On the mathematical formulation of the restricted Feynman path integrals through broken line paths

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Abstract

The restricted Feynman path integrals (RFPIs) have been proposed to study continuous quantum measurements in physics. The RFPIs are heuristically determined in terms of the usual probability amplitude multiplied by weight for each path, which contains information about the results and the resolution of the measuring device. In the present paper we will consider the RFPIs particularly for the position measurements and will prove rigorously that these RFPIs are well defined in the L^2 space and are the solutions to the non-self-adjoint Schrödinger equations. Our results in the present paper give a generalization of the results on the usual Feynman path integrals for the Schrödinger equations. Furthermore, our results are extended to quantum spin systems.

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1 Introduction

Let $T > 0$ be an arbitrary constant, $0 \le t \le T$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. First, we consider a one-particle system with mass $m > 0$ and charge $\mathfrak{q} \in \mathbb{R}$ moving in $\mathbb{R}^d$ with electric strength $E(t,x) = (E_1, \dots, E_d) \in \mathbb{R}^d$ and a magnetic strength tensor $B(t,x) = (B_{jk}(t,x))_{1 \le j<k \le d} \in \mathbb{R}^{d(d-1)/2}$. Let $(V(t,x), A(t,x)) = (V, A_1, \dots, A_d) \in \mathbb{R}^{d+1}$ be an electromagnetic potential, i.e.

$$\begin{aligned} E &= -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x}, \\ B_{jk} &= \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad (1 \le j < k \le d), \end{aligned} \tag{1.1}$$

where $\partial V/\partial x = (\partial V/\partial x_1, \dots, \partial V/\partial x_d)$. Then the Lagrangian function and the classical action are given by

$$\mathcal{L}(t,x,\dot{x}) = \frac{m}{2}|\dot{x}|^2 + \mathfrak{q}\dot{x} \cdot A(t,x) - \mathfrak{q}V(t,x), \ \dot{x} \in \mathbb{R}^d \tag{1.2}$$

and

$$S(t,s;q) = \int_s^t \mathcal{L}(\theta, q(\theta), \dot{q}(\theta)) d\theta, \quad \dot{q}(\theta) = \frac{dq(\theta)}{d\theta} \tag{1.3}$$

for a path $q(\theta) \in \mathbb{R}^d$ $(s \le \theta \le t)$, respectively. The corresponding Schrödinger equation is defined by

$$\begin{aligned} i\hbar \frac{\partial u}{\partial t}(t) &= H(t)u(t) \\ &:= \left[\frac{1}{2m} \sum_{j=1}^d \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - \mathfrak{q}A_j(t,x) \right)^2 + \mathfrak{q}V(t,x) \right] u(t), \end{aligned} \tag{1.4}$$

where $\hbar$ is the Planck constant. Throughout this paper we always consider solutions to the Schrödinger equations in the sense of distribution. Let $L^2 = L^2(\mathbb{R}^d)$ denote the space of all square integrable functions in $\mathbb{R}^d$ with the inner product $(f, g) := \int f(x)g(x)^* dx$ and the norm $\|f\|$, where $g(x)^*$ denotes the complex conjugate of $g(x)$.

Consider a continuous quantum measurement of the position of the particle in the time interval $[0, T]$. Let $\{a(t) \in \mathbb{R}^d; 0 \leq t \leq T\}$ be its result and $\delta > 0$ its resolution or error of the measuring device. The measurement gives a change of the probability amplitude of the particle, called wave-function reduction (cf. §17.5 of [10] and §1.4 of [28]). Let $f \in L^2$ be a probability amplitude of the particle at an initial time $t = 0$. Then, if we follow Feynman's postulates I and II on p. 371 of [7], the probability amplitude in the continuous measurement is heuristically given by the " sum " of $e^{i\hbar^{-1}S(t,0;q)}f(q(0))$ over a set $\Gamma(t, x; \delta)$ of all paths q satisfying $q(t) = x$ and $|q(\theta) - a(\theta)| \leq \delta$ for all $\theta \in [0, t]$, i.e.

$$\int_{\Gamma(t,x;\delta)} e^{i\hbar^{-1}S(t,0;q)} f(q(0))\mathcal{D}q. \tag{1.5}$$

An alternative Feynman path integral description has been proposed by Mensky in §4.2 of [22] and §5.1.3 of [23], written formally as

$$\int_{\Gamma(t,x)} e^{i\hbar^{-1}S(t,0;q)-c\int_0^t |q(\theta)-a(\theta)|^2 d\theta/\delta^2} f(q(0))\mathcal{D}q \tag{1.6}$$

with a constant $c > 0$, where $\Gamma(t, x)$ is a set of all paths q satisfying $q(t) = x$. See also §10.5.4 of [1], [6], §3.2 of [8], §5.1 of [21] and [24]. Both of (1.5) and (1.6) are called the restricted Feynman path integrals (RFPIs).

Our purpose in the present paper is to give a rigorous meaning in the L^2 space to each of (1.6) and a more general formula with a weight function $W(t, x) \in \mathbb{R}$ replacing $c|x - a(t)|^2/\delta^2$. Furthermore, we wil show that each of (1.6) and the more general forumula stated above is the solution to the non-self-adjoint Schrödinger equation with f at $t = 0$.

As far as the author knows, we have been able to give a rigorous meaning to (1.6) only for $A = 0$ and $V = C|x|^2$ with a constant $C \in \mathbb{R}$, where we can directly calculate (1.6) by using Gaussian integrals (cf. §4.4 and §5.4 of [22]).

We also note that there is another approach to continuous quantum position measurements of the particle. We begin by considering a sequence of n instantaneous position measurements separated by a time $\mathit{\Delta} t = T/n$ and then, determine the evolution of the measured system in the continuous limits $n \to \infty$ and so $\mathit{\Delta} t \to 0$ (cf. [3, 4, 5], Chapter 3 in [19], Chapter 2 in [22] and Chapter 2 in [23]).

Let $W(t,x)$ be a weight function and define the effective Lagrangian function under the measurement by

$$\mathcal{L}_w(t,x,\dot{x}) = \mathcal{L}(t,x,\dot{x}) + i\hbar W(t,x) \tag{1.7}$$

and the effective classical action by

$$\begin{aligned} S_w(t,s;q) &= \int_s^t \mathcal{L}_w(\theta, q(\theta), \dot{q}(\theta))d\theta \\ &= S(t,s;q) + i\hbar \int_s^t W(\theta, q(\theta))d\theta \end{aligned} \tag{1.8}$$

for a path $q(\theta) \in \mathbb{R}^d$ as in §4.2 of [22]. In the present paper we will prove for $W(t,x) = c|x-a(t)|^2/\delta^2$ and more general weight functions that the RFPIs

$$K(t,0)f := \int_{\Gamma(t,x)} e^{i\hbar^{-1}S_w(t,0;q)} f(q(0))\mathcal{D}q \tag{1.9}$$

are well defined in L^2 for $f \in L^2$. $K(t,0)f$ defined by (1.9) gives a generalization of (1.6). Furthermore, we will prove that $K(t,0)f$ satisfy the non-self-adjoint Schrödinger equations, derived from (1.7) through the Legendre transformation,

$$i\hbar\frac{\partial u}{\partial t}(t) = H_w(t)u(t)$$
$$:= \left[\frac{1}{2m}\sum_{j=1}^{d}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \mathfrak{q}A_j(t,x)\right)^2 + \mathfrak{q}V(t,x) - i\hbar W(t,x)\right]u(t) \qquad (1.10)$$

with $u(0) = f$, which was suggested for (1.6) in §4.3.1 of [23].

Next, we will generalize the above results to a one-particle spin system, where all spin components or directions may move separately in $\mathbb{R}^d$ as in the Stern-Gerlach experiment (cf. Chap. 12 of [10] and §1.1 of [28]). We generally suppose that a particle has l spin components (cf. p.12 in [2] and §2.2 of [9]) and we consider a continuous position measurement for all spin components in $[0, T]$, where $l \geq 0$ is an integer. Although we don't know the physical meaning precisely, we will study the effective Lagrangian function given by

$$\mathcal{L}_{sw}(t,x,\dot{x}) = \mathcal{L}_w(t,x,\dot{x}) - \hbar H_s(t,x) + i\hbar W_s(t,x) \qquad (1.11)$$

as a generalization of (1.7), where $\mathcal{L}_w(t,x,\dot{x})$ is the Lagrangian function defined by (1.7), $H_s(t,x)$ an $l \times l$ Hermitian matrix denoting the spin term and $W_s(t,x)$ an $l \times l$ Hermitian matrix denoting the weight term acting on the spin components. The corresponding non-self-adjoint Schrödinger equation is written as

$$i\hbar\frac{\partial u}{\partial t}(t) = \big[H_w(t)I + \hbar H_s(t,x) - i\hbar W_s(t,x)\big]u(t), \qquad (1.12)$$

where $H_w(t)$ is the Hamiltonian operator defined by (1.10). We will prove that the RFPI for (1.11) can be defined rigorously in $(L^2(\mathbb{R}^d))^l$ and is the solution to the equation (1.12). It is noted that for $W_s(t,x)$ we assume

$$|\partial_x^\alpha w_{sij}(t,x)| \leq C_\alpha, \quad i,j = 1,2,\ldots,l \qquad (1.13)$$

in $[0,T]\times\mathbb{R}^d$ for all α, where $w_{sij}(t,x)$ denotes the (i,j)-component of $W_s(t,x)$.

Finally, we consider a quantum spin system consisting of N particles with l spin components each, under a continuous position measurement for all spin components of all particles in $[0,T]$.

We note that if $W(t,x)=0$ and $W_s(t,x)=0$ in (1.7) and (1.11), all results in the present paper give the same results as for the usual Feynman path integrals in [14, 16, 18].

In the present paper the RFPIs are defined by the time-slicing method in terms of piecewise free moving paths or broken line paths. This approach to the Feynman path integrals are widely used in the physics literature (cf. §2.4 in [8], §3.2 in [22], Appendix A3 in [23], §9.1 in [25] and §5.1 in [27]).

We will prove the main theorems in the present paper, following the proofs in [13, 14, 16, 18], where the usual Feynman path integrals, i.e. with $W(t,x)=0$ and $W_s(t,x)=0$ were studied. More specifically, we first introduce the fundamental operator $\mathcal{C}(t,s)$ in §3, and then prove its stability and consistency. Combining these results and the results in [17] concerning the non-self-adjoint Schrödinger equations (1.10) and (1.12), we can complete the proofs of our results. In particular, we define the RFPIs for the spin system, following [16]. We also note that in the present paper we will use the following delicate result concerning the L^2-boundedness of pseudo-differential operators, which follows from Theorem 13.13 on p. 322 in [29].

Theorem 1.A. *Suppose* $p(x,\xi,x')\in S^0(\mathbb{R}^{3d})$, *i.e.*

$$\sup_{x,\xi,x'}|\partial_\xi^\alpha\partial_x^\beta\partial_{x'}^\gamma p(x,\xi,x')|\le C_{\alpha\beta\gamma}<\infty \tag{1.14}$$

for all multi-indices α, β *and* γ*, where* ∂_ξ^α *denotes* $(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_d)^{\alpha_d}$ *for* $\alpha = (\alpha_1, \dots, \alpha_d)$*. Let* $P(X, \mathfrak{h}D_x, X')$ *be the pseudo-differential operator defined by*

$$\int e^{ix\cdot\xi}\, đ\xi \int e^{-ix'\cdot\xi} p(x, \mathfrak{h}\xi, x') f(x') dx', \quad đ\xi = (2\pi)^{-d} d\xi \tag{1.15}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$*, where* $x \cdot \xi = \sum_{j=1}^d x_j\xi_j$*,* $\mathfrak{h} > 0$ *is a constant and* $\mathcal{S}(\mathbb{R}^d)$ *denotes the Schwartz space of all rapidly decreasing functions in* $\mathbb{R}^d$*. Then we have*

$$\|P(X, \mathfrak{h}D_x, X')\|_{L^2\to L^2} = \sup_{x,\xi} |p(x, \xi, x)| + O(\mathfrak{h}), \tag{1.16}$$

where $\|P\|_{L^2\to L^2}$ *denotes the operator norm from* L^2 *into* L^2*.*

The delicate estimate (1.16) in Theorem 1.A will be essentially used as $\mathfrak{h} = t - s$ to prove (3.28) in the present paper.

The plan of the present paper is as follows. In §2 all main results are stated in Theorems 2.1 - 2.6. In §3 and §4 we will prove the stability and the consistence of $\mathcal{C}(t, s)$ respectively. In §5 Theorems 2.1 and 2.2 will be proved. In §6 Theorems 2.3 - 2.6 will be proved. In the appendix we will give a proof of Theorem 1.A by means of Theorem 13.13 in [29].

