CONSTRUCTION OF ONE-FIXED-POINT ACTIONS ON SPHERES OF NONSOLVABLE GROUPS I

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Abstract. Let G be a finite group. It is known that if a homotopy sphere X has a one-fixed-point smooth G-action then the dimension of X is greater than or equal to 6. It is also known that there is an effective 2-pseudofree one-fixed-point smooth G-action on the sphere S^n of dimension n if and only if n is equal to 6 and G is isomorphic to the alternating group A_5 on five letters. E. Stein proved that for the group $G = \mathrm{SL}(2, 5) \times Z_m$ such that m is prime to 30, there is a 3-pseudofree one-fixed-point smooth G-action on S^7, where Z_m is a cyclic group of order m. In this article, we determine the finite groups G possessing 3-pseudofree one-fixed-point smooth G-actions on S^6. In addition, for an arbitrary finite group G isomorphic to A_5, $A_5 \times Z_2$, or $\mathrm{SL}(2, 5) \times Z_m$ such that m is prime to 30, we prove that there is a 3-pseudofree one-fixed-point smooth G-action on S^7.

1. Introduction

In this paper, G is a finite group and we read a G-manifold as a smooth manifold with a smooth G-action. Let $\mathcal{S}(G)$ denote the set of all subgroups of G and E the trivial group. The set $\mathcal{S}(G)$ is an ordered set (possibly not a totally ordered set), i.e. for H, $K \in \mathcal{S}(G)$, we say $H < K$ if H is a proper subgroup of K. For a subset A of $\mathcal{S}(G)$, let $\max(A)$ denote the set of maximal elements of A with respect to the order on A inherited from $\mathcal{S}(G)$. A real G-representation V is called *free* if $\dim V^H = 0$ for all $H \in \mathcal{S}(G) \smallsetminus \{E\}$. Let m be a non-negative integer. We call a G-action on a manifold X *m-pseudofree* if $\dim X^H \leq m$ for all $H \in \mathcal{S}(G) \smallsetminus \{E\}$. We call an m-pseudofree G-action on X *properly m-pseudofree* if there is a subgroup $H \in \mathcal{S}(G) \smallsetminus \{E\}$ such that $\dim X^H = m$. We call a G-action on X a *one-fixed-point action* if X^G consists of exactly one point. It is known that the Poincaré sphere (a homology sphere of dimension 3) admits a one-fixed-point action of the

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alternating group A_5 on five letters. However the works M. Furuta [12], S. Demichelis [9] and N. Buchdahl–S. Kwasik–R. Schultz [7] together show that any homotopy sphere of dimension ≤ 5 does not admit a one-fixed-point action of finite group. Therefore a homotopy sphere Σ possessing a one-fixed-point action of finite group satisfies $\dim \Sigma \geq 6$. The existence of one-fixed-point G-action on a homotopy sphere makes it look like there exists a one-fixed-point G-action on the same dimensional sphere.

Our present study was motivated by the following results of E. Laitinen–P. Traczyk [17]. Unless otherwise stated, let Σ be a homotopy sphere of dimension ≥ 5 equipped with a G-action and let x_0 be a G-fixed point of Σ. For a G-fixed point x of Σ, let $T_x(\Sigma)$ denote the tangential G-representation of Σ at x. The trivial real G-representation of dimension 1 will be denoted by $\mathbb{R}$.

Laitinen–Traczyk Theorem 1. *Suppose the G-action on $T_{x_0}(\Sigma)$ is 2-pseudofree. If the G-fixed-point set Σ^G contains at least 2 points then Σ^H is diffeomorphic to the k-dimensional sphere, where k is 0, 1, or 2, for any $H \in \mathcal{S}(G) \smallsetminus \{E\}$.*

They obtain the next result as a corollary to the theorem above from S. Illman [13, Theorem 5].

Laitinen–Traczyk Theorem 2. *Suppose the G-action on $T_{x_0}(\Sigma)$ is 2-pseudofree. Then for any $x \in \Sigma^G$, the tangential G-representation $T_x(\Sigma)$ is G-homeomorphic to $T_{x_0}(\Sigma)$. In addition, Σ is G-homeomorphic to the unit sphere of $\mathbb{R} \oplus V$, where $V = T_{x_0}(\Sigma)$.*

They obtained a necessary condition on 2-pseudofree one-fixed-point G-actions on homotopy spheres.

Laitinen–Traczyk Theorem 3. *If $\Sigma^G = \{x_0\}$ and $T_{x_0}(\Sigma)$ is a 2-pseudofree G-representation then $\dim \Sigma = 6$, the group G is isomorphic to A_5, and $T_{x_0}(\Sigma)$ is the direct sum of two irreducible real G-representations of dimension 3.*

We recall the following facts concerning the existence of one-fixed-point actions of finite group on spheres. Let S^n denote the sphere of dimension n.

(F1) In [32, Proposition 4.3], E. Stein showed the existence of 3-pseudofree one-fixed-point actions on S^7 of the group $\mathrm{SL}(2,5) \times Z_m$ satisfying $(m, 30) = 1$.

(F2) In [19, Theorem A], [22, Theorem A], we showed the existence of 2-pseudofree one-fixed-point actions on S^6 of A_5.

(F3) In [3, Theorem 7], A. Bak and the author showed the existence of 3-pseudofree one-fixed-point actions on S^7 of A_5.

Therefore, putting Laitinen–Traczyk Theorem 3 and (F2) together, we see that a homotopy sphere Σ admits a 2-pseudofree one-fixed-point action of finite group if and only if $\dim \Sigma = 6$.

In the present paper, we will obtain the following two theorems from Laitinen–Traczyk Theorems 1–3.

Theorem 1.1. *Suppose that Σ is of even dimension ≥ 6 and the G-action on $T_{x_0}(\Sigma)$ is 3-pseudofree. If the G-fixed-point set contains at least 2 points then Σ^G is a $\mathbb{Z}_2$-homology sphere of dimension ≤ 3, and for any $x \in \Sigma^G$, $T_x(\Sigma)$ is $\langle g \rangle$-homeomorphic to $T_{x_0}(\Sigma)$ for any $g \in G$.*

Theorem 1.2. *If Σ is of even dimension ≥ 6, $T_{x_0}(\Sigma)$ is a properly 3-pseudofree G-representation, and $\Sigma^G = \{x_0\}$, then $\dim \Sigma = 6$ and either (1) or (2) below holds.*

(1) G is isomorphic to the symmetric group S_5 on five letters, and $T_{x_0}(\Sigma)$ is an irreducible real G-representation.

(2) G is isomorphic to $A_5 \times Z$ such that Z is a group of order 2, and the real G-representations V^Z and V_Z are irreducible and 3-dimensional, where $V = T_{x_0}(\Sigma)$, V^Z is the Z-fixed-point set of V, and V_Z is the orthogonal complement of V^Z in V.

In addition, S. Tamura and the author [27] showed that S_5 does not admit a one-fixed-point action on S^7, and P. Mizerka [18] showed that $\mathrm{TL}(2,5)$ (the GAP ID is 240(89)) does not admit an effective one-fixed-point action on S^n for any $n \leq 13$.

In the present paper, we will also prove the next existence result of one-fixed-point actions of finite group on spheres.

Theorem 1.3. *For the integer n, the finite group G, and the real G-representation V described below, there is an effective one-fixed-point G-action on the sphere S of dimension n such that $T_{x_0}(S)$ is isomorphic to V as real G-representations, where x_0 is the G-fixed point of S.*

(1) $n = 6$:

(i) $G = A_5$ *and* V *is a direct sum of two irreducible real* G*-representations of dimension* 3*. In this case, the* G*-action on* V *is properly* 2*-pseudofree.*

(ii) $G = S_5$ *and* V *is an irreducible real* G*-representation of dimension* 6*. In this case, the* G*-action on* V *is properly* 3*-pseudofree.*

(iii) $G = A_5 \times Z$*, where* Z *is a group of order* 2*, and* V *has the form* $V = V^Z \oplus V_Z$ *such that* V^Z *and* V_Z *are irreducible real* G*-representations of dimension* 3*. In this case, the* G*-action on* V *is properly* 3*-pseudofree.*

(2) $n = 7$*:*

(iv) $G = A_5$ *and* V *is a direct sum of irreducible real* G*-representations of dimension* 3 *and* 4*. In this case, the* G*-action on* V *is properly* 3*-pseudofree.*

(v) $G = A_5 \times Z$*, where* Z *is a group of order* 2*, and* V *has the form* $V = V^Z \oplus V_Z$ *such that* V^Z *is an irreducible real* G*-representation of dimension* 3 *and* V_Z *is an irreducible real* G*-representation of dimension* 4*. In this case, the* G*-action on* V *is properly* 3*-pseudofree.*

(3) $n = 3 + 4k$ *with* $k \in \mathbb{N}$*:*

(vi) $G = \mathrm{SL}(2,5) \times Z_m$*, where* Z_m *is a cyclic group of order* m *satisfying* $(m, 30) = 1$*, and* V *has the form* $V = V^{Z \times Z_m} \oplus W$*, where* $Z = \mathrm{Center}(\mathrm{SL}(2,5))$*, such that* $V^{Z \times Z_m}$ *is an irreducible real* G*-representation of dimension* 3 *and* W *is a free real* G*-representation of dimension* $4k$*. In this case, the* G*-action on* V *is properly* 3*-pseudofree.*

(4) $n = 6 + 8k$ *with* $k \in \mathbb{N}$*:*

(vii) $G = \mathrm{TL}(2,5) \times Z_m$*, where* $\mathrm{TL}(2,5)$ *is the double cover of* S_5 *of minus type (the GAP ID is* 240(89)*) with* $Z = \mathrm{Center}(\mathrm{TL}(2,5))$*,* Z_m *is a cyclic group of order* m *satisfying* $(m, 30) = 1$*, and* V *has the form* $V = V^{Z \times Z_m} \oplus W$ *such that* $V^{Z \times Z_m}$ *is an irreducible real* G*-representation of dimension* 6 *and* W *is a free real* G*-representation of dimension* $8k$*. In this case, the* G*-action on* V *is properly* 6*-pseudofree.*

Concerning this result, we note that there exist a free real G-representation of dimension 4 for $G = \mathrm{SL}(2,5) \times Z_m$ and a free real G-representation of dimension 8 for $G = \mathrm{TL}(2,5) \times Z_m$ whenever $(m, 30) = 1$. We remark that Theorem 1.3 implies the facts (F1)–(F3) mentioned above.

Next note that the sphere S^n of dimension n admits a properly 3-pseudofree one-fixed-point action of finite group if $n = 6$ or $3 + 4k$ with $k \in \mathbb{N}$. There arises a question: we wonder whether the sphere S^n of dimension $n = 5 + 4k$ with $k \in \mathbb{N}$ admits a properly 3-pseudofree one-fixed-point action of finite group.

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2. Proof of Theorems 1.1 and 1.2

Let Σ be a $\mathbb{Z}$-homology sphere of even dimension ≥ 6 equipped with a 3-pseudofree G-action and with a G-fixed point x_0, let V denote the tangential G-representation $T_{x_0}(\Sigma)$ of Σ at x_0, and let G_0 denote the subgroup

$$\{g \in G \mid \text{ the transformation } g : V \to V \text{ preserves an orientation of } V\}$$

of G. Therefore $|G/G_0| = 1$ or 2.

Proposition 2.1. *Let H be a subgroup of G. If $\dim V^H = 3$ then the order of H is 2, the generator σ of H acts on V as the scalar -1, and $\sigma \notin G_0$.*

Proof. We have the decomposition $V = V^H \oplus V_H$ as real H-representations. Since the G-action on V is 3-pseudofree and $\dim V^H = 3$, V_H is a free H-representation. Since $\dim V$ is even and $\dim V^H = 3$, $\dim V_H$ is odd. Therefore $|H| = 2$ and the generator of H acts on V as the scalar -1. Since $\dim V_H$ is odd, the action of the generator reverses orientations of V_H and V. □

Proposition 2.2. *The G_0-action on V is 2-pseudofree.*

Proof. Let $H \in \mathcal{S}(G_0) \smallsetminus \{E\}$. Proposition 2.1 says $\dim V^H \leq 2$. □

Proposition 2.3. *Suppose Σ is a homotopy sphere. Then the following holds.*

(1) *If $|\Sigma^{G_0}| \geq 2$ then for any $H \in \mathcal{S}(G_0) \smallsetminus \{E\}$, Σ^H is diffeomorphic to S^k, where $0 \leq k \leq 2$.*

(2) *If $|\Sigma^{G_0}| = 1$ then G_0 is isomorphic to A_5, Σ is diffeomorphic to S^6, and $\mathrm{res}^G_{G_0} V$ is a direct sum of two irreducible real G_0-representations of dimension 3.*

Proof. This follows from Proposition 2.2 and Laitinen–Traczyk Theorems 1–3. □

Proof of Theorem 1.1. If $G = G_0$ then the G-action on V is 2-pseudofree and the theorem is clear from Laitinen–Traczyk Theorems 1 and 2. Thus it suffices to prove the theorem in the case $|G/G_0| = 2$. We suppose $|G/G_0| = 2$.

If $G_0 \neq E$ then $\Sigma^{G_0} \cong S^k$, where $0 \leq k \leq 2$, we obtain $\Sigma^G = (\Sigma^{G_0})^{G/G_0} \cong (S^k)^{G/G_0} \cong S^h$ for $h = 0$, 1, or 2. If $G_0 = E$ then G is a group of order 2, and hence Σ^G is a $\mathbb{Z}_2$-homology sphere.

Let $x \in \Sigma^G \smallsetminus \{x_0\}$ and set $W = T_x(\Sigma)$. If $\dim \Sigma^G \geq 1$ then Σ^G is connected and hence W is isomorphic to V as real G-representations.

Suppose $\dim \Sigma^G = 0$. Since Σ^G is a $\mathbb{Z}_2$-homology sphere, we have $\Sigma^G = \{x_0, x\}$. Let $g \in G$. If g belongs to G_0 or $\dim V^g \leq 2$ then Laitinen–Traczyk Theorem 2 implies that W is $\langle g \rangle$-homeomorphic to V. Finally we suppose $\dim V^g = 3$. Proposition 2.1 says that g is of order 2. Since Σ^g is a $\mathbb{Z}_2$-homology sphere, we get $\dim W^g = \dim V^g = 3$. Clearly we have $\dim W = \dim V = \dim \Sigma$. Therefore W is isomorphic to V as real $\langle g \rangle$-representations. □

Proof of Theorem 1.2. Recall that if $|\Sigma^{G_0}| \geq 2$ then Laitinen–Traczyk Theorem 1 implies that Σ^{G_0} is diffeomorphic to S^k for $k = 0$, 1, or 2. In this case Σ^G is also diffeomorphic to S^h for $h = 0$, 1, or 2, which is a contradiction. Therefore we get $\Sigma^{G_0} = \Sigma^G = \{x_0\}$, which implies $\dim \Sigma = 6$, G_0 is isomorphic to A_5, and $\mathrm{res}^G_{G_0} V$ is a direct sum of two irreducible real G_0-representations V_1 and V_2 of dimension 3. Since V is properly 3-pseudofree, there is an element $g \in G$ such that $\dim V^g = 3$. Then the order of g is 2 and $g \notin G_0$. Therefore G is isomorphic to S_5 or $A_5 \times Z$ with $Z = \langle g \rangle$. In the case $G \cong S_5$, the irreducibility of V follows from the property $\mathrm{res}^G_{G_0} V = V_1 \oplus V_2$. In the case $G \cong A_5 \times Z$, V is isomorphic to $(V_1 \otimes W_1) \oplus (V_2 \otimes W_2)$, where W_1 and W_2 are real Z-representations of dimension 1. Since V is 3-pseudofree, one of W_1 or W_2 has a nontrivial Z-action and the other has the trivial Z-action. □

3. The element β_G of the Burnside ring of G

Let G be a finite group and let $\Omega(G)$ denote the Burnside ring of G. Each element of $\Omega(G)$ is an equivalence class $[F_1] - [F_2]$ of a pair (F_1, F_2) consisting of finite G-sets F_1 and F_2. A subgroup H gives the homomorphism $\chi_H : \Omega(G) \to \mathbb{Z}$ defined by $\chi_H([F_1] - [F_2]) = |F_1^H| - |F_2^H|$.

Suppose that G is nonsolvable. Let $\mathcal{S}(G)_{\rm sol}$ be the set of all solvable subgroups of G and set $\mathcal{S}(G)_{\rm nonsol} = \mathcal{S}(G) \smallsetminus \mathcal{S}(G)_{\rm sol}$. Then by [8, (1.3.2), (1.3.3), Proposition 1.3.5], there is a unique element β $(= \beta_G)$ of $\Omega(G)$ such that

$$\chi_H(\beta) = \begin{cases} 0 & \text{for } H \in \mathcal{S}(G)_{\rm nonsol} \\ 1 & \text{for } H \in \mathcal{S}(G)_{\rm sol}. \end{cases} \tag{3.1}$$

For a subgroup H of G, we denote by $(H)_G$ the G-conjugacy class of H, i.e.

$$(H)_G = \{gHg^{-1} \mid g \in G\} \subset \mathcal{S}(G).$$

For H, $K \in \mathcal{S}(G)$, we say that H is *subconjugate* (or *G-subconjugate*) to K and write $(H)_G \le (K)_G$ if gHg^{-1} is a subgroup of K for some element $g \in G$. There is a unique subset $\mathrm{Iso}(G, \beta)$ of $\mathcal{S}(G)$ which is closed under conjugations of elements in G and satisfies

$$\beta = \sum_{(H)_G \subset \mathrm{Iso}(G,\beta)} a_{(H)_G}[G/H] \quad \text{for some integers } a_{(H)_G} \neq 0. \tag{3.2}$$

It immediately follows that $\mathrm{Iso}(G, \beta) \subset \mathcal{S}(G)_{\rm sol}$, $\max(\mathcal{S}(G)_{\rm sol}) \subset \mathrm{Iso}(G, \beta)$, and $a_{(H)_G} = 1$ holds for each $H \in \max(\mathcal{S}(G)_{\rm sol})$. By (3.1), β is an idempotent of $\Omega(G)$.

