STRICHARTZ ESTIMATES FOR MAGNETIC SCHRÖDINGER, WAVE AND KLEIN-GORDON EQUATIONS IN EXTERIOR DOMAIN AND APPLICATION TO SCATTERING THEORY

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ABSTRACT. Our purpose of this paper is to derive Strichartz estimates for solutions of magnetic Schrödinger, wave and Klein-Gordon equations in exterior to the star-shaped obstacle. For its proof we need the smoothing estimates for solutions of perturbed equations and the Strichartz estimates for solutions of free equations. Moreover as an application of them, we shall investigate the scattering theory for these equations with a power type nonlinearity in suitable space.

1. STRICHARTZ ESTIMATES

Strichartz estimates are one of the standard tool in the study of linear and nonlinear evolution equations. They are initiated by the fundamental paper of Strichartz [24]. After him, many authors extended them to some equations with variable coefficients. D'Ancona & Fanelli [9] (see also [8], [10]) have treated the magnetic potentials in whole space. On the other hand, in exterior domains, Ivanovici [12] proved the corresponding Strichartz estimates for free Schrödinger equations. Smith & Sogge studied them for free wave equation in odd dimension (see [23]), and Burq and Metcalfe extended to higher dimension $n \geq 4$ (see [3, 15]). Recently the Strichartz estimates for Schrödinger, Klein-Gordon and wave equations with potential are treated in Mochizuki & Murai [20]. Our goal in this section is to extend the above results to the case of more general potentials.

To be more precisely, throughout this paper let Ω be an exterior domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$ such that $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to the origin. The case $\Omega = \mathbb{R}^n$ is not excluded when $n \geq 3$. In case $n = 2$, we fix $r_0 > 0$ satisfying $\overline{\Omega} \subset \{x; |x| > r_0\}$. Consider in Ω the following Schrödinger, wave $(m = 0)$ and Klein-Gordon $(m > 0)$ equations:

$$i\partial_t u = Lu + G(x,t), \quad x \in \Omega, \quad t \neq 0, \tag{1.1}$$

$$\partial_t^2 u + Lu + m^2 u = G(x,t), \quad x \in \Omega, \quad t \neq 0 \tag{1.2}$$

with Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \mathbb{R}. \tag{1.3}$$

Here $L = -\Delta_b + c(x)$, Δ_b is the magnetic Laplacian

$$\Delta_b = \nabla_b \cdot \nabla_b = \sum_{j=1}^{n} \{\partial_j + ib_j(x)\}^2, \quad \partial_j = \frac{\partial}{\partial x_j},$$

$b_j(x)$ and $c(x)$ are real-valued C^1-functions of $x \in \overline{\Omega}$. $b(x) = (b_1(x), \cdots, b_n(x))$ represents a magnetic potential. Thus, the magnetic field is defined by its rotation $\nabla \times b(x) = (\partial_j b_k(x) - \partial_k b_j(x))_{j<k}$.

Let Δ_D be the Dirichlet Laplacian. We can define $(-\Delta_D)^{s/2}$ via the functional calculus of selfadjoint operators for $s \geq 0$ and $\dot{H}^s_D(\Omega)$ is a domain of $(-\Delta_D)^{s/2}$. Thus by duality and interpolation argument we define $\dot{H}^s_D(\Omega)$ for $s \in \mathbb{R}$. Similarly, define the inhomogeneous space $H^s_D(\Omega)$, $s \in \mathbb{R}$ as a domain of $(-\Delta_D + 1)^{s/2}$. In this paper, we simply write $L^p L^q(\Omega) = L^p_t(\mathbb{R}_\pm; L^q_x(\Omega))$ for $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

To achieve our purpose we will employ the following strategy: the equation (1.1) or (1.2) with $G = 0$ is rewritten to the evolution equation:

$$i\partial_t u = \Lambda u.$$

Decomposing the operator Λ into the free part and the perturbed part, that is, writing $\Lambda = \Lambda_0 + W$, the solution $u(t) = e^{-it\Lambda} f$ is represented as an integral equation:

$$u(t) = e^{-it\Lambda_0} f - i \int_0^t e^{-i(t-\tau)\Lambda_0} W u(\tau)\, d\tau \tag{1.4}$$

by Duhamel's principle. Hence, we shall concentrate to estimate each term of the right hand side of (1.4). To do so, we need the Strichartz estimates for the free solutions $e^{-it\Lambda_0} f$ and the smoothing estimates for the perturbed solutions $e^{-it\Lambda} f$. Moreover as an immediately consequence of the estimate for this solution, we can also get the estimate for the inhomogeneous term $G(x,t)$ by TT^*-argument treated in Ginibre & Velo [11].

In the following we denote $r = |x|$ and

$$[r] = \begin{cases} r, \\ r(1+\log(r/r_0)), \end{cases} \qquad [n-2] = \begin{cases} n-2, & \text{when } n \geq 3, \\ 1, & \text{when } n = 2. \end{cases}$$

We prepare two elementary lemmas.

Lemma 1.1. *Let $n \geq 2$ and $-1 \leq \gamma \leq 1$. Assume that $b(x)$ and $c(x)$ satisfy*

$$|b(x)| \leq \delta[r]^{-1}, \quad -\delta[r]^{-2} \leq c(x) \leq C[r]^{-2} \tag{1.5}$$

for some $\delta > 0$ sufficiently small and $C > 0$. Then $\|\sqrt{L}^{\gamma} g\|_{L^2(\Omega)} \simeq \|\sqrt{-\Delta_D}^{\gamma} g\|_{L^2(\Omega)}$. Moreover the inhomogeneous version also holds.

Proof. When $\gamma = 1$, we have

$$\begin{aligned}\|\sqrt{L}g\|^2_{L^2(\Omega)} &= ((\nabla + ib(x))g, (\nabla + ib(x))g)_{L^2(\Omega)} + (c(x)g, g)_{L^2(\Omega)} \\ &= \|\nabla g\|^2_{L^2(\Omega)} + 2\mathrm{Im}(b \cdot \nabla g, g)_{L^2(\Omega)} + ((|b|^2 + c)g, g)_{L^2(\Omega)},\end{aligned}$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is an inner-product of $L^2(\Omega)$. Applying assumption (1.5) and the Hardy inequality

$$\int_\Omega \frac{[n-2]^2}{4[r]^2}|g|^2\, dx \leq \int_\Omega |\nabla_b g|^2\, dx \tag{1.6}$$

with $b = 0$ (which is proved by Mochizuki [17] for $n \geq 3$ and Mochizuki & Nakazawa [22] for $n = 2$) to the second and third terms, we obtain

$$|(b \cdot \nabla g, g)_{L^2(\Omega)}| \leq \int_\Omega |\nabla g||b||g|\, dx \leq \delta \|\nabla g\|^2_{L^2(\Omega)},$$

$$((|b|^2 + c)g, g)_{L^2(\Omega)} = \int_\Omega |b|^2|g|^2\, dx + \int_\Omega c|g|^2\, dx \leq C\|\nabla g\|^2_{L^2(\Omega)}.$$

Summarizing these inequalities, we have $\|\sqrt{L}g\|_{L^2(\Omega)} \leq C\|\nabla g\|_{L^2(\Omega)}$. On the other hand, proceeding as for the upper bound, we estimate from below as follows:

$$\begin{aligned}\|\sqrt{L}g\|^2_{L^2(\Omega)} &\geq \|\nabla g\|^2_{L^2(\Omega)} + 2\mathrm{Im}(b \cdot \nabla g, g)_{L^2(\Omega)} + (cg, g)_{L^2(\Omega)} \\ &\geq (1 - C\delta)\|\nabla g\|^2_{L^2(\Omega)}\end{aligned}$$

for δ small enough. This with the fact $\|\nabla g\|_{L^2(\Omega)} \simeq \|\sqrt{-\Delta_D} g\|_{L^2(\Omega)}$ proves the case of $\gamma = 1$. By duality, we have for $\gamma = -1$. Finally we use the interpolation argument to get the required equivalence for $-1 \leq \gamma \leq 1$.

It is clear that the inhomogeneous version holds by the same argument. □

Furthermore, to estimate the integral term, the following lemma is useful.

Lemma 1.2 (Christ & Kiselev [5]). *Let $-\infty \leq a < b \leq \infty$. Given two Banach spaces X, Y and a bounded linear operator*

$$Th = \int_a^b K(t, \tau)h(\tau)\, d\tau$$

from $L^{\tilde p}((a,b);X)$ *to* $L^p((a,b);Y)$*, then its truncated version*

$$Sh = \int_a^t K(t,\tau)h(\tau)\,d\tau$$

is also bounded on the same spaces, provided $1 \le \tilde p < p \le \infty$*.*

Note that thanks to this lemma, to prove the following type inequality

$$\left\| \int_0^t e^{-i(t-\tau)\Lambda} F(\tau)\,d\tau \right\|_{L^pY} \le C\|F\|_{L^{p'}Y'},$$

it is enough to estimate the integral

$$\left\| \int_0^\infty e^{-i(t-\tau)\Lambda} F(\tau)\,d\tau \right\|_{L^pY}.$$

In our previous paper [20] the smoothing estimates for e^{-itL} and $e^{-it\sqrt{L+m^2}}$ are proved under the smallness assumption on the rotation of $b(x)$ and the negative part $c_0(x)$ of $c(x)$ as follows:

$$|\nabla \times b(x)|, |c_0(x)| \le \varepsilon[r]^{-2}, \tag{1.7}$$

where $\varepsilon \ge 0$ is a small constant. Moreover, using them, the Strichartz estimates for the solution with $b=0$ were proved along the strategy mentioned above. However, in the present case the coefficient $b(x)$ makes the argument more complicate. Actually, our method requires to regard the terms relating to $b(x)$ as a perturbation of $-\Delta$, and so we need the smoothing estimates in $H^s_D(\Omega)$ and another smallness assumption on $b(x)$ itself like (1.5) from Lemma 1.1 to derive the Strichartz estimates. To avoid this overlap condition, in this paper changing the proof of the smoothing estimates in [20] to that without (1.7), we simplify the assumption on $b(x)$ (see the proof of Proposition 1.3).

1.1. **Schrödinger equations.** Throughout this section we put $\xi(r) = (1+[r])^{-1}$ and always assume that the coefficients $b(x)$ and $c(x) = c_0(x) + c_1(x)$ satisfy

$$(AS1)\qquad \begin{gathered} |b(x)|, |\nabla \cdot b(x) + |b(x)|^2 + c_0(x)| \le \varepsilon_0 \xi(r)^2, \\ c_1(x) = o(r^{-1})\ (r\to\infty), \quad c_1(x) \ge 0, \quad \partial_r\{rc_1(x)\} \le 0, \end{gathered}$$

where $\varepsilon_0 \ge 0$ is a small constant and $\partial_r = \partial/\partial r$.

Let us derive the smoothing estimate for solution $e^{-itL}f$ which is a crucial tool for the proof of Strichartz estimate.

Proposition 1.3. *Let* $n \ge 2$*. Assume* $(AS1)$*. Then the following estimate holds.*

$$\|\xi e^{-itL} f\|_{L^2L^2(\Omega)} + \|\xi e^{-itL} f\|_{L^2 H_D^{\frac12}(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Proof. The desired estimate follows from the same argument as in [20] (see also [17], [18]). The following lemma is available.