2 Main theorems

Hereafter we suppose $\hbar = 1$ and $\mathfrak{q} = 1$ for simplicity. We first consider (1.9). Let t in $[0, T]$. For an arbitrary integer $\nu \geq 1$ we take $\tau_j \in [0, t]$ $(j = 1, 2, \dots, \nu - 1)$ satisfying $0 = \tau_0 < \tau_1 < \cdots < \tau_{\nu-1} < \tau_\nu = t$, set $\Delta := \{\tau_j\}_{j=1}^{\nu-1}$ and write $|\Delta| := \max\{\tau_{j+1} - \tau_j; j = 0, 1, \dots, \nu - 1\}$. Let $x \in \mathbb{R}^d$ be fixed. We take arbitrary points $x^{(j)} \in \mathbb{R}^d$ $(j = 0, 1, \dots, \nu - 1)$ and determine the piecewise free moving path or the piecewise straight line $q_\Delta(\theta; x^{(0)}, \dots, x^{(\nu-1)}, x) \in$

$\mathbb{R}^d$ $(0 \le \theta \le t)$ by joining $x^{(j)}$ at τ_j $(j = 0, 1, \dots, \nu, x^{(\nu)} = x)$ in order. Let $S_w(t, s; q)$ be the effective classical action defined by (1.8). Take $\chi \in C_0^\infty(\mathbb{R}^d)$, i.e. an infinitely differentiable function on $\mathbb{R}^d$ with compact support, such that $\chi(0) = 1$ and fix it through the present paper. For simplicity we suppose that χ is real-valued. We will determine the approximation of the RFPI expressed as (1.9) by

$$K_\Delta(t,0)f = \lim_{\epsilon \to 0+} \prod_{j=0}^{\nu-1} \sqrt{\frac{m}{2\pi i(\tau_{j+1} - \tau_j)}}^{\,d} \int \cdots \int_{\mathbb{R}^d} e^{iS_w(t,0;q_\Delta)} \times f(x^{(0)}) \prod_{j=1}^{\nu-1} \chi(\epsilon x^{(j)}) dx^{(0)} dx^{(1)} \cdots dx^{(\nu-1)} \tag{2.1}$$

for $f \in C_0^\infty(\mathbb{R}^d)$. The right-hand side of (2.1) is called an oscillatory integral and will be denoted by

$$\prod_{j=0}^{\nu-1} \sqrt{\frac{m}{2\pi i(\tau_{j+1} - \tau_j)}}^{\,d} \mathrm{Os} - \int \cdots \int_{\mathbb{R}^d} e^{iS_w(t,0;q_\Delta)} f(x^{(0)}) dx^{(0)} dx^{(1)} \cdots dx^{(\nu-1)}$$

(cf. p. 45 of [20]).

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ and $x \in \mathbb{R}^d$ we write $|\alpha| = \sum_{j=1}^d \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $< x > = \sqrt{1 + |x|^2}$. In the present paper we often use symbols $C, C_\alpha, C_{\alpha\beta}$, C_a and δ_α to write down constants, though these values are different in general.

Throughout the present paper we assume that $\partial_x^\alpha E_j(t,x)$ $(j = 1, 2, \dots, d)$, $\partial_x^\alpha B_{jk}(t,x)$ $(1 \le j < k \le d)$ and $\partial_x^\alpha W(t,x)$ are continuous in $[0,T] \times \mathbb{R}^d$ for all α. Then, $\partial_x^\alpha \partial_t B_{jk}(t,x)$ $(1 \le j < k \le d)$ are also continuous in $[0,T] \times \mathbb{R}^d$ for all α, because of Faraday's law $\partial_t B_{jk} = -\partial E_k / \partial x_j + \partial E_j / \partial x_k$, which follows from (1.1).

Assumption 2.A. We assume

$$|\partial_x^\alpha E_j(t,x)| \leq C_\alpha,\ |\alpha| \geq 1, \quad j = 1, 2, \ldots, d, \tag{2.2}$$

$$|\partial_x^\alpha B_{jk}(t,x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)},\ |\alpha| \geq 1, \quad 1 \leq j < k \leq d \tag{2.3}$$

in $[0,T] \times \mathbb{R}^d$ with constants $C_\alpha \geq 0$ and $\delta_\alpha > 0$.

Assumption 2.B. We assume (2.2) and

$$|\partial_x^\alpha \partial_t B_{jk}(t,x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)},\ |\alpha| \geq 1, \quad 1 \leq j < k \leq d \tag{2.4}$$

in $[0,T] \times \mathbb{R}^d$ with constants $C_\alpha \geq 0$ and $\delta_\alpha > 0$.

Assumption 2.C. We assume that $\partial_x^\alpha A_j(t,x)$ $(j = 1, 2, \ldots, d)$ and $\partial_x^\alpha V(t,x)$ are continuous in $[0,T] \times \mathbb{R}^d$ for all α and satisfy

$$|\partial_x^\alpha A_j(t,x)| \leq C_\alpha,\ |\alpha| \geq 1, \quad j = 1, 2, \ldots, d, \tag{2.5}$$

$$|\partial_x^\alpha V(t,x)| \leq C_\alpha < x >, \quad |\alpha| \geq 1 \tag{2.6}$$

in $[0,T] \times \mathbb{R}^d$ with constants $C_\alpha \geq 0$.

Assumption 2.D. We assume

$$W(t,x) \geq -C_W, \tag{2.7}$$

$$|\partial_x^\alpha W(t,x)|^{p_\alpha} \leq C_\alpha \big\{1 + C_W + W(t,x)\big\}, \quad |\alpha| \geq 1, \tag{2.8}$$

$$|\partial_x^\alpha W(t,x)| \leq C_\alpha < x >, \quad |\alpha| \geq 1 \tag{2.9}$$

in $[0,T] \times \mathbb{R}^d$ with constants $C_W \geq 0, C_\alpha \geq 0$ and $p_\alpha \geq 1$.

Example 2.1. The function $c|x - a(t)|^2/\delta^2$ in (1.6) with a continuous path $a(t) \in \mathbb{R}^d$ satisfies Assumption 2.D.

Theorem 2.1. *Suppose that Assumptions 2.A and 2.D are satisfied. Then, there exists a constant* $\rho^* > 0$ *such that the following statements hold for arbitrary potentials* (V, A) *with continuous* $V, \partial V/\partial x_j, \partial A_j/\partial t, \partial A_j/\partial x_k$ $(j, k = 1, 2, \dots, d)$ *in* $[0, T] \times \mathbb{R}^d$*, all* Δ *satisfying* $|\Delta| \le \rho^*$ *and all* $t \in [0, T]$*:*

(1) $K_\Delta(t, 0)f$ *defined on* $f \in C_0^\infty(\mathbb{R}^d)$ *by* (2.1) *is determined independently of the choice of* χ *and* $K_\Delta(t, 0)f$ *can be uniquely extended to a bounded operator on* L^2*.*

(2) For all $f \in L^2$*, as* $|\Delta| \to 0$*,* $K_\Delta(t, 0)f$ *converges in* L^2 *uniformly in* $t \in [0, T]$ *to an element* $K(t, 0)f \in L^2$*, which we call the RFPI of* f*.*

(3) For all $f \in L^2$*,* $K(t, 0)f$ *belongs to* $C_t^0([0, T]; L^2)$*, where* $C_t^j([0, T]; L^2)$ *denotes the space of all* L^2*-valued,* j*-times continuously differentiable functions in* $t \in [0, T]$*. In addition,* $K(t, 0)f$ *is the unique solution in* $C_t^0([0, T]; L^2)$ *to* (1.10) *with* $u(0) = f$*.*

(4) Let $\psi(t, x) \in C^1([0, T] \times \mathbb{R}^d)$ *be a real-valued function such that* $\partial_{x_j}\partial_{x_k}\psi(t, x)$ *and* $\partial_t\partial_{x_j}\psi(t, x)$ $(j, k = 1, 2, \dots, d)$ *are continuous in* $[0, T] \times \mathbb{R}^d$ *and consider the gauge transformation*

$$V' = V - \frac{\partial \psi}{\partial t}, \quad A'_j = A_j + \frac{\partial \psi}{\partial x_j} \quad (j = 1, 2, \dots, d). \tag{2.10}$$

We write (2.1) *for this* (V', A') *as* $K'_\Delta(t, 0)f$*. Then we have the formula*

$$K'_\Delta(t, 0)f = e^{i\psi(t,\cdot)} K_\Delta(t, 0) \left(e^{-i\psi(0,\cdot)} f\right) \tag{2.11}$$

for all $f \in L^2$ *as in the case of* $W(t, x) = 0$ *(cf. (6.16) in [14]), and we have the analogous relation between the limits* $K'(t, 0)f$ *and* $K(t, 0)f$*.*

Let us introduce the weighted Sobolev spaces

$$B^a(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d); \|f\|_a := \|f\| + \sum_{|\alpha|=a} \big(\|x^\alpha f\| + \|\partial_x^\alpha f\|\big) < \infty\}$$

$$(a = 1, 2, \dots) \tag{2.12}$$

as in [14]. We denote the dual space of B^a by B^{-a} (cf. Lemma 2.4 in [12]) and the L^2 space by B^0.

Theorem 2.2. *Suppose that either Assumption 2.A or 2.B is satisfied. In addition, we suppose Assumptions 2.C and 2.D. Then there exists another constant $\rho^* > 0$ such that the same statements (1) - (4) as in Theorem 2.1 hold for all Δ satisfying $|\Delta| \le \rho^*$ and all $t \in [0,T]$. In addition, for all $f \in B^a(\mathbb{R}^d)$ $(a = 1, 2, \dots)$ $K_\Delta(t,0)f$ belongs to B^a and as $|\Delta| \to 0$, $K_\Delta(t,0)f$ converges in B^a uniformly in $t \in [0,T]$ to $K(t,0)f$, which belongs to $C_t^0([0,T]; B^a)$.*

Next, we consider a one-particle spin system (1.11) with l spin components. Throughout the present paper we assume that $\partial_x^\alpha h_{sij}(t,x)$ and $\partial_x^\alpha w_{sij}(t,x)$ are continuous in $[0,T] \times \mathbb{R}^d$ for all α and $i, j = 1, 2, \dots, l$, where h_{sij} denotes the (i,j)-component of H_s. For a continuous path $q(\theta) \in \mathbb{R}^d$ $(s \le \theta \le t)$ we define an $l \times l$ matrix $\mathcal{F}(\theta, s; q)$ $(s \le \theta \le t)$ by the solution $\mathcal{U}(\theta)$ to

$$\frac{d}{d\theta}\mathcal{U}(\theta) = -\big\{iH_s(\theta, q(\theta)) + W_s(\theta, q(\theta))\big\}\mathcal{U}(\theta), \quad \mathcal{U}(s) = I \tag{2.13}$$

with the identity matrix I.

Let $\Delta = \{\tau_j\}_{j=1}^{\nu-1}$ be a subdivision of $[0,t]$ and $q_\Delta = q_\Delta(\theta; x^{(0)}, \dots, x^{(\nu-1)}, x) \in \mathbb{R}^d$ $(0 \le \theta \le t)$ the piecewise free moving path defined in the early part of this section. We define the probability amplitude by

$$\exp *iS_{sw}(t,0;q_\Delta) := \big(\exp iS_w(t,0;q_\Delta)\big)\mathcal{F}(t,0;q_\Delta) \tag{2.14}$$

as in the case of $W_s(t,x)=0$ (cf. §2 in [16]), where $S_w(t,0;q_\Delta)$ is the classical action defined by (1.8). For $f = {}^t(f_1,\dots,f_l) \in C_0^\infty(\mathbb{R}^d)^l$ we determine the approximation of the RFPI for this system under the measurement by

$$K_{s\Delta}(t,0)f = \lim_{\epsilon\to 0+}\prod_{j=0}^{\nu-1}\sqrt{\frac{m}{2\pi i(\tau_{j+1}-\tau_j)}}^{\,d}\int\cdots\int_{\mathbb{R}^d}\big(\exp *iS_{sw}(t,0;q_\Delta)\big) \\ \times f(x^{(0)})\prod_{j=1}^{\nu-1}\chi(\epsilon x^{(j)})dx^{(0)}dx^{(1)}\cdots dx^{(\nu-1)} \tag{2.15}$$

as in [16]. The L^2-norm of $f = {}^t(f_1,\dots,f_l) \in (L^2)^l$ is defined by $\|f\| := \sqrt{\sum_{j=1}^l \|f_j\|^2}$.

Theorem 2.3. *Besides Assumptions 2.A and 2.D we suppose that $H_s(t,x)$ and $W_s(t,x)$ satisfy*

$$|\partial_x^\alpha h_{sij}(t,x)| \le C_\alpha, \quad i,j = 1,2,\dots,l \tag{2.16}$$

and (1.13) *for all α, respectively. Let $\rho^* > 0$ be the constant determined in Theorem 2.1. Then the same statements for $K_{s\Delta}(t,0)f$ as for $K_\Delta(t,0)f$ in Theorem 2.1 hold, where $K_s(t,0)f := \lim_{|\Delta|\to 0} K_{s\Delta}(t,0)f \in C_t^0([0,T];(L^2)^l)$ for $f \in (L^2)^l$ is the unique solution in $C_t^0([0,T];(L^2)^l)$ to* (1.12) *with $u(0) = f$.*

Example 2.2. We consider a continuous quantum measurement of the positions of all spin components of a particle. Let $a^{(j)}(t) \in \mathbb{R}^d$ $(j = 1,2,\dots,l)$ be the result for the j-th spin component and $\delta > 0$ the resolution of the measuring device. Then in Theorem 2.3 we take $W(t,x) = 0$ and the diagonal matrix $W_s(t,x)$ with

$$w_{sjj}(t,x) = \Omega\left(\frac{c|x-a^{(j)}(t)|^2}{\delta^2}\right), \quad j = 1,2,\dots,l,$$

where $\Omega(\theta) \in C^\infty([0,\infty))$ is an increasing function such that $\Omega(\theta) = \theta$ if $0 \le \theta \le 1$ and $\Omega(x) = L$ if $\theta \ge 2$ with a sufficiently large constant $L > 0$. These $W(t,x)$ and $W_s(t,x)$ satisfy the assupmtions of Theorem 2.3.

