ON HEIGHT ZERO CHARACTERS OF p-SOLVABLE GROUPS

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Abstract

Let G be a finite p-solvable group and N a normal subgroup of G. Suppose that B is a p-block of G with defect group D such that $|D| > |D \cap N|$. Given $\mu \in \mathrm{Irr}(N)$, we show that the set of height zero characters in $\mathrm{Irr}(B)$ that lie over μ is either empty or contains two or more elements.

1. Introduction

Fix a prime p and let G be a finite group. Let B be a Brauer p-block of G and denote by $\mathrm{Irr}_0(B)$ the set of ordinary irreducible characters in B of height zero. If the defect of B is positive, then a result of Cliff, Plesken and Weiss [1] asserts that $|\mathrm{Irr}_0(B)| \geq 2$. (See also [7].)

Now let N be a normal subgroup of G and suppose $\mu \in \mathrm{Irr}(N)$. Let $\mathrm{Irr}(G|\mu)$ be the set of irreducible characters of G that lie over μ, and write $\mathrm{Irr}_0(B|\mu) = \mathrm{Irr}_0(B) \cap \mathrm{Irr}(G|\mu)$. The aim of this paper is to prove a relative version of the above result in case G is p-solvable.

Theorem. *Let N be a normal subgroup of a p-solvable group G, and let B be a p-block of G with defect group D such that $|D| > |D \cap N|$. Let $\mu \in \mathrm{Irr}(N)$ and suppose* $\mathrm{Irr}_0(B|\mu) \neq \emptyset$. *Then* $|\mathrm{Irr}_0(B|\mu)| \geq 2$.

2. Proof of Theorem

Fix a prime p and let B be a p-block of a group G. Let N be a normal subgroup of G and let $\mu \in \mathrm{Irr}(b)$, where b is a p-block of N. Suppose μ is an irreducible constituent of χ_N, where $\chi \in \mathrm{Irr}(B)$. By [8, Lemma 2.2], we have $\mathrm{ht}(\chi) \geq \mathrm{ht}(\mu)$. If ν is any other constituent of χ_N, then ν is G-conjugate to μ and belongs to a G-conjugate of b. Since G-conjugate blocks of N have equal defects, the difference $\mathrm{ht}(\chi) - \mathrm{ht}(\mu)$ is independent of the choice of the constituent μ.

If $\mathrm{ht}(\chi) = \mathrm{ht}(\mu)$, then the character χ is said to be of *relative height zero* with respect to N. We denote by $\mathrm{Irr}_0^{\mu}(B)$ the set of all those characters in $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\mu)$ having relative height zero with respect to N. It is clear that $\chi \in \mathrm{Irr}_0(B|\mu)$

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if and only if $\mathrm{ht}(\mu)=0$ and $\chi \in \mathrm{Irr}_0^{\mu}(B)$. Now our main theorem is a consequence of the following more general result.

Theorem 2.1. *Let $N \lhd G$, where G is p-solvable and let B be a p-block of G with defect group D such that $|D| > |D \cap N|$. Let $\mu \in \mathrm{Irr}(N)$ and assume $\mathrm{Irr}_0^{\mu}(B) \neq \emptyset$. Then $|\mathrm{Irr}_0^{\mu}(B)| \geq 2$.*

In order to prove Theorem 2.1, we need a series of preliminary results.

Lemma 2.2. *Let N be a normal subgroup of an arbitrary group G. Let $\mu \in \mathrm{Irr}(N)$ and suppose $\chi \in \mathrm{Irr}_0^{\mu}(B)$, where B is a p-block of G. Let T be the inertial group of μ in G and let $\theta \in \mathrm{Irr}(T|\mu)$ be the Clifford correspondent of χ. If B_0 is the p-block of T to which θ belongs, then B_0 and B have a common defect group, $\theta \in \mathrm{Irr}_0^{\mu}(B_0)$ and $|\mathrm{Irr}_0^{\mu}(B)| \geq |\mathrm{Irr}_0^{\mu}(B_0)|$.*

Proof. Let b be the block of N such that $\mu \in \mathrm{Irr}(b)$. Then both B and B_0 cover b by Lemma 5.5.7 of [9]. Next, as $\theta^G = \chi$, Lemma 5.3.1(ii) of [9] implies that ${B_0}^G$ is defined and ${B_0}^G = B$. By [9, Theorem 5.5.16], we can choose defect groups Q and D_0 for b and B_0, respectively, such that $Q = D_0 \cap N$. Then by [9, Lemma 5.3.3], there exists a defect group D of B such that $D_0 \subseteq D$.