Lemma 1.4 ([13, 18]). *Let $\mathcal{H}$ be a Hilbert space and let $(\Lambda - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ be the resolvent of selfadjoint operator Λ in $\mathcal{H}$. Suppose that $\mathcal{A}$ is a densely defined, closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{H}_1$. Assume that there exists $C > 0$ such that*

$$\sup_{z \notin \mathbb{R}} \|\mathcal{A}(\Lambda - z)^{-1}\mathcal{A}^* g\|_{\mathcal{H}_1} < C\|g\|_{\mathcal{H}_1}, \quad g \in \mathcal{D}(\mathcal{A}^*).$$

Then we have for each $f \in \mathcal{H}$,

$$\|\mathcal{A}e^{-it\Lambda} f\|^2_{L^2\mathcal{H}_1} \le 2C\|f\|^2_{\mathcal{H}}.$$

First we need to prove the inequality:

$$(1.8)\quad (1 + \sqrt{|z|})\|\xi(L - z)^{-1}g\|_{L^2(\Omega)} + \|\xi\nabla(L - z)^{-1}g\|_{L^2(\Omega)} \le C\|\xi^{-1}g\|_{L^2(\Omega)}$$

for the resolvent $(L - z)^{-1}$ of L, $z \in \mathbb{C} \setminus \mathbb{R}$ under our assumption (similar inequality is already proved by Mochizuki & Murai [20] under the smallness condition on the rotation of $b(x)$ instead of it on $b(x)$ itself like $(AS1)$). To do so, we represent the operator L as

$$L = -\Delta + A \cdot \nabla + B_0 + B_1$$

with

$$A = -2ib, \quad B_0 = -i\nabla \cdot b + |b|^2 + c_0, \quad B_1 = c_1,$$

and use the following identity.

$$(L - z)^{-1} = R_{B_1}(z)\left(I + (A \cdot \nabla + B_0)R_{B_1}(z)\right)^{-1},$$

where $R_{B_1}(z) = (-\Delta_D + B_1 - z)^{-1}$. Note that the assumption $(AS1)$ implies

$$|A|, |B_0| \le \varepsilon_0 \xi^2, \quad B_1 \ge 0, \quad \partial_r(rB_1) \le 0.$$

Here for the resolvent $R_{B_1}(z)$ the following inequality is proved in [20] under the assumption $(AS1)$ on B_1 and the star-shapedness of $\mathbb{R}^n \setminus \Omega$:

$$(1.9)\quad (1+\sqrt{|z|})\|\xi R_{B_1}(z)g\|_{L^2(\Omega)} + \|\xi\nabla R_{B_1}(z)g\|_{L^2(\Omega)} \le C\|\xi^{-1}g\|_{L^2(\Omega)}.$$

Hence if the operator $\xi^{-1}(I+(A\cdot\nabla+B_0)R_{B_1}(z))^{-1}\xi$ exists and bounded in $L^2(\Omega)$, then the same type inequality as above holds for the resolvent $(L - z)^{-1}$. Since $(AS1)$, using (1.9), we have

$$\begin{aligned}
&\|\xi^{-1}(A \cdot \nabla + B_0)R_{B_1}(z)g\|_{L^2(\Omega)} \\
&\le \|\xi^{-2}\max\{|A|, |B_0|\}\|_{L^\infty(\Omega)}(\|\xi\nabla R_{B_1}(z)g\|_{L^2(\Omega)} + \|\xi R_{B_1}(z)g\|_{L^2(\Omega)}) \\
&\le \varepsilon_0\|\xi^{-1}g\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, if ε_0 is sufficiently small we can invert $I+(A\cdot\nabla+B_0)R_{B_1}(z)$ by a Neumann series and get

$$\|\xi^{-1}\left(I+(A\cdot\nabla+B_0)R_{B_1}(z)\right)^{-1}g\|_{L^2(\Omega)}\leq C\|\xi^{-1}g\|_{L^2(\Omega)}.$$

As soon as the inequality (1.8) is proved, we can obtain the desired estimate for the first term by Lemma 1.4 with $\mathcal{A}=\xi$, $\Lambda=L$ and $\mathcal{H}=\mathcal{H}_1=L^2(\Omega)$. On the other hand (1.8) shows

$$\begin{aligned}&\|\xi(L-z)^{-1}g\|_{H^1_D(\Omega)}\\ \leq&\|\xi(L-z)^{-1}g\|_{L^2(\Omega)}+\|\nabla\xi(L-z)^{-1}g\|_{L^2(\Omega)}+\|\xi\nabla(L-z)^{-1}g\|_{L^2(\Omega)}\\ \leq&C\|\xi^{-1}g\|_{L^2(\Omega)},\end{aligned}\tag{1.10}$$

where we used the fact $|\nabla\xi|\leq C\xi$. By duality, we also have

$$\|\xi(L-z)^{-1}g\|_{L^2(\Omega)}\leq C\|\xi^{-1}g\|_{H^{-1}_D(\Omega)}.\tag{1.11}$$

Then interpolating these inequalities (1.10) and (1.11), we obtain

$$\|\xi(L-z)^{-1}g\|_{H^{\frac12}_D(\Omega)}\leq C\|\xi^{-1}g\|_{H^{-\frac12}_D(\Omega)}.$$

Thus, it follows from Lemma 1.4 with $\mathcal{A}=(-\Delta_D+1)^{1/4}\xi$, $\Lambda=L$ and $\mathcal{H}=\mathcal{H}_1=L^2(\Omega)$, the desired estimate for the second term can be obtained. □

Next we shall treat the Strichartz estimates. In addition to $(AS1)$, we further assume the following condition.

$$\xi(r)^{-2}c_1(x)\in L^\infty(\Omega).\tag{AS2}$$

Noting that this together with $(AS1)$ implies

$$\xi^{-2}A\in L^\infty(\Omega),\quad \xi^{-2}B\in L^\infty(\Omega),\quad B=B_0+B_1.$$

In the following exponent pairs (p,q) and $(\tilde p,\tilde q)$ always satisfy the relation:

$$2<p,\tilde p\leq\infty,\quad \frac{2}{p}+\frac{n}{q}=\frac{n}{2},\quad \frac{2}{\tilde p}+\frac{n}{\tilde q}=\frac{n}{2}.\tag{1.12}$$

The main result in this section is the following.

Theorem 1.5. *Let $n\geq 2$. Assume $(AS1)$–$(AS2)$ and that $\mathbb{R}^n\setminus\Omega$ is a strictly convex obstacle. Then for the solution u of* (1.1) *with* (1.3) *and the initial data $u(0)=f$ the following estimate holds.*

$$\|u\|_{L^pL^q(\Omega)}+\|u\|_{L^\infty L^2(\Omega)}\leq C_1\left(\|f\|_{L^2(\Omega)}+\|G\|_{L^{\tilde p'}L^{\tilde q'}(\Omega)}\right).\tag{1.13}$$

Proof. The solution u is represented as the integral form:

$$u(t) = e^{-itL}f - i\int_0^t e^{-i(t-\tau)L}G(\tau)\,d\tau.$$

Hence it is enough to derive the inequality:

$$\|e^{-itL}f\|_{L^pL^q(\Omega)} + \|e^{-itL}f\|_{L^\infty L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{1.14}$$

Indeed, the first term can be estimated by (1.14) directly. Using (1.14) again and the dual inequality

$$\left\|\int_0^\infty e^{i\tau L}h(\tau)\,d\tau\right\|_{L^2(\Omega)} \le C\|h\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)}$$

of (1.14), we have

$$\left\|\int_0^\infty e^{-i(t-\tau)L}G(\tau)\,d\tau\right\|_{L^pL^q(\Omega)} \le C\left\|\int_0^\infty e^{i\tau L}G(\tau)\,d\tau\right\|_{L^2(\Omega)} \le C\|G\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)}.$$

Finally we can apply Lemma 1.2 to get the desired estimate. Thus in the following we shall concentrate to prove (1.14). The estimate of the second term is an immediate consequence of the unitarity of e^{-itL}. To estimate the first term, we write

$$Lv = -\Delta v + \nabla\cdot(Av) - \nabla\cdot Av + Bv := -\Delta v + W_1(v) + W_2(v)$$

with $W_1(v) = \nabla\cdot(Av)$ and $W_2(v) = -\nabla\cdot Av + Bv$. Then the solution $v = e^{-itL}f$ is represented as the integral form:

$$v(t) = e^{it\Delta_D}f - i\int_0^t e^{i(t-\tau)\Delta_D}(W_1(v) + W_2(v))\,d\tau.$$

It is enough to estimate each term of the right hand side of this integral equation. The first term can be estimated by the inequality:

$$\|e^{it\Delta_D}f\|_{L^pL^q(\Omega)} \le C\|f\|_{L^2(\Omega)} \tag{1.15}$$

which is proved by Ivanovici [12] under the condition that $\mathbb{R}^n\setminus\Omega$ is strictly convex. As to the second term, thanks to Lemma 1.2 it is enough to estimate the following integral.

$$\int_0^\infty e^{i(t-\tau)\Delta_D}(W_1(v) + W_2(v))\,d\tau.$$

Using the inequality (1.15) and applying the dual inequality of Proposition 1.3:

$$\left\|\int_0^\infty e^{-i\tau\Delta_D}h(\tau)\,d\tau\right\|_{L^2(\Omega)} \le C\|\xi^{-1}h\|_{L^2H_D^{-\frac{1}{2}}(\Omega)}$$

to the term for $W_1(v)$, we have

$$\left\|\int_0^\infty e^{i(t-\tau)\Delta_D}W_1(v)\,d\tau\right\|_{L^pL^q(\Omega)} \le C\left\|\int_0^\infty e^{-i\tau\Delta_D}W_1(v)\,d\tau\right\|_{L^2(\Omega)}$$
$$\le C\|\xi^{-1}W_1(v)\|_{L^2H_D^{-\frac{1}{2}}(\Omega)}.$$

Note that

$$\xi^{-1}\nabla\cdot(Av) = \nabla\cdot(\xi^{-1}Av) - A\cdot\nabla\xi^{-1}v.$$

Then using the fact $\|\nabla\cdot\vec{f}\|_{H_D^{-\frac{1}{2}}(\Omega)} \le \|\vec{f}\|_{H_D^{\frac{1}{2}}(\Omega)}$ with $\vec{f}|_{\partial\Omega} = 0$, we have

$$\|\xi^{-1}\nabla\cdot(Av)\|_{L^2H_D^{-\frac{1}{2}}(\Omega)} \le \|\xi^{-1}Av\|_{L^2H_D^{\frac{1}{2}}(\Omega)} + \|A\cdot\nabla\xi^{-1}v\|_{L^2H_D^{-\frac{1}{2}}(\Omega)}.$$

Here note that $|\nabla\xi^{-2}| \le C\xi^{-2}$ and $|\nabla\cdot A| \le C|B|$, it follows from the assumption on A and B that

$$\begin{aligned}
&\|\xi^{-2}Av\|_{H_D^1(\Omega)}\\
&\le \|\nabla\xi^{-2}Av\|_{L^2(\Omega)} + \|\xi^{-2}\nabla\cdot Av\|_{L^2(\Omega)} + \|\xi^{-2}A\cdot\nabla v\|_{L^2(\Omega)} + \|\xi^{-2}Av\|_{L^2(\Omega)}\\
&\le C\left\{\|\xi^{-2}B\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)} + \|\xi^{-2}A\|_{L^\infty(\Omega)}\left(\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)\right\}\\
&\le C\|v\|_{H_D^1(\Omega)}
\end{aligned}$$

and

$$\|\xi^{-2}Av\|_{L^2(\Omega)} \le \|\xi^{-2}A\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}.$$