The B^a-norm of $f = {}^t(f_1,\dots,f_l) \in (B^a)^l$ is defined by $\|f\|_a := \sqrt{\sum_{j=1}^l \|f_j\|_a^2}$.

Theorem 2.4. *We suppose that either Assumption 2.A or 2.B is satisfied. In addition, we suppose Assumptions 2.C, 2.D,* (1.13) *and* (2.16)*. Let $\rho^* > 0$ be the constant determined in Theorem 2.2. Then the same statements for $K_{s\Delta}(t,0)f$ as for $K_\Delta(t,0)f$ in Theorem 2.2 hold, where we replace B^a with $(B^a)^l$.*

Finally, we consider a quantum spin system consisting of N particles which have l spin components each. We perform a continuous quantum measurement of the positions of all spin components of all particles in $[0,T]$. Denoting the coordinates of the j-th particle by $\mathbf{x}_j \in \mathbb{R}^d$ $(j = 1,2,\dots,N)$, we write $x = (\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_N) \in \mathbb{R}^{dN}$. Let $W_j(t,\mathbf{x}_j) \in \mathbb{R}$ and set

$$\begin{aligned} \mathcal{L}_w^\sharp(t,x,\dot{x}) = \sum_{j=1}^N \Big\{ & \frac{m}{2}|\dot{\mathbf{x}}_j|^2 + \dot{\mathbf{x}}_j\cdot\mathbf{A}_j(t,\mathbf{x}_j) - V_j(t,\mathbf{x}_j) \\ & + iW_j(t,\mathbf{x}_j)\Big\} - \sum_{j,k=1,j\neq k}^N V_{jk}(t,\mathbf{x}_j-\mathbf{x}_k), \end{aligned} \tag{2.17}$$

where $\mathbf{A}_j(t,\mathbf{x}_j) \in \mathbb{R}^d, V_j(t,\mathbf{x}_j) \in \mathbb{R}$ and $V_{jk}(t,\mathbf{x}_j-\mathbf{x}_k) \in \mathbb{R}$. The effective

Lagrangian function we consider is

$$\mathcal{L}^\sharp_{sw}(t,x,\dot{x}) = \mathcal{L}^\sharp_{w}(t,x,\dot{x}) + \sum_{j=1}^N I_1\otimes\cdots\otimes I_{j-1}$$

$$\otimes\big\{-H_{sj}(t,\mathbf{x}_j)+iW_{sj}(t,\mathbf{x}_j)\big\}\otimes I_{j+1}\otimes\cdots\otimes I_N, \qquad (2.18)$$

generalizing (1.11), where $H_{sj}(t,\mathbf{x}_j)$ and $W_{sj}(t,\mathbf{x}_j)$ are $l\times l$ Hermitian matrices.

For a continuous path $\mathbf{q}_j(\theta)\in\mathbb{R}^d$ $(j=1,2,\dots,N, s\le\theta\le t)$, we define $\mathcal{F}_j(\theta,s;\mathbf{q}_j)$ $(s\le\theta\le t)$ by the solution to (2.13) where $H_s=H_{sj}$ and $W_s=W_{sj}$. For the piecewise free moving path $q_\Delta=\big(\mathbf{q}_{1\Delta}(\theta;\mathbf{x}_1^{(0)},\dots,\mathbf{x}_1^{(\nu-1)},\mathbf{x}_1),\dots,\mathbf{q}_{N\Delta}(\theta;\mathbf{x}_N^{(0)},\dots,\mathbf{x}_N^{(\nu-1)},\mathbf{x}_N)\big)\in\mathbb{R}^{Nd}$ $(0\le\theta\le t)$, we define the probability amplitude by

$$\begin{aligned}&\exp *iS^\sharp_{sw}(t,0;q_\Delta)\\ &:=\big(\exp iS^\sharp_w(t,0;q_\Delta)\big)\mathcal{F}_1(t,0;\mathbf{q}_{1\Delta})\otimes\cdots\otimes\mathcal{F}_N(t,0;\mathbf{q}_{N\Delta})\end{aligned} \qquad (2.19)$$

in terms of the tensor product of matrices, where $S^\sharp_w(t,0;q_\Delta)$ is the classical action defined from (2.17). Then we determine the approximation $K^\sharp_{s\Delta}(t,0)f$ of the RFPI by (2.15), where $\exp *iS_{sw}(t,0;q_\Delta)$ and $f\in C_0^\infty(\mathbb{R}^d)^l$ are replaced with $\exp *iS^\sharp_{sw}(t,0;q_\Delta)$ and $f=f_1\otimes\cdots\otimes f_N$ $\big(f_j\in C_0^\infty(\mathbb{R}^d)^l, j=1,2,\dots,N\big)$, respectively.

Writing $A(t,x)=\big(\mathbf{A}_1(t,\mathbf{x}_1),\dots,\mathbf{A}_N(t,\mathbf{x}_N)\big)\in\mathbb{R}^{dN}$ and $V(t,x)=\sum_{j=1}^N V_j(t,\mathbf{x}_j)+\sum_{j,k=1,j\ne k}V_{jk}(t,\mathbf{x}_j-\mathbf{x}_k)\in\mathbb{R}$, we define $E(t,x)\in\mathbb{R}^{dN}$ and $B_{jk}(t,x)\in\mathbb{R}$ $(1\le j<k\le dN)$ by (1.1). Then we have the following.

Theorem 2.5. *Suppose that Assumptions 2.A is satisfied. In addition, we assume that each* $W_j(t,\mathbf{x}_j)$ $(j=1,2,\dots,N)$ *satisfies 2.D and that each* $W_{sj}(t,\mathbf{x}_j)$ *and* $H_{sj}(t,\mathbf{x}_j)$ *satisfies* (1.13) *and* (2.16), *respectively. Let* $(L^2)^l\otimes$

$\cdots \otimes (L^2)^l$ *denote the tensor product of* N *copies of* $L^2(\mathbb{R}^d)^l$. *Then, there exits a constant* $\rho'^* > 0$ *such that the same statements for* $K^\sharp_{s\Delta}(t,0)f$ *as for* $K_\Delta(t,0)f$ *in Theorem 2.1 hold, where* $K^\sharp_s(t,0)f := \lim_{|\Delta|\to 0} K^\sharp_{s\Delta}(t,0)f \in C^0_t([0,T];(L^2)^l \otimes \cdots \otimes (L^2)^l)$ *for* $f \in (L^2)^l \otimes \cdots \otimes (L^2)^l$ *is the unique solution in* $C^0_t([0,T];(L^2)^l \otimes \cdots \otimes (L^2)^l)$ *to*

$$
\begin{aligned}
i\frac{\partial u}{\partial t}(t) = \Bigg[\sum_{j=1}^{N}\Bigg\{\frac{1}{2m}\left|\frac{1}{i}\frac{\partial}{\partial \mathbf{x}_j} - \mathbf{A}_j(t,\mathbf{x}_j)\right|^2 + V_j(t,\mathbf{x}_j) - iW_j(t,\mathbf{x}_j) \\
+ I_1 \otimes \cdots \otimes I_{j-1} \otimes \big\{H_{sj}(t,\mathbf{x}_j) - iW_{sj}(t,\mathbf{x}_j)\big\} \otimes I_{j+1} \otimes \cdots \otimes I_N\Bigg\} \\
+ \sum_{j,k=1, j\neq k}^{N} V_{jk}(t,\mathbf{x}_j - \mathbf{x}_k)\Bigg] u(t) \qquad (2.20)
\end{aligned}
$$

with $u(0) = f$.

We see that the N-fold tensor product $L^2(\mathbb{R}^d)^l \otimes \cdots \otimes L^2(\mathbb{R}^d)^l$ is equal to $\big(L^2(\mathbb{R}^d) \otimes \cdots \otimes L^2(\mathbb{R}^d)\big)^{l^N}$ because we have

$$
\begin{pmatrix} g_1(\mathbf{x}_1) \\ g_2(\mathbf{x}_1) \end{pmatrix} \otimes \begin{pmatrix} h_1(\mathbf{x}_2) \\ h_2(\mathbf{x}_2) \end{pmatrix} = \sum_{i,j=1}^{2} g_i(\mathbf{x}_1) h_j(\mathbf{x}_2) e_i \otimes e_j
$$

with $e_1 = {}^t(1,0)$ and $e_2 = {}^t(0,1)$, for examle, when $N = 2$ and $l = 2$. This shows

$$
L^2(\mathbb{R}^d)^l \otimes \cdots \otimes L^2(\mathbb{R}^d)^l = L^2(\mathbb{R}^{dN})^{l^N}
$$

because of $L^2(\mathbb{R}^d) \otimes \cdots \otimes L^2(\mathbb{R}^d) = L^2(\mathbb{R}^{Nd})$ (cf. II.10 on p.52 in [26]). In the same way we can define a subspace $B^a(\mathbb{R}^{dN})^{l^N}$ $(a = 1, 2, \dots)$ in $L^2(\mathbb{R}^d)^l \otimes \cdots \otimes L^2(\mathbb{R}^d)^l$. Then we have the following.

Theorem 2.6. *Suppose that either Assumption 2.A or 2.B is satisfied. In addition, we suppose Assumption 2.C for* $(V(t,x), A(t,x))$ *and that each*

$W_j(t,\mathbf{x}_j), W_{sj}(t,\mathbf{x}_j)$ and $H_{sj}(t,\mathbf{x}_j)$ satisfies the assumptions stated in Theorem 2.5. Then, there exits another constant $\rho'^* > 0$ such that the same statements for $K^\sharp_{s\Delta}(t,0)f$ as for $K_\Delta(t,0)f$ in Theorem 2.2 hold, where we replace $B^a(\mathbb{R}^d)$ with $B^a(\mathbb{R}^{dN})^{l^N}$.

Remark 2.1. We consider polynomially growing potentials

$$V(t,x) = |x|^{2(l_0+1)} + \sum_{|\alpha|\leq 2l_0+1} a_\alpha(t)x^\alpha, \tag{2.21}$$

$$A_j(t,x) = \sum_{|\alpha|\leq l_0} b_{j\alpha}(t)x^\alpha \quad (j = 1,2,\ldots,d) \tag{2.22}$$

with an integer $l_0 \geq 1$ and functions $a_\alpha(t) \in \mathbb{R}, b_{j\alpha}(t) \in \mathbb{R}$ in $C^1([0,T])$. These potentials $V(t,x)$ and $A(t,x)$ do not satisfy either Assumption 2.A, 2.B or 2.C. We suppose Assumption 2.D, (1.13) and (2.16) for $W(t,x), W_s(t,x)$ and $H_s(t,x)$ respectively, where we replace (2.9) with

$$|\partial_x^\alpha W(t,x)| \leq C_\alpha < x >^{l_0+1}, \quad |\alpha| \geq 1. \tag{2.23}$$

We define $K_\Delta(t,0)f$ and $K_{s\Delta}(t,0)f$ by (2.1) and (2.15) respectively. Then we have the same statements (1) - (4) as in Theorems 2.1 and 2.3, following the proofs of Theorems 2.1 and 2.3 (cf. [18]). In the present paper we don't prove their statements. Their proofs will be published elsewhere in the general form.

3 Stability of $\mathcal{C}(t,s)$

Let $S(t,s;q)$ and $S_w(t,s;q)$ be the classical actions defined by (1.3) and (1.8), respectively. Let $q^{t,s}_{x,y}$ and $\gamma^{t,s}_{x,y}$ be the straight lines defined by

$$q^{t,s}_{x,y}(\theta) = y + \frac{\theta - s}{t - s}(x - y), \quad s \leq \theta \leq t \tag{3.1}$$

and

$$\gamma_{x,y}^{t,s}(\theta) = (\theta, q_{x,y}^{t,s}(\theta)) \in \mathbb{R}^{d+1}, \ s \le \theta \le t, \tag{3.2}$$

respectively. Throughout the present paper we often write $\rho = t - s$. Then we have

$$\begin{aligned}
S(t,s;q_{x,y}^{t,s}) &= \frac{m|x-y|^2}{2(t-s)} + \int_{\gamma_{x,y}^{t,s}} \left(A \cdot dx - V dt\right) \\
&= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot \int_0^1 A(s+\vartheta\rho, y+\vartheta(x-y))d\vartheta \\
&\quad - \int_s^t V\left(\theta, y + \frac{\theta - s}{t-s}(x-y)\right) d\theta \\
&= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot \int_0^1 A(t-\vartheta\rho, x-\vartheta(x-y))d\vartheta \\
&\quad - \rho \int_0^1 V(t-\vartheta\rho, x-\vartheta(x-y))d\vartheta,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
S_w(t,s;q_{x,y}^{t,s}) &= S(t,s;q_{x,y}^{t,s}) + i\int_s^t W\left(\theta, y + \frac{\theta - s}{t-s}(x-y)\right) d\theta \\
&= S(t,s;q_{x,y}^{t,s}) + i\rho \int_0^1 W(t-\vartheta\rho, x-\vartheta(x-y))d\vartheta.
\end{aligned} \tag{3.4}$$

Let $M \ge 0$ be an integer and suppose that $p(x,w) \in C^\infty(\mathbb{R}^{2d})$ satisfies

$$|\partial_w^\alpha \partial_x^\beta p(x,w)| \le C_{\alpha\beta} < x; w >^M, \ (x,w) \in \mathbb{R}^{2d} \tag{3.5}$$

for all α and β, where $< x; w >= \sqrt{1+|x|^2+|w|^2}$. For $f \in C_0^\infty(\mathbb{R}^d)$ we define

$$P(t,s)f = \begin{cases} \sqrt{m/(2\pi i\rho)}^{\,d} \displaystyle\int \left(\exp iS_w(t,s;q_{x,y}^{t,s})\right) \\ \qquad \times\, p(x,(x-y)/\sqrt{\rho})f(y)dy, \quad s < t, \\ \sqrt{m/(2\pi i)}^{\,d} \mathrm{Os} - \displaystyle\int (\exp im|w|^2/2) \\ \qquad \times\, p(x,w)dwf(x), \quad s = t. \end{cases} \tag{3.6}$$

Then the formal adjoint operator $P(t,s)^\dagger$ of $P(t,s)$ on $C_0^\infty(\mathbb{R}^d)$ is given by

$$P(t,s)^\dagger f = \begin{cases} \sqrt{im/(2\pi\rho)}^{\,d} \displaystyle\int \left(\exp iS_w(t,s;q^{t,s}_{y,x})\right)^* \\ \qquad\qquad \times\, p(y,(y-x)/\sqrt{\rho})^* f(y)dy, \quad s<t, \\ \sqrt{im/(2\pi)}^{\,d} \mathrm{Os} - \displaystyle\int (\exp -im|w|^2/2) \\ \qquad\qquad \times\, p(x,w)^* dw f(x), \quad s=t. \end{cases}$$

We can prove the following as in the proof of Lemma 2.1 of [14].