The subgroup lattice of A_5 up to conjugations is as in Diagram 3.1.

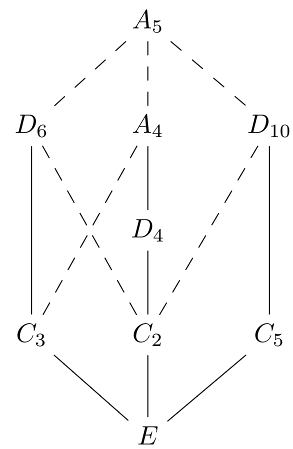


Diagram 3.1

In Diagram 3.1, C_m and D_n denote a cyclic group of order m and a dihedral group of order n, respectively, and A_4 denote the alternating group on four letters. There a real line between two subgroups H and K indicates that $gHg^{-1} \lhd K$ holds for some $g \in G$, and a dotted line indicates that $gHg^{-1} < K$ holds for some $g \in G$ and $gHg^{-1} \lhd K$ does not hold for any $g \in G$.

Proposition 3.1. *Let G be A_5. Then the idempotent β_G in $\Omega(G)$ has the form*

$$\beta_G = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/E], \tag{3.3}$$

and therefore

$$\mathrm{Iso}(G, \beta_G) = (A_4)_G \cup (D_{10})_G \cup (D_6)_G \cup (C_3)_G \cup (C_2)_G \cup (E)_G. \tag{3.4}$$

Proof. We tabulate the data $|(G/H)^K|$ necessary to determine β_G in Table 3.1. The proposition is readily follows from Table 3.1. □

K	G	A_4	D_{10}	D_6	C_5	D_4	C_3	C_2	E
G/G	1	1	1	1	1	1	1	1	1
G/A_4	0	1	0	0	0	1	2	1	5
G/D_{10}	0	0	1	0	1	0	0	2	6
G/D_6	0	0	0	1	0	0	1	2	10
G/C_5	0	0	0	0	2	0	0	0	12
G/D_4	0	0	0	0	0	3	0	3	15
G/C_3	0	0	0	0	0	0	2	0	20
G/C_2	0	0	0	0	0	0	0	2	30
G/E	0	0	0	0	0	0	0	0	60

Table 3.1

The subgroup lattice of S_5 up to conjugations is as in Diagram 3.2.

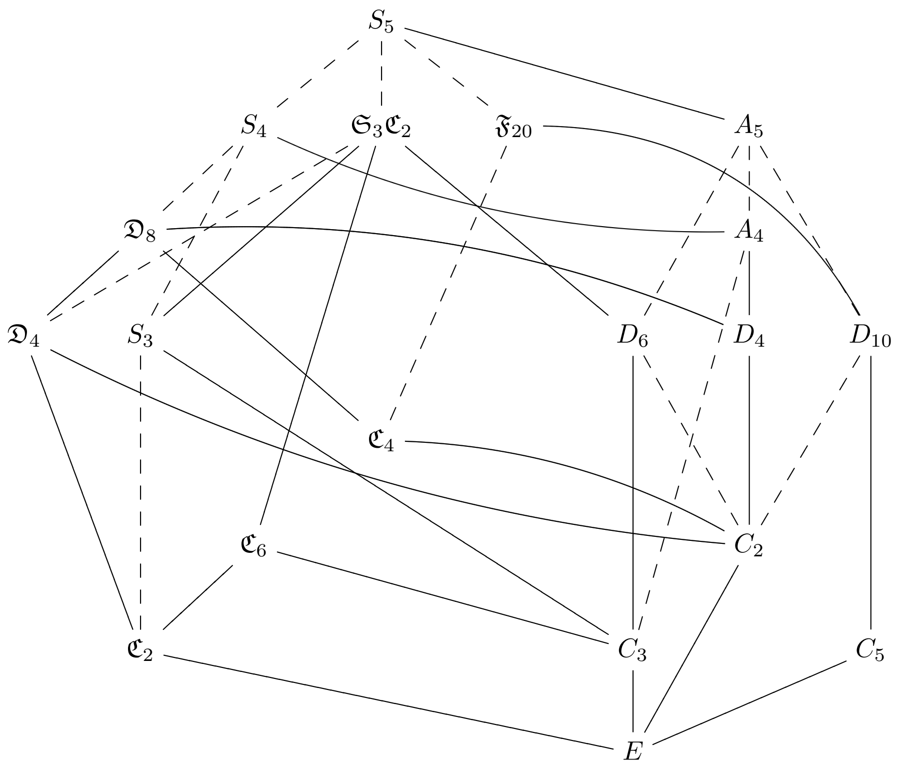


Diagram 3.2

Here $\mathfrak{C}_m$ and $\mathfrak{D}_n$ are a cyclic subgroup and a dihedral subgroup (not of A_5 but) of S_5 of order m and n, respectively, $\mathfrak{F}_{20}$ is a subgroup of order 20, S_3 is a subgroup isomorphic to the symmetric group on 3 letters, $\mathfrak{S}_3\mathfrak{C}_2$ is a subgroup of order 12 isomorphic to $\mathfrak{S}_3 \times \mathfrak{C}_2$, where $\mathfrak{S}_3$ is a subgroup conjugate to S_3 in S_5.

Proposition 3.2. *Let G be S_5. Then the idempotent β_G in $\Omega(G)$ has the form*

$$\beta_G = [G/S_4] + [G/\mathfrak{F}_{20}] + [G/\mathfrak{S}_3\mathfrak{C}_2] - [G/S_3] - [G/\mathfrak{D}_4] - [G/\mathfrak{C}_4] + [G/\mathfrak{C}_2] \tag{3.5}$$

and hence

$$\mathrm{Iso}(G,\beta_G) = (S_4)_G \cup (\mathfrak{F}_{20})_G \cup (G/\mathfrak{S}_3\mathfrak{C}_2)_G \cup (S_3)_G \cup (\mathfrak{D}_4)_G \cup (\mathfrak{C}_4)_G \cup (\mathfrak{C}_2)_G. \tag{3.6}$$

Proof. The proposition is easily obtained from Table 3.2 of the numbers $|(G/H)^K|$. □

K	G	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	D_6	$\mathfrak{C}_6$	S_3	C_5	$\mathfrak{C}_4$	$\mathfrak{D}_4$	D_4	C_3	C_2	$\mathfrak{C}_2$	E
G/G	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
G/A_5	0	2	0	0	0	2	2	0	2	0	0	2	0	0	2	2	2	0	2
G/S_4	0	0	1	0	0	1	0	1	0	0	2	0	1	1	1	2	1	3	5
$G/\mathfrak{F}_{20}$	0	0	0	1	0	0	1	0	0	0	0	1	2	0	0	0	2	0	6
$G/\mathfrak{S}_3\mathfrak{C}_2$	0	0	0	0	1	0	0	0	1	1	1	0	0	2	0	1	2	4	10
G/A_4	0	0	0	0	0	2	0	0	0	0	0	0	0	0	2	4	2	0	10
G/D_{10}	0	0	0	0	0	0	2	0	0	0	0	2	0	0	0	0	4	0	12
$G/\mathfrak{D}_8$	0	0	0	0	0	0	0	1	0	0	0	0	1	1	3	0	3	3	15
G/D_6	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	2	4	0	20
$G/\mathfrak{C}_6$	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	2	0	2	20
G/S_3	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	2	0	6	20
G/C_5	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	24
$G/\mathfrak{C}_4$	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2	0	30
$G/\mathfrak{D}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	2	6	30
G/D_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	6	0	30
G/C_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	40
G/C_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	60
$G/\mathfrak{C}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	60
G/E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	120

TABLE 3.2

Remark 3.1.

(1) For the case $G = A_5 \times Z$ with $|Z| = 2$, β_G is obtained as $f^*\beta_{A_5}$, where $f : G \to A_5$ is the canonical projection.
(2) For the case $G = \mathrm{SL}(2,5) \times Z_m$, β_G is obtained as $g^*\beta_{A_5}$, where $g : \mathrm{SL}(2,5) \times Z_m \to A_5$ is an epimorphism.
(3) For the case $G = \mathrm{TL}(2,5) \times Z_m$, β_G is obtained as $h^*\beta_{S_5}$, where $h : \mathrm{TL}(2,5) \times Z_m \to S_5$ is an epimorphism.

Let V be a real G-representation. For the connected-sum operation on G-framed maps with the target manifold $D(V)$ or $S(\mathbb{R} \oplus V)$, we need the next property for V.

Definition 3.1. We say that V is *ample* for β_G if $\mathrm{Iso}(G, \beta_G) \smallsetminus \max(\mathcal{S}_{\mathrm{sol}}(G))$ is contained in $\mathrm{Iso}(G, V \smallsetminus \{0\})$.

Proposition 3.3. *In the following cases, V is ample for β_G.*

(1) Case $G = A_5$ and V containing an irreducible real G-representation of dimension 3.
(2) Case $G = S_5$ and V containing an irreducible real G-representation of dimension 6.
(3) Case $G = A_5 \times Z$, where $|Z| = 2$, and V such that V^Z contains an irreducible real G-representation of dimension 3.
(4) Case $G = \mathrm{SL}(2,5) \times Z_m$, where $(m, 30) = 1$, and V such that $V^{Z \times Z_m}$ contains an irreducible real G-representation of dimension 3, where Z is the center of $\mathrm{SL}(2,5)$.
(5) $G = \mathrm{TL}(2,5) \times Z_m$, where $(m, 30) = 1$, and V such that $V^{Z \times Z_m}$ contains an irreducible real G-representation of dimension 6, where Z is the center of $\mathrm{TL}(2,5)$.

Proof. First we consider the case $G = A_5$. Clearly we have

$$\max(\mathcal{S}(G)_{\mathrm{sol}}) = (A_4)_G \cup (D_{10})_G \cup (D_6)_G.$$

Let W be an irreducible real G-representation of dimension 3. Then the H-fixed-point set W^H, $H \in \mathcal{S}(G)$, has the dimension as in Table 3.3.

H	C_5	C_3	C_2	E	Non Cyclic
$\dim W^H$	1	1	1	3	0

TABLE 3.3

Table 3.3 shows

$$\mathrm{Iso}(G, W \smallsetminus \{0\}) = (E)_G \cup (C_2)_G \cup (C_3)_G \cup (C_5)_G. \tag{3.7}$$

It follows from (3.4) and (3.7) that W is ample for β_G.

Second we consider the case $G = S_5$. It follows readily that

$$\max(\mathcal{S}(G)_{\mathrm{sol}}) = (S_4)_G \cup (\mathfrak{F}_{20})_G \cup (\mathfrak{S}_3\mathfrak{C}_2)_G.$$

Let W be an irreducible real G-representation of dimension 6. Then the H-fixed-point set W^H, $H \in \mathcal{S}(G)$, has the dimension as in Table 3.4.

H	S_3	$\mathfrak{C}_6$	C_5	$\mathfrak{D}_4$	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	2	1	1	2	3	2	6	0

TABLE 3.4

where $\mathcal{K} = \{G, A_5, S_4, \mathfrak{S}_3\mathfrak{C}_2, \mathfrak{F}_{20}, A_4, D_{10}, \mathfrak{D}_8, D_6, D_4\}$. Table 3.4 shows

$$\begin{aligned}\mathrm{Iso}(G, W \smallsetminus \{0\}) =& (E)_G \cup (C_2)_G \cup (\mathfrak{C}_2)_G \cup (C_3)_G \\ & \cup (\mathfrak{C}_4)_G \cup (\mathfrak{D}_4)_G \cup (C_5)_G \cup (\mathfrak{C}_6)_G \cup (S_3)_G.\end{aligned} \tag{3.8}$$

It follows from (3.6) and (3.8) that W is ample for β_G.

The ampleness of V for β_G in the cases (3), (4) and (5) follows from that in the cases (1) and (2). □

4. DEFINITION OF G-FRAMED MAPS

Let G be a finite nonsolvable group and let I denote the closed unit interval $[0, 1]$. For a space A and a map $g : P \to Q$, we denote by $A \times g$ the map $id_A \times g : A \times P \to A \times Q$. For a space A and a pair $\boldsymbol{g} = (g, c)$ of maps $g : P \to Q$ and $c : S \to T$, we denote by $A \times \boldsymbol{g}$ the pair $(A \times g, A \times c)$. In this paper, we mean by a G-framed map $\boldsymbol{f}$ a pair (f, b) consisting of a G-map $f : (X, \partial X) \to (Y, \partial Y)$ between G-manifolds X and Y with boundaries ∂X and ∂Y, respectively, where the case $\partial X = \partial Y = \emptyset$ is possible, and a G-bundle isomorphism $b : \tau_X \to f^*\tau_Y$, where $\tau_X = \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^\ell)$

and $\tau_Y = \varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \varepsilon_Y(\mathbb{R}^\ell)$ and we suppose $\ell \geq \dim X + 2$. In this situation, the equality $\dim X^H = \dim Y^H$ holds for all $H \in \mathcal{S}(G)$ such that $X^H \neq \emptyset$, because $\dim X^H$ is equal to the fiber dimension of the real vector bundle $T(X)^H$ and it is true for X replaced by Y.

In this paper, unless otherwise stated, for G-framed maps $\boldsymbol{f} = (f, b)$, $\boldsymbol{f}' = (f', b')$, $\boldsymbol{f}'' = (f'', b'')$, ..., the source manifolds of f, f', f'', ..., are $(X, \partial X)$, $(X', \partial X')$, $(X'', \partial X'')$, ... and the target manifolds of them are same $(Y, \partial Y)$, and we suppose that $\partial X = \partial X' = \partial X'' = \cdots = \partial Y$. A homotopy $\boldsymbol{F}$ from $\boldsymbol{f}$ to $\boldsymbol{f}'$ means a pair (F, B) consisting of a G-map $F : I \times X \to I \times Y$ and a G-bundle isomorphism

$$B : T(I \times X) \oplus \varepsilon_{I\times X}(\mathbb{R}^\ell) \to F^*T(I \times Y) \oplus \varepsilon_{I\times X}(\mathbb{R}^\ell)$$

satisfying the following conditions.

(1) $p_I(F(t, x)) = t$ for all $t \in I$ and $x \in X$, where p_I is the projection $I \times Y \to I$.
(2) The restriction of $\boldsymbol{F}$ to $\{0\} \times X$ coincides with $\{0\} \times \boldsymbol{f}$.
(3) The restriction of $\boldsymbol{F}$ to $\{1\} \times X$ coincides with $\{1\} \times \boldsymbol{f}'$.
(4) The restriction of $\boldsymbol{F}$ to $I \times \partial X$ coincides with $I \times \boldsymbol{f}|_{\partial X}$, where $\boldsymbol{f}|_{\partial X}$ is the restriction of $\boldsymbol{f}$ to ∂X.

A G-framed cobordism $\boldsymbol{F}$ from $\boldsymbol{f}$ to $\boldsymbol{f}'$ rel. boundary (or rel. ∂) means a pair (F, B) consisting of a G-map

$$F : (W, \partial_0 W, \partial_1 W, \partial_{01} W) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z), \tag{4.1}$$

and a G-bundle isomorphism

$$B : T(W) \oplus \varepsilon_W(\mathbb{R}^\ell) \to F^*T(Z) \oplus \varepsilon_W(\mathbb{R}^\ell),$$

where $\partial_0 W$, $\partial_1 W$, and $\partial_{01} W$ are G-manifolds canonically identified with X, X', and $I \times \partial Y$, respectively, and $Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$ and $\partial_{01} Z = I \times \partial Y$, satisfying the following conditions.

(1) $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_{01} W$, $\partial_0 W \cap \partial_1 W = \emptyset$, $\partial_0 W \cap \partial_{01} W = \partial(\partial_0 W)$, $\partial_1 W \cap \partial_{01} W = \partial(\partial_1 W)$, and $\partial(\partial_{01} W) = \partial(\partial_0 W) \amalg \partial(\partial_1 W)$.
(2) The restriction of $\boldsymbol{F}$ to $\partial_0 W$ coincides with $\boldsymbol{f}$ up to homotopies of G-framed maps rel. ∂.
(3) The restriction of $\boldsymbol{F}$ to $\partial_1 W$ coincides with $\boldsymbol{f}'$ up to homotopies of G-framed maps rel. ∂.

(4) The restriction of $\boldsymbol{F}$ to $\partial_{01}W$ coincides with $I \times \boldsymbol{id}_Y|_{\partial Y}$ $(= I \times \boldsymbol{id}_X|_{\partial X})$, where $\boldsymbol{id}_Y|_{\partial Y}$ is the restriction of $\boldsymbol{id}_Y$ to ∂Y.