Then interpolating these inequalities, we get

$$\|\xi^{-1}Av\|_{H_D^\gamma(\Omega)} \le C\|\xi v\|_{H_D^\gamma(\Omega)}$$

for $0 \le \gamma \le 1$. Using this inequality with $\gamma = 1/2$ and Proposition 1.3, we get

$$\|\xi^{-1}Av\|_{L^2H_D^{\frac{1}{2}}(\Omega)} \le C\|\xi v\|_{L^2H_D^{\frac{1}{2}}(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

On the other hand, since $|\nabla\xi^{-1}| \le C\xi^{-1}[r]^{-\frac{1}{2}}$ and $|\nabla([r]^{1/2}\xi\nabla\xi^{-1})| \le C[r]^{-1}$, we have by using the Hardy inequarity (1.6) with $b = 0$

$$\begin{aligned}
\|[r]^{\frac{1}{2}}\xi\nabla\xi^{-1}g\|_{H_D^1(\Omega)} &\le \|\nabla([r]^{\frac{1}{2}}\xi\nabla\xi^{-1})g\|_{L^2(\Omega)} + \|[r]^{\frac{1}{2}}\xi\nabla\xi^{-1}\cdot\nabla g\|_{L^2(\Omega)}\\
&\quad + \|[r]^{\frac{1}{2}}\xi\nabla\xi^{-1}g\|_{L^2(\Omega)}\\
&\le C\|g\|_{H_D^1(\Omega)}
\end{aligned}$$

and

$$\|[r]^{\frac{1}{2}}\xi\nabla\xi^{-1}g\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}.$$

Hence by the interpolation and the duality argument show

$$\|\nabla\xi^{-1}g\|_{H_D^{-\gamma}(\Omega)} \le C\|\xi^{-1}[r]^{-\frac{1}{2}}g\|_{H_D^{-\gamma}}, \quad 0 \le \gamma \le 1.$$

This inequality with $\gamma = 1/2$ implies

$$\|A \cdot \nabla \xi^{-1} v\|_{L^2 H_D^{-\frac{1}{2}}(\Omega)} \le C \|\xi^{-1} [r]^{-\frac{1}{2}} A v\|_{L^2 H_D^{-\frac{1}{2}}(\Omega)}.$$

Moreover using the dual of Hardy inequality:

$$\|[r]^{-\frac{1}{2}} v\|_{L^2(\Omega)} \le \|v\|_{H_D^{\frac{1}{2}}(\Omega)}$$

(which is proved by interpolating (1.6) with $b = 0$ and the trivial one $\|v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}$) and Proposition 1.3 again, we obtain

$$\begin{aligned} \|\xi^{-1}[r]^{-\frac{1}{2}} A v\|_{L^2 H_D^{-\frac{1}{2}}(\Omega)} &\le \|\xi^{-1} A v\|_{L^2 L^2(\Omega)} \\ &\le \|\xi^{-2} A\|_{L^\infty(\Omega)} \|\xi v\|_{L^2 L^2(\Omega)} \le C \|f\|_{L^2(\Omega)}. \end{aligned}$$

These inequalities give

$$\|\xi^{-1} W_1(v)\|_{L^2 H_D^{-\frac{1}{2}}(\Omega)} \le C \|f\|_{L^2(\Omega)}.$$

On the other hand the dual of Proposition 1.3:

$$\left\| \int_0^\infty e^{-i\tau \Delta_D} h(\tau)\, d\tau \right\|_{L^2(\Omega)} \le C \|\xi^{-1} h\|_{L^2 L^2(\Omega)}$$

shows that the term for $W_2(v)$ can be estimated as

$$\begin{aligned} \left\| \int_0^\infty e^{i(t-\tau)\Delta_D} W_2(v)\, d\tau \right\|_{L^p L^q(\Omega)} &\le C \|\xi^{-1} W_2(v)\|_{L^2 L^2(\Omega)} \\ &\le C \|\xi^{-2} B\|_{L^\infty(\Omega)} \|\xi v\|_{L^2 L^2(\Omega)} \le C \|f\|_{L^2(\Omega)}. \end{aligned}$$

Thus, summarizing the above argument, the desired estimate is now proved. □

1.2. **Wave equations.** Throughout this section let $\varphi = \varphi(r) > 0$ be a smooth function of $r > 0$ and let $\mu = \mu(r) > 0$ be a smooth $L^1(\mathbb{R}_+)$ function of $r > 0$ satisfying

$$\mu'(r) \le 0, \quad \mu'(r)^2 \le 2\mu(r)\mu''(r). \tag{1.16}$$

In the wave case, we always assume that the coefficients $b(x)$ and $c(x) = c_0(x) + c_1(x)$ satisfy

$(AW1)$

$$|b(x)|,\ r|\nabla \cdot b(x) + |b(x)|^2 + c_0(x)| \le \varepsilon_0 \mu(r),$$
$$c_1(x) = o(r^{-1})\ (r \to \infty), \quad c_1(x) \ge 0, \quad \partial_r\{\varphi(r) c_1(x)\} \le 0,$$

where $\varepsilon_0 \ge 0$ is a small constant.

Let us first state the smoothing estimate.

Proposition 1.6. *Let $n \geq 3$. Assume $(AW1)$ and (1.5). Then the following estimate holds for $0 \leq \gamma \leq 1$.*

$$\|\sqrt{\mu} e^{-it\sqrt{L}} f\|_{L^2 \dot{H}^{\gamma}_D(\Omega)} \leq C\|f\|_{\dot{H}^{\gamma}_D(\Omega)}.$$

Proof. In this proof we simply write $\partial_t u = u_t$ where u is a solution of (1.2) with $G = 0$ and also rewrite the equation as

$$u_{tt} - \Delta u + A \cdot \nabla u + B_0 u + B_1 u = 0, \tag{1.17}$$

where A, B_0 and B_1 are same as Schrödinger case. Then the assumption $(AW1)$ is also rewritten as

$$|A| \leq \varepsilon_0 \mu, \quad |B_0| \leq \varepsilon_0 r^{-1} \mu, \quad B_1 \geq 0, \quad \partial_r(\varphi B_1) \leq 0.$$

Multiplying by $\varphi\left(\overline{\tilde{x} \cdot \nabla u} + \frac{n-1}{2r}\overline{u}\right)$ $(\tilde{x} = x/r)$ on each term in (1.17) and take a real part, we have

$$\partial_t X + \frac{1}{2}\nabla \cdot Y + Z = 0, \tag{1.18}$$

where we put

$$X = \varphi \mathrm{Re}\left(u_t \overline{\tilde{x} \cdot \nabla u} + \frac{n-1}{2r} u_t \overline{u}\right),$$

$$\begin{aligned} Y = \tilde{x}\varphi(-|u_t|^2 + |\nabla u|^2 + B_1|u|^2) \\ - 2\varphi \mathrm{Re} \nabla u \left(\overline{\tilde{x} \cdot \nabla u} + \frac{n-1}{2r}\overline{u}\right) + \tilde{x}\left(\varphi \frac{n-1}{2r}\right)'|u|^2, \end{aligned}$$

$$\begin{aligned} Z = \left(\frac{\varphi}{r} - \varphi'\right)\left\{|\nabla u|^2 - |\tilde{x} \cdot \nabla u|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right\} \\ + \frac{1}{2}\varphi'(|u_t|^2 + |\nabla u|^2) - \varphi'' \frac{n-1}{4r}|u|^2 - \frac{1}{2}\partial_r(\varphi B_1)|u|^2 \\ + \varphi \mathrm{Re}\left\{(A \cdot \nabla u + B_0 u)\left(\overline{\tilde{x} \cdot \nabla u} + \frac{n-1}{2r}\overline{u}\right)\right\}. \end{aligned}$$

Integrating (1.18) over $\Omega \times [s, t]$, we have

$$\int_\Omega X(t)\, dx - \int_\Omega X(s)\, dx + \frac{1}{2}\int_s^t \int_{\partial\Omega} Y(\tau) \cdot \nu \, dS d\tau + \int_s^t \int_\Omega Z(\tau)\, dx d\tau = 0, \tag{1.19}$$

where $\nu = \nu(x)$ is the outer normal vector at $x \in \partial\Omega$. We now estimate the integrals of X and Y in (1.19). Let $\varphi(r) = \int_0^r \mu(\sigma)\, d\sigma$. It follows from the Hardy inequality (1.6) and Young inequality, we have

$$\left|\int_\Omega X(\tau)\, dx\right| \leq C(\|\mu\|_{L^1(\mathbb{R}_+)}) \int_\Omega \left(|u_\tau|^2 + |\nabla u|^2\right)\, dx.$$

Here by Lemma 1.1 the integral of the right hand side is equivalent to

$$\int_\Omega \left(|u_t|^2 + |\nabla_b u|^2 + c|u|^2\right)\,dx,$$

and this integral is conserved in time by the energy conservation low. Hence, we have

$$\left|\int_\Omega X(\tau)\,dx\right| \le C\left(\|\nabla u(0)\|^2_{L^2(\Omega)} + \|u_t(0)\|^2_{L^2(\Omega)}\right) \tag{1.20}$$

for any $\tau \in \mathbb{R}$. As to Y, the boundary condition and assumption on the star-shapedness of $\mathbb{R}^n \setminus \Omega$ assure that the integral in $Y(\tau)\cdot\nu$ can be neglected, since

(1.21)
$$\begin{aligned}\int_{\partial\Omega} Y(\tau)\cdot\nu\,dS &= \int_{\partial\Omega}\{\varphi(\tilde{x}\cdot\nu)(-|u_\tau|^2 + |\nu\cdot\nabla u|^2) - 2\varphi(\tilde{x}\cdot\nu)|\nu\cdot\nabla u|^2\}\,dS\\ &= -\int_{\partial\Omega}\varphi(\tilde{x}\cdot\nu)(|u_\tau|^2 + |\nu\cdot\nabla u|^2)\,dS \ge 0.\end{aligned}$$

Hence it follows from (1.19)–(1.21) that

$$\int_s^t\int_\Omega Z(\tau)\,dxd\tau \le C\left(\|\nabla u(0)\|^2_{L^2(\Omega)} + \|u_t(0)\|^2_{L^2(\Omega)}\right).$$

Finally as to Z, since the assumption $(AW1)$, the term for B_1 can be neglected, and the term for A and B_0 can be estimated by Young inequality and the boundedness of $\varphi(r)$

$$\begin{aligned}&\left|\varphi\mathrm{Re}\left\{(A\cdot\nabla u + B_0 u)\left(\overline{\tilde{x}\cdot\nabla u} + \frac{n-1}{2r}\overline{u}\right)\right\}\right|\\ &\le C\varepsilon_0\left(\mu|\nabla u|^2 + \frac{\mu}{r}|u||\nabla u| + \frac{\mu}{r^2}|u|^2\right)\\ &\le C\varepsilon_0\left(\mu|\nabla u|^2 + \mu\frac{(n-2)^2}{4r^2}|u|^2\right).\end{aligned} \tag{1.22}$$

Applying the weighted Hardy inequality:

(1.23)
$$\begin{aligned}\int_s^t\int_\Omega \mu\frac{(n-2)^2}{4r^2}|u|^2\,dxd\tau &\le \int_s^t\int_\Omega |\nabla(\sqrt{\mu}u)|^2\,dxd\tau\\ &\le \int_s^t\int_\Omega\left(\mu|\nabla u|^2 - \mu'\frac{n-1}{2r}|u|^2\right)\,dxd\tau\end{aligned}$$

which is proved in Mochizuki [16] under (1.16) to the second term of the right hand side in (1.22), we have

$$\int_s^t \int_\Omega \left| \varphi \mathrm{Re} \left\{ (A \cdot \nabla u + B_0 u) \left(\overline{\tilde{x} \cdot \nabla u} + \frac{n-1}{2r} \overline{u} \right) \right\} \right| dx d\tau \\ \le C\varepsilon_0 \int_s^t \int_\Omega \left(\mu |\nabla u|^2 - \mu' \frac{n-1}{2r} |u|^2 \right) dx d\tau.$$