Lemma 3.1. *Let $p(x,w)$ be a function satisfying* (3.5). *We assume* (2.7). *In addition, we assume that $\partial_x^\alpha V(t,x)$ and $\partial_x^\alpha A_j(t,x)$ $(j=1,2,\dots,d)$ are continuous in $[0,T]\times\mathbb{R}^d$ for all α and that there exists a constant $M'\geq 0$ satisfying*

$$|\partial_x^\alpha V(t,x)| + \sum_{j=1}^d |\partial_x^\alpha A_j(t,x)| + |\partial_x^\alpha W(t,x)| \leq C_\alpha < x >^{M'}$$

in $[0,T]\times\mathbb{R}^d$ for all α. Then, for $f\in\mathcal{S}$ $\partial_x^\alpha(P(t,s)f)$ are continuous in $0\leq s\leq t\leq T$ and $x\in\mathbb{R}^d$ for all α.

In particular, when $p(x,w)=1$, we write $P(t,s)f$ as $\mathcal{C}(t,s)f$. That is,

$$\mathcal{C}(t,s)f = \begin{cases} \sqrt{m/(2\pi i\rho)}^{\,d} \displaystyle\int \left(\exp iS_w(t,s;q^{t,s}_{x,y})\right) f(y)dy, \quad s<t, \\ f, \qquad s=t. \end{cases} \tag{3.7}$$

Then, from (2.1) we can write

$$K_\Delta(t,0)f = \lim_{\epsilon\to 0+} \mathcal{C}(t,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}(\tau_1,0)f \tag{3.8}$$

for $f\in C_0^\infty(\mathbb{R}^d)$.

For a weight function $W(t,x)$ we set

$$c_w(t,s;x,y) = \exp\left(-\rho\int_0^1 W(t-\theta\rho, x-\theta(x-y))d\theta\right), \ \rho = t-s. \tag{3.9}$$

Lemma 3.2. *Let $p(x,w)$ be a function satisfying (3.5). We assume that $\partial_x^\alpha V(t,x), \partial_x^\alpha A_j(t,x)$ and $\partial_x^\alpha \partial_t A_j(t,x)$ are continuous in $[0,T]\times\mathbb{R}^d$ for $|\alpha|\le 1$ and $j=1,2,\dots,d$. Let $f\in C_0^\infty(\mathbb{R}^d)$. Then for any $0<\epsilon\le 1$ and $0\le s<t\le T$ we have*

$$
\begin{aligned}
P(t,s)^\dagger \chi(\epsilon\cdot)^2 P(t,s) f = {} & \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2 \\
& \times \left(\exp i(x-y)\cdot\frac{m\Phi}{t-s}\right) c_w(t,s;z,x) c_w(t,s;z,y) \\
& \times p\left(z, \frac{z-x}{\sqrt{t-s}}\right)^* p\left(z,\frac{z-y}{\sqrt{t-s}}\right) dz, \qquad (3.10)
\end{aligned}
$$

$$
\Phi = \Phi(t,s;x,y,z) = (\Phi_1,\dots,\Phi_d),
$$

$$
\begin{aligned}
\Phi_j = {} & z_j - \frac{x_j+y_j}{2} + \frac{t-s}{m}\int_0^1 A_j(s, x+\theta(y-x))d\theta \\
& - \frac{(t-s)^2}{m}\int_0^1\int_0^1 \sigma_1 E_j(\tau(\sigma),\zeta(\sigma))d\sigma_1 d\sigma_2 \\
& - \frac{t-s}{m}\sum_{k=1}^d (z_k-x_k)\int_0^1\int_0^1 \sigma_1 B_{jk}(\tau(\sigma),\zeta(\sigma))d\sigma_1 d\sigma_2 \qquad (3.11)
\end{aligned}
$$

or

$$
\begin{aligned}
\Phi_j = {} & z_j - \frac{x_j+y_j}{2} + \frac{t-s}{m}\int_0^1 A_j(s,x+\theta(y-x))d\theta \\
& - \frac{(t-s)^2}{m}\int_0^1\int_0^1 \sigma_1 E_j(\tau(\sigma),\zeta(\sigma))d\sigma_1 d\sigma_2 - \frac{(t-s)^2}{m}\int_0^1 d\theta \sum_{k=1}^d (z_k-x_k) \\
& \times \int_0^1\int_0^1 \sigma_1(1-\sigma_1)\frac{\partial B_{jk}}{\partial t}(s+\theta(1-\sigma_1)\rho, \zeta(\sigma))d\sigma_1 d\sigma_2, \qquad (3.12)
\end{aligned}
$$

where

$$
\big(\tau(\sigma),\zeta(\sigma)\big) = \big(t-\sigma_1(t-s), z+\sigma_1(x-z)+\sigma_1\sigma_2(y-x)\big) \in \mathbb{R}^{d+1}. \qquad (3.13)
$$

Proof. From (3.4), (3.6) and (3.9) we can write

$$P(t,s)f = \sqrt{m/(2\pi i\rho)}^{\,d}\int \big(\exp iS(t,s;q^{t,s}_{x,y})\big)c_w(t,s;x,y) \times p\left(x,\frac{x-y}{\sqrt{t-s}}\right)f(y)dy, \quad \rho = t-s>0. \tag{3.14}$$

Hence, we can easily prove Lemma 3.2 from Lemma 5.2 in [18]. □

Lemma 3.3. *We assume that* $\partial_x^\alpha A_j(t,x)$ $(j=1,2,\dots,d)$ *are continuous for all* α *and satisfy* (2.5) *in* $[0,T]\times\mathbb{R}^d$.
(1) Suppose that Assumption 2.A is satisfied. Let $\Phi_j(t,s;x,y,z)$ $(j=1,2,\dots,d)$ *be the functions defined by* (3.11). *Then, there exist a constant* $\rho^*>0$ *such that for all fixed* $0\le t-s\le\rho^*$ *and* $(x,y)\in\mathbb{R}^{2d}$, *the map:* $\mathbb{R}^d \ni z\to\xi = \Phi(t,s;x,y,z)\in\mathbb{R}^d$ *is a homeomorphism, whose inverse will be denoted by the map:* $\mathbb{R}^d\ni\xi\to z = z(t,s;x,\xi,y)\in\mathbb{R}^d$, *and we have*

$$\sum_{j=1}^d |\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma z_j(t,s;x,\xi,y)| \le C_{\alpha\beta\gamma}, \ |\alpha+\beta+\gamma|\ge 1, \tag{3.15}$$

$$\det\frac{\partial z}{\partial\xi}(t,s;x,\xi,y) = 1+(t-s)h(t,s;x,\xi,y)>0, \tag{3.16}$$

$$|\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma h(t,s;x,\xi,y)|\le C_{\alpha\beta\gamma}<\infty \tag{3.17}$$

for all α,β *and* γ *in* $0\le t-s\le\rho^*$ *and* $(x,\xi,y)\in\mathbb{R}^{3d}$.
(2) Suppose that Assumption 2.B is satisfied. Let $\Phi_j(t,s;x,y,z)$ $(j=1,2,\dots,d)$ *be the functions defined by* (3.12). *Then we have the same statements as in (1).*

Proof. We have already proved (1) in Lemma 3.2, (3.9) and (3.10) of [14] (cf. Lemma 3.6, (3.18) and (3.19) of [13]).

We will prove (2). Let us write the 5-th term on the right-hand side of (3.12) as $-(t-s)^2 B'(t,s;x,y,z)/m$. Then, from the assumption (2.4) we can prove

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma B_j'(t,s;x,\xi,y)| \le C_{\alpha\beta\gamma}, \quad |\alpha+\beta+\gamma| \ge 1 \tag{3.18}$$

in $0 \le s \le t \le T$ and $(x,y,z) \in \mathbb{R}^{3d}$ as in the proof of (3.15) of [13], where $B' = (B_1', \cdots, B_d') \in \mathbb{R}^d$. Thereby we can prove (2) as in the proof of (1). □

From now on we fix $\rho^* > 0$ determined in Lemma 3.3 throughout the present paper. The following lemma is crucial in the present paper.

Lemma 3.4. *Assume* (2.7) *and* (2.8) *where we take* $C_W = 0$. *Let* $c_w(t,s;x,y)$ *be the function defined by* (3.9). *Then we have*

$$0 \le c_w(t,s;x,y) \le 1, \ |\partial_x^\alpha \partial_y^\beta c_w(t,s;x,y)| \le C_{\alpha\beta}, \ |\alpha+\beta| \ge 1 \tag{3.19}$$

for $0 \le s \le t \le T$ *and* $(x,y) \in \mathbb{R}^{2d}$ *with constants* $C_{\alpha\beta} \ge 0$.

Proof. It is clear from (2.7) and (3.9) that the first inequality of (3.19) holds. For $a \ge 0$ we can easily see

$$\sup_{r \ge 0} e^{-r}(T+r)^a = C_a' < \infty \tag{3.20}$$

with constants $C_a' \ge 0$. Let $|\alpha| \ge 1$. We note $p_\alpha \ge 1$. Using Hölder's inequality in the case of $p_\alpha > 1$ and (2.8), we have

$$\begin{aligned}
&\rho \int_0^1 |(\partial_x^\alpha W)(t-\theta\rho, x-\theta(x-y))|d\theta \\
&\le \rho \Big(\int_0^1 |(\partial_x^\alpha W)(t-\theta\rho, x-\theta(x-y))|^{p_\alpha} d\theta \Big)^{1/p_\alpha} \\
&\le C_\alpha^{1/p_\alpha} \rho \Big[\int_0^1 \big\{ 1 + W(t-\theta\rho, x-\theta(x-y)) \big\} d\theta \Big]^{1/p_\alpha} \\
&= C_\alpha^{1/p_\alpha} \rho^{1-1/p_\alpha} \Big\{ \rho + \rho \int_0^1 W(t-\theta\rho, x-\theta(x-y)) d\theta \Big\}^{1/p_\alpha} \\
&\le C_\alpha^{1/p_\alpha} \rho^{1-1/p_\alpha} \Big\{ T + \rho \int_0^1 W(t-\theta\rho, x-\theta(x-y)) d\theta \Big\}^{1/p_\alpha}. \qquad (3.21)
\end{aligned}$$

Hence, letting $\alpha = (1, 0, \dots, 0) \in \mathbb{R}^d$, by (2.7) and (3.20) we have

$$\begin{aligned}
&|\partial_{x_1} c_w(t, s; x, y)| \le \Big(\exp -\rho \int_0^1 W(t-\theta\rho, x-\theta(x-y)) d\theta \Big) \\
&\times C_\alpha^{1/p_\alpha} T^{1-1/p_\alpha} \Big\{ T + \rho \int_0^1 W(t-\theta\rho, x-\theta(x-y)) d\theta \Big\}^{1/p_\alpha} \\
&\le C_\alpha^{1/p_\alpha} T^{1-1/p_\alpha} C'_{1/p_\alpha} < \infty. \qquad (3.22)
\end{aligned}$$

In the same way we can complete the proof of the second inequality of (3.19), using (3.20) and (3.21). □

Proposition 3.5. *Suppose that either Assumption 2.A or 2.B is satisfied. In addition, we suppose Assumption 2.C,* (2.7) *and* (2.8)*, where we take* $C_W = 0$ *and* (2.6) *is replaced with*

$$|\partial_x^\alpha V(t, x)| \le C_\alpha < x >^{M_1}, \quad |\alpha| \ge 1 \qquad (3.23)$$

with an integer $M_1 \ge 1$ *independent of* α*. Let* $\rho^* > 0$ *be the constant determined in Lemma 3.3 and* $\mathcal{C}(t, s)$ *the operator defined by* (3.7)*. Then there* *exists a constant* $K_0 \geq 0$ *such that*

$$\|\mathcal{C}(t,s)f\| \leq e^{K_0(t-s)}\|f\|, \quad 0 \leq t-s \leq \rho^* \tag{3.24}$$

for all $f \in L^2$.