Here the G-cobordism W from X and X' is not necessarily diffeomorphic to $I \times X$. For a subset of A of X, if $I \times A \subset W$ and the restriction of $\boldsymbol{F}$ to $I \times A$ coincides with $I \times \boldsymbol{f}|_A$ up to G-homotopies of G-framed maps then we call $\boldsymbol{F}$ a G-framed cobordism rel. A. Let $\mathcal{F}$ be a G-conjugation-invariant set of subgroups of G, i.e. if $H \in \mathcal{F}$ then $(H)_G \subset \mathcal{F}$. If $\boldsymbol{F}$ is a G-framed map rel. a G-regular neighborhood of $\bigcup_{K \in \mathcal{F}} X^K$, we say that $\boldsymbol{F}$ is a G-framed map rel. $\mathcal{F}$. For a G-framed map $\boldsymbol{F} = (F, B)$, the map F in (4.1) will be written as $F : W \to I \times Y$ for simplicity of notation when the context is clear.

Let M be a subgroup of G. Hereafter $\boldsymbol{F}_M = (F_M, B_M)$, $\boldsymbol{F}'_M = (F'_M, B'_M)$, $\boldsymbol{F}''_M = (F''_M, B''_M)$, ..., are M-framed cobordisms with M-maps $F_M : W_M \to I \times Y$, $F'_M : W'_M \to I \times Y$, $F''_M : W''_M \to I \times Y$, ..., respectively.

Let V be a real G-representation being $\mathcal{S}(G)_{\text{nonsol}}$-free, i.e.

$$\dim V^H = 0 \ \text{ for all } H \in \mathcal{S}(G)_{\text{nonsol}}.$$

Hereafter, unless otherwise stated, Y will be the unit disk $D(V)$ of V with respect to a G-invariant inner product. Clearly, the boundary of Y is obviously the unit sphere $S(V)$. Remark that if V is faithful then the G-action Y is effective and therefore the G-action on X is also effective. We assume that a G-framed map $\boldsymbol{f} = (f, b)$, where $f : (X, \partial X) \to (Y, \partial Y)$, satisfies the boundary condition that $\partial X = \partial Y$ and there is a G-collar neighborhood C of ∂X in X such that the restriction $\boldsymbol{f}|_C = (f|_C, b|_C)$ of $\boldsymbol{f}$ to C is the identity G-framed map on C. This clearly requires that C is also a G-collar neighborhood of ∂Y in Y.

5. G-connected sums of G-framed maps

Let $\boldsymbol{f} = (f, b)$ be a G-framed map with target $Y = D(V)$. We have the canonical G-bundle isomorphisms $f^*\varepsilon_Y(\mathbb{R}) \to \varepsilon_X(\mathbb{R})$, $f^*\varepsilon_Y(\mathbb{R}^\ell) \to \varepsilon_X(\mathbb{R}^\ell)$, $T(Y) \to \varepsilon_Y(V)$, and $f^*T(Y) \to \varepsilon_X(V)$. Let $\mathfrak{o}^1$ and $\mathfrak{o}^\ell$ be the canonical orientations of $\mathbb{R}$ and $\mathbb{R}^\ell$, respectively. For a subgroup H of G, we get the induced orientations $\mathfrak{o}^1_{Y^H}$, $\mathfrak{o}^\ell_{Y^H}$, $\mathfrak{o}^1_{X^H}$, $\mathfrak{o}^\ell_{X^H}$, of $\varepsilon_{Y^H}(\mathbb{R})$, $\varepsilon_{Y^H}(\mathbb{R}^\ell)$, $\varepsilon_{X^H}(\mathbb{R})$, $\varepsilon_{X^H}(\mathbb{R}^\ell)$, respectively. Note that $T(Y^H) = T(Y)^H = \varepsilon_{Y^H}(V^H)$, $(f^*T(Y))^H = f^{H*}T(Y^H)$. Let $\tau^H_X = \varepsilon_{X^H}(\mathbb{R}) \oplus T(X^H) \oplus \varepsilon_{X^H}(\mathbb{R}^\ell)$. There are two possibilities in choice of an orientation of $sV^H = \mathbb{R} \oplus V^H \oplus \mathbb{R}^\ell$ even if $\dim V^H = 0$. Fix an orientation $\mathfrak{o}_{sV^H}$ of sV^H. This induces the orientation $\mathfrak{o}_{\tau^H_Y}$ of $\tau^H_Y =$

$\varepsilon_{Y^H}(\mathbb{R}) \oplus T(Y^H) \oplus \varepsilon_{Y^H}(\mathbb{R}^\ell)$, and $\mathfrak{o}_{\tau_X^H}$ of τ_X^H via b^H. In this paper we refer to $\mathfrak{o}_{\tau_Y^H}$ and $\mathfrak{o}_{\tau_X^H}$ as orientations of Y^H and X^H, respectively. Without loss of any generality, we can assume that the restriction $\mathfrak{o}_{\tau_Y^H}|_{y_0}$ of $\mathfrak{o}_{\tau_Y^H}$ to $y_0 = 0 \in Y$ coincides with $\mathfrak{o}^1 \cup \mathfrak{o}'$ for some orientation $\mathfrak{o}'$ of $V^H \oplus \mathbb{R}^\ell$.

Let $\Sigma(X,Y)$ denote the union $X \cup_\partial Y$ of X and Y glued along the boundary $\partial Y = \partial X$. Here $\Sigma(X,Y)$ is a G-manifold. We have the G-map $\Sigma(f, id_Y) : \Sigma(X,Y) \to Y$ such that the restrictions $\Sigma(f, id_Y)|_X$ and $\Sigma(f, id_Y)|_Y$ are f and id_Y, respectively. We call $\Sigma(X,Y)$ and $\Sigma(f, id_Y)$ the *quasisphericalizations* of X and f, respectively. For a while let Z be the quasisphericalization of X. The stable tangent bundle $\tau_Z^H = \varepsilon_{Z^H}(\mathbb{R}) \oplus T(Z^H) \oplus \varepsilon_{Z^H}(\mathbb{R}^\ell)$ of Z^H has the orientation $\mathfrak{o}_{\tau_Z^H}$ extending $\mathfrak{o}_{\tau_X^H}$ such that the restriction $\mathfrak{o}_{\tau_Z^H}|_{y_0}$ of $\mathfrak{o}_{\tau_Z^H}$ to y_0 coincides with $(-\mathfrak{o}^1) \cup \mathfrak{o}'$. The restriction of $\Sigma(f, id_Y)^H$ to a small disk-neighborhood of y_0 in $\Sigma(X,Y)^H$ is orientation reversing.

Let $D_X(x)$ and $D_Y(y_0)$ be small H- and G-disk-neighborhoods of x and y_0 in X and Y, respectively. For a subset A of X or Y, the interior of A in X or Y is denoted by $\mathring{A}$. Suppose that the isotropy subgroup of G at x is H and the restriction $f|_{D_X(x)} : D_X(x) \to D_Y(y_0)$ of f is an H-diffeomorphism such that $f^K : X^K \to Y^K$ is locally orientation preserving at x for any $K \leq H$. Then $\psi = f|_{G \cdot D_X(x)} : G \cdot D_X(x) \to G \times_H D_Y(y_0)$ is a G-diffeomorphism. The G-manifold

$$\begin{aligned} X &\#_{G,H,x,y_0} (G \times_H \Sigma(X,Y)) \\ &= (X \smallsetminus G \cdot \mathring{D}_X(x)) \cup_\varphi G \times_H (\Sigma(X,Y) \smallsetminus \mathring{D}_Y(y_0)), \end{aligned}$$

where

$$\varphi : G \times_H \partial D_Y(y_0) \to G \cdot \partial D_X(x)$$

is the restriction of ψ^{-1}, is called the *G-connected sum* of X and $\Sigma(X,Y)$ of isotropy type $(H)_G$ with respect to points x and y_0. For any subgroup K of G, the manifold $(X \#_{G,H,x,y_0} (G \times_H \Sigma(X,Y)))^K$ has the orientation of which the restriction to $(X \smallsetminus G \cdot \mathring{D}_X(x))^K$ coincides with the restriction of $\mathfrak{o}_{\tau_X^K}$ to $(X \smallsetminus G \cdot \mathring{D}_X(x))^K$. We get the G-map $f \#_{G,H,x,y_0} G \times_H \Sigma(f, id_Y)$ gluing the restriction of f to $X \smallsetminus G \cdot \mathring{D}_X(x)$ and the restriction of $G \times_H \Sigma(f, id_Y)$ to $G \times_H (\Sigma(X,Y) \smallsetminus \mathring{D}_Y(y_0))$. We mean by $([G/G]+[G/H])X$ and $([G/G]+[G/H])f$ the G-manifold $X \#_{G,H,x,y_0} (G \times_H \Sigma(X,Y))$ and the G-map $f \#_{G,H,x,y_0} (G \times_H \Sigma(f, id_Y))$, respectively.

On the other hand, the G-manifold

$$X \#_{G,H,x,x} (G \times_H -\Sigma(X,Y)) = (X \smallsetminus G \cdot \mathring{D}_X(x)) \cup_\iota G \times_H (\Sigma(X,Y) \smallsetminus \mathring{D}_X(x)),$$

where

$$\iota : G \times_H \partial D_X(x) \to G \cdot \partial D_X(x)$$

is the canonical map, is called the *G-connected sum* of X and $-\Sigma(X,Y)$ of isotropy type $(H)_G$ with respect to points x ($\in X$) and x ($\in -\Sigma(X,Y)$). For any subgroup K of G, the manifold $(X\#_{G,H,x,x}(G \times_H -\Sigma(X,Y)))^K$ has the orientation of which the restriction to $(X \smallsetminus G \cdot \mathring{D}_X(x))^K$ coincides with the restriction of $\mathfrak{o}_{\tau_X^K}$ to $(X \smallsetminus G \cdot \mathring{D}_X(x))^K$. We get the G-map $f\#_{G,H,x,x}G \times_H -\Sigma(f, id_Y)$ gluing the restriction of f to $X \smallsetminus G \cdot \mathring{D}_X(x)$ and the restriction of $G \times_H \Sigma(f, id_Y)$ to $\Sigma(X,Y) \smallsetminus \mathring{D}_X(x)$. We mean by $([G/G] - [G/H])X$ and $([G/G] - [G/H])f$ the G-manifold $X \#_{G,H,x,x} (G \times_H -\Sigma(X,Y))$ and the G-map $f \#_{G,H,x,x} (G \times_H -\Sigma(f, id_Y))$, respectively.

Let γ_0 and $\gamma = \gamma_0 + [G/H]$ (resp. $\gamma = \gamma_0 - [G/H]$) be elements of the Burnside ring $\Omega(G)$. Suppose that $\mathrm{Iso}(G,\gamma_0) \cup \mathrm{Iso}(G,\gamma) \subset \mathrm{Iso}(G, V \smallsetminus \{0\})$. As an inductive step, we assume that we have obtained $\gamma_0 X$ and $\gamma_0 f$. Suppose there is $x \in (\gamma_0 f)^{-1}(y_0)$ with $G_x = H$ such that $\gamma_0 f$ is transverse regular to $\{y_0\}$ in Y and $(\gamma_0 f)^K$ is locally orientation preserving at x for every $K \le H$. Then similarly to the construction above of $([G/G] \pm [G/H])X$ and $([G/G] \pm [G/H])f$, we can obtain the equivariant connected sums

$$\begin{aligned} \gamma X &= \gamma_0 X \#_{G,H,x,y_0} (G \times_H \Sigma(X,Y)), \\ \gamma f &= \gamma_0 f \#_{G,H,x,y_0} (G \times_H \Sigma(f, id_Y)), \\ (\text{resp. } \gamma X &= \gamma_0 X \#_{G,H,x,x} (G \times_H -\Sigma(X,Y)), \\ \gamma f &= \gamma_0 f \#_{G,H,x,x} (G \times_H -\Sigma(f, id_Y)))\,. \end{aligned} \tag{5.1}$$

6. Basic lemmas on the reflection method

Let $M \in \mathcal{S}(G)_{\mathrm{sol}}$, $\boldsymbol{f} = (f, b)$ a G-framed map and $\boldsymbol{F}_M = (F_M, B)$ a G-framed cobordism from $\mathrm{res}^G_M \boldsymbol{f}$ to $\mathrm{res}^G_M \boldsymbol{id}_Y$ rel. ∂. Here we recall that $Y = D(V)$, $f : (X, \partial X) \to (Y, \partial Y)$, and $F_M : W_M \to I \times Y$. For a submanifold Z of X and an embedding $\Psi : I \times Z \to W_M$, we call Ψ a *product embedding* if

(1) $\Psi(t,x) = (t,x)$ in $\partial_{01} W_M$ for all $x \in Z \cap \partial X$ and $t \in I$,
(2) $\Psi(t,x) = (t,x)$ in a collar neighborhood $C_X = [0,\delta] \times X$ of $\{0\} \times X$ in W_M for all $t \in [0,\delta]$ and $x \in Z$, and

(3) $\Psi(1-t,x)=(1-t,\psi(x))$ in a collar neighborhood $C_Y=[1-\delta,1]\times Y$ of $\{1\}\times Y$ in W_M for all $t\in[0,\delta]$ and $x\in Z$, for some embedding $\psi: Z\to Y$.

Here δ is a small positive real number and $[0,\delta]$ and $[1-\delta,1]$ are the closed intervals $\subset\mathbb{R}$. For $K\in\mathcal{S}(G)$ and a K-subcomplex Z of X with respect to a smooth G-triangulation of X, let $N_K(Z,X)$ denote a K-regular neighborhood of Z in X. Therefore for $H\in\mathcal{S}(G)$, $N_K(X^H,X)$ is a K-tubular neighborhood of X^H, where $K=N_G(H)$. By virtue of the G-isomorphism b, the restriction $f^H: X^H\to Y^H$ of f is K-homotopic to a diffeomorphism if and only if the restriction $f|_{N_K(X^H,X)}: N_K(X^H,X)\to N_K(Y^H,Y)$ of f is K-homotopic to a diffeomorphism, where $K=N_G(H)$. For a subgroup H of G, we denote by $\mathcal{U}_G(H)$ the set of subgroups K of G satisfying $H<K$. For $H\in\mathcal{S}(M)$, we call the set

$$X^{>H}=\bigcup_{K\in\mathcal{U}_G(H)}X^K$$

the *G-singular set of X at H*.

Definition 6.1. Let H be a subgroup of G satisfying $N_G(H)\subset M$. We say that (X,Y,W_M) has the *(G,M)-tame singular set at H* (or $X^{>H}$ is (G,M)-*tame in* (X,W_M)) if there is a product M-embedding $\Psi_M: I\times N_M(M\cdot X^{>H},X)\to W_M$ such that $\mathrm{Im}(\Psi_M)^{>H}={W_M}^{>H}$.

For a subgroup $K\in\mathcal{S}(G)$, let

$$\begin{aligned}\mathcal{V}_G(K)&=\mathcal{S}(G)\smallsetminus\bigcup_{L\in(K)_G}\mathcal{S}(L),\ \text{and}\\ \mathcal{V}_{M,G}(K)&=\mathcal{S}(M)\smallsetminus\bigcup_{L\in(K)_G}\mathcal{S}(M\cap L).\end{aligned}\tag{6.1}$$

We remark that if $H\in\mathcal{S}(M)$, $N_G(H)\subset M$, and $(H)_G\cap\mathcal{S}(M)=(H)_M$ then

$$\{g\in G\mid gHg^{-1}\subset M\}\subset M.\tag{6.2}$$

The modification of G-framed maps by following Lemmas 6.1, 6.2, and 6.4 is called the *reflection method* in G-surgery theory.

Lemma 6.1. *Let $M\in\mathcal{S}(G)^*_{\mathrm{sol}}$ and $H\in\mathcal{S}(M)$ satisfying $N_G(H)\subset M$. Suppose the following.*

(i) (X,Y,W_M) has the (G,M)-tame singular set at H with respect to a product M-embedding $\Psi_M: I\times N_M(M\cdot X^{>H},X)\to W_M$.

(ii) There is an M-homotopy

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ *such that* $\mathbb{H}_M|_{W_M \times \{0\}}$ *coincides with* F_M *and* $\mathbb{H}_M|_{\mathrm{Im}(\Psi_M)\times\{1\}}$ *is a diffeomorphism.*

Then there are

- *a G-framed map* $\boldsymbol{f}'$ *rel.* ∂, *where* $\boldsymbol{f}' = (f', b')$ *and* $f' : (X', \partial X') \to (Y, \partial Y)$,
- *a G-framed cobordism* $\boldsymbol{F}_G$ *from* $\boldsymbol{f}$ *to* $\boldsymbol{f}'$ *rel.* ∂ *and* $\mathcal{V}_G(H)$,
- *an M-framed cobordism* $\mathbb{F}_M$ *from* $\mathrm{res}^G_M \boldsymbol{F}_G \cup_{\mathrm{res}^G_M \boldsymbol{f}} \boldsymbol{F}_M$ *to* $\boldsymbol{F}'_M$ *rel.* ∂ *and* $\mathcal{V}_M(H)$, *where* $\boldsymbol{F}'_M = (F'_M, B'_M)$ *is an M-framed cobordism from* $\mathrm{res}^G_M \boldsymbol{f}'$ *to* $\mathrm{res}^G_M \boldsymbol{id}_Y$ *rel.* ∂ *and* $\mathcal{V}_{M,G}(H)$, *and*

$$F'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

- *a product M-embedding* $\Phi'_M : I \times N_M(M \cdot X'^H, X') \to W'_M$ *with* $\mathrm{Im}(\Phi'_M) = N_M(M \cdot {W'_M}^H, W'_M)$, *and*
- *an M-homotopy*

$$\mathbb{H}'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W'_M \cup \partial_{01} W'_M$

possessing the following properties.