Since $\varphi(r) \ge r\mu(r) = r\varphi'(r)$ and $|\tilde{x} \cdot \nabla u| \le |\nabla u|$, the integral of Z can be estimated if ε_0 is sufficiently small as

$$\int_s^t \int_\Omega Z(\tau)\, dx d\tau \ge C \int_s^t \int_\Omega \left\{ \mu(|u_\tau|^2 + |\nabla u|^2) - \mu' \frac{n-1}{2r} |u|^2 \right\} dx d\tau.$$

Thus, the above argument shows the inequality:

$$\int_s^t \int_\Omega \left\{ \mu(|u_\tau|^2 + |\nabla u|^2) - \mu' \frac{n-1}{2r} |u|^2 \right\} dx d\tau \\ \le C \left(\|\nabla u(0)\|^2_{L^2(\Omega)} + \|u_t(0)\|^2_{L^2(\Omega)} \right). \tag{1.24}$$

To derive the desired estimate, we focus on the first term of the left hand side and choose $(u(0), u_t(0)) = (0, f)$ or $= (\sqrt{L}^{-1} f, 0)$ in (1.24) to obtain

$$\|\sqrt{\mu} \cos(t\sqrt{L}) f\|_{L^2 L^2(\Omega)} \le C\|f\|_{L^2(\Omega)},$$
$$\|\sqrt{\mu} \sin(t\sqrt{L}) f\|_{L^2 L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

These inequalities imply the case of $\gamma = 0$. Furthermore we put $u = e^{-it\sqrt{L}} f$ in (1.24). Then the argument of (1.23) reads us to the case of $\gamma = 1$. Hence the desired estimate is now proved by the interpolation. □

In the following, we put $\mu(r) = (1 + r)^{-1-\delta}$, $\delta > 0$. In order to state the result for Strichartz estimate, we further assume the following condition.

$$(AW2) \qquad r\mu(r)^{-1} c_1(x) \in L^\infty(\Omega).$$

This together with $(AW1)$ implies

$$\mu^{-1} A \in L^\infty(\Omega), \quad r\mu^{-1} B \in L^\infty(\Omega), \quad B = B_0 + B_1.$$

And let the exponents p, q, $\tilde{p}$, $\tilde{q}$ and κ always satisfy the following relations.

$$\begin{gathered} 2 < p, \tilde{p} \le \infty, \quad \frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad \frac{2}{\tilde{p}} + \frac{n-1}{\tilde{q}} = \frac{n-1}{2}, \\ \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \kappa = \frac{1}{\tilde{p}'} + \frac{n}{\tilde{q}'} - 2. \end{gathered} \tag{1.25}$$

Remark that the final equality of (1.25) makes the relations of the exponents p, q, $\tilde{p}$, $\tilde{q}$ more restrictive as the following.

$$\frac{1}{\tilde{p}'} - \frac{1}{p} = \frac{1}{\tilde{q}'} - \frac{1}{q} = \frac{2}{n+1}, \quad 2 < p < \frac{2(n+1)}{n-3}. \tag{1.26}$$

Our main result in this section is as follows.

Theorem 1.7. *Let $n \geq 3$. Assume $(AW1)$–$(AW2)$ and that $\mathbb{R}^n \backslash \Omega$ is a strictly convex obstacle. Then for the solution u of (1.2) with (1.3) and the initial data $(u(0), \partial_t u(0)) = (f_1, f_2)$ the following estimate holds.*

$$\begin{aligned}\|u\|_{L^pL^q(\Omega)} &+ \|u\|_{L^\infty \dot{H}^\kappa_D(\Omega)} + \|\partial_t u\|_{L^\infty \dot{H}^{\kappa-1}_D(\Omega)} \\ &\leq C_2 \left(\|f_1\|_{\dot{H}^\kappa_D(\Omega)} + \|f_2\|_{\dot{H}^{\kappa-1}_D(\Omega)} + \|G\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)} \right).\end{aligned}$$

Proof. The solution u is represented as the integral form:

$$u(t) = \cos(t\sqrt{L})f_1 + \frac{\sin(t\sqrt{L})}{\sqrt{L}} f_2 + \int_0^t \frac{\sin\{(t-\tau)\sqrt{L}\}}{\sqrt{L}} G(\tau)\, d\tau.$$

In order to estimate the each term of the right hand side, it suffices to derive the inequalities:

$$\|e^{-it\sqrt{L}} f\|_{L^\infty \dot{H}^\kappa_D(\Omega)} \leq C \|f\|_{\dot{H}^\kappa_D(\Omega)}, \tag{1.27}$$

$$\|e^{-it\sqrt{L}} f\|_{L^pL^q(\Omega)} \leq C \|f\|_{\dot{H}^\kappa_D(\Omega)}. \tag{1.28}$$

The case of $G = 0$ obviously follows from these inequalities and Lemma 1.1. In fact, we have

$$\begin{aligned}&\|u\|_{L^\infty \dot{H}^\kappa_D(\Omega)} + \|\partial_t u\|_{L^\infty \dot{H}^{\kappa-1}_D(\Omega)} \\ &= \left\| \cos(t\sqrt{L})f_1 + \frac{\sin(t\sqrt{L})}{\sqrt{L}} f_2 \right\|_{L^\infty \dot{H}^\kappa_D(\Omega)} \\ &\qquad + \left\| -\sqrt{L}\sin(t\sqrt{L})f_1 + \cos(t\sqrt{L})f_2 \right\|_{L^\infty \dot{H}^{\kappa-1}_D(\Omega)} \\ &\leq C \left(\|f_1\|_{\dot{H}^\kappa_D(\Omega)} + \|\sqrt{L}^{-1} f_2\|_{\dot{H}^\kappa_D(\Omega)} + \|\sqrt{L} f_1\|_{\dot{H}^{\kappa-1}_D(\Omega)} + \|f_2\|_{\dot{H}^{\kappa-1}_D(\Omega)} \right) \\ &\leq C \left(\|f_1\|_{\dot{H}^\kappa_D(\Omega)} + \|f_2\|_{\dot{H}^{\kappa-1}_D(\Omega)} \right).\end{aligned}$$

Moreover

$$\begin{aligned}\|u\|_{L^pL^q(\Omega)} &= \left\|\cos(t\sqrt{L})f_1 + \frac{\sin(t\sqrt{L})}{\sqrt{L}}f_2\right\|_{L^pL^q(\Omega)} \\ &\le C\left(\|f_1\|_{\dot{H}^\kappa_D(\Omega)} + \|\sqrt{L}^{-1}f_2\|_{\dot{H}^\kappa_D(\Omega)}\right) \\ &\le C\left(\|f_1\|_{\dot{H}^\kappa_D(\Omega)} + \|f_2\|_{\dot{H}^{\kappa-1}_D(\Omega)}\right).\end{aligned}$$

As to the integral term, using (1.27) and (1.28), we have

$$\begin{aligned}&\left\|\int_0^\infty \frac{e^{-i(t-\tau)\sqrt{L}}}{\sqrt{L}}G\,d\tau\right\|_{L^pL^q(\Omega)} + \left\|\int_0^\infty \frac{e^{-i(t-\tau)\sqrt{L}}}{\sqrt{L}}G\,d\tau\right\|_{L^\infty\dot{H}^\kappa_D(\Omega)} \\ &\quad + \left\|\int_0^\infty e^{-i(t-\tau)\sqrt{L}}G\,d\tau\right\|_{L^\infty\dot{H}^{\kappa-1}_D(\Omega)} \le C\left\|\int_0^\infty e^{i\tau\sqrt{L}}G\,d\tau\right\|_{\dot{H}^{\kappa-1}_D(\Omega)}.\end{aligned}$$

Here we used Lemma 1.1. On the other hand, by duality (1.28) implies

$$\left\|\int_0^\infty e^{i\tau\sqrt{L}}h(\tau)\,d\tau\right\|_{\dot{H}^{-\tilde{\kappa}}_D(\Omega)} \le C\|h\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)}, \quad \tilde{\kappa} = n\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right)-\frac{1}{\tilde{p}}.$$

Now choosing $\kappa - 1 = -\tilde{\kappa}$ which is equivalent to the final equality in the relations (1.25) and using Lemma 1.2, these inequalities show

$$\begin{aligned}&\left\|\int_0^t \frac{e^{-i(t-\tau)\sqrt{L}}}{\sqrt{L}}G\,d\tau\right\|_{L^pL^q(\Omega)} + \left\|\int_0^t \frac{e^{-i(t-\tau)\sqrt{L}}}{\sqrt{L}}G\,d\tau\right\|_{L^\infty\dot{H}^\kappa_D(\Omega)} \\ &\qquad + \left\|\int_0^t e^{-i(t-\tau)\sqrt{L}}G\,d\tau\right\|_{L^\infty\dot{H}^{\kappa-1}_D(\Omega)} \le C\|G\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)}.\end{aligned}$$

Summarizing the above argument, the desired estimate can be obtained. Hence we shall concentrate to prove the inequalities (1.27) and (1.28).

The first inequality (1.27) is an immediately consequence of the unitarity of $e^{-it\sqrt{L}}$ and Lemma 1.1. Indeed, we have

$$\begin{aligned}\|e^{-it\sqrt{L}}f\|_{L^\infty\dot{H}^\kappa_D(\Omega)} &\simeq \|\sqrt{L}^\kappa e^{-it\sqrt{L}}f\|_{L^\infty L^2(\Omega)} \\ &= \|e^{-it\sqrt{L}}\sqrt{L}^\kappa f\|_{L^\infty L^2(\Omega)} \le \|\sqrt{L}^\kappa f\|_{L^2(\Omega)} \simeq \|f\|_{\dot{H}^\kappa_D(\Omega)}.\end{aligned}$$

Next let us prove the second inequality (1.28). We need the following lemma.