Since θ lies over μ, we have $\mathrm{ht}(\theta) \geq \mathrm{ht}(\mu)$, and so $\theta(1)_p = |T : D_0|_p p^{\mathrm{ht}(\theta)} \geq |T : D_0|_p p^{\mathrm{ht}(\mu)}$. Then $\chi(1)_p = |G : T|_p \theta(1)_p \geq |G : D_0|_p p^{\mathrm{ht}(\mu)}$. On the other hand, as $\chi \in \mathrm{Irr}_0^{\mu}(B)$, we have that $\chi(1)_p = |G : D|_p p^{\mathrm{ht}(\chi)} = |G : D|_p p^{\mathrm{ht}(\mu)}$. It follows that $|D_0| \geq |D|$. Now, as $D_0 \subseteq D$, we conclude that $D = D_0$, thereby proving the first assertion. Then we get that $\theta(1)_p = |T : D_0|_p p^{\mathrm{ht}(\mu)}$, which implies that $\mathrm{ht}(\theta) = \mathrm{ht}(\mu)$. Then $\theta \in \mathrm{Irr}_0^{\mu}(B_0)$, as needed.

Suppose $\xi \in \mathrm{Irr}_0^{\mu}(B_0)$. Then $\mathrm{ht}(\xi) = \mathrm{ht}(\mu)$ and by Theorem 3.3.8 and Lemma 5.3.1 of [9], $\xi^G \in \mathrm{Irr}(B) \cap \mathrm{Irr}(G|\mu)$. Next

$$\xi^G(1)_p = |G : T|_p \xi(1)_p = |G : T|_p |T : D|_p p^{\mathrm{ht}(\xi)} = |G : D|_p p^{\mathrm{ht}(\mu)},$$

which shows that $\xi^G \in \mathrm{Irr}_0^{\mu}(B)$. So the correspondence $\xi \mapsto \xi^G$ defines a map from $\mathrm{Irr}_0^{\mu}(B_0)$ to $\mathrm{Irr}_0^{\mu}(B)$. Since this map is injective by [9, Theorem 3.3.8], we conclude that $|\mathrm{Irr}_0^{\mu}(B)| \geq |\mathrm{Irr}_0^{\mu}(B_0)|$. This completes the proof of the Lemma. □

Let π be a prime set with complement π' in the set of all prime numbers. Suppose G is a (finite) π-separable group. An irreducible character χ of G is said to be π-*special* if $\chi(1)$ is a π-number and for every subnormal subgroup H of G, the determinantal order $o(\theta)$ of every irreducible constituent θ of χ_H is a π-number. (See Section 2A in [2].)

By [2, Theorem 2.2], the product of any π-special character of G times a π'-special character is irreducible. An irreducible character χ of G is said to be *π-factored* if $\chi = \alpha\beta$, where α is π-special and β is π'-special. If $\chi \in \mathrm{Irr}(G)$ is π-factored, then the π-special and π'-special factors of χ are uniquely determined (by Theorem 2.2 in [2]), and are denoted by χ_π and $\chi_{\pi'}$, respectively. In case $\pi = \{p\}$, a single prime, we shall simply write p-special, p'-special, χ_p and $\chi_{p'}$ instead of $\{p\}$-special, $\{p\}'$-special, $\chi_{\{p\}}$ and $\chi_{\{p\}'}$, respectively.

Suppose now that χ is an arbitrary irreducible character of G. One can associate with χ a canonical pair (W, γ), where W is a subgroup of G, $\gamma \in \mathrm{Irr}(W)$ is π-factored and $\gamma^G = \chi$. This pair, which turns out to be uniquely determined up to G-conjugacy, is called a *nucleus* for χ. In case χ is π-factored, then the pair (G, χ) is the single nucleus of χ. (See Section 4A in [2] for the precise definition of a nucleus of a character.)