(1) $N_M(M \cdot X'^{>H}, X') = N_M(M \cdot X^{>H}, X)$, $N_M(M \cdot {W'_M}^{>H}, W'_M) = N_M(M \cdot {W_M}^{>H}, W_M)$, *and* $\Phi'_M|_{I\times N} = \Psi_M|_{I\times N}$ *for* $N = N_M(M \cdot X'^H, X') \cap N_M(M \cdot X^{>H}, X)$.

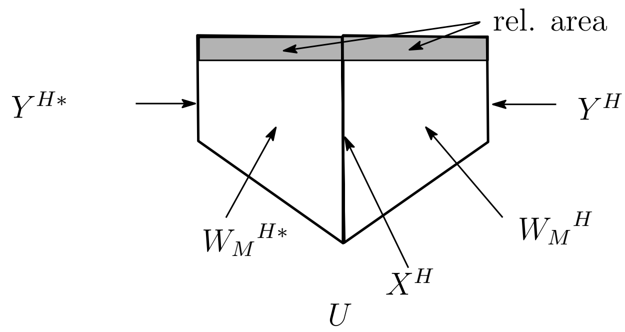
(2) $\mathbb{H}'_M|_{W'_M \times \{0\}}$ *coincides with* F'_M, $\mathbb{H}'_M|_{N_M(M\cdot {W'_M}^H, W'_M)\times\{1\}}$ *is a diffeomorphism, and* $\mathbb{H}'_M|_{\Phi'_M(I\times N)\times I}$ *coincides with* $\mathbb{H}_M|_{\Psi_M(I\times N)\times I}$ *for* N *above.*

In particular, X'^H *is* $N_G(H)$*-diffeomorphic rel.* ∂ *to* Y^H *and* $f'^H : X'^H \to Y^H$ *is* $N_G(H)$*-homotopic rel.* ∂ *to a diffeomorphism.*

Remark 6.1. If $(H)_G|_M = (H)_M$, where $(H)_G|_M = (H)_G \cap \mathcal{S}(M)$, then the properties (1) and (2) in Lemma 6.1 are true for H replaced by arbitrary $H' \in (H)_G|_M$.

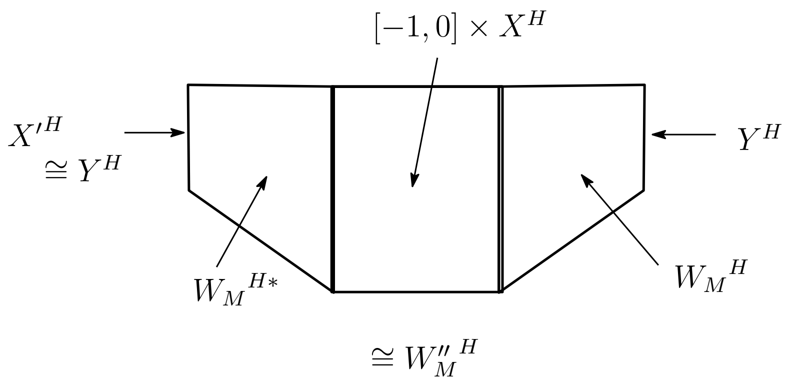
Proof. By virtue of Ψ_M, we can regard ${W_M}^H$ is an $N_G(H)$-cobordism from X^H to Y^H rel. $X^{>H} \cup \partial X^H$. Let ${W_M}^{H*}$ be a copy of ${W_M}^H$ and let Y^{H*} and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^{H*}$ be the copies of Y^H and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^H$, respectively, in ${W_M}^{H*}$. Then the union $U = {W_M}^{H*} \cup_{X^H} {W_M}^H$ of

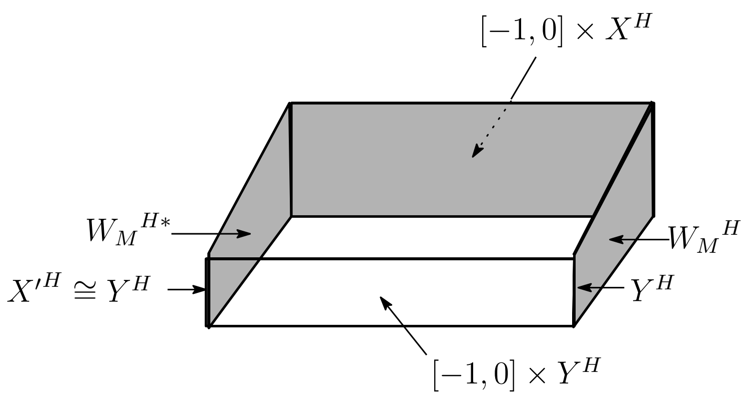
${W_M}^{H*}$ and ${W_M}^H$ attached along X^H can be regarded as an $N_G(H)$-cobordism rel. ∂ and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^{H*}$ from Y^{H*} to Y^H.



In addition, the associated map $f^{H*} : Y^{H*} \to Y^H$ is a copy of the identity map on Y^H. Let $\boldsymbol{F}_G = (F_G, B_G)$, where $F_G : W_G \to I \times Y$, be the G-framed cobordism from $\boldsymbol{f}$ to $\boldsymbol{f}'$ rel. ∂ obtained by G-surgeries on X of isotropy type $(H)_G$ such that ${W_G}^H = {W_M}^{H*}$. Then $\boldsymbol{f}'$ is a desired G-framed map.

Let us observe $\boldsymbol{F}_G$ above. Set $W''_M = W_G \cup_X W_M$ and $\boldsymbol{F}''_M = (F''_M, B''_M)$, where $F''_M = F_G \cup_f F_M$ and $B''_M = B_G \cup_b B_M$. The following two pictures





show that ${W''_M}^H = {W_G}^H \cup_{X^H} {W_M}^H$ is $N_M(H)$-cobordant rel. ∂ to the product cobordism $I \times Y^H$. Therefore $\boldsymbol{F}''_M$ is M-framed cobordant rel. ∂ to an M-framed cobordism $\boldsymbol{F}'_M = (F'_M, B'_M)$, where $F'_M : W'_M \to I \times Y$, such that ${W'_M}^H$ is $N_G(H)$-diffeomorphic rel. ∂ to $I \times Y^H$ and ${F'_M}^H : {W'_M}^H \to I \times Y^H$ is $N_G(H)$-homotopic to a diffeomorphism. We can formalize the above observation to Lemma 6.1. □

Lemma 6.2. *Let M, H and Z be as in Lemma 6.1. Invoke the following hypotheses (i)–(iii).*

(i) $(\mathrm{res}^G_M X, \mathrm{res}^G_M Y, W_M)$ *has the (M,M)-tame singular set at H with respect to a product M-embedding* $\Psi_M : I \times N_M(M \cdot (\mathrm{res}^G_M X)^{>H}, \mathrm{res}^G_M X) \to W_M$.

(ii) There is an M-homotopy

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ *such that* $\mathbb{H}_M|_{W_M \times \{0\}}$ *coincides with* F_M *and* $\mathbb{H}_M|_{\mathrm{Im}(\Psi_M) \times \{1\}}$ *is a diffeomorphism.*

(iii) There is $K \in \mathcal{U}_M(H)$ *such that* $\dim Y^H = \dim Y^K > 0$ *and* $X^{>H} = X^K$.

Then the conclusion same as Lemma 6.1 holds. In particular, ${X'}^H = {X'}^K = X^K$, ${f'}^H = {f'}^K = f^K$, *and* ${W'_M}^H = {W'_M}^K = W_M^K$ *for some* $K \in \mathcal{U}_M(H)$.

Proof. If K_1 and K_2 both satisfy the conditions required for K in (iii) then so does $K_1 \cap K_2$. Let K be the smallest subgroup satisfying the conditions in (iii). Then we have $X^{>H} = X^K$ and $X^H = X^K \amalg X^{=H}$. In addition ${W_M}^H = {W_M}^{>H} \amalg {W_M}^{=H} = {W_M}^K \amalg {W_M}^{=H}$ follows from (i) and (iii). Let W^*_M be a copy of W_M. Then ${W^*_M}^H \cup_{X^H} {W_M}^H$ is $N_M(H)$-cobordant rel. ∂ to ${W^*_M}^K \cup_{X^H} {W_M}^K$ by M-surgeries of isotropy type $(H)_M$. Therefore, we can remove $X^{=H}$ and ${W_M}^{=H}$ by G-surgeries on $\boldsymbol{f}$ and M-surgeries on $\boldsymbol{F}_M$ of isotropy types $(H)_G$ and $(H)_M$, respectively. □

Define $\mathcal{Y}(G, M, H)$ by

$$\mathcal{Y}(G, M, H) = \{K \in \mathcal{U}_G(H) \mid K \cap M = H\}. \tag{6.3}$$

Let Z be a G-manifold. We say that Z satisfies the *primitive gap condition for* (G, M, H) if the following conditions are satisfied.

(1) $\dim Z^H_\alpha > \dim Z^K_\beta$ for all $K \in \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$ and $\beta \in \pi_0(Z^K)$ with $Z_\beta \subset Z_\alpha$.

(2) $\dim Z^K = 0$ for all $K \in \mathcal{Y}(G, M, H)$.

Lemma 6.3. *Let $M \in \mathcal{S}(G)_{\mathrm{sol}}$ and $H \in \mathcal{S}(M)$ such that $N_G(H) \subset M$. Suppose the following conditions are fulfilled.*

(1) $(\mathrm{res}^G_M X, \mathrm{res}^G_M Y, W_M)$ has the (M, M)-tame singular set at H.

(2) X satisfies the primitive gap condition for (G, M, H).

(3) ${W_M}^H$ is connected.

Then (X, Y, W_M) has the (G, M)-tame singular set at H.

Proof. The set $X(\mathcal{Y}) = \bigcup_{K \in \mathcal{Y}(G,M,H)} X^K$ is a finite set. Therefore it is easy to obtain a product $N_M(H)$-embedding $I \times X(\mathcal{Y}) \to {W_M}^H \smallsetminus {W_M}^{>H}$ and to obtain a product M-embedding $\Psi_M : I \times N_M(M \cdot X^{>H}, X) \to W_M$ such that $\mathrm{Im}(\Psi_M)^{>H} = {W_M}^{>H}$. □

The next lemma follows from Lemmas 6.1 and 6.3.

Lemma 6.4. *Let M, H, Z be as in Lemma 6.1. Suppose the following (i)–(iv).*

(i) $(\mathrm{res}^G_M X, \mathrm{res}^G_M Y, W_M)$ has the (M, M)-tame singular set with respect to a product M-embedding $\Psi_M : I \times N_M(M \cdot (\mathrm{res}^G_M X)^{>H}, X) \to W_M$.

(ii) There is an M-homotopy

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M|_{W_M \times \{0\}}$ coincides with F_M and $\mathbb{H}_M|_{\mathrm{Im}(\Psi_M) \times \{1\}}$ is a diffeomorphism.

(iii) X satisfies the primitive gap condition for (G, M, H).

(iv) ${W_M}^H$ is connected.

Then the conclusion same as Lemma 6.1 holds.

In the rest of this section we give a remark on the (G, M)-tame singularity. Let Z be a G-manifold. We say that Z satisfies the *gap condition at H* if

$$2 \dim Z^K_\beta < \dim Z^H_\alpha \tag{6.4}$$

holds for all $K \in \mathcal{U}_G(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z^K_\beta \subset Z^H_\alpha$, where Z^H_α and Z^K_β stand for the underlying spaces of α and β. We say that Z satisfies the *cobordism gap condition at H* if

(1) $\dim Z^K_\beta + \dim Z^L_\gamma + 1 < \dim Z^H_\alpha$ holds for all $K \in \mathcal{U}_G(H) \smallsetminus \mathcal{U}_M(H)$, $L \in \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z^K_\beta \subset Z^H_\alpha$, $\gamma \in \pi_0(Z^L)$ with $Z^L_\gamma \subset Z^H_\alpha$, and

(2) $2\dim Z^K_\beta + 1 < \dim Z^H_\alpha$ holds for all $K \in \mathcal{U}_G(H) \smallsetminus \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z^K_\beta \subset Z^H_\alpha$.

Remark 6.2. Suppose the following conditions are fulfilled.

(1) $(\mathrm{res}^G_M X, \mathrm{res}^G_M Y, W_M)$ has the (M,M)-tame singular set at H.
(2) Y satisfies the cobordism gap condition at H.
(3) $f^H : X^H \to Y^H$ and ${F_M}^H : {W_M}^H \to I \times Y^H$ are connected up to the middle dimensions, respectively.

Then (X, Y, W_M) has the (G,M)-tame singular set at H.

7. Remarks on specific representations

Let $\mathcal{F}$ and $\mathcal{H}$ be sets of subgroups of G such that $\mathcal{F} \subset \mathcal{H}$. We call $\mathcal{F}$ *upper closed* in $\mathcal{H}$ if K belongs to $\mathcal{F}$ whenever $H \in \mathcal{F}$, $K \in \mathcal{H}$, and $H \subset K$.

Definition 7.1. Let $\mathcal{F}$ be a subset of $\mathcal{S}(G)_{\mathrm{sol}}$ which is G-conjugation invariant and upper closed in $\mathcal{S}(G)_{\mathrm{sol}}$. We say that $\mathcal{F}$ is *G-simply organized* (for equivariant surgeries) if there are a complete set $\mathcal{F}^*$ of representatives of $\mathcal{F}$ and a map $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$, satisfying the following conditions.

(1) $H \subset N_G(H) \subset \rho_{\max}(H)$ for any $H \in \mathcal{F}^*$.
(2) $\rho_{\max}(K^*) = \rho_{\max}(H)$ for any $H \in \mathcal{F}^*$ and $K \in \mathcal{U}_{\rho_{\max}(H)}(H)$, where K^* is the representative of $(K)_G$ in $\mathcal{F}^*$.
(3) $(H)_G \cap \mathcal{S}(\rho_{\max}(H)) = (H)_{\rho_{\max}(H)}$ for any $H \in \mathcal{F}^*$.

We remark that if $\mathcal{F}$ is G-simply organized as above then by (6.2) we have

$$\{g \in G \mid gHg^{-1} \subset \rho_{\max}(H)\} \subset \rho_{\max}(H).$$

for all $H \in \mathcal{F}^*$, and furthermore if $\mathcal{F}'$ is a subset of $\mathcal{F}$ such that $\mathcal{F}'$ is G-invariant and upper closed in $\mathcal{S}(G)_{\mathrm{sol}}$ then $\mathcal{F}'$ is G-simply organized.

Let H be a subgroup of G and Z a G-manifold. We say that Z satisfies the *weak gap condition* at H if

$$2\dim Z^K_\delta \le \dim Z^H_\gamma \tag{7.1}$$

holds for all $\gamma \in \pi_0(Z^H)$, $K \in \mathcal{U}_G(H)$, and $\delta \in \pi_0(Z^K)$ with $Z^K_\delta \subset Z^H_\gamma$. For $\gamma \in \pi_0(Z^H)$, let $\overline{H}_\gamma$ denote the set of elements g of $\overline{H} = N_G(H)/H$ such that $g\gamma = \gamma$, and let $\Pi(H,\gamma)_{1/2}$ denote the set of pairs (K,δ) of $K \in \mathcal{U}_G(H)$ and $\delta \in \pi_0(Z^K)$ such that $Z^K_\delta \subset Z^H_\gamma$ and $2\dim Z^K_\delta = \dim Z^H_\gamma$. We say that Z

satisfies the *modified weak gap condition* at H if the following conditions are fulfilled.

(1) Z satisfies the weak gap condition at H.

(2) For all $\gamma \in \pi_0(Z^H)$ with $\dim Z^H_\gamma > 0$ and $(K, \delta) \in \Pi(H, \gamma)_{1/2}$,

  (a) $K \subset N_G(H)$ and $K/H \subset \overline{H}_\gamma$,

  (b) $|(K/H) \cap \overline{H}_\gamma(2)| \leq 1$, where $\overline{H}_\gamma(2)$ is the set of elements in $\overline{H}_\gamma$ of order 2, and

  (c) $\dim Z^L_\omega + 1 < \dim Z^K_\delta$ for all $L \in \mathcal{U}_G(K)$ and $\omega \in \pi_0(Z^L)$ with $Z^L_\omega \subset Z^K_\delta$.

(3) For all $\gamma \in \pi_0(Z^H)$ with $\dim Z^H_\gamma > 0$ and (K_1, δ_1), $(K_2, \delta_2) \in \Pi(H, \gamma)_{1/2}$, the smallest subgroup $\langle K_1, K_2 \rangle$ of G containing $K_1 \cup K_2$ is solvable.

Let S_5 (resp. A_5) denote the symmetric group (resp. the alternating group) on the five letters 1, 2, ..., 5. We fix subgroups of S_5 as follows.

S_4 (resp. A_4) the symmetric group (resp. the alternating group) on the letters 2, 3, 4, 5.

S_3 the symmetric group on the letters 1, 2, 3.

$\mathfrak{C}_2 = \langle (4, 5) \rangle$, $\mathfrak{C}_4 = \langle (2, 4, 3, 5) \rangle$, and $\mathfrak{C}_6 = \langle (1, 2, 3)(4, 5) \rangle$ (cyclic groups).