Lemma 1.8 (Smith & Sogge [23], Burq [3], Metcalfe [15])**.** *Let* $n \geq 3$. *Assume that* $\mathbb{R}^n \setminus \Omega$ *is a strictly convex obstacle. Then*

$$\|e^{-it\sqrt{-\Delta_D}} f\|_{L^pL^q(\Omega)} \leq C\|f\|_{\dot{H}^\kappa_D(\Omega)}.$$

$v = e^{-it\sqrt{L}} f$ satisfies the following problem.

$$\begin{cases} \partial_t^2 v - \Delta v = -Wv, \\ v(0) = f, \quad \partial_t v(0) = -i\sqrt{L}f, \\ v|_{\partial\Omega} = 0, \end{cases}$$

where

$$W = A \cdot \nabla + B.$$

By Duhamel's principle the solution v is represented as follows.

$$v(t) = \cos(t\sqrt{-\Delta_D})f - i\frac{\sin(t\sqrt{-\Delta_D})}{\sqrt{-\Delta_D}}\sqrt{L}f - \int_0^t \frac{\sin\{(t-\tau)\sqrt{-\Delta_D}\}}{\sqrt{-\Delta_D}} Wv \, d\tau.$$

The first and second terms can be estimated by Lemma 1.8 and Lemma 1.1. In fact, we have

$$\left\| \cos(t\sqrt{-\Delta_D})f - i\frac{\sin(t\sqrt{-\Delta_D})}{\sqrt{-\Delta_D}}\sqrt{L}f \right\|_{L^pL^q(\Omega)}$$
$$\leq C\left(\|f\|_{\dot{H}^\kappa_D(\Omega)} + \left\| \frac{\sqrt{L}f}{\sqrt{-\Delta_D}} \right\|_{\dot{H}^\kappa_D(\Omega)} \right) \leq C\|f\|_{\dot{H}^\kappa_D(\Omega)}.$$

As to the third term, using Lemma 1.8 and the dual of the inequality from Proposition 1.6 with $b = c = 0$:

$$\left\| \int_0^\infty e^{i\tau\sqrt{-\Delta_D}} h(\tau)\, d\tau \right\|_{\dot{H}^{-\gamma}_D(\Omega)} \leq C\|\sqrt{\mu}^{-1} h\|_{L^2\dot{H}^{-\gamma}_D(\Omega)},$$

we have for $0 \leq \kappa \leq 1$

$$\left\| \int_0^\infty \frac{e^{-i(t-\tau)\sqrt{-\Delta_D}}}{\sqrt{-\Delta_D}} Wv \, d\tau \right\|_{L^pL^q(\Omega)} \leq C \left\| \int_0^\infty e^{i\tau\sqrt{-\Delta_D}} Wv \, d\tau \right\|_{\dot{H}^{\kappa-1}_D(\Omega)}$$
$$\leq C\|\sqrt{\mu}^{-1} Wv\|_{L^2\dot{H}^{\kappa-1}_D(\Omega)}.$$

Now it is enough to prove the inequality:

$$\|\sqrt{\mu}^{-1} Wv\|_{L^2\dot{H}^{\kappa-1}_D(\Omega)} \leq C\|f\|_{\dot{H}^\kappa_D(\Omega)}$$

for $\kappa = 0,\ 1$. Then interpolating these inequalities, we can get the desired estimate. Since the assumptions $(AW1)$ and $(AW2)$ we have

$$\|\sqrt{\mu}^{-1} Wv\|_{L^2L^2(\Omega)}$$
$$\leq \max\{\|\mu^{-1}A\|_{L^\infty(\Omega)}, \|\mu^{-1}rB\|_{L^\infty(\Omega)}\} \left(\|\sqrt{\mu}\nabla v\|_{L^2L^2(\Omega)} + \|\sqrt{\mu}r^{-1}v\|_{L^2L^2(\Omega)} \right).$$

Applying the inequality (1.23) to the second term and using the inequality (1.24), we obtain

$$\|\sqrt{\mu}^{-1}Wv\|_{L^2L^2(\Omega)} \le C\|f\|_{\dot{H}^1_D(\Omega)}.$$

This implies the case of $\kappa = 1$. If $\kappa = 0$, write

$$Wv = \nabla \cdot (Av) - \nabla \cdot Av + Bv,$$

and hence

$$\sqrt{\mu}^{-1}Wv = \nabla \cdot (\sqrt{\mu}^{-1}Av) - A \cdot \nabla\sqrt{\mu}^{-1}v - \sqrt{\mu}^{-1}(\nabla \cdot A - B)v.$$

It follows from the fact $\|\nabla \cdot \vec{f}\|_{\dot{H}^{-1}_D(\Omega)} \le \|\vec{f}\|_{L^2(\Omega)}$ for $\vec{f}|_{\partial\Omega} = \vec{0}$ that

$$\|\nabla \cdot (\sqrt{\mu}^{-1}Av)\|_{L^2\dot{H}^{-1}_D(\Omega)} \le \|\sqrt{\mu}^{-1}Av\|_{L^2L^2(\Omega)}.$$

Noting $|\nabla\sqrt{\mu}^{-1}| \le Cr^{-1}\sqrt{\mu}^{-1}$ and $|\nabla \cdot A| \le C|B|$, we have by Hardy inequality

$$\|A\cdot\nabla\sqrt{\mu}^{-1}v\|_{L^2\dot{H}^{-1}_D(\Omega)} \le C\|r^{-1}\sqrt{\mu}^{-1}Av\|_{L^2\dot{H}^{-1}_D(\Omega)} \le C\|\sqrt{\mu}^{-1}Av\|_{L^2L^2(\Omega)},$$

$$\|\sqrt{\mu}^{-1}(\nabla \cdot A - B)v\|_{L^2\dot{H}^{-1}_D(\Omega)} \le C\|\sqrt{\mu}^{-1}rBv\|_{L^2L^2(\Omega)}.$$

Hence using the assumptions $(AW1)$ and $(AW2)$, we can estimate

$$\begin{aligned}&\|\sqrt{\mu}^{-1}Wv\|_{L^2\dot{H}^{-1}_D(\Omega)}\\ \le\, & C\left(\|\sqrt{\mu}^{-1}Av\|_{L^2L^2(\Omega)} + \|\sqrt{\mu}^{-1}rBv\|_{L^2L^2(\Omega)}\right)\\ \le\, & C\max\{\|\mu^{-1}A\|_{L^\infty(\Omega)}, \|\mu^{-1}rB\|_{L^\infty(\Omega)}\}\|\sqrt{\mu}v\|_{L^2L^2(\Omega)}.\end{aligned}$$

Finally using Proposition 1.6 with $\gamma = 0$, we get the case of $\kappa = 0$.

Thus summarizing the above argument, we obtain (1.28) and the proof of Theorem 1.7 is now finished. □

1.3. **Klein-Gordon equations.** Throughout this section we put $\xi(r) = (1 + [r])^{-1}$. In the Klein-Gordon case, we always assume that the coefficients $b(x)$ and $c(x)$ satisfy $(AS1)$ and $(AS2)$ from Schrödinger case. Let the exponents p, q, $\tilde{p}$, $\tilde{q}$ and κ always satisfy the following relations.

$$\begin{aligned}&2 < p, \tilde{p} \le \infty, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad \frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2},\\ &\frac{1}{p} + \frac{3n}{2q} = \frac{3n}{4} - \kappa = \frac{1}{\tilde{p}'} + \frac{3n}{2\tilde{q}'} - 2.\end{aligned} \tag{1.29}$$

Remark that the final equality makes the relations for p, q, $\tilde{p}$, $\tilde{q}$ more restrictive as the following.

$$\frac{1}{\tilde{p}'} = \frac{1}{p} + \frac{1}{2}, \quad \frac{1}{\tilde{q}'} = \frac{1}{q} + \frac{1}{n}, \quad 2 < p < \infty. \tag{1.30}$$

The main result in this section is the following.

Theorem 1.9. *Let $n \geq 2$. Assume $(AS1)$ and $(AS2)$. Then for the solution u of* (1.2) *with* (1.3) *and the initial data $(u(0), \partial_t u(0)) = (f_1, f_2)$ the following estimate holds.*

$$\|u\|_{L^pL^q(\Omega)} + \|u\|_{L^\infty H^\kappa_D(\Omega)} + \|\partial_t u\|_{L^\infty H^{\kappa-1}_D(\Omega)} \leq C_3 \left(\|f_1\|_{H^\kappa_D(\Omega)} + \|f_2\|_{H^{\kappa-1}_D(\Omega)} + \|G\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)} \right).$$

We first prepare the Strichartz estimate for the free solutions which is proved in Mochizuki & Murai [20].

Lemma 1.10 (Mochizuki & Murai [20])**.** *Let $n \geq 2$. Then the following estimate holds.*

$$\|e^{-it\sqrt{-\Delta_D+m^2}} f\|_{L^pL^q(\Omega)} \leq C\|f\|_{H^\kappa_D(\Omega)}.$$

In the Klein-Gordon case, the weighted energy method used in the proof of Proposition 1.6 does not work well, and it is difficult to obtain the same type estimate as Proposition 1.6. For this reason we need to employ the another strategy. Let us first derive the estimate for the solution with $c = 0$. So we prepare the following smoothing estimate treated in Mochizuki & Murai [21].

Lemma 1.11. *Let $n \geq 2$. Assume $(AS1)$ and* (1.5)*. Then the following estimate holds for $0 \leq \gamma \leq 1$.*

$$\|\xi e^{-it\sqrt{-\Delta_b+m^2}} f\|_{L^2H^\gamma_D(\Omega)} \leq C\|f\|_{H^\gamma_D(\Omega)}.$$

Proof. The proof is based on the following estimate proved in Mochizuki & Murai [21].

$$\|\xi\nabla_b u_b\|_{L^2L^2(\Omega)} + m\|\xi u_b\|_{L^2L^2(\Omega)} + \|\xi\partial_t u_b\|_{L^2L^2(\Omega)} \leq C\left(\|\nabla_b f_1\|_{L^2(\Omega)} + m\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\right),$$

where $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to the origin and u_b is the solution of (1.2) with $c = G = 0$ (In [21], this inequality is proved by using (1.8) under the smallness condition on the rotation of b. Hence similar to the proof of (1.8), this assumption can be replaced with our one). Similar to the wave case, choosing $(f_1, f_2) = (f, 0)$ or $= (0, \sqrt{-\Delta_b + m^2} f)$, we obtain

$$\|\xi e^{-it\sqrt{-\Delta_b+m^2}} f\|_{L^2L^2(\Omega)} + \|\xi\nabla_b e^{-it\sqrt{-\Delta_b+m^2}} f\|_{L^2L^2(\Omega)} \leq C\|f\|_{H^1_D(\Omega)}.$$

This shows the case of $\gamma = 1$. Furthermore we can choose $(f_1, f_2) = (0, f)$ or $= (\sqrt{-\Delta_b + m^2}^{-1} f, 0)$ to obtain

$$\|\xi \cos(t\sqrt{-\Delta_b + m^2}) f\|_{L^2L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

$$\|\xi \sin(t\sqrt{-\Delta_b + m^2})f\|_{L^2L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

These inequalities imply the case of $\gamma = 0$. Thus the assertion follows from the interpolation argument. □

The following proposition can be obtained.

Proposition 1.12. *Under the same conditions as Theorem 1.9 the following estimate holds.*

$$\|e^{-it\sqrt{-\Delta_b+m^2}}f\|_{L^pL^q(\Omega)} \le C\|f\|_{H^\kappa_D(\Omega)}.$$

Proof. It can be proved by the similar line of the proof of Theorem 1.7. $v_b = e^{-it\sqrt{-\Delta_b+m^2}}f$ satisfies the following problem.

$$\begin{cases} \partial_t^2 v_b - \Delta v_b + m^2 v_b = -Wv_b, \\ v_b(0) = f, \quad \partial_t v_b(0) = -i\sqrt{-\Delta_b + m^2}f, \\ v_b|_{\partial\Omega} = 0, \end{cases}$$

where

$$W = -2ib \cdot \nabla_b - i\nabla \cdot b - |b|^2.$$

By Duhamel's principle the solution v_b is represented as

$$\begin{aligned} v_b(t) = \cos(t\sqrt{-\Delta_D + m^2})f - i\frac{\sin(t\sqrt{-\Delta_D + m^2})}{\sqrt{-\Delta_D + m^2}}\sqrt{-\Delta_b + m^2}f \\ - \int_0^t \frac{\sin\{(t-\tau)\sqrt{-\Delta_D + m^2}\}}{\sqrt{-\Delta_D + m^2}} Wv_b\, d\tau. \end{aligned}$$

As to the first and second terms, it suffices to apply Lemma 1.10 and Lemma 1.1. Using Lemma 1.10 again and the dual inequality of Lemma 1.11 with $b = 0$, we have for the integral term

$$\left\| \int_0^\infty \frac{e^{-i(t-\tau)\sqrt{-\Delta_D+m^2}}}{\sqrt{-\Delta_D + m^2}} Wv_b\, d\tau \right\|_{L^pL^q(\Omega)} \le C\|\xi^{-1}Wv_b\|_{L^2H_D^{\kappa-1}(\Omega)}.$$