Lemma 2.3. *Let $N \lhd G$, where G is p-solvable and let $\mu \in \mathrm{Irr}(N)$ be G-invariant. Choose a nucleus (W, γ) for μ and let $S = \mathrm{N}_G((W, \gamma))$ be the stabilizer of (W, γ) in G. Then $G = NS$ and $W = N \cap S$.*

Proof. This follows from Lemma 3.6 of [4]. □

Lemma 2.4. *Let G be an arbitrary group with normal subgroup N, and let $\mu \in \mathrm{Irr}(N)$ be G-invariant. Suppose $G = NS$ for a subgroup S and write $W = N \cap S$. Assume $\gamma \in \mathrm{Irr}(W)$ is S-invariant and $\gamma^N = \mu$. Then*

(a) *Character induction defines a bijection from $\mathrm{Irr}(S|\gamma)$ onto $\mathrm{Irr}(G|\mu)$.*

Furthermore, assuming $\chi \in \mathrm{Irr}_0^\mu(B)$ where B is a p-block of G, if θ is the character in $\mathrm{Irr}(S|\gamma)$ such that $\theta^G = \chi$ and B_0 is the p-block of S to which θ belongs, we have

(b) $\theta \in \mathrm{Irr}_0^\gamma(B_0)$;
(c) *B_0 has a defect group D_0 contained in a defect group D of B and* $|D : D \cap N| = |D_0 : D_0 \cap W|$;
(d) $|\mathrm{Irr}_0^\gamma(B_0)| \leq |\mathrm{Irr}_0^\mu(B)|$.

Proof. Part (a) follows from Lemma 2.11(b) in [2].

Now suppose $\chi \in \mathrm{Irr}_0^\mu(B)$ where B is a p-block of G. Let θ be the character in $\mathrm{Irr}(S|\gamma)$ such that $\theta^G = \chi$ and let B_0 be the p-block of S to which θ belongs.

Since $\theta^G = \chi$, [9, Lemma 5.3.1] tells us that ${B_0}^G$ is defined and equals B. Then by Lemma 5.3.3 of [9], B_0 has a defect group D_0 contained in some defect group D of B.

As $\mathrm{ht}(\chi)=\mathrm{ht}(\mu)$, we have $\chi(1)_p=|G:D|_p p^{\mathrm{ht}(\chi)}=|G:D|_p p^{\mathrm{ht}(\mu)}$. Also, since θ lies over γ, we have $\mathrm{ht}(\theta)\geq\mathrm{ht}(\gamma)$, and so $\theta(1)_p=|S:D_0|_p p^{\mathrm{ht}(\theta)}\geq|S:D_0|_p p^{\mathrm{ht}(\gamma)}$. It follows that $|G:D|_p p^{\mathrm{ht}(\mu)}\geq|G:D_0|_p p^{\mathrm{ht}(\gamma)}$, as $\chi(1)_p=|G:S|_p\theta(1)_p$. Therefore,

$$p^{\mathrm{ht}(\mu)}\geq|D:D_0|p^{\mathrm{ht}(\gamma)}. \tag{1}$$

Let b be the block of N to which μ belongs, and let b_0 be the block of W to which γ belongs. Since μ is invariant in G and γ is invariant in S, we have that b is G-stable and b_0 is S-stable. It follows by [9, Theorem 5.5.16(ii)] that $D\cap N$ is a defect group of b, and $D_0\cap W$ is a defect group of b_0. Therefore, $\mu(1)_p=|N:D\cap N|_p p^{\mathrm{ht}(\mu)}$ and $\gamma(1)_p=|W:D_0\cap W|_p p^{\mathrm{ht}(\gamma)}$. Since $\mu=\gamma^N$, we have $\mu(1)_p=|N:W|_p\gamma(1)_p$, and hence $|N:D\cap N|_p p^{\mathrm{ht}(\mu)}=|N:D_0\cap W|_p p^{\mathrm{ht}(\gamma)}$. Therefore,