Proof. We first suppose that Assumption 2.A is satisfied. Then, letting Φ_j be defined by (3.11), from (3.10) we have

$$\mathcal{C}(t,s)^\dagger \chi(\epsilon\cdot)^2 \mathcal{C}(t,s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2 \times \left(\exp i(x-y)\cdot \frac{m\Phi}{t-s}\right) c_w(t,s;z,x)c_w(t,s;z,y)dz$$

for $f \in \mathcal{S}$. We will use (1) in Lemma 3.3. Letting $0 \leq t-s \leq \rho^*$ and making the change of variables: $\mathbb{R}^d \ni z \to \xi = \Phi(t,s;x,y,z) \in \mathbb{R}^d$ in the above equation, we have

$$\mathcal{C}(t,s)^\dagger \chi(\epsilon\cdot)^2 \mathcal{C}(t,s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2 \times \left(\exp i(x-y)\cdot \frac{m\xi}{t-s}\right) c_w(t,s;z,x)c_w(t,s;z,y) \det \frac{\partial z}{\partial \xi}(t,s;x,\xi,y)d\xi$$

with $z = z(t,s;x,\xi,y)$, which shows

$$\mathcal{C}(t,s)^\dagger \chi(\epsilon\cdot)^2 \mathcal{C}(t,s)f = \left(\frac{1}{2\pi}\right)^d \int e^{i(x-y)\cdot\eta}d\eta \int \chi(\epsilon z)^2 c_w(t,s;z,x) \times c_w(t,s;z,y) \det \frac{\partial z}{\partial \xi}(t,s;x,(t-s)\eta/m,y)f(y)dy, \; 0 \leq t-s \leq \rho^* \tag{3.25}$$

with $z = z(t,s;x,(t-s)\eta/m,y)$. Hence, noting (3.15) and (3.19), we can easily prove

$$\lim_{\epsilon\to 0+} \mathcal{C}(t,s)^\dagger \chi(\epsilon\cdot)^2 \mathcal{C}(t,s)f = \left(\frac{1}{2\pi}\right)^d \int e^{i(x-y)\cdot\eta}d\eta \int c_w(t,s;z,x)c_w(t,s;z,y) \times \det \frac{\partial z}{\partial \xi}(t,s;x,(t-s)\eta/m,y)f(y)dy, \; 0 \leq t-s \leq \rho^* \tag{3.26}$$

with $z = z(t,s;x,(t-s)\eta/m,y)$ in the topology of $\mathcal{S}$, which we write as $\mathcal{C}(t,s)^\dagger\,\mathcal{C}(t,s)f$ formally.

Let $z = z(t,s;x,(t-s)\eta/m,y)$. Then from (3.16) we can write

$$
\begin{aligned}
&c_w(t,s;z,x)c_w(t,s;z,y)\det\frac{\partial z}{\partial \xi}(t,s;x,(t-s)\eta/m,y)\\
&= c_w(t,s;z,x)c_w(t,s;z,y) + (t-s)c_w(t,s;z,x)c_w(t,s;z,y)h(t,s;x,(t-s)\eta/m,y)\\
&\equiv p_1(t,s;x,(t-s)\eta,y) + (t-s)p_2(t,s;x,(t-s)\eta,y). \qquad (3.27)
\end{aligned}
$$

Noting (3.15), (3.17) and (3.19), apply Theorem 1.A in the introduction as $\mathfrak{h} = t-s$ to $P_1(t,s;X,(t-s)D_x,X')$. Then we have

$$
\|P_1(t,s;X,(t-s)D_x,X')f\| \le \{1+K(t-s)\}\|f\|,\ 0\le t-s\le\rho^* \qquad (3.28)
$$

with a constant $K\ge 0$. In the same way we have

$$
\|P_2(t,s;X,(t-s)D_x,X')f\| \le K'\|f\|,\ 0\le t-s\le\rho^* \qquad (3.29)
$$

with a constant $K'\ge 0$. Thus, from (3.26) and (3.27) we obtain

$$
\begin{aligned}
\|\mathcal{C}(t,s)^\dagger\,\mathcal{C}(t,s)f\| &\le \big\{1+2K_0(t-s)\big\}\|f\|\\
&\le e^{2K_0(t-s)}\|f\|,\quad 0\le t-s\le\rho^* \qquad (3.30)
\end{aligned}
$$

with a constant $K_0\ge 0$. Consequently we have

$$
\begin{aligned}
\|\mathcal{C}(t,s)f\|^2 &\le \lim_{\epsilon\to 0+}\big(\chi(\epsilon\cdot)\mathcal{C}(t,s)f,\chi(\epsilon\cdot)\mathcal{C}(t,s)f\big)\\
&= \lim_{\epsilon\to 0+}\big(\mathcal{C}(t,s)^\dagger\chi(\epsilon\cdot)^2\mathcal{C}(t,s)f,f\big)\\
&= \big(\mathcal{C}(t,s)^\dagger\,\mathcal{C}(t,s)f,f\big) \le e^{2K_0(t-s)}\|f\|^2
\end{aligned}
$$

for $f\in\mathcal{S}$ by using Fatou's lemma, which shows (3.24).

Next we suppose that Assumption 2.B is satisfied. Letting Φ_j be defined by (3.12), we can also prove (3.26) and (3.30) as in the proof of the first case. Consequently we can prove (3.24). $\square$

Proposition 3.6. *Let $p(x, w)$ be a function satisfying* (3.5) *and $P(t, s)$ the operator defined by* (3.6). *Then, under the assumptions of Proposition 3.5 we have*

$$\|P(t, s)f\|_a \leq C_a\|f\|_{M+aM_1}, \quad 0 \leq t - s \leq \rho^* \tag{3.31}$$

for $a = 0, 1, 2, \ldots$ and all $f \in B^{M+aM_1}$ with constants $C_a \geq 0$, where M_1 is the integer in (3.23).

Proof. Setting

$$p'(t, s; x, w) := p(x, w)c_w(t, s; x, x - \sqrt{\rho}w), \tag{3.32}$$

from (3.14) we have

$$P(t, s)f = \sqrt{m/(2\pi i\rho)}^{\,d} \int (\exp iS(t, s; q^{t,s}_{x,y}))p'\left(t, s; x, \frac{x - y}{\sqrt{t - s}}\right) f(y)dy,$$

$$\rho = t - s > 0 \tag{3.33}$$

and also from (3.19)

$$|\partial_w^\alpha \partial_x^\beta p'(t, s; x, w)| \leq C_{\alpha\beta} < x; w >^M \tag{3.34}$$

for all α and β.

At first we suppose that Assumption 2.A is satisfied. Then, using (3.33) and (3.34), we can prove (3.31) from Theorem 4.4 of [14]. We can also prove (3.31) under Assumption 2.B, noting (3.18) and following the proof of Theorem 4.4 in [14]. $\square$

4 Consistency of $\mathcal{C}(t, s)$

Lemma 4.1. *Let $H_w(t)$ be the operator defined by* (1.10)*. We assume that for all α $\partial_x^\alpha V(t,x), \partial_x^\alpha A_j(t,x)$ $(j = 1, 2, \dots, d)$ and $\partial_x^\alpha \partial_t A_j(t,x)$ are continuous in $[0,T] \times \mathbb{R}^d$ and satisfy*

$$
\begin{aligned}
&|\partial_x^\alpha V(t,x)| + \sum_{j=1}^{d} \big(|\partial_x^\alpha A_j(t,x)| + |\partial_x^\alpha \partial_t A_j(t,x)|\big) + |\partial_x^\alpha W(t,x)| \\
&\le C_\alpha < x >^{M'}
\end{aligned}
$$

with constants $C_\alpha \ge 0$ and $M' \ge 0$, where M' is independent of α. Then, there exists a function $r(t, s; x, w)$ satisfying (3.5) *for an integer $M \ge 0$ such that $\partial_w^\alpha \partial_x^\beta r(t, s; x, w)$ are continuous in $0 \le s \le t \le T$ and $(x, w) \in \mathbb{R}^{2d}$ for all α, β and we have*

$$
\left\{ i\frac{\partial}{\partial t} - H_w(t) \right\} \mathcal{C}(t,s) f = \sqrt{t-s} R(t,s) f \tag{4.1}
$$

for $f \in C_0^\infty(\mathbb{R}^d)$.

Proof. We note (1.8) and (3.7). Then, replacing $V(t,x)$ with $V(t,x) - iW(t,x)$ in the proof of Lemma 4.1 of [15], we can complete the proof of Lemma 4.1. □

Proposition 4.2. *Besides the assumptions of Proposition 3.5 we assume*

$$
|\partial_x^\alpha W(t,x)| \le C_\alpha < x >^{M''}
$$

in $[0,T] \times \mathbb{R}^d$ for all α with constants $C_\alpha \ge 0$ and $M'' \ge 0$, where M'' is independent of α. Then, there exists a function $r(t, s; x, w)$ satisfying the properties stated in Lemma 4.1 and we have

$$
\|R(t,s)f\|_a \le C_a \|f\|_{M + aM_1}, \quad 0 \le t - s \le \rho^* \tag{4.2}
$$

for $a = 0, 1, 2, \dots$ and all $f \in B^{M+aM_1}$, where M_1 is the integer in (3.23)*.*

Proof. From (1.1) we have $\partial_t A_j = -E_j - \partial_{x_j} V$. Hence we see that $\partial_x^\alpha \partial_t A_j(t,x)$ are continuous in $[0,T]\times \mathbb{R}^d$ for all α from the assumptions. In addition, from (2.2) and (3.23) we have

$$|\partial_x^\alpha \partial_t A_j(t,x)| \leq |\partial_x^\alpha E_j(t,x)| + |\partial_x^\alpha \partial_{x_j} V(t,x)| \leq C_\alpha < x >^{M_1}, \ |\alpha| \geq 1.$$

Consequently the assumptions of Lemma 4.1 are satisfied. Then, applying Proposition 3.6 to $R(t,s)f$ in Lemma 4.1, we get (4.2). □

Making the change of variables: $\mathbb{R}^d \ni y \to w = (x-y)/\sqrt{\rho} \in \mathbb{R}^d$ in (3.6), from (3.3) and (3.4) we have

$$P(t,s)f = \sqrt{\frac{m}{2\pi i}}^{\,d} \int e^{i\phi(t,s;x,w)} p(x,w) f(x-\sqrt{\rho}w)dw, \ \rho = t-s > 0 \tag{4.3}$$

for $f \in C_0^\infty(\mathbb{R}^d)$ as in the proof of (2.9) of [14], where

$$\phi(t,s;x,w) = \frac{m}{2}|w|^2 + \sqrt{\rho}w \cdot \int_0^1 A(t-\theta\rho, x-\theta\sqrt{\rho}w)d\theta$$
$$-\rho\int_0^1 V(t-\theta\rho, x-\theta\sqrt{\rho}w)d\theta + i\rho\int_0^1 W(t-\theta\rho, x-\theta\sqrt{\rho}w)d\theta. \tag{4.4}$$

Lemma 4.3. *Suppose Assumption 2.C and* (2.9)*. Let* $\mathcal{C}(t,s)$ *be the operator defined by* (3.7)*. Then, for an arbitrary multi-index* κ *both of commutators* $[\partial_x^\kappa, \mathcal{C}(t,s)]f$ *and* $[x^\kappa, \mathcal{C}(t,s)]f$ *for* $f \in C_0^\infty(\mathbb{R}^d)$ *are written in the form*

$$(t-s)\sum_{|\gamma|<|\kappa|} \widetilde{P}_\gamma(t,s)(\partial_x^\gamma f) := (t-s)\sum_{|\gamma|<|\kappa|} \sqrt{\frac{m}{2\pi i}}^{\,d}$$
$$\times \int e^{i\phi(t,s;x,w)} p_\gamma(t,s;x,\sqrt{\rho}w)(\partial_x^\gamma f)(x-\sqrt{\rho}w)dw, \tag{4.5}$$

where $p_\gamma(t,s;x,\zeta)$ *satisfy*

$$|\partial_\zeta^\alpha \partial_x^\beta p_\gamma(t,s;x,\zeta)| \leq C_{\alpha\beta} < x;\zeta >^{|\kappa|-|\gamma|} \tag{4.6}$$

for all α *and* β.

Proof. We note (2.9). Then we can prove Lemma 4.3 as in the proof of Lemma 3.2 of [15], replacing $V(t,x)$ with $V(t,x) - iW(t,x)$. $\square$

Proposition 4.4. *Suppose that the assumptions of Theorem 2.2 are satisfied, where we take $C_W = 0$. Then, for $a = 0, 1, 2, \dots$ there exist constants $K_a \geq 0$ such that*

$$\|\mathcal{C}(t,s)f\|_a \leq e^{K_a(t-s)}\|f\|_a, \quad 0 \leq t - s \leq \rho^* \tag{4.7}$$

for all $f \in B^a$.