This norm can be estimated by using the assumption on $b(x)$ and Lemma 1.11 as follows.

$$\begin{aligned} &\|\xi^{-1}Wv_b\|_{L^2L^2(\Omega)} \\ &\le \|\xi^{-1}(b \cdot \nabla_b v_b + \nabla \cdot b v_b + |b|^2 v_b)\|_{L^2L^2(\Omega)} \\ &\le \max\{\|\xi^{-2}b\|_{L^\infty(\Omega)}, \|\xi^{-2}(\nabla \cdot b + |b|^2)\|_{L^\infty(\Omega)}\} \\ &\qquad\qquad \left(\|\xi\nabla_b v_b\|_{L^2L^2(\Omega)} + \|\xi v_b\|_{L^2L^2(\Omega)}\right) \\ &\le C\|f\|_{H^1_D(\Omega)}. \end{aligned}$$

Similarly we can write

$$Wv_b = -2i\nabla \cdot (bv_b) + i\nabla \cdot bv_b + |b|^2 v_b$$

to get

$$
\begin{aligned}
&\|\xi^{-1}Wv_b\|_{L^2H_D^{-1}(\Omega)}\\
&\le \|\xi^{-1}bv_b\|_{L^2L^2(\Omega)} + \|\xi^{-1}(\nabla\cdot b+|b|^2)v_b\|_{L^2L^2(\Omega)}\\
&\le \max\{\|\xi^{-2}b\|_{L^\infty(\Omega)}, \|\xi^{-2}(\nabla\cdot b+|b|^2)\|_{L^\infty(\Omega)}\}\|\xi v_b\|_{L^2L^2(\Omega)}\\
&\le C\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

Here we used the fact $|\nabla\xi^{-1}| \le C\xi^{-1}$ in the first step. Interpolating the above inequalities, we obtain the estimate for the integral term and then conclude the proof of Proposition 1.12. □

Proof of Theorem 1.9. Similar to the wave case, it is enough to prove the following inequalities.

$$
\|e^{-it\sqrt{L+m^2}}f\|_{L^\infty H_D^\kappa(\Omega)} \le C\|f\|_{H_D^\kappa(\Omega)}, \tag{1.31}
$$

$$
\|e^{-it\sqrt{L+m^2}}f\|_{L^pL^q(\Omega)} \le C\|f\|_{H_D^\kappa(\Omega)}. \tag{1.32}
$$

The inequality (1.31) follows from the unitarity of $e^{-it\sqrt{L+m^2}}$ and Lemma 1.1. On the other hand, $v = e^{-it\sqrt{L+m^2}}f$ satisfies the problem

$$
\begin{cases}
\partial_t^2 v - \Delta_b v + m^2 v = -cv,\\
v(0) = f, \quad \partial_t v(0) = -i\sqrt{L+m^2}f,\\
v|_{\partial\Omega} = 0,
\end{cases}
$$

and is represented as

$$
\begin{aligned}
v(t) = \cos(t\sqrt{-\Delta_b+m^2})f - i\frac{\sin(t\sqrt{-\Delta_b+m^2})}{\sqrt{-\Delta_b+m^2}}\sqrt{L+m^2}f\\
- \int_0^t \frac{\sin\{(t-\tau)\sqrt{-\Delta_b+m^2})\}}{\sqrt{-\Delta_b+m^2}}cv\,d\tau.
\end{aligned} \tag{1.33}
$$

Proposition 1.12 and the dual inequality of Lemma 1.11 give

$$
\begin{aligned}
\left\|\int_0^\infty \frac{e^{-i(t-\tau)\sqrt{-\Delta_b+m^2}}}{\sqrt{-\Delta_b+m^2}}cv\,d\tau\right\|_{L^pL^q(\Omega)} &\le C\left\|\int_0^\infty e^{i\tau\sqrt{-\Delta_b+m^2}}cv\,d\tau\right\|_{L^2(\Omega)}\\
&\le C\|\xi^{-1}cv\|_{L^2L^2(\Omega)}
\end{aligned}
$$

since $\kappa < 1$. It follows from the assumption on $c(x)$ and the smoothing estimate

$$
\|\xi e^{-it\sqrt{L+m^2}}f\|_{L^2L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}
$$

which can be obtained along the same arugument in [20] based on the inequality (1.8) that

$$
\|\xi^{-1}cv\|_{L^2L^2(\Omega)} \le \|\xi^{-2}c\|_{L^\infty(\Omega)}\|\xi v\|_{L^2L^2(\Omega)} \le C\|f\|_{L^2(\Omega)} \le C\|f\|_{H_D^\kappa(\Omega)}
$$

since $\kappa \geq 0$. Summarizing the above inequalities, we obtain

$$\left\| \int_0^t \frac{\sin\{(t-\tau)\sqrt{-\Delta_b + m^2}\}}{\sqrt{-\Delta_b + m^2}} cv \, d\tau \right\|_{L^p L^q(\Omega)} \leq C\|f\|_{H_D^\kappa(\Omega)}.$$

We can apply this inequality and Proposition 1.12 to each term of (1.33) to obtain (1.32), and thus Theorem 1.9 is now proved. □

2. Application to nonlinear problem

Consider in Ω the Schrödinger, wave and Klein-Gordon equations:

$$i\partial_t u = Lu + G(u), \tag{2.1}$$

$$\partial_t^2 u + Lu + m^2 u + G(u) = 0, \tag{2.2}$$

with a linear or a nonlinear term G depending on the solution. L satisfies the assumptions from Theorem 1.5, 1.7 or 1.9. As an application of the Strichartz estimates, we shall investigate the global existence and scattering theory for the solution of the above equations with boundary condition (1.3). Roughly speaking, our purpose in this section is to prove that the solution $u(t)$ behaves like the corresponding free ($G = 0$) solution $u^\pm(t)$ to the problem:

$$\begin{cases} i\partial_t u^\pm = Lu^\pm, \\ u^\pm(0) = f^\pm, \\ u^\pm|_{\partial\Omega} = 0, \end{cases} \quad \textit{or} \quad \begin{cases} \partial_t^2 u^\pm + Lu^\pm + m^2 u^\pm = 0, \\ (u^\pm(0), \partial_t u(0)) = (f_1^\pm, f_2^\pm), \\ u^\pm|_{\partial\Omega} = 0 \end{cases} \tag{2.3}$$

when the time tends to infinity, and hence the scattering operator can be defined in an appropriate space.

The equations (2.1) and (2.2) with the initial data f are rewritten as an integral equation:

$$u(t) = \Gamma(t)f + \int_0^t \mathcal{K}(t-\tau)G(u(\tau)) \, d\tau, \tag{2.4}$$

where

$$\Gamma(t)f = \begin{cases} i\mathcal{K}(t)f, \\ \partial_t \mathcal{K}(t)f_1 + \mathcal{K}(t)f_2, \end{cases}$$

$$\mathcal{K}(t) = \begin{cases} -ie^{-itL}, & \text{(Schrödinger)} \\ \dfrac{\sin(t\sqrt{L+m^2})}{\sqrt{L+m^2}}, & \text{(wave or Klein-Gordon)} \end{cases}$$

We will consider the solution to (2.4) in the following space. Denote $\mathcal{H} = L^2(\Omega)$ (Schrödinger), $\dot{H}_D^\kappa(\Omega) \times \dot{H}_D^{\kappa-1}(\Omega)$ (wave) or $H_D^\kappa(\Omega) \times H_D^{\kappa-1}(\Omega)$ (Klein-Gordon). We set

$$Y_T = \{u \in L_T^\infty \mathcal{H} \cap L_T^p L^q(\Omega); \|u\|_{Y_T} = \|u\|_{L_T^\infty \mathcal{H}} + \|u\|_{L_T^p L^q(\Omega)} \leq K\},$$

where $L^p_T = L^p(0,T)$. When the wave case, $u \in \mathcal{H}$ means that $u \in \dot{H}^\kappa_D(\Omega)$, $\partial_t u \in \dot{H}^{\kappa-1}_D(\Omega)$ and its norm is defined by

$$\|u\|_{\mathcal{H}} := \|u\|_{\dot{H}^\kappa_D(\Omega)} + \|\partial_t u\|_{\dot{H}^{\kappa-1}_D(\Omega)}.$$

When Klein-Gordon case we should replace the homogeneous Sobolev space with the inhomogeneous one. $K > 0$ is a constant depending on the initial data which will be defined later. Denote Y_T with $T = \pm\infty$ by Y.

To show our result, we impose the following hypothesis on the nonlinearity.

(H) $G(0) = 0$ and for $u_1, u_2 \in Y_T$

$$\|G(u_1) - G(u_2)\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)} \le C_4 T^\theta (\|u_1\|^{\alpha-1}_{Y_T} + \|u_2\|^{\alpha-1}_{Y_T})\|u_1 - u_2\|_{Y_T},$$

where $\alpha \ge 1$, $\theta = \theta(p, \tilde{p}, \alpha, n) \ge 0$.

Main result in this section is as follows.

Theorem 2.1. *Under the hypothesis (H), the following assertions hold.*

(i) When $\theta > 0$ for any initial data $f \in \mathcal{H}$, there exists $T > 0$ and a unique local solution u of (2.4) in Y_T.

(ii) When $\theta = 0$, if we restrict as $C_4 K^{\alpha-1} \ll 1$, then, there exists a unique global solution u of (2.4). Moreover, for any $f \in \mathcal{H}$ as above there exists the scattering data $f^+ \in \mathcal{H}$ such that the corresponding solution $u^+(t)$ of (2.3) satisfies

$$\|u(t) - u^+(t)\|_{\mathcal{H}} \to 0$$

as $t \to \infty$. Conversely, if $f^- \in \mathcal{H}$ is sufficiently small and $u^-(t)$ is a corresponding solution of (2.3), then there exists a solution of (2.4) satisfies

$$\|u(t) - u^-(t)\|_{\mathcal{H}} \to 0$$

as $t \to -\infty$. Thus, the scattering operator $S : f^- \to f^+$ is well-defined in a neighborhood of the origin in $\mathcal{H}$.

Proof. We construct the contraction map in Y_T. Put

$$\Phi[u](t) = \Gamma(t)f + \int_0^t \mathcal{K}(t-\tau)G(u(\tau))\,d\tau.$$

Using Theorem 1.5, 1.7 or 1.9, we have

$$\|\Phi[u]\|_{Y_T} \le \tilde{C}\left(\|f\|_{\mathcal{H}} + \|G(u)\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)}\right), \quad \tilde{C} = C_1,\ C_2 \text{ or } C_3$$

and applying (H) to the second term of the right hand side, we obtain

$$\|\Phi[u]\|_{Y_T} \le \tilde{C}\|f\|_{\mathcal{H}} + \tilde{C}C_4 T^\theta K^\alpha. \tag{2.5}$$

Similarly, we have

$$\|\Phi[u_1] - \Phi[u_2]\|_{Y_T} \le 2\tilde{C} C_4 T^\theta K^{\alpha-1} \|u_1 - u_2\|_{Y_T}, \quad u_1, u_2 \in Y_T. \tag{2.6}$$

Choosing $K = 2\tilde{C}\|f\|_{\mathcal{H}}$ and $T^\theta < 1/(4\tilde{C} C_4 K^{\alpha-1})$, we obtain

$$\|\Phi[u]\|_{Y_T} < K, \quad \|\Phi[u_1] - \Phi[u_2]\|_{Y_T} < \frac{1}{2}\|u_1 - u_2\|_{Y_T} \tag{2.7}$$

which imply the existence of the unique local solution in Y_T via the well-known fixed point argument.