$$p^{\mathrm{ht}(\mu)}=|D\cap N:D_0\cap W|p^{\mathrm{ht}(\gamma)}. \tag{2}$$

Now, in view of (1), we get that $|D_0:D_0\cap W|\geq|D:D\cap N|$, and consequently

$$|N:W|\geq|DN:D_0W|. \tag{3}$$

Since $W=N\cap S$, and $D_0\subseteq S$, we have $D_0W=D_0(N\cap S)=(D_0N)\cap S$. Also, as $G=NS$, it is clear that $G=(D_0N)S$. Therefore, $|G|=|D_0N||S||(D_0N)\cap S|^{-1}=|D_0N||S||D_0W|^{-1}$. Now since $|G|=|N||S||W|^{-1}$, we conclude that

$$|N:W|=|D_0N:D_0W|. \tag{4}$$

Using (3) now, it follows that $|D_0N|\geq|DN|$. On the other hand, we know that $D_0\subseteq D$. Therefore $D_0N=DN$, and hence, in light of (4), we get that $|N:W|=|DN:D_0W|$. Then $|D:D\cap N|=|D_0:D_0\cap W|$, which finishes the proof of (c).

Next, using (2), we have that

$$p^{\mathrm{ht}(\mu)}=|D:D_0|p^{\mathrm{ht}(\gamma)}. \tag{5}$$

Now $\chi(1)_p=|G:D|_p p^{\mathrm{ht}(\mu)}=|G:D_0|_p p^{\mathrm{ht}(\gamma)}$, and thus, as $\chi(1)_p=|G:S|_p\theta(1)_p$ and $\theta(1)_p=|S:D_0|_p p^{\mathrm{ht}(\theta)}$, it follows that $p^{\mathrm{ht}(\theta)}=p^{\mathrm{ht}(\gamma)}$, which clearly proves (b).

Finally, we show (d). Suppose $\xi\in\mathrm{Irr}_0^{\gamma}(B_0)$. Then $\mathrm{ht}(\xi)=\mathrm{ht}(\gamma)$. Also, by (a), $\xi^G\in\mathrm{Irr}(G|\mu)$. Since ${B_0}^G=B$, we have that $\xi^G\in\mathrm{Irr}(B)$ (by [9, Lemma 5.3.1]), and so $\xi^G(1)_p=|G:D|_p p^{\mathrm{ht}(\xi^G)}$. On the other hand, we also have $\xi^G(1)_p=|G:S|_p\xi(1)_p$. Therefore

$$\begin{aligned} p^{\mathrm{ht}(\xi^G)} &= (|S|_p)^{-1}|D|\xi(1)_p=(|S|_p)^{-1}|D||S:D_0|_p p^{\mathrm{ht}(\xi)} \\ &= |D:D_0|p^{\mathrm{ht}(\xi)}=|D:D_0|p^{\mathrm{ht}(\gamma)}=p^{\mathrm{ht}(\mu)}, \end{aligned}$$

where the last equality is (5). We have thus shown that $\xi^G \in \mathrm{Irr}_0^\mu(B)$. Now, in light of (a), part (d) of the Lemma follows. $\square$

Lemma 2.5. *Let $N \lhd G$, where G is p-solvable and let μ be a G-invariant p-factored character of N. Let B be a p-block of G of maximal defect such that $\mathrm{Irr}_0^\mu(B) \neq \emptyset$. Then $|\mathrm{Irr}_0^\mu(B)| = |\mathrm{Irr}_0^{\mu_{p'}}(B)|$.*

Proof. Since $\mathrm{Irr}_0^\mu(B) \neq \emptyset$ and B has maximal defect, Theorem 2.3 in [8] implies that μ extends to PN for some Sylow p-subgroup P of G. Then, by Theorem 4.1 in [6], μ_p extends to a p-special character δ of G, and the correspondence $\theta \mapsto \delta\theta$ defines a bijection from $\mathrm{Irr}(G|\mu_{p'})$ onto $\mathrm{Irr}(G|\mu)$. Now to prove the assertion of the lemma, it suffices to show that the above bijection maps $\mathrm{Irr}_0^{\mu_{p'}}(B)$ onto $\mathrm{Irr}_0^\mu(B)$.