$\mathfrak{S}_3\mathfrak{C}_2 = \langle (1, 2), (1, 2, 3), (4, 5) \rangle$ ($\cong S_3 \times \mathfrak{C}_2$).

$C_2 = \langle (2, 3)(4, 5) \rangle$, $C_3 = \langle (1, 2, 3) \rangle$, and $C_5 = \langle (1, 2, 3, 4, 5) \rangle$ (cyclic groups).

$D_4 = \langle (2, 3)(4, 5), (2, 4)(3, 5) \rangle$, $D_6 = \langle (1, 2, 3), (2, 3)(4, 5) \rangle$, and $D_{10} = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4) \rangle$ (dihedral groups).

$\mathfrak{D}_4 = \langle (2, 3), (2, 3)(4, 5) \rangle$, and $\mathfrak{D}_8 = \langle (2, 4, 3, 5), (2, 3) \rangle$ (dihedral groups).

The normalizers of subgroups H of $G = A_5$ are as in Table 7.1.

H	A_4	D_{10}	D_6	D_4	C_5	C_3	C_2	E
$N_G(H)$	A_4	D_{10}	D_6	A_4	D_{10}	D_6	D_4	G

Table 7.1

We assign $\rho_{\max}(H)$ to H as in Table 7.2.

H	A_4	D_{10}	D_6	D_4	C_5	C_3	C_2
$\rho_{\max}(H)$	A_4	D_{10}	D_6	A_4	D_{10}	D_6	A_4

Table 7.2

We immediately obtain the proposition:

Proposition 7.1. *Let $G = A_5$, $\mathcal{F} = \mathcal{S}(A_5)_{\rm sol} \smallsetminus \{E\}$, and $\mathcal{F}^* = \{A_4, D_{10}, D_6, D_4, C_5, C_3, C_2\}$. Then $\mathcal{F}$ is G-simply organized with respect to $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{F})^*$ given by Table 7.2.*

The next result follows from Table 3.3.

Proposition 7.2. *Let $G = A_5$. Let W_3 and W_3' be irreducible real G-representations of dimension 3 and let $W = W_3 \oplus W_3'$. Then*

(1) $\dim W_3^H = 0$ *for $H = A_4, D_{10}, D_6, D_4$,*

(2) $\dim W_3^H = 1$, *W_3 satisfies the gap condition at H for $H = C_5, C_3, C_2$,*

(3) $\dim W^H = 2$ *and W satisfies the primitive gap condition for $(G, \rho_{\max}(H), H)$ (as well as the gap condition at H) for $H = C_5, C_3, C_2$, and*

(4) W satisfies the gap condition at $H = E$.

Let $G = A_5$. Let W_3 and W_4 be irreducible real G-representations of dimensions 3 and 4, respectively, and let $W = W_3 \oplus W_4$. Then the dimensions of the H-fixed-point sets W^H are as in the next table.

H	G	A_4	D_{10}	D_6	C_5	D_4	C_3	C_2	E
$\dim W^H$	0	1	0	1	1	1	3	3	7

Table 7.3

We immediately obtain the proposition:

Proposition 7.3. *Let $G = A_5$. Let W_3 and W_4 be irreducible real G-representations of dimensions 3 and 4, respectively, and let $W = W_3 \oplus W_4$. Then*

(1) $\dim W^H = 0$ *for $H = D_{10}$,*

(2) $\dim W^H = 1$ *for $H = A_4, D_6, C_5, D_4$, and W satisfies the gap condition at H for $H = A_4, D_6, C_5$,*

(3) $\dim W^H = 3$ *and* W *satisfies the gap condition at* H *for for* $H = C_3$, C_2, *and*

(4) W *satisfies the gap condition at* $H = E$.

Next we consider the case $G = S_5$. The normalizers of subgroups H of S_5 are as in the next table

H	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$N_G(H)$	G	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
$N_G(H)$	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	S_4	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{D}_8$	G

TABLE 7.4

We assign $\rho_{\max}(H)$ to H as follows.

H	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$\rho_{\max}(H)$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	C_2
$\rho_{\max}(H)$	$\mathfrak{F}_{20}$	S_4	S_4	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	S_4

TABLE 7.5

We immediately obtain the proposition.

Proposition 7.4. *Let* $G = S_5$, $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$, *and* $\mathcal{F}^*$ *the set of subgroups* H *in Table 7.5. Then* $\mathcal{F}$ *is* G*-simply organized with respect to* $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\text{sol}})^*$ *given by Table 7.5.*

The next result follows from Table 3.4.

Proposition 7.5. *Let* $G = S_5$ *and let* W *be an irreducible real* G*-representation of dimension* 6. *Then*

(1) $\dim W^H = 0$ *for* $H = S_4$, $\mathfrak{F}_{20}$, $\mathfrak{S}_3\mathfrak{C}_2$, A_4, D_{10}, $\mathfrak{D}_8$, D_6, D_4,

(2) $\dim W^H = 1$ *and* W *satisfies the gap condition at* H *for* $H = S_3$, $\mathfrak{C}_6$, $\mathfrak{D}_4$, $\mathfrak{C}_4$,

(3) $\dim W^H = 2$ *and* W *satisfies the primitive gap condition for* $(G, \rho_{\max}(H), H)$ *for* $H = C_5$, C_3, C_2,

(4) $\dim W^H = 3$ *and* W *satisfies the gap condition at* H *for* $H = \mathfrak{C}_2$*, and*

(5) W *satisfies the modified weak gap condition at* $H = E$*.*

Now let $G = A_5 \times Z$, where Z is a group of order 2. We identify subgroups $H \in \mathcal{S}(A_5)$ with $H \times \{e\} \in \mathcal{S}(G)$, respectively, and Z with $\{e\} \times Z \in \mathcal{S}(G)$. Let $\mathcal{C}_2$ be the subgroup of order 2 belonging to $\mathcal{S}(C_2Z) \smallsetminus \{C_2, Z\}$. Let $\mathcal{D}_{2n}$ be the dihedral subgroup of order $2n$ generated by C_n and $\mathcal{C}_2$. We tabulate subgroups H giving a complete set of representatives of conjugacy classes of subgroups of $G = A_5 \times Z$ and the normalizers of subgroups H in Table 7.6.

H	G	A_5	A_4Z	$D_{10}Z$	D_6Z	A_4	$\mathcal{D}_{10}$	D_{10}	C_5Z	D_4Z	C_3Z
$N_G(H)$	G	G	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	$\mathcal{D}_6$	D_6	C_5	$\mathcal{D}_4$	C_2Z	D_4	C_3	$\mathcal{C}_2$	C_2	Z	E
$N_G(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	D_4Z	A_4Z	D_6Z	D_4Z	D_4Z	G	G

TABLE 7.6

In the case $G = A_5 \times Z$ above, we assign $\rho_{\max}(H)$ to H as in Table 7.7.

H	A_4Z	$D_{10}Z$	D_6Z	A_4	$\mathcal{D}_{10}$	D_{10}	C_5Z	D_4Z	C_3Z
$\rho_{\max}(H)$	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	$\mathcal{D}_6$	D_6	C_5	$\mathcal{D}_4$	C_2Z	D_4	C_3	C_2
$\rho_{\max}(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	A_4Z	A_4Z	D_6Z	A_4Z

TABLE 7.7

We immediately obtain the next proposition.

Proposition 7.6. *Let* $G = A_5 \times Z$*, where* Z *is a group of order* 2*,* $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus (\{E, Z\} \cup (\mathcal{C}_2)_G)$*, and* $\mathcal{F}^*$ *the set of subgroups* H *in Table 7.7. Then* $\mathcal{F}$ *is* G*-simply organized with respect to* $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\rm sol})^*$ *given by Table 7.7.*

Let W_3 and W_3' be irreducible real A_5-representations of dimension 3 and let $\mathbb{R}$ and $\mathbb{R}_\pm$ be 1-dimensional real Z-representations with trivial and nontrivial Z-actions, respectively. The dimensions of the H-fixed-point sets W^H of $W = (W_3 \otimes \mathbb{R}) \oplus (W_3' \otimes \mathbb{R}_\pm)$ are as in Table 7.8.

H	$\mathcal{D}_{10}$	C_5Z	C_3Z	$\mathcal{D}_6$	C_5	$\mathcal{D}_4$	C_2Z	C_3	$\mathcal{C}_2$	C_2	Z	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	1	1	2	1	1	2	3	2	3	6	0

TABLE 7.8

where $\mathcal{K} = \{G, A_5, A_4Z, D_{10}Z, D_6Z, A_4, D_{10}, D_4Z, D_6, D_4\}$. The next proposition follows.

Proposition 7.7. *Let $G = A_5 \times Z$, where Z is a group of order 2, and let $W = (W_3 \otimes \mathbb{R}) \oplus (W_3' \otimes \mathbb{R}_\pm)$ be a real G-representation of dimension 6 described above. Then*

(1) $\dim W^H = 0$ *for* $H = A_4Z$, $D_{10}Z$, D_6Z, A_4, D_{10}, D_4Z, D_6, D_4,

(2) $\dim W^H = 1$ *and W satisfies the gap condition at H for $H = \mathcal{D}_{10}$, C_5Z, C_3Z, $\mathcal{D}_6$, $\mathcal{D}_4$, C_2Z,*

(3) $\dim W^H = 2$ *and W satisfies the primitive gap condition for $(G, \rho_{\max}(H), H)$ for $H = C_5$, C_3, C_2,*

(4) $\dim W^H = 3$ *and W satisfies the gap condition at H for $H = \mathcal{C}_2$, Z, and*

(5) W satisfies the modified weak gap condition at $H = E$.

Next we consider the case where $W = (W_3 \otimes \mathbb{R}) \oplus (W_4 \otimes \mathbb{R}_\pm)$, where W_4 is an irreducible real A_5-representation of dimension 4. Then the dimensions of the H-fixed-point sets W^H of W are as in the next table.

H	A_4	C_5Z	C_3Z	D_6	$\mathcal{D}_6$	C_5	$\mathcal{D}_4$	D_4	C_2Z	C_3	$\mathcal{C}_2$	C_2	Z	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	1	1	1	1	1	1	1	3	3	3	3	7	0

TABLE 7.9

where $\mathcal{K} = \{G, A_5, A_4Z, D_{10}Z, D_6Z, \mathcal{D}_{10}, D_{10}, D_4Z\}$. The next proposition follows.

Proposition 7.8. *Let $G = A_5 \times Z$, where Z is a group of order 2, and let $W = (W_3 \otimes \mathbb{R}) \oplus (W_4 \otimes \mathbb{R}_\pm)$ be a real G-representation of dimension 7 described above. Then*

(1) $\dim W^H = 0$ *for* $H = A_4Z$, $D_{10}Z$, D_6Z, $\mathcal{D}_{10}$, D_{10}, D_4Z,

(2) $\dim W^H = 1$ *and* W *satisfies the gap condition at* H *for* $H = A_4$, C_5Z, $\mathcal{D}_6$, D_6, C_3Z, C_5, $\mathcal{D}_4$, C_2Z,

(3) $\dim W^H = 3$ *and* W *satisfies the gap condition at* H *for* $H = C_3$, $\mathcal{C}_2$, C_2, Z,

(4) W *satisfies the gap condition at* $H = E$.

8. G-surgery obstructions of isotropy type $(H)_G$

Let $\boldsymbol{f} = (f, b)$ be a G-framed map as in the previous section. Recall that $f : (X, \partial X) \to (Y, \partial Y)$, $Y = D(V)$, $b : \tau_X \to f^*\tau_Y$, and $\partial f : \partial X \to \partial Y$ is the identity map on $\partial X = \partial Y$. Hence the mapping degree of $f^H : (X^H, \partial X^H) \to (Y^H, \partial Y^H)$ is 1 whenever $H \in \mathcal{S}(G)$ and $\dim V^H > 0$.

Let H be a subgroup and set $\overline{H} = N_G(H)/H$. Let $G(2)$ denote the set of elements of order 2 in G. Thus $\overline{H}(2)$ is the set of elements of order 2 in $\overline{H}$. For a principal ideal domain R satisfying $a^2 = a$ in $R/2R$ for all $a \in R$, let $A_{\overline{H}} = R[\overline{H}]$ denote the group algebra of $\overline{H}$ over R. Therefore $A_{\overline{H}} = \mathrm{Map}(\overline{H}, R)$. Let $w_{\overline{H}} : \overline{H} \to \{1, -1\}$ denote the orientation homomorphism of V^H with $\overline{H}$-action. Set $n_H = \dim V^H$, let k_H be the integer satisfying $n_H = 2k_H$ or $2k_H + 1$, and set $\lambda_H = (-1)^{k_H}$. $A_{\overline{H}}$ has the involution $- : A_{\overline{H}} \to A_{\overline{H}}$; $x \mapsto \overline{x}$, defined by

$$\overline{\sum_{g\in\overline{H}} r_g\, g} = \sum_{g\in\overline{H}} r_g w_{\overline{H}}(g) g^{-1}, \tag{8.1}$$

where $r_g \in R$. Depending on $\varepsilon \in \{1, -1\}$, we define the submodule $\min_\varepsilon(A_{\overline{H}})$ of $A_{\overline{H}}$ by

$$\min_\varepsilon(A_{\overline{H}}) = \{x - \varepsilon\overline{x} \mid x \in A_{\overline{H}}\} \quad \text{(see (8.1))}.$$

Case $n_H = 2k_H \geq 6$. Let $Q_{\overline{H}}$ (resp. $S_{\overline{H}}$) denote the set of elements $g \in \overline{H}(2)$ satisfying $\dim(V^H)^g = k_H - 1$ (resp. $\dim(V^H)^g = k_H$). Let

$$A_{\overline{H},s} = R[S_{\overline{H}}],$$
$$\Gamma_{\overline{H}} = \min_{-\lambda_H}(A_{\overline{H}}) + R[S_{\overline{H}}],$$
$$\Lambda_{\overline{H}} = \min_{\lambda_H}(A_{\overline{H}}) + R[Q_{\overline{H}}],$$

where $R[S_{\overline{H}}] = \mathrm{Map}(S_{\overline{H}}, R)$ and $R[Q_{\overline{H}}] = \mathrm{Map}(Q_{\overline{H}}, R)$. We call

$$\boldsymbol{A}_{\overline{H}} = (A_{\overline{H}}, (-, \lambda_H), \Gamma_{\overline{H}}, \overline{H}, A_{\overline{H},s}, A_{\overline{H},s} + \Lambda_{\overline{H}})$$

the *double parameter algebra* of the $\overline{H}$-manifold Y^H, see [6, Definition 2.5] and [6, p. 538].

Let $\Theta_{\overline{H},2}$ be the set of all generators of $H_{k_H}((Y^H)^K, \partial(Y^H)^K; \mathbb{Z}_2) \cong \mathbb{Z}_2$, where K runs over $\mathcal{S}(\overline{H})$ such that $\dim(Y^H)^K = k_H$, and $\widetilde{\Theta}_{\overline{H}}$ the set of all generators of $H_{k_H}((Y^H)^K,\ \partial(Y^H)^K;\mathbb{Z}) \cong \mathbb{Z}$, where K runs over $\mathcal{S}(\overline{H})$ such that $\dim(Y^H)^K = k_H$. The canonical map $pr_{\overline{H}} : \widetilde{\Theta}_{\overline{H}} \to \Theta_{\overline{H},2}$ is a double covering. We have the map $\rho_{\overline{H}} : \Theta_{\overline{H},2} \to \mathfrak{P}(S_{\overline{H}})$, where $\mathfrak{P}(S_{\overline{H}})$ is the set of subsets of $S_{\overline{H}}$, defined by $\rho_{\overline{H}}(t) = K \cap S_{\overline{H}}$ for a generator t of $H_{k_H}((Y^H)^K, \partial(Y^H)^K; \mathbb{Z}_2)$ with $\dim(Y^H)^K = k_H$. We call

$$\boldsymbol{\Theta}_{\overline{H}} = (pr_{\overline{H}} : \widetilde{\Theta}_{\overline{H}} \to \Theta_{\overline{H},2},\ \rho_{\overline{H}} : \Theta_{\overline{H},2} \to \mathfrak{P}(S_{\overline{H}}))$$

the *positioning data* of the $\overline{H}$-manifold Y^H, see [6, pp. 533, 538]. By the definition [6, p. 545], we obtain the abelian group

$$\mathcal{L}_{V,H}(R[\overline{H}]) = W_{n_H}(R, \overline{H}, Q_{\overline{H}}, S_{\overline{H}}, \boldsymbol{\Theta}_{\overline{H}})_{\text{free}}.$$

Case $n_H = 2k_H + 1 \geq 3$. Let $Q_{\overline{H}}$ denote the set of elements g with order 2 of $\overline{H}$ satisfying $\dim(V^H)^g = k_H$ and

$$\Lambda_{\overline{H}} = \min\nolimits_{\lambda_H}(A_{\overline{H}}) + R[Q_{\overline{H}}].$$

We call

$$\boldsymbol{A}_{\overline{H}} = (A_{\overline{H}}, (-, \lambda_H), \Lambda_{\overline{H}})$$

the *form algebra* of the $\overline{H}$-manifold Y^H. By [20, Definition 1.5], we obtain the abelian group

$$\mathcal{L}_{V,H}(R[\overline{H}]) = W_1^{\lambda_H}(A_{\overline{H}}, \Lambda_{\overline{H}}).$$

Suppose V is $\mathcal{S}(G)_{\text{nonsol}}$-free, i.e. $V^L = \{0\}$ for all $L \in \mathcal{S}(G)_{\text{nonsol}}$. Let $H \in \mathcal{S}(G)_{\text{sol}}$. We obtain the $\overline{H}$-framed map $\boldsymbol{f}^H = (f^H, b^H)$ from the G-framed map $\boldsymbol{f}$, where $f^H : (X^H, \partial X^H) \to (Y^H, \partial Y^H)$ and $b^H : \tau_{X^H} \to {f^H}^* \tau_{Y^H}$.