(*ii*) In this case it should be restricted that $4\tilde{C} C_4 K^{\alpha-1} < 1$ with same K as above. Then we obtain (2.7) for any T. Thus the same argument as (*i*) shows the unique global existence of the solution. The asymptotic behavior follows from this result. In fact, for given small data $f^- \in \mathcal{H}$ let $u^-(t)$ be the solution of (2.1) or (2.2) with $G = 0$. Then there exists the solution $u(t)$ of the form

$$u(t) = u^-(t) + \int_{-\infty}^t K(t-\tau) G(u)\, d\tau,$$

and since the solution u is global in Y, Theorem 1.5, 1.7 or 1.9 and (H) show

$$\begin{aligned} \left\| \int_{-\infty}^t K(t-\tau) G(u)\, d\tau \right\|_{L^\infty \mathcal{H}} &\le \tilde{C} \|G(u)\|_{L^{\tilde{p}'} L^{\tilde{q}'}(\Omega)} \\ &\le \tilde{C} C_4 \|u\|_Y^\alpha < \infty. \end{aligned}$$

Hence we have

$$\|u(t) - u^-(t)\|_{\mathcal{H}} \to 0$$

as $t \to -\infty$. Similarly if $u(t)$ is the solution with small data $f \in \mathcal{H}$, then there exists a solution $u^+(t)$ with data $f^+ \in \mathcal{H}$ satisfies

$$\|u(t) - u^+(t)\|_{\mathcal{H}} \to 0$$

as $t \to \infty$. This concludes the proof of Theorem 2.1. □

We will give an example for the nonlinearity satisfying our hypothesis (H) in the next subsections.

2.1. **Nonlinear Schrödinger equation.**

Let $V(x,t)$ be a time-dependent complex function belonging to $L_T^\nu L^\rho(\Omega)$ where

$$0 \le \frac{1}{\rho} < \frac{2}{n}, \quad \frac{1}{\nu} \le 1 - \frac{n}{2\rho}.$$

For this function we set a nonlinear term as $G(u) = V(x,t)|u|^{\alpha-1}u$. The local solution exists when the power α satisfies

$$1 \le \alpha \le 1 + \frac{4}{n}\left(1 - \frac{1}{\nu} - \frac{n}{2\rho}\right), \tag{2.8}$$

while the global result is proved if and only if the endpoint case. To explain this fact, we need to prove that the above G satisfies our hypothesis (H) which is shown in the following lemma.

Lemma 2.2. *Let G satisfy* (2.8)*. Then for u_1, $u_2 \in Y_T$ the following inequality holds.*

$$\begin{aligned}&\|G(u_1) - G(u_2)\|_{L_T^{\tilde{p}'}L^{\tilde{q}'}(\Omega)}\\ &\quad\le C_V T^{\frac{1}{\tilde{p}'}-\frac{\alpha}{p}-\frac{1}{\nu}}\left(\|u_1\|^{\alpha-1}_{L_T^pL^q(\Omega)} + \|u_2\|^{\alpha-1}_{L_T^pL^q(\Omega)}\right)\|u_1-u_2\|_{L_T^pL^q(\Omega)}.\end{aligned}$$

Proof. We shall prove only the case of $\tilde{p} = p$ and $\tilde{q} = q$. More general case can be proved along the similar line. Using Hölder inequality we have

$$\begin{aligned}&\|V|u_1|^{\alpha-1}u_1 - V|u_2|^{\alpha-1}u_2\|_{L_T^{p'}L^{q'}(\Omega)}\\ &\le C\left(\|V|u_1|^{\alpha-1}\|_{L_T^{\frac{p}{p-2}}L^{\frac{q}{q-2}}(\Omega)} + \|V|u_2|^{\alpha-1}\|_{L_T^{\frac{p}{p-2}}L^{\frac{q}{q-2}}(\Omega)}\right)\|u_1-u_2\|_{L_T^pL^q(\Omega)}.\end{aligned}$$

Hence it sufficies to estimate the term $\|V|u|^{\alpha-1}\|_{L_T^{\frac{p}{p-2}}L^{\frac{q}{q-2}}(\Omega)}$. Using Hölder inequality again, we have

$$\|V|u|^{\alpha-1}\|_{L_T^{\frac{p}{p-2}}L^{\frac{q}{q-2}}(\Omega)} \le \|V\|_{L_T^{\nu}L^{\rho}(\Omega)}\||u|^{\alpha-1}\|_{L_T^aL^b(\Omega)}$$

with

$$1 - \frac{2}{p} = \frac{1}{\nu} + \frac{1}{a}, \quad 1 - \frac{2}{q} = \frac{1}{\rho} + \frac{1}{b}. \tag{2.9}$$

Here choosing

$$p \ge a(\alpha-1), \quad q = b(\alpha-1), \tag{2.10}$$

we obtain

$$\|V|u|^{\alpha-1}\|_{L_T^{\frac{p}{p-2}}L^{\frac{q}{q-2}}(\Omega)} \le T^{1-\frac{\alpha+1}{p}-\frac{1}{\nu}}\|V\|_{L_T^{\nu}L^{\rho}(\Omega)}\|u\|^{\alpha-1}_{L_T^pL^q(\Omega)}$$

which implies the desired inequality. The conditions (2.9) and (2.10) together with the admissibility (1.12) imply

$$\alpha \le 1 + \frac{4}{n}\left(1 - \frac{1}{\nu} - \frac{n}{2\rho}\right).$$

This corresponds to the condition $\theta \geq 0$, and hence we obtain the global solution and the scattering results when the equality holds in the above inequality. □

Remark 1. According to this lemma, when $\alpha = 1$, that is, (2.1) is a linear equation with time-dependent potential, the global behavior and scattering occurs under the smallness condition

$$2C_1\|V\|_{L^\nu L^\rho(\Omega)} < 1, \quad \frac{1}{\nu} = 1 - \frac{n}{2\rho}.$$

This is the same result of Mochizuki & Motai [19] which is treated the case of $b = c = 0$ and $\Omega = \mathbb{R}^n$.

Remark 2. In the case when V is a constant, $b = c = 0$ and $\Omega = \mathbb{R}^n$, there are so many results (see [7] for details). Focusing on that relating to the Strichartz estimates in an exterior domain, we refer to Blair & Smith & Sogge [2], Burq & Gérard & Tzvetkov [4], Cassano & D'Ancona [6] and the references therein. Especially the magnetic potential is treated in [6] under the assumption that the Strichartz estimate for e^{-itL} are valid.

2.2. **Nonlinear wave equation.** Let $V(x,t)$ be a time-dependent complex function belonging to $L^\nu_T L^\rho(\Omega)$ where $(n+1)/2 \leq \rho, \nu \leq \infty$ and

$$\frac{(n-1)^2+4}{\nu} \leq \frac{(n-1)(n-3)}{\rho} + 4. \tag{2.11}$$

We set $G(u) = V(x,t)|u|^{\alpha-1}u$. For this term the following situations are considerable: when $\alpha = 1$, then $G(u) = V(x,t)u$, that is, (2.2) is a linear equation with time-dependent potnetial. In this case it can be easily checked that the relation (1.26) and Hölder inequality give

$$\|Vu_1 - Vu_2\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)} \leq T^{\frac{2}{n+1}-\frac{1}{\nu}}\|V\|_{L^\nu_T L^{\frac{n+1}{2}}(\Omega)}\|u_1 - u_2\|_{L^p_T L^q(\Omega)}.$$

This implies the hypothesis (H), and then the local existence result can be proved by the previous argument. Moreover if we impose the smallness condition

$$2C_2\|V\|_{L^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)} < 1 \tag{2.12}$$

instead of that for the initial data, then the global result can be also proved.

On the other hand when $\alpha \neq 1$, we restrict $\nu \neq (n+1)/2$, $\rho \neq (n+1)/2$. The local existence of the solution for (2.2) can be proved under the condition:

$$1+\frac{4(n-1)}{(n-1)^2+4}\left(1-\frac{n+1}{2\rho}\right)<\alpha \tag{2.13}$$
$$<\min\left\{1+\frac{4}{n-1}\left(1-\frac{1}{\nu}-\frac{n-1}{2\rho}\right), 1+\frac{4(n-1)}{(n-1)^2-4}\left(1-\frac{n+1}{2\rho}\right)\right\}$$
$$\text{if}\quad \frac{n-3}{2(n-1)}<\kappa<\min\left\{\frac{1-\frac{n+1}{2\nu}}{2\left(1-\frac{1}{\nu}-\frac{n-1}{2\rho}\right)}, \frac{n+1}{2(n-1)}\right\}.$$

Moreover in addition to (2.11) we restrict

$$\frac{n-3}{2\nu}\geq\frac{n-1}{2\rho}-\frac{2}{n+1}. \tag{2.14}$$

Then the global existence and scattering results can be obtained under the condition:

(2.15)
$$1+\frac{4}{n-1}\left(1-\frac{1}{\nu}-\frac{n-1}{2\rho}\right)\leq\alpha<1+\frac{4(n-1)}{n^2-2n-1}\left(1-\frac{1}{2\nu}-\frac{n}{2\rho}\right)$$
$$\text{if}\quad \frac{1-\frac{n+1}{2\nu}}{2\left(1-\frac{1}{\nu}-\frac{n-1}{2\rho}\right)}\leq\kappa<\frac{n+1}{2(n-1)}.$$

Here to keep the consistency of the inequalities (2.13) and (2.15) the restrictions (2.11) and (2.14) are needed. Remark that when V is a complex constant, we put $\nu=\rho=\infty$ and the conditions (2.13) and (2.15) are rewritten as

$$\frac{(n+1)^2}{(n-1)^2+4}<\alpha<\frac{n+3}{n-1}\quad\text{if}\quad\frac{n-3}{2(n-1)}<\kappa<\frac{1}{2},$$

$$\frac{n+3}{n-1}\leq\alpha<\frac{n^2+2n-5}{n^2-2n-1}\quad\text{if}\quad\frac{1}{2}\leq\kappa<\frac{n+1}{2(n-1)},$$

respectively. These conditions coincide with those of the case of $\Omega=\mathbb{R}^n$ (see Lindblad & Sogge [14]). For the case when $\Omega\neq\mathbb{R}^n$ and $\kappa=1$, we refer to [1] and references therein.

Now let us prove that $G(u)$ with (2.13) or (2.15) satisfies our hypothesis (H) in the following lemmas.

