e_{2m-5} + g(S\phi X, \phi e_{2m-5})\phi e_{2m-5} \\
 &= g(\phi X, Se_1)e_1 + g(\phi X, S\phi e_1)\phi e_1 + \cdots + g(\phi X, Se_{p-1})e_{p-1}
 \end{aligned}$$

$$\begin{aligned}
& + g(\phi X, S\phi e_{p-1})\phi e_{p-1} + g(\phi X, S e_{p+1})e_{p+1} + g(\phi X, S\phi e_{p+1})\phi e_{p+1} \\
& + \cdots + g(\phi X, S e_{2m-5})e_{2m-5} + g(\phi X, S\phi e_{2m-5})\phi e_{2m-5} \\
= & g(\phi X, e_1)e_1 + g(\phi X, \phi e_1)\phi e_1 + \cdots + g(\phi X, e_{p-1})e_{p-1} \\
& + g(\phi X, \phi e_{p-1})\phi e_{p-1} - g(\phi X, e_{p+1})e_{p+1} - g(\phi X, \phi e_{p+1})\phi e_{p+1} \\
& - \cdots - g(\phi X, e_{2m-5})e_{2m-5} - g(\phi X, \phi e_{2m-5})\phi e_{2m-5} \\
= & -g(X, \phi e_1)e_1 - g(X, \phi^2 e_1)\phi e_1 + \cdots - g(X, \phi e_{p-1})e_{p-1} \\
& - g(X, \phi^2 e_{p-1})\phi e_{p-1} + g(X, \phi e_{p+1})e_{p+1} + g(X, \phi^2 e_{p+1})\phi e_{p+1} \\
& + \cdots + g(X, \phi e_{2m-5})e_{2m-5} + g(X, \phi^2 e_{2m-5})\phi e_{2m-5} \\
= & -g(X, \phi e_1)e_1 + g(X, e_1)\phi e_1 + \cdots - g(X, \phi e_{p-1})e_{p-1} + g(X, e_{p-1})\phi e_{p-1} \\
& + g(X, \phi e_{p+1})e_{p+1} - g(X, e_{p+1})\phi e_{p+1} \\
& + \cdots + g(X, \phi e_{2m-5})e_{2m-5} - g(X, e_{2m-5})\phi e_{2m-5} \\
= & -g(X, e_2)e_1 + g(X, e_1)e_2 + \cdots - g(X, e_p)e_{p-1} + g(X, e_{p-1})e_p \\
& + g(X, e_{p+2})e_{p+1} - g(X, e_{p+1})e_{p+2} \\
& + \cdots + g(X, e_{2m-4})e_{2m-5} - g(X, e_{2m-5})e_{2m-4} \\
= & g(X, e_1)e_2 + \cdots + g(X, e_{p-1})e_p + g(X, e_{p+2})e_{p+1} + \cdots + g(X, e_{2m-4})e_{2m-5} \\
& - g(X, e_2)e_1 - \cdots - g(X, e_p)e_{p-1} - g(X, e_{p+1})e_{p+2} - \cdots - g(X, e_{2m-5})e_{2m-4}
\end{aligned}$$

for any tangent vector field $X \in \mathcal{Q}$. From (6.7) and (6.8) we see that the shape operator S of M commutes with the structure tensor ϕ on M , that is, $S\phi = \phi S$ on \mathcal{Q} .

Bearing in mind $S\xi = SA\xi = SAN = 0$, together with $\phi^2 A\xi = -A\xi$, we naturally obtain that the commuting property $S\phi = \phi S$ holds on $[\xi] \oplus \mathcal{Q}^\perp$.

From these facts, we conclude that the shape operator S commutes with the structure tensor ϕ on M , when the Reeb function α identically vanishes on M . It completes the proof of our proposition. \square

Consequently, Proposition 6.3 and (6.3) assure that *the affine Killing Reeb vector field ξ of M must be Killing*. Moreover, by virtue of this fact and Lemma 4.1 we give a complete proof of Theorem 1.4.

7. PROOF OF COROLLARY 1.5

As a generalized notion of Killing vector field of Riemannian manifolds (\bar{M}, g) , we introduced conformal vector fields in the introduction. In fact, in (1.7)

$$(\mathcal{L}_V g)(X, Y) = 2\delta g(X, Y)$$

for any vector fields X and Y and the smooth function $\delta = 0$ on \bar{M} implies that any conformal Killing vector field V satisfies the Killing property (1.1). From this, it assures:

Fact A. *Any Killing vector field V of \bar{M} becomes conformal Killing with $\delta = 0$.*

Motivated by this assertion, let us consider the converse problem of Fact A. That is, we prove that if the Reeb vector field ξ of a Hopf real hypersurface in Q^m is conformal, it satisfies the Killing property as follows:

Lemma 7.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 3$, with conformal Killing Reeb vector field ξ . Then, the Reeb vector field ξ becomes Killing. That is, the shape operator S of M commutes with the structure tensor ϕ on M .*

Proof. From the assumption of the Reeb vector field being conformal Killing, (1.8) gives

$$(7.1) \quad \phi SX - S\phi X = 2\delta X$$

for any tangent vector field X on M . Substituting $X = \xi$ in this equation and using M being Hopf implies the smooth function δ identically vanishes on M . From this, the conformal Killing property (7.1) becomes for any tangent vector field X on M

$$\phi SX - S\phi X = 0,$$

which means that the vector field ξ is Killing. \square

From Lemma 7.1, together with Theorem A, we give a complete proof of Corollary 1.5.

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