We say that $\boldsymbol{f}$ is $\mathcal{P}$-*adjusted* at H if $f^K : X^K \to Y^K$ is a $\mathbb{Z}_p$-homology equivalence for every prime p and every $K \in \mathcal{U}_{N_G(H)}(H)$ such that $|K/H|$ is a power of p. We suppose that Y satisfies the modified weak gap condition at H and $\boldsymbol{f}$ is $\mathcal{P}$-adjusted at H. The G-framed map $\boldsymbol{f}$ is G-framed cobordant rel. ∂ by G-surgeries of isotropy type $(H)_G$ to $\boldsymbol{f}' = (f', b')$, where $f' : (X', \partial X') \to (Y, \partial Y)$, such that $f : X' \to Y$ is k_H-connected, where $\dim Y^H = 2k_H$ or $2k_H + 1$. Suppose $f^H : X^H \to Y^H$ is k_H-connected. We define the *surgery* *kernel* $L(f^H;R)$ to be the $\overline{H}$-module

$$\mathrm{Ker}[{f^H}_* : H_{k_H}(X^H;R) \to H_{k_H}(Y^H;R)] = H_{k_H}(X^H;R)$$
$$\text{if } \dim Y^H = 2k_H \geq 6,$$

(8.2) $K_{k_H+1}({X^H}_0, \partial \overline{H}\, U) \otimes_{\mathbb{Z}} R$

$$\text{if } \dim Y^H = 2k_H + 1 \geq 5, \text{ see [20, Diagram 4.2], and}$$

$K_2({X^H}_0, \partial \overline{H}\, U; R)$ if $\dim Y^H = 3$, see [24, Diagram 3.1],

where U is a submanifold of $\overline{H}$-manifold X^H and ${X^H}_0 = X^H \setminus \overline{H}\, \mathring{U}$.

Lemma 8.1. *Let $R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for a prime p. Suppose the following (i)–(iii).*

(i) V satisfies the modified weak gap condition at H.

(ii) $\boldsymbol{f}$ is $\mathcal{P}$-adjusted at H.

(iii) $f^H : X^H \to Y^H$ is k_H-connected.

If the surgery kernel $L(f^H;R)$ is stably free over $R[\overline{H}]$ then there is an element

$$\sigma_{G,H}(\boldsymbol{f})\ (= \sigma_{\overline{H}}(\boldsymbol{f}^H)) \text{ of } \mathcal{L}_{V,H}(R[\overline{H}])$$

having the property: if $\sigma_{G,H}(\boldsymbol{f}) = 0$ then $\boldsymbol{f}$ is G-framed cobordant rel. ∂ by G-surgeries of isotropy type $(H)_G$ to $\boldsymbol{f}' = (f', b')$, where $f' : (X', \partial X') \to (Y, \partial Y)$, such that

(1) ${X'}^H$ is 1-connected and R-acyclic if $\dim V^H \geq 5$, and

(2) ${X'}^H$ is (connected and) R-acyclic if $\dim V^H = 3$.

Proof. The lemma follows from the proofs of [6, Theorems 1.1 and 1.2], [20, Theorem A], and [24, Theorem 1.1]. □

9. Construction of G-framed maps

Let G be a nonsolvable group, $\beta = \beta_G$ the idempotent of $\Omega(G)$ defined by (3.1), and V a real G-representation of positive dimension being $\mathcal{S}(G)_{\mathrm{nonsol}}$-free and ample for β_G. Recall that $V^L = \{0\}$ for all $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$. Let $Z = S(\mathbb{R} \oplus V)$ and $Z^+ = Z \amalg \{pt\}$. The sphere Z is the union of the hemispheres $S_+ = \{(u,v) \in S(\mathbb{R} \oplus V) \mid u \geq 0\}$ and $S_- = \{(u,v) \in S(\mathbb{R} \oplus V) \mid u \leq 0\}$, where $u \in \mathbb{R}$ and $v \in V$. Let $y_+ = (1,0) \in S(\mathbb{R} \oplus V)$ and $y_- = (-1,0) \in S(\mathbb{R} \oplus V)$, where $\pm 1 \in \mathbb{R}$ and $0 \in V$. We have the canonical G-diffeomorphism $S_+ \to D(V)$, which carries y_+ to $y_0 = 0$, and identify S_+ with $D(V)$ via the diffeomorphism. Recall the generalized cohomology

$$\omega_G^0(Z) = \lim_{m \to \infty} [Z^+ \wedge M^\bullet, M^\bullet]_0^G,$$

where $M^\bullet$ is the one-point compactification of $M = \mathbb{R}[G]^m$. For $\alpha = 1 - \beta$, the set $S = \{\alpha\}$ is multiplicatively closed subset of $\Omega(G)$ and the restriction map

(9.1)

$$\begin{array}{ccc} S^{-1}\omega_G^0(Z) & \longrightarrow & S^{-1}\omega_G^0(Z^G) \\ & & \Big\downarrow = \\ & & S^{-1}\omega_G^0(\{y_+\}) \oplus S^{-1}\omega_G^0(\{y_-\}) \xrightarrow[\cong]{} S^{-1}\Omega(G) \oplus S^{-1}\Omega(G) \end{array}$$

is an isomorphism. The module $\Omega(G) \oplus \Omega(G)$ contains the element $(\alpha, 0)$. By the arguments in [23] and [26, Section 4], originally due to T. Petrie [30, Sections 1 and 2], we obtain the next lemma.

Lemma 9.1. *There are a G-framed map $\boldsymbol{f} = (f, b)$, where $Y = D(V)$, $f : (X, \partial X) \to (Y, \partial Y)$ with $\partial f = id_{\partial Y}$ and $b : \tau_X \to f^*\tau_Y$, and M-framed cobordisms $\boldsymbol{F}_M = (F_M, B_M)$ from $\mathrm{res}^G_M \boldsymbol{f}$ to $\mathrm{res}^G_M \boldsymbol{id}_Y$ rel. ∂, where $F_M : W_M \to I \times Y$ is an M-map and $B_M : T(W_M) \oplus \varepsilon_{W_M}(\mathbb{R}^\ell) \to {F_M}^* T(I \times Y) \oplus \varepsilon_{W_M}(\mathbb{R}^\ell)$ is an M-bundle isomorphism, for all $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, satisfying the following conditions* (C1)–(C3)*.*

- (C1) *$X^L = \emptyset$ for any $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$.*
- (C2) *$f^{-1}(y_0)^H$ consists of one point, say x_H, $f : X \to Y$ is transverse regular at x_H to y_0 in Y, and $f^K : X^K \to Y^K$ is locally an orientation-preserving diffeomorphism from a neighborhood of x_H in X^K to a neighborhood of y_0 in Y^K, for any $H \in \max(\mathcal{S}(G)_{\mathrm{sol}})$ with $\dim V^H = 0$ and $K \in \mathcal{S}(H)$.*
- (C3) *$f^{-1}(y_0)^{=H} = \emptyset$ for each $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \max(\mathcal{S}(G)_{\mathrm{sol}})$ with $\dim V^H = 0$, where $f^{-1}(y_0)^{=H}$ is the subset of $f^{-1}(y_0)^H$ consisting of points with isotropy subgroup H.*

In the lemma above, it holds that

- (C4) $\mathrm{Iso}(G, X) \supset \mathrm{Iso}(G, Y \smallsetminus \{y_0\}) \cup \max(\mathcal{S}(G)_{\mathrm{sol}}) \supset \mathrm{Iso}(G, \beta)$, and
- (C5) $\deg[f^H : (X^H, \partial X^H) \to (Y^H, \partial Y^H)] = 1$ for any $H \in \mathcal{S}(G)$ with $\dim V^H > 0$.

Lemma 9.2. *For the M-framed cobordism $\boldsymbol{F}_M$, where $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, in Lemma 9.1, we can adjust it so as to satisfy the following conditions.*

(C6) *X^M is diffeomorphic to Y^M and ${W_M}^M$ is a product cobordism, i.e. diffeomorphic to $I \times Y^M$, and furthermore*

$$ {F_M}^M : ({W_M}^M, \partial_0 {W_M}^M, \partial_1 {W_M}^M, \partial_{01} {W_M}^M) \to (Z^M, \partial_0 Z^M, \partial_1 Z^M, \partial_{01} Z^M), $$

where $Z = I \times Y$, is homotopic rel. $\partial_1 {W_M}^M \cup \partial_{01} {W_M}^M$ to a diffeomorphism. Therefore $f^M : X^M \to Y^M$ is homotopic rel. ∂ to a diffeomorphism.

(C7) *If $H \in \mathcal{S}(M)$ and $\dim V^H = 0$ then ${W_M}^H = {W_M}^M$ (and ${W_M}^H$ is diffeomorphic to the closed interval $[0, 1]$).*

Proof. The properties in (C6) is readily achieved by the reflection method.

To show (C7), let $H \in \mathcal{S}(M)$ with $\dim V^H = 0$. If $X^{=H} \neq \emptyset$ then we get $H = M$ by (C2) and (C3). In the case $H = M$, we get $X^H = \{x_M\}$ and $\dim {W_M}^H = 1$ and it holds that one of the connected components of $\dim {W_M}^H$ is diffeomorphic to $[0, 1]$ and the others are diffeomorphic to the circle. It is easy to convert W_M by M-surgeries of isotropy type $(H)_M$ $(H = M)$ so that ${W_M}^H$ is diffeomorphic to $[0, 1]$. Therefore it suffices to consider the case $H < M$. As an inductive assumption, suppose that ${W_M}^K = {W_M}^M$ for all $K \in \mathcal{U}_M(H)$. Then each connected component of ${W_M}^H \smallsetminus {W_M}^{>H}$ $({W_M}^{>H} = {W_M}^M)$ is diffeomorphic to the circle. We can readily remove those undesired connected components of ${W_M}^H$ by M-surgeries of isotropy type $(H)_M$ to obtain the property ${W_M}^H = {W_M}^M$. □

In the following sections, we assume that $\boldsymbol{f}$ and $\boldsymbol{F}_M$ are adjusted by Lemma 9.2.

Proposition 9.3. *Let H be a solvable subgroup of G. Suppose the G-framed map $\boldsymbol{f} = (f, b)$ above satisfies the modified weak gap condition at H and the condition that*

($\mathcal{G}_1$) *X^K is $\mathbb{Z}$-acyclic for all $K \in \mathcal{U}_G(H)$ such that $H \triangleleft K$ and K/H is a hyper-elementary group.*

Set $n_H = \dim V^H$ and let k_H be the integer satisfying $n_H = 2k_H$ or $2k_H + 1$. Suppose $n_H \geq 5$ (resp. $n_H = 3$) and X^H is $(k_H - 1)$-connected. Then $([G/G] - \beta_G)\boldsymbol{f}$ is G-framed cobordant rel. ∂ to $\boldsymbol{f}' = (f', b')$, where $f' : (X', \partial X') \to (Y, \partial Y)$ and $b' : \tau_{X'} \to f'^* \tau_Y$, by G-surgeries of isotropy type $(H)_G$, such that X'^H is contractible (resp. $\mathbb{Z}$-acyclic).

Here we remark that the equalities $X^L = \emptyset = X'^L$ and $\dim X^H = \dim Y^H = \dim X'^H$ hold for $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$ and $H \in \mathcal{S}(G)_{\mathrm{sol}}$, respectively.

Proof. Note that X^H is 1-connected and $f^H : X^H \to Y^H$ is k_H-connected (resp. X^H is connected and $f^H_\# : \pi_1(X^H) \to \pi_1(Y^H)$ is surjective). Let $L(f^H;\mathbb{Z})$ be the surgery kernel. By the condition $(\mathcal{G}_1)$ above, $L(f^H;\mathbb{Z})$ is stably free over $\mathbb{Z}[\overline{H}]$, where $\overline{H} = N_G(H)/H$. By Lemma 8.1, we obtain the obstruction $\sigma_{G,H}(\boldsymbol{f};\mathbb{Z})$ in $\mathcal{L}_{V,H}(\mathbb{Z}[\overline{H}])$ to convert $\boldsymbol{f}$ so that $f^H : X^H \to Y^H$ would be a homotopy equivalence (resp. a $\mathbb{Z}$-homology equivalence) by G-surgeries rel. ∂ of isotropy type $(H)_G$. Note the property

$$\sigma_{G,H}(([G/G] - \beta_G)\,\boldsymbol{f};\mathbb{Z}) = ([\overline{H}/\overline{H}] - {\beta_G}^H)\,\sigma_{G,H}(\boldsymbol{f};\mathbb{Z}),$$

where ${\beta_G}^H$ is the element $[X_1^H] - [X_2^H] \in \Omega(\overline{H})$ if $\beta = [X_1] - [X_2]$ for finite G-sets X_1 and X_2. Recall the induction theory of equivariant-surgery-obstruction groups, see [10, 11], [2], [14, Corollary 1.4], and [25, Theorems 1.1 and 13.5]. If $H \trianglelefteq K \in \mathcal{S}(G)$ and K/H is solvable, then K is solvable and

$$\mathrm{res}^{\overline{H}}_{K/H}([G/G] - \beta_G)^H = (\mathrm{res}^G_K([G/G] - \beta_G))^H = 0 \ \text{ in } \Omega(K/H).$$

It follows that

$$\mathrm{res}^{\overline{H}}_{K/H}(([\overline{H}/\overline{H}] - {\beta_G}^H)\,\sigma_{G,H}(\boldsymbol{f};\mathbb{Z})) = 0$$

for all $K/H \in \mathcal{S}(\overline{H})_{\mathrm{sol}}$ and

$$([\overline{H}/\overline{H}] - {\beta_G}^H)\,\sigma_{G,H}(\boldsymbol{f};\mathbb{Z}) = 0.$$

Therefore, $([G/G] - \beta_G)\,\boldsymbol{f}$ is G-framed cobordant rel. ∂ to $\boldsymbol{f}'$ stated in the proposition by G-surgeries of isotropy type $(H)_G$. □

10. Simply organized families and G-surgeries

Let G be a nonsolvable group and V an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation. Set

$$\text{(10.1)} \qquad \mathcal{H}(G,V,0) = \{H \in \mathcal{S}(G)_{\mathrm{sol}} \mid \dim V^H = 0\}.$$

Let $\boldsymbol{f} = (f,b)$ and $\boldsymbol{F}_M = (F_M, B_M)$ be the G-framed map and the M-framed cobordisms, where $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, obtained in Lemma 9.1. Let $Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$, and $\partial_{01} Z = I \times \partial Y$. We suppose that $\boldsymbol{f}$ and $\boldsymbol{F}_M$ are adjusted by Lemma 9.2. In this situation, for every $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, ${W_M}^M$ is diffeomorphic to $I \times Y^M$, X^M is diffeomorphic to Y^M, $f^M : X^M \to Y^M$ is homotopic rel. ∂ to a diffeomorphism, ${W_M}^M$ is diffeomorphic to $I \times Y^M$, and ${F_M}^M : {W_M}^M \to I \times Y^M$ is homotopic rel.

$\partial_1 {W_M}^M \cup \partial_{01} {W_M}^M$ to a diffeomorphism. In addition, we have $X^{=H} = \emptyset$ and $W_M^{=H} = \emptyset$ for all $H \in \mathcal{H}(G, V, 0) \smallsetminus \max(\mathcal{S}(G)_{\mathrm{sol}})$ and $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$ such that $H \subset M$.

For a subset $\mathcal{H}$ of $\mathcal{S}(G)$, let $X(\mathcal{H})$ denote the union of X^H, where H ranges over $\mathcal{H}$. Let $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$ and set $\mathcal{H}_M = \{M\} \cup \mathcal{H}(G, V, 0)$. Let $N_M(X(\mathcal{H}_M), X)$ be an M-regular neighborhood of $X(\mathcal{H}_M)$ in X. In this section we set $X^{(0)} = X$, $\boldsymbol{f}^{(0)} = \boldsymbol{f}$, $W_M^{(0)} = W_M$, $\boldsymbol{F}_M^{(0)} = \boldsymbol{F}_M$, for $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, and $\boldsymbol{F}_G^{(0)} = I \times \boldsymbol{f}$. It is easy to obtain a product M-embedding $\Phi_M^{(0)} : I \times N_M(X(\mathcal{H}_M), X) \to W_M$ and an M-homotopy

$$\mathbb{H}_M^{(0)} : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M^{(0)}|_{W_M \times \{0\}} = F_M$ and $\mathbb{H}_M^{(0)}|_{\mathrm{Im}(\Phi_M^{(0)}) \times \{1\}}$ is a diffeomorphism to its image.

Now let $\mathcal{F}$ be a G-conjugation-invariant and upper-closed subset of $\mathcal{S}(G)_{\mathrm{sol}}$ and suppose $\mathcal{F}$ is G-simply organized with respect to $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$. By Definition 7.1, the equality

$$X(\mathcal{U}_G(H)) = X(\mathcal{U}_M(H)) \cup X(\mathcal{Y}(G, M, H)) \tag{10.2}$$

holds for $H \in \mathcal{F}^*$ and $M = \rho_{\max}(H)$, where $\mathcal{Y}(G, M, H)$ is the set of subgroups $K \in \mathcal{U}_G(H)$ such that $K \cap M = H$. Here we note that $X(\mathcal{U}_G(H)) = X^{>H}$ and $X(\mathcal{U}_M(H)) = (\mathrm{res}_M^G X)^{>H}$.