Lemma 2.3. *Let G satisfy* (2.13)*. Then for u_1, $u_2 \in W_T$ the following inequality holds.*

$$\|G(u_1) - G(u_2)\|_{L_T^{\tilde{p}'} L^{\tilde{q}'}(\Omega)} \tag{2.16}$$
$$\le C_V T^{\frac{2}{n+1} - \frac{\alpha-1}{p} - \frac{1}{\nu}} \left(\|u_1\|^{\alpha-1}_{L_T^p L^q(\Omega)} + \|u_2\|^{\alpha-1}_{L_T^p L^q(\Omega)} \right) \|u_1 - u_2\|_{L_T^p L^q(\Omega)}.$$

Proof. Noticing the relation (1.26), the Hölder inequality gives

$$\|V|u_1|^{\alpha-1}u_1 - V|u_2|^{\alpha-1}u_2\|_{L_T^{\tilde{p}'} L^{\tilde{q}'}(\Omega)}$$
$$\le \left(\|V|u_1|^{\alpha-1}\|_{L_T^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)} + \|V|u_2|^{\alpha-1}\|_{L_T^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)} \right) \|u_1 - u_2\|_{L_T^p L^q(\Omega)}.$$

Hence, it is enough to estimate the term $\|V|u|^{\alpha-1}\|_{L_T^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)}$. Choosing

$$p \ge \frac{\nu(n+1)}{2\nu - n - 1}(\alpha - 1), \quad q = \frac{\rho(n+1)}{2\rho - n - 1}(\alpha - 1), \tag{2.17}$$

we can get by Hölder inequality again

$$\|V|u|^{\alpha-1}\|_{L_T^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)} \le \|V\|_{L_T^\nu L^\rho(\Omega)} \||u|^{\alpha-1}\|_{L_T^{\frac{\nu(n+1)}{2\nu-n-1}} L^{\frac{\rho(n+1)}{2\rho-n-1}}(\Omega)}$$
$$\le T^{\frac{2}{n+1} - \frac{\alpha-1}{p} - \frac{1}{\nu}} \|V\|_{L_T^\nu L^\rho(\Omega)} \|u\|^{\alpha-1}_{L_T^p L^q(\Omega)}.$$

This proves the inequality (2.16). Eliminating p and q from (2.17), we obtain

$$\alpha \le 1 + \frac{4}{n-1} \left(1 - \frac{1}{\nu} - \frac{n-1}{2\rho} \right).$$

Here note that the global result can be obtained if the equality holds. On the other hand, the second equality of (2.17) together with the restriction by the admissibility (1.26) gives

$$1 + \frac{4(n-1)}{(n-1)^2 + 4} \left(1 - \frac{n+1}{2\rho} \right) < \alpha < 1 + \frac{4(n-1)}{(n-1)^2 - 4} \left(1 - \frac{n+1}{2\rho} \right).$$

Moreover, the parameter κ is represented as

$$\kappa = \frac{n+1}{4} - \left(1 - \frac{n+1}{2\rho} \right) \frac{1}{\alpha - 1}$$

by the second equality of (2.17) and (1.25). Thus, summarizing the above inequalities, the condition (2.13) can be obtained. □

Lemma 2.4. *Let G satisfy* (2.15)*. Then the following inequality holds for* $0 \le \beta \le \alpha - 1$*.*

$$\begin{aligned}(2.18)\quad & \|G(u_1) - G(u_2)\|_{L^{\tilde{p}'} L^{\tilde{q}'}(\Omega)} \\ & \le C_V \left(\|u_1\|^{\alpha-\beta-1}_{L^\infty \dot{H}^\kappa_D(\Omega)} \|u_1\|^{\beta}_{L^p L^q(\Omega)} + \|u_2\|^{\alpha-\beta-1}_{L^\infty \dot{H}^\kappa_D(\Omega)} \|u_2\|^{\beta}_{L^p L^q(\Omega)} \right) \\ & \qquad \times \|u_1 - u_2\|_{L^p L^q(\Omega)}.\end{aligned}$$

Proof. Similar to the case of the previous lemma, it sufficies to estimate the term $\|V|u|^{\alpha-1}\|_{L^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)}$. Using the Gagliardo-Nirenberg inequality, we obtain for $0 \le \beta \le \alpha - 1$

$$\||u|^{\alpha-1}\|_{L^{\frac{\nu(n+1)}{2\nu-n-1}} L^{\frac{\rho(n+1)}{2\rho-n-1}}(\Omega)} \le \|u\|^{\alpha-\beta-1}_{L^\infty L^{\frac{2n}{n-2\kappa}}(\Omega)} \|u\|^{\beta}_{L^p L^q(\Omega)},$$

where

$$(2.19)\qquad \frac{2}{n+1} - \frac{1}{\nu} = \frac{\beta}{p}, \quad \frac{2}{n+1} - \frac{1}{\rho} = \left(\frac{1}{2} - \frac{\kappa}{n}\right)(\alpha - \beta - 1) + \frac{\beta}{q}.$$

Finally applying Sobolev embedding

$$(2.20)\qquad \dot{H}^\kappa_D(\Omega) \subset L^{\frac{2n}{n-2\kappa}}(\Omega)$$

to the first term, we obtain

$$\|V|u|^{\alpha-1}\|_{L^{\frac{n+1}{2}} L^{\frac{n+1}{2}}(\Omega)} \le \|V\|_{L^\nu L^\rho(\Omega)} \|u\|^{\alpha-\beta-1}_{L^\infty \dot{H}^\kappa_D(\Omega)} \|u\|^{\beta}_{L^p L^q(\Omega)}$$

which implies the inequality (2.18). It follows from (2.19) and the admissibility (1.25) that we have for α and β

$$\frac{\alpha-1}{n}\left(\frac{2}{n-1} - \frac{n+1}{\nu(n-1)}\right) = \beta\left(\frac{\alpha-1}{2} - \frac{2}{n} + \frac{1}{n\nu} + \frac{1}{\rho}\right).$$

Substituting this to $0 \le \beta \le \alpha - 1$, we obtain

$$(2.21)\qquad \alpha \ge 1 + \frac{4}{n-1}\left(1 - \frac{1}{\nu} - \frac{n-1}{2\rho}\right).$$

Moreover, (2.19) with (1.25) shows

$$\alpha = 1 + \frac{4}{n-2\kappa}\left(1 - \frac{1}{2\nu} - \frac{n}{2\rho}\right).$$

Substituting this to (2.21) and using the restriction by (1.26), the relation (2.15) can be obtained. $\square$

2.3. **Nonlinear Klein-Gordon equation.** Let $V(x,t)$ be a time-dependent complex function belonging to $L^\nu_T L^\rho(\Omega)$, $2 \le \nu \le \infty$, $n \le \rho \le \infty$ and we set $G(u) = V(x,t)|u|^{\alpha-1}u$. Similar to the wave case, when $\alpha = 1$ the hypothesis (H) follows from (1.30) and Hölder inequality, and hence the global behavior can be obtained under the smallness condition

$$2C_3\|V\|_{L^2L^n(\Omega)} < 1. \tag{2.22}$$

When $\alpha \neq 1$ we set $\nu \neq 2$, $\rho \neq n$, and the local result holds under

(2.23)
$$1+\frac{2}{n}\left(1-\frac{n}{\rho}\right) < \alpha < \min\left\{1+\frac{4}{n}\left(1-\frac{1}{\nu}-\frac{n}{2\rho}\right), 1+\frac{2}{n-2}\left(1-\frac{n}{\rho}\right)\right\}$$
$$\text{if} \quad 0 < \kappa < \min\left\{\frac{n\left(1-\frac{2}{\nu}\right)}{4\left(1-\frac{1}{\nu}-\frac{n}{2\rho}\right)}, 1\right\}.$$

The global one also holds under

$$1+\frac{4}{n}\left(1-\frac{1}{\nu}-\frac{n}{2\rho}\right) \le \alpha < 1+\frac{2}{n-2}\left(1-\frac{n}{\rho}\right) \tag{2.24}$$
$$\text{if} \quad \frac{n\left(1-\frac{2}{\nu}\right)}{4\left(1-\frac{1}{\nu}-\frac{n}{2\rho}\right)} \le \kappa < 1$$

with the restriction
$$\frac{n-2}{\nu} \ge \frac{n}{\rho} + \frac{n-4}{2}.$$

When V is a constant, we put $\nu = \rho = \infty$ and then the conditions (2.23) and (2.24) are rewritten as follows:

$$1+\frac{2}{n} < \alpha < \min\left\{1+\frac{4}{n}, \frac{n}{n-2}\right\} \quad \text{if} \quad 0 < \kappa < \min\left\{\frac{n}{4}, 1\right\},$$
$$1+\frac{4}{n} \le \alpha < \frac{n}{n-2} \quad \text{if} \quad \frac{n}{4} \le \kappa < 1.$$

For this nonlinearity the following inequalities can be obtained.

Lemma 2.5. *Let $G(u) = V(x,t)|u|^{\alpha-1}u$. Then we have*

$$\begin{aligned} &\|G(u_1) - G(u_2)\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)} \\ &\le C_V T^{\frac{1}{2}-\frac{\alpha-1}{p}-\frac{1}{\nu}} \left(\|u_1\|^{\alpha-1}_{L^p_T L^q(\Omega)} + \|u_2\|^{\alpha-1}_{L^p_T L^q(\Omega)}\right) \|u_1 - u_2\|_{L^p_T L^q(\Omega)}, \end{aligned} \tag{2.25}$$

when G satisfies (2.23)*, and for* $0 \le \beta \le \alpha - 1$

$$\|G(u_1) - G(u_2)\|_{L^{\tilde{p}'}L^{\tilde{q}'}(\Omega)} \le C_V \left(\|u_1\|^{\alpha-\beta-1}_{L^\infty H^\kappa_D(\Omega)} \|u_1\|^{\beta}_{L^pL^q(\Omega)} + \|u_2\|^{\alpha-\beta-1}_{L^\infty H^\kappa_D(\Omega)} \|u_2\|^{\beta}_{L^pL^q(\Omega)} \right) \times \|u_1 - u_2\|_{L^pL^q(\Omega)}, \tag{2.26}$$

when G satisfies (2.24).

Proof. Noticing the relation (1.30), the Hölder inequality gives

$$\|G(u_1) - G(u_2)\|_{L^{\tilde{p}'}_TL^{\tilde{q}'}(\Omega)} \le C \left(\|V|u_1|^{\alpha-1}\|_{L^2_TL^n(\Omega)} + \|V|u_2|^{\alpha-1}\|_{L^2_TL^n(\Omega)} \right) \|u_1 - u_2\|_{L^p_TL^q(\Omega)}.$$

Choosing

$$p \ge \frac{2\nu}{\nu-2}(\alpha-1), \quad q = \frac{n\rho}{\rho - n}(\alpha - 1) \tag{2.27}$$

which together with (1.29) is equivalent to the condition (2.23), we have by Hölder inequality

$$\|V|u|^{\alpha-1}\|_{L^2_TL^n(\Omega)} \le T^{\frac{1}{2}-\frac{\alpha-1}{p}-\frac{1}{\nu}} \|V\|_{L^\nu_TL^\rho(\Omega)} \|u\|^{\alpha-1}_{L^p_TL^q(\Omega)}.$$

This implies the inequality (2.25).

On the other hand, Gagliardo-Nirenberg inequality shows

$$\||u|^{\alpha-1}\|_{L^{\frac{2\nu}{\nu-2}}L^{\frac{n\rho}{\rho-n}}(\Omega)} \le \|u\|^{\alpha-\beta-1}_{L^\infty L^{\frac{2n}{n-2\kappa}}(\Omega)} \|u\|^{\beta}_{L^pL^q(\Omega)}$$

for $0 \le \beta \le \alpha - 1$ where

$$\frac{1}{2} - \frac{1}{\nu} = \frac{\beta}{p}, \quad \frac{1}{n} - \frac{1}{\rho} = \left(\frac{1}{2} - \frac{\kappa}{n}\right)(\alpha - \beta - 1) + \frac{\beta}{q}.$$

This condition together with (1.29) imply (2.24). Finally using Sobolev embedding

$$H^\kappa_D(\Omega) \subset L^{\frac{2n}{n-2\kappa}}(\Omega), \tag{2.28}$$

we obtain

$$\|V|u|^{\alpha-1}\|_{L^2L^n(\Omega)} \le \|V\|_{L^\nu L^\rho(\Omega)} \|u\|^{\alpha-\beta-1}_{L^\infty H^\kappa_D(\Omega)} \|u\|^{\beta}_{L^pL^q(\Omega)}.$$

Thus, the inequality (2.26) holds. □

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