Let $M = \mathrm{O}_{p'}(G)$. Since δ is p-special, then the irreducible constituents of δ_M are all p-special, and so, as M is a p'-group, they must all be the principal character 1_M of M. It follows by [10, Theorem 10.20] that δ belongs to the principal block of G.

Suppose $\theta \in \mathrm{Irr}_0^{\mu_{p'}}(B)$. Then $\mathrm{ht}(\theta) = \mathrm{ht}(\mu_{p'}) = 0$, as $\mu_{p'}(1)$ is a p'-number. Now, since B has maximal defect, it follows that $\theta(1)$ is a p'-number. Then, in view of [11, Lemma 2.9], we have $\delta\theta \in \mathrm{Irr}(B)$. Next, by [11, Lemma 2.10] (for instance), μ belongs to a block of N of maximal defect. Then

$$p^{\mathrm{ht}(\delta\theta)} = (\delta\theta)(1)_p = \delta(1) = \mu_p(1) = \mu(1)_p = p^{\mathrm{ht}(\mu)},$$

and thus $\delta\theta \in \mathrm{Irr}_0^\mu(B)$.

Now let $\chi \in \mathrm{Irr}_0^\mu(B)$. Then $\chi = \delta\eta$ for some $\eta \in \mathrm{Irr}(G|\mu_{p'})$. Since $\mathrm{ht}(\chi) = \mathrm{ht}(\mu)$, we have $\chi(1)_p = \mu(1)_p$. It follows that $\eta(1)$ is a p'-number, as $\delta(1) = \mu(1)_p$. Now [11, Lemma 2.9] tells us that $\eta \in \mathrm{Irr}(B)$. Finally, since $\mathrm{ht}(\eta) = 0 = \mathrm{ht}(\mu_{p'})$, we conclude that $\eta \in \mathrm{Irr}_0^{\mu_{p'}}(B)$. The proof of the lemma is now complete. $\square$

Suppose μ is a p'-special character of a normal subgroup N of a p-solvable group G. Two characters $\chi, \chi' \in \mathrm{Irr}(G|\mu)$ are said to be linked if they are linked in the sense of Brauer, i.e., if there is $\varphi \in \mathrm{IBr}(G)$ such that the decomposition numbers $d_{\chi\varphi}$ and $d_{\chi'\varphi}$ are nonzero. The equivalence classes defined by the transitive extension of this linking are called *relative blocks of G with respect to* (N, μ) (see [3, Section 3]). In particular, if B is any block of G covering the block of N to which μ belongs, then $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\mu)$ is a union of some relative blocks with respect to (N, μ).

We should mention that a notion of defect group associated with a relative block was introduced in [3, Section 4]. The defect groups of a relative block form a single G-conjugacy class of p-subgroups of G.

If $\mathcal{B}$ is a relative block of G with respect to (N,μ) and D is a defect group of $\mathcal{B}$, then the relative height (with respect to (N,μ)) of $\chi \in \mathcal{B}$ is defined as $\mathrm{h}_\mu(\chi) = \chi(1)_p|D|(|G|_p)^{-1}$. It turns out that $\mathrm{h}_\mu(\chi) = p^n$, where n is some nonnegative integer. (See [3, Section 4].)

Lemma 2.6. *Let N be a normal subgroup of a p-solvable group G such that $|G:N|_p > 1$, and let $\mu \in \mathrm{Irr}(N)$ be p'-special. Let B be a p-block of G of maximal defect and suppose $\mathrm{Irr}_0^\mu(B) \neq \emptyset$. Then $|\mathrm{Irr}_0^\mu(B)| \geq 2$.*

Proof. Let $\chi \in \mathrm{Irr}_0^\mu(B)$ and let b be the block of N to which μ belongs. Since μ has p'-degree, b has maximal defect and $\mathrm{ht}(\mu) = 0$. Therefore $\mathrm{ht}(\chi) = 0$, and so, as B has maximal defect, the character χ has p'-degree.