Proof. Let $|\kappa| = a$. Using Proposition 3.6 and Lemma 4.3, we have

$$\|x^\kappa(\mathcal{C}(t,s)f)\| \leq \|\mathcal{C}(t,s)(x^\kappa f)\| + (t-s)\sum_{|\gamma|<a}\|\tilde{P}_\gamma(t,s)(\partial_x^\gamma f)\|$$

$$\leq \|\mathcal{C}(t,s)(x^\kappa f)\| + C(t-s)\sum_{|\gamma|<a}\|\partial_x^\gamma f\|_{a-|\gamma|}$$

$$\leq \|\mathcal{C}(t,s)(x^\kappa f)\| + C'(t-s)\|f\|_a.$$

Here we used $\|\partial_x^\gamma f\|_{a-|\gamma|} \leq \text{Const.}\|f\|_a$ from (4.21) in [14]. Hence from Proposition 3.5 we have

$$\|x^\kappa(\mathcal{C}(t,s)f)\| \leq e^{K_0(t-s)}\|x^\kappa f\| + C'(t-s)\|f\|_a. \tag{4.8}$$

In the same way we have

$$\|\partial_x^\kappa(\mathcal{C}(t,s)f)\| \leq e^{K_0(t-s)}\|\partial_x^\kappa f\| + C''(t-s)\|f\|_a. \tag{4.9}$$

Since $\|f\|_a$ is defined by (2.12), from (3.24), (4.8) and (4.9) we obtain

$$\|\mathcal{C}(t,s)f\|_a = \|\mathcal{C}(t,s)f\| + \sum_{|\kappa|=a}\Big(\|x^\kappa(\mathcal{C}(t,s)f)\| + \|\partial_x^\kappa(\mathcal{C}(t,s)f)\|\Big)$$

$$\leq e^{K_0(t-s)}\|f\|_a + K_0'(t-s)\|f\|_a = \Big(e^{K_0(t-s)} + K_0'(t-s)\Big)\|f\|_a$$

$$\leq e^{(K_0+K_0')(t-s)}\|f\|_a,$$

which shows (4.7). □

Theorem 4.5. *Suppose Assumption 2.C,* (2.7) *and* (2.9)*. Then for any* $u_0 \in B^a$ $(a = 0, \pm 1, \pm 2, \dots)$ *there exists the unique solution* $u(t)$ *in* $C_t^0([0,T]; B^a) \cap C_t^1([0,T]; B^{a-2})$ *with* $u(0) = u_0$ *to the equation* (1.10)*. This solution* $u(t)$ *satisfies*

$$\|u(t)\|_a \leq C_a \|u_0\|_a, \ 0 \leq t \leq T. \tag{4.10}$$

Proof. The results corresponding to Theorem 4.5 have been proved in (1) of Theorem 2.1 of [17], where $\|f\|_a$ was defined by $\|f\| + \sum_{|\alpha|=2a} \big(\|x^\alpha f\| + \|\partial_x^\alpha f\|\big)$ differently from (2.12). Following the proof of (1) of Theorem 2.1 of [17], we can prove Theorem 4.5 as below. We set $\chi_\epsilon(x,\xi) := \chi\big(\epsilon(<x> + <\xi>)\big)$ $(0 < \epsilon \leq 1)$ and $\lambda(x,\xi) := \mu + <x> + <\xi>$, where $\mu > 0$ is the constant such that there exist a function $w(x,\xi)$ satisfying

$$W(X, D_x) f = (\mu + <X> + <D_x>)^{-1} f$$

for $f \in \mathcal{S}$ and

$$|\partial_\xi^\alpha \partial_x^\beta w(x,\xi)| \leq C_{\alpha\beta} (1 + <x> + <\xi>)^{-1}$$

for all α and β (cf. Lemma 2.3 of [12]).

We set

$$Q_\epsilon(X, D_x) = \Big[\Lambda(X, D_x), X_\epsilon(X, D_x)^\dagger H_w(t) X_\epsilon(X, D_x)\Big] \Lambda(X, D_x)^{-1}$$

as in (4.3) of [17]. Then, noting Assumption (2.C) and (2.9), we can prove

$$|\partial_\xi^\alpha \partial_x^\beta q_\epsilon(x,\xi)| \leq C_{\alpha\beta} < \infty$$

for all α and β with constants $C_{\alpha\beta}$ independent of $0 < \epsilon \le 1$ as in the proof of Lemma 4.1 of [17] and Lemma 3.1 of [12]. Therefore we obtain the results corresponding to Lemma 4.1 of [17]. Then we can complete the remaining proof of Theorem 4.5, following the proof of Theorem 2.1 of [17]. □

Proposition 4.6. *We suppose the same assumptions as in Proposition 4.4. Let $U(t,s)f$ be the solution to* (1.10) *found in Theorem 4.5. Then there exists an integer $M \ge 0$ such that we have*

$$\|\mathcal{C}(t,s)f - U(t,s)f\|_a \le C_a \rho^{3/2} \|f\|_{M+a}, \quad 0 \le t-s \le \rho^* \tag{4.11}$$

for $a = 0, 1, 2, \dots$.

Proof. Using (4.1), we can write

$$\begin{aligned} i\big\{\mathcal{C}(t,s)f - f\big\} &= i\big\{\mathcal{C}(s+\rho,s)f - f\big\} = i\rho \int_0^1 \frac{\partial \mathcal{C}}{\partial t}(s+\theta\rho,s) f d\theta \\ &= \rho \int_0^1 \Big\{ H_w(s+\theta\rho)\mathcal{C}(s+\theta\rho,s)f + \sqrt{\theta\rho} R(s+\theta\rho,s)f \Big\} d\theta \end{aligned}$$

and so

$$\begin{aligned} i\frac{\mathcal{C}(t,s)f - f}{\rho} - H_w(s)f &= \sqrt{\rho}\int_0^1 \sqrt{\theta} R(s+\theta\rho,s)f d\theta + \int_0^1 H_w(s+\theta\rho) \\ &\cdot \big\{\mathcal{C}(s+\theta\rho,s)f - f\big\} d\theta + \int_0^1 \big\{ H_w(s+\theta\rho)f - H_w(s)f\big\} d\theta. \end{aligned}$$

Using

$$\begin{aligned} \mathcal{C}(s+\theta\rho,s)f - f &= \theta\rho \int_0^1 \frac{\partial \mathcal{C}}{\partial t}(s+\theta'\theta\rho,s) f d\theta' \\ &= \frac{\theta\rho}{i}\int_0^1 \big\{ H_w(s+\theta'\theta\rho)\mathcal{C}(s+\theta'\theta\rho,s)f + \sqrt{\theta'\theta\rho} R(s+\theta'\theta\rho,s)f \big\} d\theta', \end{aligned}$$

we have

$$
\begin{aligned}
i\frac{\mathcal{C}(t,s)f-f}{\rho} - H_w(s)f &= \sqrt{\rho}\int_0^1 \sqrt{\theta}R(s+\theta\rho,s)fd\theta + \frac{\rho}{i}\int_0^1 \theta H_w(s+\theta\rho)d\theta \\
&\cdot \int_0^1 \big\{H_w(s+\theta'\theta\rho)\mathcal{C}(s+\theta'\theta\rho,s)f + \sqrt{\theta'\theta\rho}R(s+\theta'\theta\rho,s)f\big\}d\theta' \\
&+ \int_0^1 \big\{H_w(s+\theta\rho)f - H_w(s)f\big\}d\theta. \qquad (4.12)
\end{aligned}
$$

In the same way

$$
\begin{aligned}
i\frac{U(t,s)f-f}{\rho} - H_w(s)f &= \frac{\rho}{i}\int_0^1 \theta H_w(s+\theta\rho)d\theta \\
&\cdot \int_0^1 \big\{H_w(s+\theta'\theta\rho)U(s+\theta'\theta\rho,s)f\big\}d\theta' \\
&+ \int_0^1 \big\{H_w(s+\theta\rho)f - H_w(s)f\big\}d\theta. \qquad (4.13)
\end{aligned}
$$

Taking difference between (4.12) and (4.13), we have

$$
\begin{aligned}
i\left\{\mathcal{C}(t,s)f - U(t,s)f\right\} &= \rho^{3/2}\int_0^1 \sqrt{\theta}R(s+\theta\rho,s)fd\theta + \frac{\rho^2}{i}\int_0^1 \theta H_w(s+\theta\rho)d\theta \\
&\cdot \int_0^1 \big\{H_w(s+\theta'\theta\rho)\mathcal{C}(s+\theta'\theta\rho,s)f + \sqrt{\theta'\theta\rho}R(s+\theta'\theta\rho,s)f\big\}d\theta' \\
&- \frac{\rho^2}{i}\int_0^1 \theta H_w(s+\theta\rho)d\theta \int_0^1 H_w(s+\theta'\theta\rho)U(s+\theta'\theta\rho,s)fd\theta'. \qquad (4.14)
\end{aligned}
$$

Consequently, noting (2.5), (2.6) and (2.9), and applying (4.2) with $M_1 = 1$, (4.7) and (4.10) to (4.14), we obtain

$$
\begin{aligned}
\|\mathcal{C}(t,s)f - U(t,s)f\|_a &\le C_1\rho^{3/2}\|f\|_{M+a} + C_2\rho^2(\|f\|_{4+a} + \|f\|_{M+2+a}) \\
&+ C_3\rho^2\|f\|_{4+a} \qquad (4.15)
\end{aligned}
$$

with constants C_j $(j = 1, 2, 3)$, which shows (4.11). □

5 Proofs of Theorems 2.1 and 2.2

Lemma 5.1. *We suppose the same assumptions as in Proposition 3.5. Let* $K_\Delta(t,0)f$ *and* $\mathcal{C}(t,s)f$ *be the operators defined by* (2.1) *and* (3.7) *respectively. Then we have*

$$K_\Delta(t,0)f = \mathcal{C}(t,\tau_{\nu-1})\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\cdots\mathcal{C}(\tau_1,0)f \tag{5.1}$$

for all $f \in L^2$ *and all* Δ *such that* $|\Delta| \le \rho^*$.

Proof. From (3.8) we could write

$$K_\Delta(t,0)f = \lim_{\epsilon\to 0}\mathcal{C}(t,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}(\tau_1,0)f$$

for $f \in C_0^\infty(\mathbb{R}^d)$. Then from (3.24) we have

$$\begin{aligned}
&\|\mathcal{C}(t,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}(\tau_1,0)f \\
&\quad - \mathcal{C}(t,\tau_{\nu-1})\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\cdots\mathcal{C}(\tau_1,0)f\| \\
&= \Bigg\|\sum_{j=1}^{\nu-1}\mathcal{C}(t,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}(\tau_{j+1},\tau_j) \\
&\quad \cdot\{\chi(\epsilon\cdot)-1\}\mathcal{C}(\tau_j,\tau_{j-1})\mathcal{C}(\tau_{j-1},\tau_{j-2})\cdots\mathcal{C}(\tau_1,0)f\Bigg\| \\
&\le C\sum_{j=1}^{\nu-1}\|\{\chi(\epsilon\cdot)-1\}\mathcal{C}(\tau_j,\tau_{j-1})\mathcal{C}(\tau_{j-1},\tau_{j-2})\cdots\mathcal{C}(\tau_1,0)f\|
\end{aligned} \tag{5.2}$$

for $f \in L^2$ with a constant $C \ge 0$ independent of $0 < \epsilon \le 1$, which shows (5.1) for $f \in L^2$. □

Now we will prove Theorems 2.1 and 2.2. We can easily see that we may assume $C_W = 0$ in Assumption 2.D without loss of generality, because we have only to take $W(t,x)+C_W$ in place of $W(t,x)$ in (1.10) and (2.1). Hence we assume $C_W=0$ hereafter in this section.

We will first prove Theorem 2.2. Using (5.1), we write

$$K_\Delta(t,0)f-U(t,0)f=\mathcal{C}(t,\tau_{\nu-1})\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\cdots\mathcal{C}(\tau_1,0)f-U(t,\tau_{\nu-1})$$
$$\cdot U(\tau_{\nu-1},\tau_{\nu-2})\cdots U(\tau_1,0)f=\sum_{j=1}^{\nu}\mathcal{C}(t,\tau_{\nu-1})\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\cdot\cdots\mathcal{C}(\tau_{j+1},\tau_j)$$
$$\cdot\{\mathcal{C}(\tau_j,\tau_{j-1})-U(\tau_j,\tau_{j-1})\}U(\tau_{j-1},\tau_{j-2})\cdots U(\tau_1,0)f=\sum_{j=1}^{\nu}\mathcal{C}(t,\tau_{\nu-1})$$
$$\cdot\,\mathcal{C}(\tau_{\nu-1},\tau_{\nu-2})\cdots\mathcal{C}(\tau_{j+1},\tau_j)\{\mathcal{C}(\tau_j,\tau_{j-1})-U(\tau_j,\tau_{j-1})\}U(\tau_{j-1},0)f. \tag{5.3}$$

Hence, using (4.7), (4.10) and (4.11), we have

$$\|K_\Delta(t,0)f-U(t,0)f\|_a\le\sum_{j=1}^{\nu}C_ae^{K_at}(\tau_j-\tau_{j-1})^{3/2}\|U(\tau_{j-1},0)f\|_{M+a}$$
$$\le C'_a\sqrt{|\Delta|}e^{K_aT}T\|f\|_{M+a}. \tag{5.4}$$

Let $f\in B^a$ be arbitrary. For any $\epsilon>0$ we take a $g\in B^{a+M}$ such that

$$\|g-f\|_a<\epsilon. \tag{5.5}$$

Using (4.7), (4.10) and (5.5), we have

$$\|K_\Delta(t,0)f-U(t,0)f\|_a\le\|K_\Delta(t,0)g-U(t,0)g\|_a$$
$$+\|K_\Delta(t,0)(f-g)\|_a+\|U(t,0)(f-g)\|_a$$
$$\le C'_a\sqrt{|\Delta|}e^{K_aT}T\|g\|_{M+a}+\big(e^{K_aT}+C_a\big)\epsilon. \tag{5.6}$$

Hence

$$\overline{\lim}_{|\Delta|\to 0}\|K_\Delta(t,0)f-U(t,0)f\|_a\le\big(e^{K_aT}+C_a\big)\epsilon,$$

which shows

$$\lim_{|\Delta|\to 0} \|K_\Delta(t,0)f - U(t,0)f\|_a = 0. \tag{5.7}$$

Now consider the gauge transformation (2.10). From (1.2), (1.7) and (1.8) we can easily see

$$S'_w(t,s;q^{t,s}_{x,y}) = S_w(t,s;q^{t,s}_{x,y}) + \psi(t,x) - \psi(s,y) \tag{5.8}$$

(cf. p. 1024 in [14]), which shows

$$\mathcal{C}'(t,s)f = e^{i\psi(t,\cdot)}\mathcal{C}(t,s)e^{-i\psi(s,\cdot)}f. \tag{5.9}$$

Consequently we can prove (2.11) from (5.1). Thus we could complete the proof of Theorem 2.2.