Lemma 10.1. *Suppose $\mathcal{F}$ contains $\mathcal{F}^{(0)} = \max(\mathcal{S}(G)_{\mathrm{sol}}) \cup \mathcal{H}(G, V, 0)$. In addition suppose the next condition is fulfilled.*

(D1) $\dim V^K = 0$ *for all* $H \in (\mathcal{F}^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathcal{F}^{(0)}$ *and* $K \in \mathcal{Y}(G, \rho_{\max}(H), H)$.

Then there are a G-framed map $\boldsymbol{f}'$ rel. ∂, a G-framed cobordism $\boldsymbol{F}_G$ from $\boldsymbol{f}$ to $\boldsymbol{f}'$ rel. ∂ and $\mathcal{S}(M)_{\mathrm{nonsol}}$, and an M-framed cobordism $\boldsymbol{F}'_M$ from $\mathrm{res}_M^G \boldsymbol{f}'$ to $\mathrm{res}_M^G \boldsymbol{id}_Y$ rel. ∂ for each $M \in \max(\mathcal{F})^*$ having the following properties.

(1) ${X'}^H$ is diffeomorphic to Y^H and ${f'}^H : {X'}^H \to Y^H$ is $N_G(H)$-homotopic rel. ∂ to a diffeomorphism for all $H \in \mathcal{F}$.

(2) For each $M \in \max(\mathcal{F})^*$, there is an M-homotopy

$$\mathbb{H}'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W'_M \cup \partial_{01} W'_M$ such that $\mathbb{H}'_M|_{W'_M \times \{0\}}$ coincides with F'_M and $\mathbb{H}'_M|_{{W'_M}^H \times \{1\}}$ is a diffeomorphism to its image for every $H \in \mathcal{F}^*$ with $\rho_{\max}(H) = M$.

Proof. We can write $\mathcal{F}$ in the form

$$\mathcal{F} = \mathcal{F}^{(0)} \amalg (H_1)_G \amalg (H_2)_G \amalg \cdots \amalg (H_m)_G,$$

where $H_i \in \mathcal{F}^*$ for $1 \le i \le m$, satisfying the condition that if $|H_i| > |H_j|$ then $i < j$. Set $M_i = \rho_{\max}(H_i)$. Let H_i be one of the subgroups above such that $(H_i)_G$ is a maximal conjugacy class in $\mathcal{F} \smallsetminus \mathcal{F}^{(0)}$. For $H = H_i$ and $M = \rho_{\max}(H)$, since $X^{>H} \subset X(\mathcal{F}^{(0)})$, we will adopt a restriction of $\Phi^{(0)}_M$ as a product M-embedding $\Psi^{(i)}_i : I \times N_M(M \cdot X^{>H}, X) \to W_M$.

For $k = 1, \ldots, m$, we inductively define $\mathcal{F}^{(k)}$ by $\mathcal{F}^{(k)} = \mathcal{F}^{(k-1)} \amalg (H_k)_G$. We prove the lemma by induction on $k = 1, \ldots, m$. Recall that for integers i and j, we mean by $[i..j]$ the set of integers t such that $i \le t \le j$. Suppose that (for fixed k) we have obtained inductively,

- G-framed maps $\boldsymbol{f}^{(i)}$ rel. ∂, where $f^{(i)} : (X^{(i)}, \partial X^{(i)}) \to (Y, \partial Y)$,
- G-framed cobordisms $\boldsymbol{F}^{(i)}_G$ rel. ∂ and $\mathcal{V}_G(H_i)$, from $\boldsymbol{f}^{(i-1)}$ to $\boldsymbol{f}^{(i)}$, where
  $$F^{(i)}_G : (W^{(i)}_G, \partial_0 W^{(i)}_G, \partial_1 W^{(i)}_G, \partial_{01} W^{(i)}_G) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$
- M-framed cobordisms $\boldsymbol{F}^{(i)}_M$ rel. ∂ from $\mathrm{res}^G_M \boldsymbol{f}^{(i)}$ to $\mathrm{res}^G_M \boldsymbol{id}_Y$, where
  $$F^{(i)}_M : (W^{(i)}_M, \partial_0 W^{(i)}_M, \partial_1 W^{(i)}_M, \partial_{01} W^{(i)}_M) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$
  such that $F^{(i)}_M$ is obtained by M-surgeries rel. $\partial_1 W^{(i-1)}_M \cup \partial_{01} W^{(i-1)}_M$ on $F^{(i-1)}_M$ of isotropy types $(K)_M$, where K runs over $\{L \cap M \mid L \in (H_i)_G\}$,

for $i \in [0..(k-1)]$ and $M \in \max(\mathcal{F})^*$,

- product M_j-embeddings $\Psi^{(i)}_j : I \times N_{M_j}(M_j \cdot (\mathrm{res}^G_{M_j} X^{(i-1)})^{>H_j}, X^{(i-1)}) \to W^{(i-1)}_{M_j}$ such that $\Psi^{(i)}_j = \Psi^{(i-1)}_j$ whenever $j \le i-1$,
- product M_j-embeddings $\Phi^{(i)}_j : I \times N_{M_j}(M_j \cdot (X^{(i)})^{H_j}, X^{(i)}) \to W^{(i)}_{M_j}$ such that $\Phi^{(i)}_j = \Phi^{(i-1)}_j$ whenever $j \le i-1$ and that ${\Psi^{(i)}_j}^{>H_j} = \bigcup_L {\Phi^{(i)}_j}^L$, where L runs over $\mathcal{U}_{M_j}(H_j)$, and
- M_j-homotopies
  $$\mathbb{H}^{(i)}_j : (W^{(i)}_{M_j}, \partial_0 W^{(i)}_{M_j}, \partial_1 W^{(i)}_{M_j}, \partial_{01} W^{(i)}_{M_j}) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$
  rel. $\partial_1 W^{(i)}_{M_j} \cup \partial_{01} W^{(i)}_{M_j}$ such that $\mathbb{H}^{(i)}_{M_j}|_{W^{(i)}_{M_j} \times \{0\}}$ coincides with $F^{(i)}_{M_j}$ and $\mathbb{H}^{(i)}_j|_{\mathrm{Im}(\Phi^{(i)}_j) \times \{1\}}$ is a diffeomorphism to its image,

for $i \in [1..(k-1)]$ and $j \in [1..i]$.

Note that $\dim V^{H_k} > 0$.

Case 1: $H_k \notin \mathrm{Iso}(G, V)$. By (10.2), there is a subgroup $K \in \mathcal{U}_{M_k}(H_k)$ such that $\dim V^K = \dim V^{H_k} > 0$. It follows that $X^{>H_k} \subset X^K$ and $X^{H_k} =$

$X^K \amalg X^{=H_k}$. By Lemma 6.2, we can obtain $\boldsymbol{f}^{(k)}$, $\boldsymbol{F}^{(k)}_{M_k}$, $\Phi^{(k)}_k$, and $\mathbb{H}^{(k)}_k$. Let $M \in \max(\mathcal{F})^* \smallsetminus \{M_k\}$ and set $\boldsymbol{F}^{(k)'}_M = \boldsymbol{F}_G \cup_{\boldsymbol{f}^{(k-1)}} \boldsymbol{F}^{(k-1)}_M$. Note that $W_G{}^{H_j}$ is a product cobordism for each $j \in [1..(k-1)]$. Therefore, for $j \in [1..(k-1)]$, by deforming $\boldsymbol{F}^{(k)'}_{M_j}$, we can obtain desired $\boldsymbol{F}^{(k)}_{M_j}$, $\Psi^{(k)}_j$, $\Phi^{(k)}_j$, and $\mathbb{H}^{(k)}_j$, where $W^{(k)}_{M_j}$ is M_j-homeomorphic to $W^{(k)}_G \cup_{X^{(k-1)}} W^{(k-1)}_{M_j}$. For $t \in [(k+1)..m]$, we adopt $\boldsymbol{F}^{(k)'}_{M_t}$ as $\boldsymbol{F}^{(k)}_{M_t}$.

Case 2: $H_k \in \mathrm{Iso}(G,V)$. In this case we have $\dim V^K < \dim V^{H_k}$ for all $K \in \mathcal{U}_G(H_k)$. By performing G-surgeries of isotropy type $(H_k)_G$ on $\boldsymbol{f}^{(k-1)}$ (resp. M_k-surgeries of isotropy type $(H_k)_{M_k}$ on $\boldsymbol{F}^{(k-1)}_{M_k}$), we can assume without any loss of generality that $X^{(k-1)H_k}$ (resp. $W^{(k-1)}_{M_k}$) is connected. We can obtain an M_k-product embedding $\Psi^{(k)}_k : I \times N_{M_k}(M_k \cdot (\mathrm{res}^G_{M_k} X^{(k-1)})^{>H_k}, \mathrm{res}^G_{M_k} X^{(k-1)}) \to W^{(i-1)}_{M_k}$ from $\Phi^{(0)}_{M_k}$ and $\Phi^{(j)}_k$, where j runs over the set

$$J_k = \{j \in [1..(k-1)] \mid \rho_{\max}(H_j) = M_k\}.$$

Recall the condition that $\dim V^K = 0$ for $K \in \mathcal{Y}(G, M_k, H_k)$ is fulfilled. By Lemma 6.4, we can obtain $\boldsymbol{f}^{(k)}$, $\boldsymbol{F}^{(k)}_{M_k}$, $\Phi^{(k)}_k$, and $\mathbb{H}^{(k)}_k$. Moreover we can obtain $\boldsymbol{F}^{(k)}_M$ for $M \in \max(\mathcal{F})^* \smallsetminus \{M_k\}$, and $\Psi^{(k)}_j$, $\Phi^{(k)}_j$, and $\mathbb{H}^{(k)}_j$ for $j \in [1..(k-1)]$ quite similarly to Case 1.

Putting Cases 1 and 2 together, we set $\boldsymbol{f}' = \boldsymbol{f}^{(m)}$,

$$\boldsymbol{F}_G = \boldsymbol{F}^{(1)}_G \cup_{\boldsymbol{f}^{(1)}} \boldsymbol{F}^{(2)}_G \cup_{\boldsymbol{f}^{(2)}} \cdots \cup_{\boldsymbol{f}^{(m-1)}} \boldsymbol{F}^{(m)}_G,$$

and $\boldsymbol{F}'_M = \boldsymbol{F}^{(m)}_M$ and $\mathbb{H}'_M = \mathbb{H}^{(m)}_j$, where $M = M_j$. Then the conclusion of Lemma 10.1 follows. □

11. Construction theorems of one-fixed-point actions on spheres

In the present section, let G be a nonsolvable group, let $\mathcal{F}$ and $\mathcal{H}$ be G-conjugation-invariant and upper-closed subsets of $\mathcal{S}(G)_{\mathrm{sol}}$ such that $\mathcal{F}$ is G-simply organized with respect to $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$, and

$$\max(\mathcal{S}(G)_{\mathrm{sol}}) \cup \mathcal{H}(G,V,0) \subset \mathcal{F} \subset \mathcal{H}, \tag{11.1}$$

let β_G be the element of $\Omega(G)$ defined in (3.1), and let V be an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation. Suppose V is ample for β_G and satisfy the condition (D1) in Lemma 10.1. Let $\boldsymbol{f}$ and $\boldsymbol{F}_M$ be a G-framed map and M-framed cobordisms, where $M \in \max(\mathcal{S}(G)_{\rm sol})^*$, obtained in Lemma 9.1. In this section we suppose that $\boldsymbol{f}$ and $\boldsymbol{F}_M$ are adjusted by Lemmas 9.2 and 10.1.

Theorem 11.1. *Further suppose V satisfies*

(D2) $\dim V^H = 3$ *or* $\dim V^H \geq 5$ *for* $H \in \mathcal{H} \smallsetminus \mathcal{F}$, *and*

(D3) *the modified weak gap condition at* H, *for* $H \in \mathcal{H} \smallsetminus \mathcal{F}$.

Then there exists a G-framed map $\boldsymbol{f}' = (f', b')$, where $f' : (X', \partial X') \to (Y, \partial Y)$, satisfying the following conditions.

(1) $\boldsymbol{f}'$ *is G-framed cobordant rel. ∂ and $\mathcal{S}(G)_{\rm nonsol}$ to $\boldsymbol{f}_m$, where $\boldsymbol{f}_i = ([G/G] - \beta_G)\boldsymbol{f}_{i-1}$ ($i \in [1..m]$) and $\boldsymbol{f}_0 = \boldsymbol{f}$, for some $m \in \mathbb{N}$. Therefore ${X'}^G$ is the empty set.*

(2) ${f'}^H : {X'}^H \to Y^H$ *is $N_G(H)$-homotopic rel. ∂ to a diffeomorphism for $H \in \mathcal{F}$.*

(3) ${f'}^H : {X'}^H \to Y^H$ *is a homotopy equivalence rel. ∂ for $H \in \mathcal{H}$ with* $\dim V^H \neq 3$.

(4) ${f'}^H : {X'}^H \to Y^H$ *is a $\mathbb{Z}$-homology equivalence rel. ∂ for $H \in \mathcal{H}$ with* $\dim V^H = 3$.

Proof. Inductively applying Proposition 9.3 to $H \in \mathcal{H} \smallsetminus \mathcal{F}$, we obtain the theorem. □

Theorem 11.2. *In the situation of Theorem 11.1, suppose $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$ and* $\dim V > 5$. *Then there exists a one-fixed-point G-action on the standard sphere S such that $T_{x_0}(S) \cong V$ as real G-representations, where x_0 is the G-fixed point of S.*

Proof. Let X' be the G-manifold obtained in Theorem 11.1 and set $\Sigma = D(V) \cup_\partial X'$. It is clear that Σ is a homotopy sphere with exactly one G-fixed point, say x_0, and $T_{x_0}(\Sigma) \cong_G V$. Let S be the G-connected sum $([G/G] - \beta_G)\Sigma$ with respect to the expression (3.2) of β_G. Then S is the standard sphere with exactly one G-fixed point, cf. [16, Proposition 1.3]. □

Let $\widetilde{G}$ be an extension of G by a finite solvable group N, i.e. we have the exact sequence

$$E \longrightarrow N \longrightarrow \widetilde{G} \xrightarrow{\pi} G \longrightarrow E.$$

A subgroup $\widetilde{H}$ of $\widetilde{G}$ is solvable if and only if $\pi(\widetilde{H})$ is solvable. It follows that

$$\beta_{\widetilde{G}} = \pi^* \beta_G \ \text{ and } \ \mathcal{S}(\widetilde{G})_{\rm sol} = \pi^{-1}(\mathcal{S}(G)_{\rm sol}).$$

Let $\widetilde{U}$ be a free real $\widetilde{G}$-representation and set

$$\widetilde{V} = \widetilde{U} \oplus \pi^* V.$$

Let $\widetilde{Y}$ be the unit disk of $\widetilde{V}$. There are a $\widetilde{G}$-framed map $\widetilde{\boldsymbol{f}} = (\widetilde{f}, \widetilde{b})$ rel. ∂, where $\widetilde{f} : (\widetilde{X}, \partial\widetilde{X}) \to (\widetilde{Y}, \partial\widetilde{Y})$, $\widetilde{b} : \tau_{\widetilde{X}} \to \widetilde{f}^*\tau_{\widetilde{Y}}$, $\tau_{\widetilde{X}} = \varepsilon_{\widetilde{X}}(\mathbb{R}) \oplus T(\widetilde{X}) \oplus \varepsilon_{\widetilde{X}}(\mathbb{R}^\ell)$, and $\tau_{\widetilde{Y}} = \varepsilon_{\widetilde{Y}}(\mathbb{R}) \oplus T(\widetilde{Y}) \oplus \varepsilon_{\widetilde{Y}}(\mathbb{R}^\ell)$, and $\widetilde{M}$-framed cobordisms $\widetilde{\boldsymbol{F}}_{\widetilde{M}} = (\widetilde{F}_{\widetilde{M}}, \widetilde{B}_{\widetilde{M}})$, where $M \in \max(\mathcal{S}(G)_{\rm sol})$, $\widetilde{M} = \pi^{-1}(M)$, $\widetilde{F}_{\widetilde{M}} : \widetilde{W}_{\widetilde{M}} \to I \times \widetilde{Y}$, and

$$\widetilde{B}_{\widetilde{M}} : T(\widetilde{W}_{\widetilde{M}}) \oplus \varepsilon_{\widetilde{W}_{\widetilde{M}}}(\mathbb{R}^\ell) \to \widetilde{F}^*_{\widetilde{M}}\Big(T(I \times \widetilde{Y})\Big) \oplus \varepsilon_{\widetilde{W}_{\widetilde{M}}}(\mathbb{R}^\ell)$$

such that

$$\widetilde{\boldsymbol{f}}^{\,N} = \boldsymbol{f} \quad \text{and} \quad {\widetilde{\boldsymbol{F}}_{\widetilde{M}}}^{\,N} = \boldsymbol{F}_M.$$

Theorem 11.3. *In the situation of Theorem 11.1, suppose $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$. Let $\widetilde{G}$ and $\widetilde{U}$ be as above. Suppose the condition that*

(D4) $\dim \widetilde{U} > \dim V$ *and* $\dim \widetilde{U} + \dim V > 5$

is fulfilled. Then there exists a one-fixed-point $\widetilde{G}$-action on the standard sphere $\widetilde{S}$ such that $T_{x_0}(\widetilde{S}) \cong \widetilde{U} \oplus \pi^* V$ as real $\widetilde{G}$-representations, where x_0 is the $\widetilde{G}$-fixed point of $\widetilde{S}$.