Now let $\mathcal{B}$ be the relative block of G with respect to (N,μ) such that $\chi \in \mathcal{B}$. Then, if D is a defect group of $\mathcal{B}$, we have $|D|(|G|_p)^{-1} = \mathrm{h}_\mu(\chi) = p^n$ for some integer $n \geq 0$. It follows that D is a Sylow p-subgroup of G.

By Theorem 3.1 and Lemma 4.7 of [3], there exist a group H, a block A of H and a bijection Ψ of $\mathcal{B}$ onto $\mathrm{Irr}(A)$ such that $\mathrm{h}_\mu(\theta) = p^{\mathrm{ht}(\Psi(\theta))}$ for every $\theta \in \mathcal{B}$. Also, [3, Theorem 4.2] implies that $\mathcal{B}$ has a defect group D' such that the quotient group $(D'N)/N$ is isomorphic to some defect group $\widetilde{D}$ of A. Since D', being G-conjugate to D, is a Sylow p-subgroup of G, we get that $|\widetilde{D}| = |G:N|_p > 1$. It follows that $|\mathrm{Irr}_0(A)| \geq 2$.

Now let ζ be any character in $\mathrm{Irr}_0(A)$. Then $\Psi^{-1}(\zeta) \in \mathcal{B}(\subseteq \mathrm{Irr}(B))$ and $\mathrm{h}_\mu(\Psi^{-1}(\zeta)) = 1$. It follows that $\Psi^{-1}(\zeta)$ has p'-degree, and hence, as a character of the block B, $\Psi^{-1}(\zeta)$ is of height zero. Now, being in $\mathcal{B}$, the character $\Psi^{-1}(\zeta)$ lies over μ, and we have $\Psi^{-1}(\zeta) \in \mathrm{Irr}_0^\mu(B)$. Finally, since $|\mathrm{Irr}_0(A)| \geq 2$ and Ψ^{-1} is a bijection from $\mathrm{Irr}(A)$ onto $\mathcal{B}$, it follows that $|\mathrm{Irr}_0^\mu(B)| \geq 2$, as needed to be shown. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We proceed by induction on $|G|$. Let $M = \mathrm{O}_{p'}(G)$ and write $L = MN$. Next let $\chi \in \mathrm{Irr}_0^\mu(B)$, and choose a character $\theta \in \mathrm{Irr}(L)$ lying under χ and over μ. Let b be the block of L to which θ belongs. Since $\mathrm{ht}(\chi) \geq \mathrm{ht}(\theta) \geq \mathrm{ht}(\mu)$ and $\mathrm{ht}(\chi) = \mathrm{ht}(\mu)$, it is clear that $\chi \in \mathrm{Irr}_0^\theta(B)$ and $\theta \in \mathrm{Irr}_0^\mu(b)$.

Choose a block b_0 of M covered by b, and let ν be the unique character in $\mathrm{Irr}(b_0)$. Then both θ and χ lie over ν. Next, let T be the inertial group of ν in G. Then T is the inertial group of b_0, also.

First, suppose $T < G$. Let B' and b' be the respective Fong-Reynolds correspondents of B and b with respect to b_0. Next, choose a defect group D' of B'. By [9, Theorem 5.5.10], D' is a defect group of B, and so $|D'| > |D' \cap N|$. Also, as L/N is a p'-group, we have that $D' \cap L = D' \cap N$, and it follows that $|D'| > |D' \cap (T \cap L)|$.

By [9, Theorem 5.5.10], there is a unique character $\theta' \in \mathrm{Irr}(b')$ such that $(\theta')^L = \theta$ and $\mathrm{ht}(\theta') = \mathrm{ht}(\theta)$. Similarly, there is a unique character $\chi' \in \mathrm{Irr}(B')$ such that $(\chi')^G = \chi$ and $\mathrm{ht}(\chi') = \mathrm{ht}(\chi)$. Since χ' and θ' both lie over ν, and χ lies over θ, it follows by [5, Lemma 2.6] that χ' lies over θ'. Now, as $\chi \in \mathrm{Irr}_0^{\theta}(B)$, we get that $\chi' \in \mathrm{Irr}_0^{\theta'}(B')$. Therefore, in particular, $\mathrm{Irr}_0^{\theta'}(B') \neq \emptyset$.