Next we will prove Theorem 2.1 by using Theorem 2.2, where we will use only the results in L^2. We are supposing Assumption 2.A. Consequently, using Lemma 6.1 in [14], we can find a potential (V', A') satisfying (2.5) and (2.6). From Theorem 2.2 we have (5.7) with $a = 0$ for $K'_\Delta(t,0)f$ and $U'(t,0)f$ with this potential (V', A'). Let (V, A) be an arbitrary potential stated in Theorem 2.1. Then, from the proof of Theorem in [14] on p.1023 we can find a real-valued function $\psi(t,x)$ with continuous $\partial_{x_j}\partial_{x_k}\psi$ and $\partial_t\partial_{x_j}\psi$ $(j,k = 1,2,\dots,d)$ in $[0,T]\times\mathbb{R}^d$ satisfying (2.10). Then from Theorem 2.2 we have

$$K_\Delta(t,0)f = e^{-i\psi(t,\cdot)}K'_\Delta(t,0)e^{i\psi(s,\cdot)}f, \tag{5.10}$$

which shows

$$\lim_{|\Delta|\to 0} K_\Delta(t,0)f = e^{-i\psi(t,\cdot)}U'(t,0)e^{i\psi(s,\cdot)}f = U(t,0)f \quad \text{in } L^2 \tag{5.11}$$

for $f \in L^2$ because of $U(t,0)f = e^{-i\psi(t,\cdot)}U'(t,0)e^{i\psi(s,\cdot)}f$.

Next consider the gauge transformation

$$V'' = V - \frac{\partial \varphi}{\partial t}, \quad A''_j = A_j + \frac{\partial \varphi}{\partial x_j} \quad (j = 1, 2, \dots, d)$$

stated in Theorem 2.1. Then we have

$$V' = V'' - \frac{\partial(\psi - \varphi)}{\partial t}, \quad A'_j = A''_j + \frac{\partial(\psi - \varphi)}{\partial x_j}$$

together with (2.10). Hence from (5.10) we have

$$\begin{aligned} K''_\Delta(t,0)f &= e^{-i\psi(t,\cdot)+i\varphi(t,\cdot)} K'_\Delta(t,0) e^{i\psi(s,\cdot)-i\varphi(s,\cdot)} f \\ &= e^{i\varphi(t,\cdot)} K_\Delta(t,0) e^{-i\varphi(s,\cdot)} f, \end{aligned}$$

which shows (2.11). Thus we could complete the proof of Theorem 2.1.

6 Proofs of Theorems 2.3 - 2.6

Let C_W be the constant in Assumption 2.D and $W_s(t,x)$ the Hermitian matrix in Theorems 2.3 and 2.4. As in the proofs of Theorems 2.1 and 2.2 we may assume

$$C_W = 0, \quad W_s(t,x) \geq 0 \tag{6.1}$$

in the proofs of Theorems 2.3 and 2.4, because we are assuming (1.13).

Using $\mathcal{F}(t,s;q^{t,s}_{x,y})$ defined by the solution to (2.13), we define

$$\mathcal{C}_s(t,s)f = \begin{cases} \sqrt{m/(2\pi i \rho)}^{\,d} \displaystyle\int \left(\exp iS_w(t,s;q^{t,s}_{x,y})\right) \\ \qquad \times \mathcal{F}(t,s;q^{t,s}_{x,y}) f(y) dy, \quad s < t, \\ f, \quad s = t \end{cases} \tag{6.2}$$

for $f \in C_0^\infty(\mathbb{R}^d)^l$, which is corresponding to $\mathcal{C}(t,s)$ defined by (3.7). Then we can write $K_{s\Delta}(t,0)f$ defined by (2.15) as

$$K_{s\Delta}(t,0)f = \lim_{\epsilon\to 0+} \mathcal{C}_s(t,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}_s(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}_s(\tau_1,0)f \quad (6.3)$$

for $f \in C_0^\infty(\mathbb{R}^d)^l$ in the same way as we did (3.8), using

$$\begin{aligned}&\mathcal{F}(t,0;q_\Delta)\\&= \mathcal{F}\big(t,\tau_{\nu-1};q^{t,\tau_{\nu-1}}_{x,x^{(\nu-1)}}\big)\mathcal{F}\big(\tau_{\nu-1},\tau_{\nu-2};q^{\tau_{\nu-1},\tau_{\nu-2}}_{x^{(\nu-1)},x^{(\nu-2)}}\big)\cdots\mathcal{F}\big(\tau_1,0;q^{\tau_1,0}_{x^{(1)},x^{(0)}}\big) \quad (6.4)\end{aligned}$$

which has been easily proved in Lemma 2.1 of [16].

Lemma 6.1. *(1) Assume* $W_s(t,x) \geq 0$ *in* $[0,T]\times\mathbb{R}^d$. *Let* $q(\theta) \in \mathbb{R}^d$ $(s \leq \theta \leq t)$ *be a continuous path. Then we have*

$$0 \leq \mathcal{F}(t,s;q)^\dagger\mathcal{F}(t,s;q) \leq 1, \quad (6.5)$$

$$\sum_{i=1}^l |\mathcal{F}_{ij}(t,s;q)|^2 \leq 1,\ \ j = 1,2,\dots,l, \quad (6.6)$$

where $\mathcal{F}_{ij}(t,s;q)$ *denotes the* (i,j)*-component of* $\mathcal{F}(t,s;q)$. *(2) Assume* $W_s(t,x) \geq 0$ *in* $[0,T]\times\mathbb{R}^d$ *and*

$$|\partial_x^\alpha h_s(t,x)| \leq C_\alpha,\ \ |\alpha| \geq 1, \quad (6.7)$$

$$|\partial_x^\alpha w_s(t,x)| \leq C_\alpha,\ \ |\alpha| \geq 1 \quad (6.8)$$

in $[0,T]\times\mathbb{R}^d$, *where* $|\Omega|$ *denotes the Hilbert-Schmidt norm* $\big(\sum_{i,j=1}^l |\Omega_{ij}|^2\big)^{1/2}$ *of a matrix* $\Omega = (\Omega_{ij}; i\downarrow j \to 1,2,\dots,l)$. *Then we have*

$$|\partial_x^\alpha\partial_y^\beta\mathcal{F}(t',s';q^{t,s}_{x,y})| \leq C_{\alpha\beta}(t'-s'),\ \ |\alpha+\beta| \geq 1 \quad (6.9)$$

for $0 \leq s \leq s' \leq t' \leq t \leq T$ *and* $(x,y) \in \mathbb{R}^{2d}$.

Proof. (1) We set $\mathcal{U}(t) = \mathcal{F}(t,s;q)$. From (2.13) we have

$$\frac{d}{d\theta}\mathcal{U}(\theta)^\dagger\mathcal{U}(\theta) = \mathcal{U}(\theta)^\dagger\big\{iH_s(\theta,q(\theta)) - W_s(\theta,q(\theta))\big\}\mathcal{U}(\theta)$$
$$-\mathcal{U}(\theta)^\dagger\big\{iH_s(\theta,q(\theta)) + W_s(\theta,q(\theta))\big\}\mathcal{U}(\theta)$$
$$= -2\mathcal{U}(\theta)^\dagger W_s(\theta,q(\theta))\mathcal{U}(\theta) \le 0. \tag{6.10}$$

Hence we have (6.5) because of $\mathcal{U}(s) = I$. Taking $e_1 = {}^t(1,0,\ldots,0) \in \mathbb{R}^l$, from (6.5) we have

$$1 \ge <\mathcal{U}(t)^\dagger\mathcal{U}(t)e_1, e_1> = \sum_{i=1}^{l} |\mathcal{F}_{i1}(t,s;q)|^2,$$

where $< \cdot, \cdot >$ denotes the usual inner product of $\mathbb{R}^l$. In the same way we can prove (6.6).

(2) From (2.13) we can easily see

$$\frac{\partial}{\partial x_j}\mathcal{F}(t',s';q^{t,s}_{x,y}) = -\int_{s'}^{t'} \mathcal{F}(t',\theta;q^{t,s}_{x,y})\left[\frac{\partial}{\partial x_j}\big\{iH_s(\theta,q^{t,s}_{x,y}(\theta)) + W_s(\theta,q^{t,s}_{x,y}(\theta))\big\}\right]$$
$$\times\, \mathcal{F}(\theta,s';q^{t,s}_{x,y})d\theta \tag{6.11}$$

(cf. (3.3) in [16]). Then, noting (3.1), from (6.6) - (6.8) we can prove

$$|\partial_{x_j}\mathcal{F}(t',s';q^{t,s}_{x,y})| \le C(t'-s')$$

with a constant $C \ge 0$. In the same way we can prove (6.9) from (6.11) by induction. □

Lemma 6.2. *Assume $W_s(t,x) \ge 0$, (1.13) and (2.16). Then we have*

$$\left|\partial_x^\alpha\partial_y^\beta\left\{\mathcal{F}(t,s;q^{t,s}_{x,y}) - I\right\}\right| \le C_{\alpha,\beta}(t-s) \tag{6.12}$$

in $0 \le s \le t \le T$ and $(x,y) \in \mathbb{R}^{2d}$ for all α and β.

Proof. From (2.13) we have

$$\mathcal{F}(t', s; q^{t,s}_{x,y}) - I = -\int_s^{t'} \big\{iH_s(\theta, q^{t,s}_{x,y}(\theta)) + W_s(\theta, q^{t,s}_{x,y}(\theta))\big\}\mathcal{F}(\theta, s; q^{t,s}_{x,y})d\theta.$$

Hence, by (1.13) , (2.16) and (6.6) we see

$$\big|\mathcal{F}(t, s; q^{t,s}_{x,y}) - I\big| \le C(t-s)$$

in $0 \le s \le t \le T$ and $(x, y) \in \mathbb{R}^{2d}$ with a constant $C \ge 0$. The inequalities (6.12) for $|\alpha + \beta| \ge 1$ follow from (6.9). □

Proposition 6.3. *Besides the assumptions of Proposition 3.5 we suppose* $W_s(t, x) \ge 0$, *(1.13) and (2.16). Let* $\rho^* > 0$ *be the constant determined in Lemma 3.3 and* $\mathcal{C}_s(t, s)$ *the operator defined by (6.2). Then there exists a constant* $K_0' \ge 0$ *such that*

$$\|\mathcal{C}_s(t, s)f\| \le e^{K_0'(t-s)}\|f\|, \quad 0 \le t - s \le \rho^* \tag{6.13}$$

for all $f \in (L^2)^l$.

Proof. Using $\mathcal{C}(t, s)$ defined by (3.7), we write

$$\begin{aligned}\mathcal{C}_s(t, s)f = \mathcal{C}(t, s)f + \sqrt{\frac{m}{2\pi i\rho}}^{\,d} \int \big(\exp iS_w(t, s; q^{t,s}_{x,y})\big) \\ \times \big\{\mathcal{F}(t, s; q^{t,s}_{x,y}) - I\big\} f(y)dy \end{aligned}\tag{6.14}$$

for $f \in \mathcal{S}^l$. Noting (6.12), from Proposition 3.6 we see that the L^2-norm of the second term on the right-hand side of (6.14) is bounded by $C(t-s)\|f\|$ from above with a constant $C \ge 0$. Proposition 3.5 is showing (3.24). Hence we have

$$\begin{aligned}\|\mathcal{C}_s(t, s)f\| &\le \mathrm{e}^{K_0(t-s)}\|f\| + C(t-s)\|f\| \\ &\le \mathrm{e}^{(K_0+C)(t-s)}\|f\|, \quad 0 \le t - s \le \rho^*.\end{aligned}$$

Consequently, we can prove (6.13) with a constant $K_0' \geq 0$. □

Lemma 6.4. *Besides the assumptions of Lemma 4.1 we assume* $W_s(t,x) \geq 0$ *and* (6.7) - (6.8). *Then, there exist functions* $r_{ij}(t,s;x,w)$ $(i,j = 1,2,\ldots,l)$ *satisfying* (3.5) *for an integer* $M \geq 0$ *such that* $\partial_w^\alpha \partial_x^\beta r_{ij}(t,s;x,w)$ *are continuous in* $0 \leq s \leq t \leq T$ *and* $(x,w) \in \mathbb{R}^{2d}$ *for all* α *and* β, *and we have*

$$\left\{ i\frac{\partial}{\partial t} - H_w(t) - H_s(t,x) + iW_s(t,x) \right\} \mathcal{C}_s(t,s)f$$
$$= \sqrt{t-s}\Big(R_{ij}(t,s); i \downarrow j \rightarrow 1,2,\ldots,l\Big) f \equiv \sqrt{t-s}R(t,s)f \tag{6.15}$$

for $f \in C_0^\infty(\mathbb{R}^d)^l$, *where* $R_{ij}(t,s)$ *are the operators defined by* (3.6).