Proof. Let $\boldsymbol{f}'$ be the G-framed map rel. ∂ stated in Theorem 11.1. There is a $\widetilde{G}$-framed map $\widetilde{\boldsymbol{f}}' = (\widetilde{f}', \widetilde{b}')$ rel. ∂, where $\widetilde{f}' : (\widetilde{X}', \partial\widetilde{X}') \to (\widetilde{Y}, \partial\widetilde{Y})$, such that $\widetilde{\boldsymbol{f}}'^{\,N} = \boldsymbol{f}'$. Then $\widetilde{f}'^K$ is a $\mathbb{Z}$-homology equivalence for every $K \in \mathcal{S}(\widetilde{G})_{\rm sol} \smallsetminus \{E\}$. By the condition (D4), $\widetilde{X}'$ satisfies the gap condition at E, because

$$2\dim \widetilde{X}'^H = 2\dim \widetilde{U}^H + 2\dim V^{\pi(H)} = 2\dim V^{\pi(H)}$$
$$\le 2\dim V < \dim \widetilde{U} + \dim V = \dim \widetilde{X}'$$

for $H \in \mathcal{S}(\widetilde{G}) \smallsetminus \{E\}$ such that $\widetilde{X}'^H \neq \emptyset$. Without any loss of generality, we can suppose $\widetilde{f}'$ is connected up to the middle dimension. We have the $\widetilde{G}$-surgery obstruction $\sigma_{\widetilde{G},E}(\widetilde{\boldsymbol{f}}')$ of isotropy type $(E)_{\widetilde{G}}$ in $\mathcal{L}_{\widetilde{V},E}(\mathbb{Z}[\widetilde{G}])$. Recall Proposition 9.3. Performing $\widetilde{G}$-surgeries rel. ∂ of isotropy type $(E)_{\widetilde{G}}$ on $([\widetilde{G}/\widetilde{G}] - \beta_{\widetilde{G}})\widetilde{\boldsymbol{f}}'$, we can obtain a $\widetilde{G}$-framed map $\widetilde{\boldsymbol{f}}'' = (\widetilde{f}'', \widetilde{b}'')$, where $\widetilde{f}'' : (\widetilde{X}'', \partial\widetilde{X}'') \to (\widetilde{Y}, \partial\widetilde{Y})$, such that $\widetilde{X}''^L = \emptyset$ for all $L \in \mathcal{S}(\widetilde{G})_{\rm nonsol}$ and $\widetilde{f}''$ is a homotopy equivalence. Then $\widetilde{\Sigma} = D(\widetilde{V}) \cup_\partial \widetilde{X}''$ is a homotopy sphere with exactly one $\widetilde{G}$-fixed point, say x_0. We have $T_{x_0}(\widetilde{\Sigma}) \cong \widetilde{V}$ as real $\widetilde{G}$-representations. Let $\widetilde{S}$ be the $\widetilde{G}$-connected sum $([\widetilde{G}/\widetilde{G}] - \beta_{\widetilde{G}})\widetilde{\Sigma}$ with respect to the expression of $\beta_{\widetilde{G}}$ induced from the expression (3.2) of β_G. Then $\widetilde{S}$ is the standard sphere with exactly one $\widetilde{G}$-fixed point, cf. [16, Proposition 1.3]. □

12. Proof of Theorem 1.3

In this section we prove Theorem 1.3 on a case-by-case basis. Before the proof, we recall that the condition (D1) in Lemma 10.1 (concerning the primitive gap condition for $(G, \rho_{max}(H), H)$) will be requested for $H \in \mathcal{F} \smallsetminus (\max(\mathcal{S}(G)_{\rm sol}) \cup \mathcal{H}(G, V, 0))$, and that the conditions (D2) and (D3) in Theorem 11.1 (concerning the modified weak gap condition at H) will be requested for $H \in \mathcal{S}(G)_{\rm sol} \smallsetminus \mathcal{F}$. We will give Diagrams 12.1–12.5 to help readers follow the arguments. In the diagrams, we adopt the following conventions.

(1) For a subgroup H, $H^{(m)}$ indicates $\dim V^H = m$.
(2) For subgroups H and K of G, an arrow (resp. a dotted arrow) from $H^{(m_1)}$ to $K^{(m_2)}$ indicates $\rho_{\max}(H) = K$ and $H \triangleleft K$ (resp. $\rho_{\max}(H) = K$ and $H \ntriangleleft K$).

Proof in Case $n = 6$ (i). Here $G = A_5$ and V has the form $V = V_3 \oplus V_3'$ for irreducible real G-representations V_3 and V_3' of dimension 3. The element β_G has the form (3.3). The fixed-point-set dimensions of V for A_5 are as in Diagram 12.1.

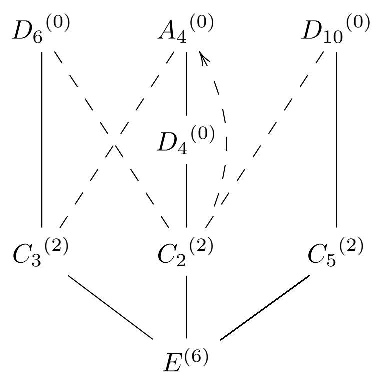


Diagram 12.1

By Proposition 3.3 (1), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus \{E\}$ and $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$. By Proposition 7.1, $\mathcal{F}$ is G-simply organized. By Proposition 7.2, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G-action on S^6. $\square$

Proof in Case $n = 6$ (ii). Here $G = S_5$ and V is an irreducible real G-representation of dimension 6. The element β_G has the form (3.5). The fixed-point-set dimensions of V for S_5 are as in Diagram 12.2.

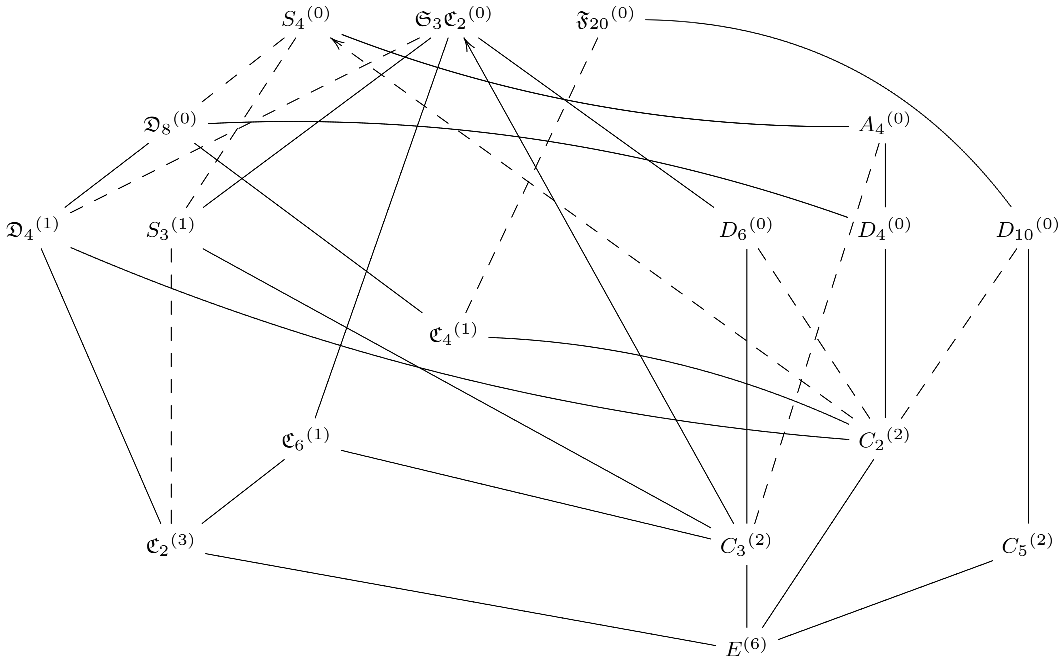


Diagram 12.2

By Proposition 3.3 (2), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$. By Proposition 7.4, $\mathcal{F}$ is G-simply organized. By Proposition 7.5, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G-action on S^6. □

Proof in Case $n = 6$ (iii). Here $G = A_5 \times Z$, where $|Z| = 2$, and V has the form $V = V^Z \oplus V_Z$ such that V^Z and V_Z are irreducible real G-representations of dimension 3. The element β_G has the form $\beta_G = \pi^*\beta_L$, where $L = A_5$ and $\pi : G \to L$ is an epimorphism. The fixed-point-set dimensions of V for $A_5 \times Z$ are as in Diagram 12.3.

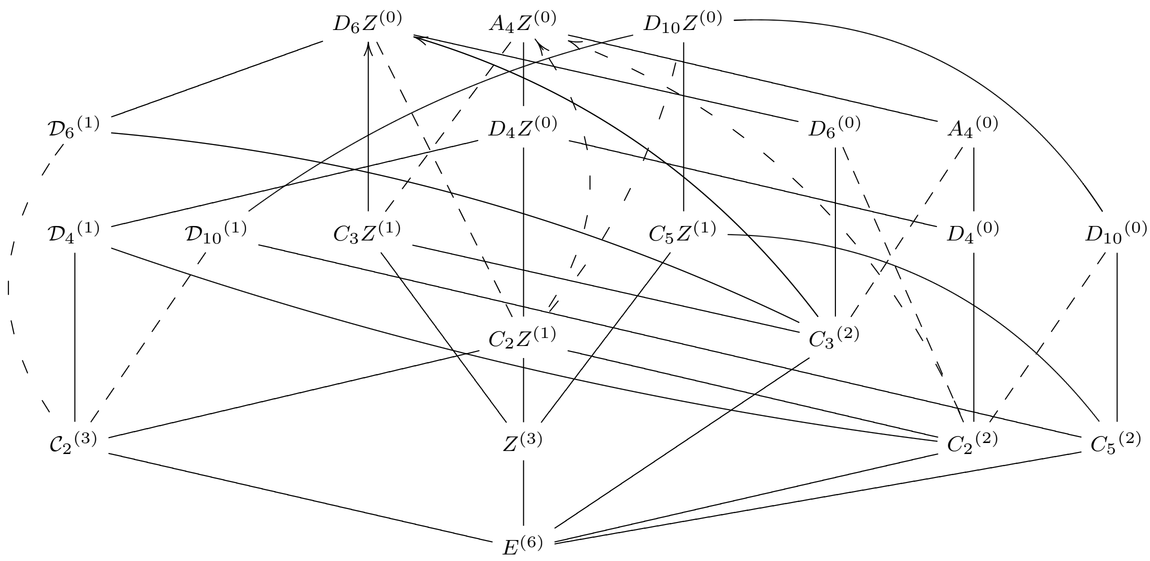


Diagram 12.3

By Proposition 3.3 (3), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E, Z\} \cup (\mathcal{C}_2)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\mathrm{sol}}$. By Proposition 7.6, $\mathcal{F}$ is G-simply organized. By Proposition 7.7, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G-action on S^6. □

Proof in Case $n = 7$ (iv). Here $G = A_5$ and V has the form $V = V_3 \oplus V_4$, where V_3 and V_4 are irreducible real G-representations of dimension 3 and 4, respectively. The fixed-point-set dimensions of V for A_5 are as in Diagram 12.4.

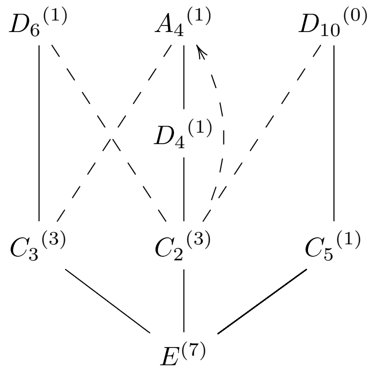


Diagram 12.4

By Proposition 3.3 (1), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (C_2)_G \cup (C_3)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\mathrm{sol}}$. By Proposition 7.1, $\mathcal{F}$ is G-simply organized. By Proposition 7.3, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G-action on S^7. □

Proof in Case $n = 7$ (v). Here $G = A_5 \times Z$, where $|Z| = 2$, and V has the form $V = V^Z \oplus V_Z$ such that V^Z and V_Z are irreducible real G-representations of dimension 3 and 4, respectively. The fixed-point-set dimensions of V for $A_5 \times Z$ are as in Diagram 12.5.

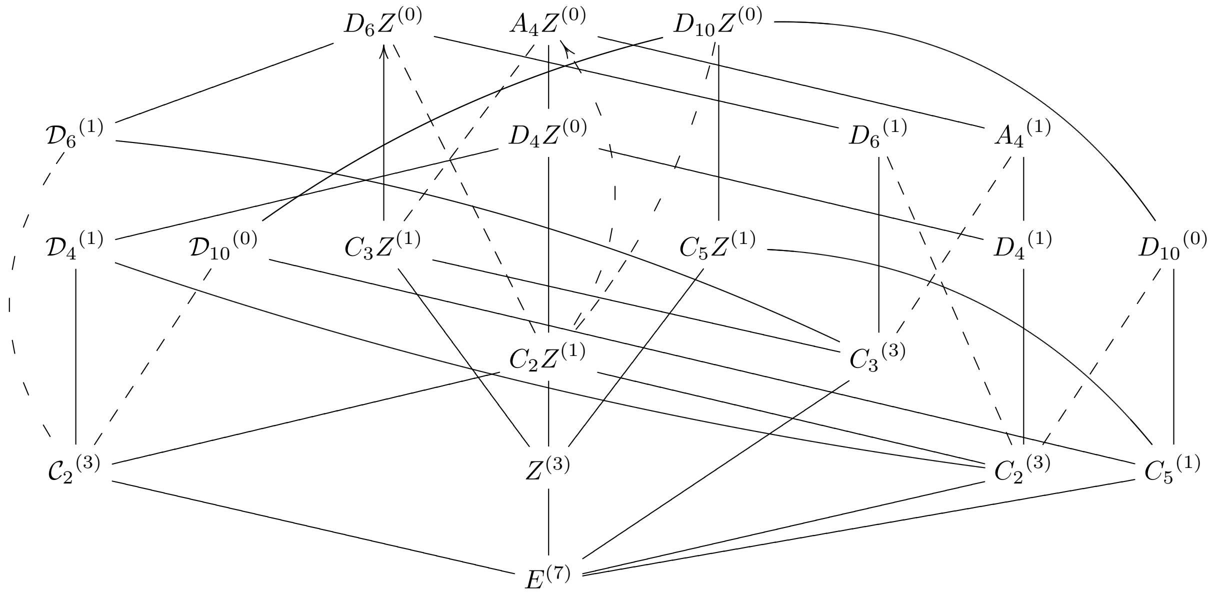


Diagram 12.5

By Proposition 3.3 (3), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E, Z\} \cup (C_2)_G \cup (\mathcal{C}_2)_G \cup (C_3)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\mathrm{sol}}$. By Proposition 7.6, $\mathcal{F}$ is G-simply organized. By Proposition 7.8, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G-action on S^7. □

Proof in Case $n = 3 + 4k$ (vi). Changing notation, let $\widetilde{G} = \mathrm{SL}(2,5) \times Z_m$ and $G = A_5$. Let $\pi : \widetilde{G} \to G$ be an epimorphism. Changing notation, let V be an irreducible real G-representation of dimension 3, let $\widetilde{U}$ be a free real $\widetilde{G}$-representation of dimension $4k$, and set $\widetilde{V} = \widetilde{U} \oplus \pi^* V$. The kernel N of π is $Z \times Z_m$, where $Z = \mathrm{Center}(\mathrm{SL}(2,5))$. The element $\beta_{\widetilde{G}}$ has the form $\beta_{\widetilde{G}} = \pi^* \beta_G$.

By Proposition 3.3 (1), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus \{E\}$ and $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$. By Proposition 7.1, $\mathcal{F}$ is G-simply organized. By Proposition 7.2, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.3 gives a desired one-fixed-point $\widetilde{G}$-action on S^{3+4k}. □

Proof in Case $n = 6 + 8k$ (vi). Changing notation, let $\widetilde{G} = \mathrm{TL}(2,5) \times Z_m$ and $G = S_5$. Let $\pi : \widetilde{G} \to G$ be an epimorphism. Changing notation, let V be an irreducible real G-representation of dimension 6, let $\widetilde{U}$ be a free real $\widetilde{G}$-representation of dimension $8k$, and set $\widetilde{V} = \widetilde{U} \oplus \pi^*V$. The kernel N of π is $Z \times Z_m$, where $Z = \mathrm{Center}(\mathrm{TL}(2,5))$. The element $\beta_{\widetilde{G}}$ has the form $\beta_{\widetilde{G}} = \pi^*\beta_G$. By Proposition 3.3 (2), V is ample for β_G. Let $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\rm sol}$. By Proposition 7.4, $\mathcal{F}$ is G-simply organized. By Proposition 7.5, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.3 gives a desired one-fixed-point $\widetilde{G}$-action on S^{6+8k}. □

We remark that the real G-representation V in Theorem 1.3 is faithful and therefore the G-action on V is effective. Since $T_{x_0}(S) \cong V$, the G-action on S obtained in Theorem 1.3 is effective.

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