Since $T < G$ and $|D'| > |D' \cap (T \cap L)|$, the inductive hypothesis guarantees that $|\mathrm{Irr}_0^{\theta'}(B')| \geq 2$. It follows by [9, Theorem 5.5.10] and [5, Lemma 2.6] that $|\mathrm{Irr}_0^{\theta}(B)| \geq 2$. Now, as $\theta \in \mathrm{Irr}_0^{\mu}(b)$, we conclude that $|\mathrm{Irr}_0^{\mu}(B)| \geq 2$, as desired.

We may now assume that $T = G$. Since χ lies over ν and $\chi \in \mathrm{Irr}(B)$, Theorem 10.20 in [10] tells us that the defect groups of B are the Sylow p-subgroups of G.

Let I be the inertial group of μ in G and let $\theta \in \mathrm{Irr}(I|\mu)$ be the Clifford correspondent of χ. Next, let B_0 be the block of I to which θ belongs. Then by Lemma 2.2, B and B_0 have a common defect group D_0, $\theta \in \mathrm{Irr}_0^{\mu}(B_0)$ and $|\mathrm{Irr}_0^{\mu}(B)| \geq |\mathrm{Irr}_0^{\mu}(B_0)|$. Also, note that D_0 is a Sylow p-subgroup of I and that $|D_0| > |D_0 \cap N|$.

Choose a nucleus (W, γ) for μ and let S be the stabilizer of (W, γ) in I. Then $\mu = \gamma^N$ and by Lemma 2.3, we have $I = NS$ and $W = N \cap S$. Next, by Lemma 2.4(a), there is a unique character $\xi \in \mathrm{Irr}(S|\gamma)$ such that $\xi^I = \theta$. Let B_1 be the block of S to which ξ belongs. Since $\theta \in \mathrm{Irr}_0^{\mu}(B_0)$, Lemma 2.4 implies that $\xi \in \mathrm{Irr}_0^{\gamma}(B_1)$, B_1 has a defect group D_1 with $|D_1 : D_1 \cap W| = |D_0 : D_0 \cap N|$, and $|\mathrm{Irr}_0^{\gamma}(B_1)| \leq |\mathrm{Irr}_0^{\mu}(B_0)|$. We claim that D_1 is a Sylow p-subgroup of S.

Since D_0 is a Sylow p-subgroup of I, we have

$$|D_1 : D_1 \cap W| = |D_0 : D_0 \cap N| = |D_0N : N| = |D_0N|_p/|N|_p = |I|_p/|N|_p.$$

Since $I = NS$ and $W = N \cap S$, we have that $S/W \cong I/N$, and hence $|I|_p/|N|_p = |S|_p/|W|_p$. It follows that

$$|D_1 : D_1 \cap W| = |S|_p/|W|_p. \tag{1}$$

Let A be the block of W to which γ belongs. Since γ is p-factored, [11, Lemma 2.10] tells us that the defect groups of A are the Sylow p-subgroups of W. Next, as ξ lies over γ, the block B_1 covers A and [9, Theorem 5.5.16(ii)] implies that $|D_1 \cap W| = |W|_p$. It follows from (1) that $|S|_p = |D_1|$, thus proving our claim.

Since γ is an S-invariant p-factored character of the normal subgroup W of S, $\mathrm{Irr}_0^{\gamma}(B_1) \neq \emptyset$ and B_1 has maximal defect, then, in light of Lemma 2.5, we have $|\mathrm{Irr}_0^{\gamma_{p'}}(B_1)| = |\mathrm{Irr}_0^{\gamma}(B_1)| > 0$. Furthermore, as $|S : W|_p = |D_0 : D_0 \cap N| > 1$, Lemma 2.6 says that $|\mathrm{Irr}_0^{\gamma_{p'}}(B_1)| \geq 2$. Finally,

$$|\mathrm{Irr}_0^{\mu}(B)| \geq |\mathrm{Irr}_0^{\mu}(B_0)| \geq |\mathrm{Irr}_0^{\gamma}(B_1)| = |\mathrm{Irr}_0^{\gamma_{p'}}(B_1)| \geq 2,$$

and the proof of the theorem is complete. □

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