Proof. We note that (6.6) and (6.9) hold under our assumptions. Consequently, replacing $V(t,x)$ and $H_s(t,x)$ with $V(t,x) - iW(t,s)$ and $H_s(t,x) - iW_s(t,x)$, respectively in the proof of Proposition 3.5 of [16], we can complete the proof of Lemma 6.4. In particular, see (3.21) and (3.22) of [16]. □

Proposition 6.5. *Besides the assumptions of Proposition 4.2, we suppose* $W_s(t,s) \geq 0$ *and* (6.7) - (6.8). *Then, there exist functions* $r_{ij}(t,s;x,w)$ $(i,j = 1,2,\ldots,l)$ *satisfying the properties stated in Lemma 6.4 and we have*

$$\|R(t,s)f\|_a \leq C_a \|f\|_{M+aM_1}, \quad 0 \leq t-s \leq \rho^* \tag{6.16}$$

for $a = 0,1,2,\ldots$ *and all* $f \in (B^{M+aM_1})^l$, *where* M_1 *is the integer in* (3.23).

Proof. As in the proof of Proposition 4.2, we can easily see that the assumptions of Lemma 6.4 hold. Hence, using Lemma 6.4, from Proposition 3.6 we can prove (6.16). □

Proposition 6.6. *Besides the assumptions of Theorem 2.2 we suppose* (1.13), (2.16) *and* (6.1). *Then, for* $a = 0, 1, 2, \ldots$ *there exist constants* $K'_a \geq 0$ *such that*

$$\|C_s(t,s)f\|_a \leq e^{K'_a(t-s)}\|f\|_a, \quad 0 \leq t-s \leq \rho^* \tag{6.17}$$

for all $f \in (B^a)^l$.

Proof. We note that (6.12) hold. Then, applying Proposition 3.6 as $M_1 = 1$ to the second term on the right-hand side of (6.14), its B^a-norm is bounded by $C_a(t-s)\|f\|_a$ from above with a constant $C_a \geq 0$ for $0 \leq t-s \leq \rho^*$. Hence, using (4.7), from (6.14) we can prove

$$\begin{aligned}\|C_s(t,s)f\|_a &\leq e^{K_a(t-s)}\|f\|_a + C_a(t-s)\|f\|_a \\ &\leq e^{(K_a+C_a)(t-s)}\|f\|_a, \quad 0 \leq t-s \leq \rho^*,\end{aligned}$$

which shows (6.17). □

Proofs of Theorems 2.3 and 2.4. Since we are assuming (1.13) and (2.16), we obtain the same results as in Theorem 4.5 for the equation (1.12) from (1) of Theorem 2.1 of [17]. We write the solution to (1.12) with $u(s) = f$ as $U_s(t,s)f$. Then, using Proposition 6.5, we can prove

$$\|\mathcal{C}_s(t,s)f - U_s(t,s)f\|_a \leq C_a\rho^{3/2}\|f\|_{M+a}, \quad 0 \leq t-s \leq \rho^* \tag{6.18}$$

for $a = 0, 1, 2, \ldots$ as in the proof of (4.11). Thereby, following the proofs of Theorems 2.1 and 2.2 in Sect. 5, we can complete the proofs of Theorems 2.3 and 2.4 together with (6.13) and (6.17).

Proofs of Theorems 2.5 and 2.6. We may assume $W_j(t,\mathbf{x}_j) \geq 0$ and $W_{sj}(t,\mathbf{x}_j) \geq 0$ $(j = 1,2,\dots,N)$ without loss of generality as in the proofs of Theorems 2.3 and 2.4. For a continuous path $q(\theta) = \big(\mathbf{q}_1(\theta),\dots,\mathbf{q}_N(\theta)\big) \in \mathbb{R}^{dN}$ $(s \leq \theta \leq t)$ we define $\mathcal{F}^\sharp(\theta,s;q)$ $(s \leq \theta \leq t)$ by the solution to

$$\frac{d}{d\theta}\mathcal{U}^\sharp(\theta) = -\Big[\sum_{j=1}^N I_1 \otimes \cdots \otimes I_{j-1} \otimes \big\{iH_{sj}(\theta,\mathbf{q}_j(\theta)) + W_{sj}(\theta,\mathbf{q}_j(\theta))\big\}$$

$$\otimes I_{j+1} \otimes \cdots \otimes I_N\Big]\mathcal{U}^\sharp(\theta), \quad \mathcal{U}^\sharp(s) = I_1 \otimes \cdots \otimes I_N \tag{6.19}$$

in the same way as we do $\mathcal{F}(\theta,s;q)$ from (2.13).

We consider $\mathcal{F}_j(\theta,s;\mathbf{q}_j)$ in (2.19). Then from the simple properties of the tensor products (cf. 4.2.1 and 4.2.10 in §4.2 of [11], and §VIII.10 of [26]) we can easily have

$$\frac{d}{d\theta}\mathcal{F}_1(\theta,s;\mathbf{q}_1) \otimes \cdots \otimes \mathcal{F}_N(\theta,s;\mathbf{q}_N) = \sum_{j=1}^N \mathcal{F}_1(\theta,s;\mathbf{q}_1) \otimes \cdots \otimes \mathcal{F}_{j-1}(\theta,s;\mathbf{q}_{j-1})$$

$$\otimes \frac{d}{d\theta}\mathcal{F}_j(\theta,s;\mathbf{q}_j) \otimes \mathcal{F}_{j+1}(\theta,s;\mathbf{q}_{j+1}) \otimes \cdots \mathcal{F}_N(\theta,s;\mathbf{q}_N) = -\sum_{j=1}^N \mathcal{F}_1(\theta,s;\mathbf{q}_1)\otimes$$

$$\cdots \otimes \mathcal{F}_{j-1}(\theta,s;\mathbf{q}_{j-1}) \otimes \big\{iH_{sj}(\theta,\mathbf{q}_j(\theta)) + W_{sj}(\theta,\mathbf{q}_j(\theta))\big\}\mathcal{F}_j(\theta,s;\mathbf{q}_j)$$

$$\otimes \mathcal{F}_{j+1}(\theta,s;\mathbf{q}_{j+1}) \otimes \cdots \mathcal{F}_N(\theta,s;\mathbf{q}_N) = -\sum_{j=1}^N \Big[I_1 \otimes \cdots \otimes I_{j-1} \otimes \big\{iH_{sj}(\theta,\mathbf{q}_j(\theta))$$

$$+ W_{sj}(\theta,\mathbf{q}_j(\theta))\big\} \otimes I_{j+1} \otimes \cdots \otimes I_N\Big]\mathcal{F}_1(\theta,s;\mathbf{q}_1) \otimes \cdots \otimes \mathcal{F}_N(\theta,s;\mathbf{q}_N). \tag{6.20}$$

Consequenlty we have

$$\mathcal{F}^\sharp(\theta,s;q) = \mathcal{F}_1(\theta,s;\mathbf{q}_1) \otimes \cdots \otimes \mathcal{F}_N(\theta,s;\mathbf{q}_N), \tag{6.21}$$

which follows from uniqueness of the solutions to (6.19). Hence we can write (2.19) as

$$\exp *iS^\sharp_{sw}(t,0;q_\Delta) = \big(\exp iS^\sharp_w(t,0;q_\Delta)\big)\mathcal{F}^\sharp(\theta,s;q_\Delta), \tag{6.22}$$

which corresponds to (2.14) for one particle system. We set

$$H_s^\sharp(t,x) := \sum_{j=1}^N I_1 \otimes \cdots \otimes I_{j-1} \otimes H_{sj}(t,\mathbf{x}_j) \otimes I_{j+1} \otimes \cdots \otimes I_N, \tag{6.23}$$

$$W_s^\sharp(t,x) := \sum_{j=1}^N I_1 \otimes \cdots \otimes I_{j-1} \otimes W_{sj}(t,\mathbf{x}_j) \otimes I_{j+1} \otimes \cdots \otimes I_N, \tag{6.24}$$

$$W^\sharp(t,x) = \sum_{j=1}^N W_j(t,\mathbf{x}_j). \tag{6.25}$$

Then we can write (2.18) in the form of (1.11) as

$$\mathcal{L}_{sw}^\sharp(t,x,\dot{x}) = \mathcal{L}_w^\sharp(t,x,\dot{x}) - H_s^\sharp(t,x) + iW_s^\sharp(t,x). \tag{6.26}$$

We can easily see that both of $H_s^\sharp(t,x)$ and $W_s^\sharp(t,x)$ are written as $l^N \times l^N$ Hermitian matrices (cf. 4.2.5 in §4.2 of [11] and §VIII.10 of [26]) and satisfy (1.13), (2.16) and $W_s^\sharp(t,x) \geq 0$. Hence we can obtain the same results as in Lemma 6.1 for $\mathcal{F}^\sharp(t,s;q)$.

We consider $C_{w^\sharp}(t,s;x,y)$ defined by (3.9) where $W = W^\sharp(t,x)$. Then we can easily see the same estimates as in (3.19) for $C_{w^\sharp}(t,s;x,y)$ because of $C_{w^\sharp}(t,s;x,y) = \prod_{j=1}^N C_{w_j}(t,s;\mathbf{x}_j,\mathbf{y}_j)$ and $W_j(t,\mathbf{x}_j) \geq 0$ $(j = 1,2,\ldots,N)$. We also note that $W^\sharp(t,x)$ satisfies (2.7) and (2.9). Hence, using the results stated above for $\mathcal{F}^\sharp(t,s;q)$ and $C_{w^\sharp}(t,s;x,y)$ and following the proofs of Theorems 2.5 and 2.6, we can complete the proofs of Theorems 2.5 and 2.6 from (6.22).

A A proof of Theorem 1.A

We will prove Theorem 1.A in the introduction from Theorem 13.13 in [29]. Let $q(x,\xi)$ be a function satisfying (1.14) and set

$$q_w(x,\xi,x') = q((x+x')/2,\xi). \tag{A.1}$$

Then Theorem 13.13 in [29] says

$$\|Q_w(X, \mathfrak{h}D_x, X')\|_{L^2\to L^2} = \sup_{x,\xi} |q(x,\xi)| + O(\mathfrak{h}). \tag{A.2}$$

We set

$$q_{wL}(x, \mathfrak{h}\xi) = \mathrm{Os} - \iint e^{-iy\cdot\eta} q_w(x, \mathfrak{h}(\xi+\eta), x+y) dy đ\eta. \tag{A.3}$$

Then, applying Theorem 2.5 in Chapter 2 of [20] as

$$p(x,\xi,x',\xi') = q_w(x,\mathfrak{h}\xi,x') \tag{A.4}$$

to (A.3), we have

$$Q_{wL}(X, \mathfrak{h}D_x) = Q_w(X, \mathfrak{h}D_x, X'). \tag{A.5}$$

Applying Theorem 3.1 in Chapter 2 of [20] as (A.4) to (A.3), we have

$$q_{wL}(x,\mathfrak{h}\xi) = q_w(x,\mathfrak{h}\xi,x) + \mathfrak{h}r(x,\mathfrak{h}\xi)$$

with a function $r(x,\xi)$ satisfying (1.14), which shows

$$q_{wL}(x,\mathfrak{h}\xi) = q(x,\mathfrak{h}\xi) + \mathfrak{h}r(x,\mathfrak{h}\xi) \tag{A.6}$$

together with (A.1). Applying the usual Calderón-Vaillaincourt theorem (cf. Theorem 1.6 in Chapter 7 of [20]) to $R(X, \mathfrak{h}D_x)$, we have

$$\|R(X,\mathfrak{h}D_x)\|_{L^2\to L^2} \le C < \infty \tag{A.7}$$

with a constatnt C independent of $0 < \mathfrak{h} \le 1$. Hence from (A.5) and (A.6) we have

$$\|Q(X,\mathfrak{h}D_x)\|_{L^2\to L^2} = \|Q_{wL}(X,\mathfrak{h}D_x)\|_{L^2\to L^2} + O(\mathfrak{h})$$
$$= \|Q_w(X,\mathfrak{h}D_x,X')\|_{L^2\to L^2} + O(\mathfrak{h}).$$

Therefore we have obtained

$$\|Q(X,\mathfrak{h}D_x)\|_{L^2\to L^2} = \sup_{x,\xi}|q(x,\xi)| + O(\mathfrak{h}) \tag{A.8}$$

from (A.2).

Now let $p(x,\xi,x')$ be a function in Theorem 1.A. We set

$$p_L(x,\mathfrak{h}\xi) = \mathrm{Os} - \iint e^{-iy\cdot\eta} p(x,\mathfrak{h}(\xi+\eta),x+y)dy đ\eta \tag{A.9}$$

as we set $q_{wL}(x,\mathfrak{h}\xi)$ in (A.3). Then we have

$$P_L(X,\mathfrak{h}D_x) = P(X,\mathfrak{h}D_x,X'), \tag{A.10}$$

$$\begin{aligned} p_L(x,\mathfrak{h}\xi) &= p(x,\mathfrak{h}\xi,x) + \mathfrak{h}r'(x,\mathfrak{h}\xi) \\ &\equiv \widehat{p}(x,\mathfrak{h}\xi) + \mathfrak{h}r'(x,\mathfrak{h}\xi) \end{aligned} \tag{A.11}$$

with $r'(x,\xi)$ satisfying (1.14) as in the proofs of (A.5) and (A.6). Thus, using (A.8), we have

$$\begin{aligned} &\|P(X,\mathfrak{h}D_x,X')\|_{L^2\to L^2} = \|P_L(X,\mathfrak{h}D_x)\|_{L^2\to L^2} \\ &= \|\widehat{P}(X,\mathfrak{h}D_x)\|_{L^2\to L^2} + O(\mathfrak{h}) = \sup_{x,\xi}|p(x,\xi,x)| + O(\mathfrak{h}), \end{aligned} \tag{A.12}$$

which shows Theorem 1.A.

Data availability statement. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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