TWIST LEFT-VEERING OPEN BOOKS, OVERTWISTEDNESS, LOOSENESS AND VIRTUAL LOOSENESS

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Abstract. We introduce twist left-veering mapping classes of punctured surfaces. We prove that a twist left-veering open book supports an overtwisted contact structure and determine when the closed braid coming from the punctures is loose or virtually loose.

1. Introduction

A pair (S, ϕ) of a compact oriented surface S with boundary and diffeomorphism $\phi \in \mathrm{Diff}^+(S, \partial S)$ is called an *(abstract) open book*. For an open book (S, ϕ) one obtains a contact 3-manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$ [29]. Open books play significant role in the study of contact structures thanks to the Giroux correspondence, [11], see also [5, 7]: There is a one-to-one correspondence between the set of contact 3-manifolds up to isotopy and the set of open books up to stabilization and equivalence. Here we say that two open books (S, ϕ) and (S', ϕ') are *equivalent* if $h \circ \phi = \phi' \circ h$ for some orientation preserving diffeomorphism $h : S \to S'$ fixing the boundary [5].

Let $\mathcal{T}$ be a transverse link in $(M, \xi) \simeq (M_{(S,\phi)}, \phi_{(S,\phi)})$, where $\simeq$ means contactomorphic. According to Bennequin [2] (when $(M, \xi) \simeq (S^3, \xi_{std})$, the standard contact 3-sphere) and Pavelescu [27, 28] (for general (M, ξ)), $\mathcal{T}$ is transversely isotopic to some closed n-braid L with respect to the open book (S, ϕ). Choose a set P of n points in the interior of S near the boundary ∂S. We can find a diffeomorphism $\phi_L \in \mathrm{Diff}^+(S, P, \partial S)$ such that L and the closed braid $P \times [0, 1]/(x, 1) \sim (\phi_L(x), 0)$ can be identified up to some braid isotopy.

It is a fundamental problem to understand geometry and topology of $(M, \xi, \mathcal{T})$ in terms of the corresponding mapping classes $[\phi] \in \mathcal{MCG}(S)$ and $[\phi_L] \in \mathcal{MCG}(S, P)$. The latter is called the *distinguished monodromy* for L. (In the following the bracket $[\cdot]$ will be omitted for simplicity, and a diffeomorphism and its mapping class will be denoted by the same symbol.) In this paper, we are particularly interested in

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detecting tight/overtwisted-ness of (M, ξ) and non-loose/loose-ness of $\mathcal{T}$. Three earlier works done by Goodman [12], Wand [30], Honda, Kazez and Matić [16] are relevant to our goal:

In [12] Goodman introduced a *sobering arc* for an open book (S, ϕ) which is a properly embedded arc α in S such that the intersection of α and its image $\phi(\alpha)$ satisfies certain numerical conditions. He showed that (S, ϕ) *supports an overtwisted contact structure if and only if* (S, ϕ) *is stably equivalent to an open book* (S', ϕ') *admitting a sobering arc.* Here we say that two open books are *stably equivalent* if they admit stabilizations that are equivalent.

A contact 3-manifold (M, ξ) is overtwisted if M contains an overtwisted disk, an embedded disk D whose boundary is tangent to ξ. A *transverse overtwisted disk* is a disk in an open book admitting a certain type of open book foliation [17, Definition 4.1]. It has been shown that the existence of a *transverse overtwisted disk* is equivalent to the existence of a usual overtwisted disk [17, Proposition 4.2, Corollary 4.6]. In this paper, one may simply understand a transverse overtwsited disk as an embedded disk D whose boundary is transverse to ξ with the self-linking number $sl(\partial D) = 1$ with respect to a trivialization of the 2-plane bundle $\xi|_D \to D$.

In [30] Wand introduced an *overtwisted region* in (S, ϕ), which can be seen as a generalization of Goodman's sobering arcs to arc systems. It is a $2N$-gon formed by an N-arc system $\Gamma \subset S$ and its image $\phi(\Gamma)$ that satisfies certain conditions (in Definition 6.1). Here, an N-arc system is a collection of pairwise disjoint N arcs. He showed that (S, ϕ) *supports an overtwisted contact structure if and only if* ϕ *is inconsistent*; that is, there exist some arc system $\Gamma \subset S$ and *some stabilization* (S', ϕ') of (S, ϕ) such that Γ and $\phi'(\Gamma)$ form an overtwisted region. This observation lead him to prove that Legendrian surgery preserves tightness.

There is an alternative refinement of Goodman's sobering arc criterion by Honda, Kazez and Matić [16], which is the *non-right-veering* arc criterion. Instead of looking at the whole picture of $\gamma \cap \phi(\gamma)$ they found the importance of focussing on $\gamma \cup \phi(\gamma)$ restricted to a neighborhood of the boundary ∂S. They proved that *an open book is overtwisted if and only if it is stably equivalent to a non-right-veering open book.*

Among the three criteria, the non-right-veering criterion is the most practically easy to check since it is essentially equivalent to the condition that the fractional Dehn twist coefficient (FDTC) is non-positive, see Propositions 3.1 and 3.2 in [16].

Note that the above three overtwistedness criteria are not at all claiming that every open book (S, ϕ) supporting an overtwisted contact structure admits sobering arcs, overtwisted regions or non-right-veering arcs.

We also note that punctured open books are not considered in the three criteria. In [22] we studied open books (S, ϕ) with a set of punctures P where ϕ is a diffeomorphism of $(S, P, \partial S)$. We continue the study of punctured open books in this paper. Since the punctures in P can be permuted by ϕ, it gives rise to a closed braid in the manifold $M_{(S,\phi)}$ that can be identified with a transverse link in the contact manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$. As a consequence, the problem of detecting overtwisted structure is converted to a problem of detecting loose-links.

It turned out that extending the results for non-punctured open books to punctured open books is not routine. Even though the notion of right-veering can naturally be extended to punctured open books, the extended non-right-veering property does not imply looseness of the transverse link. To solve this problem, in [22] a notion of non-*quasi*-right-veering is introduced for punctured open books. This condition appears to be a 'correct' generalization of non-right-veering as it implies looseness of the transverse link.

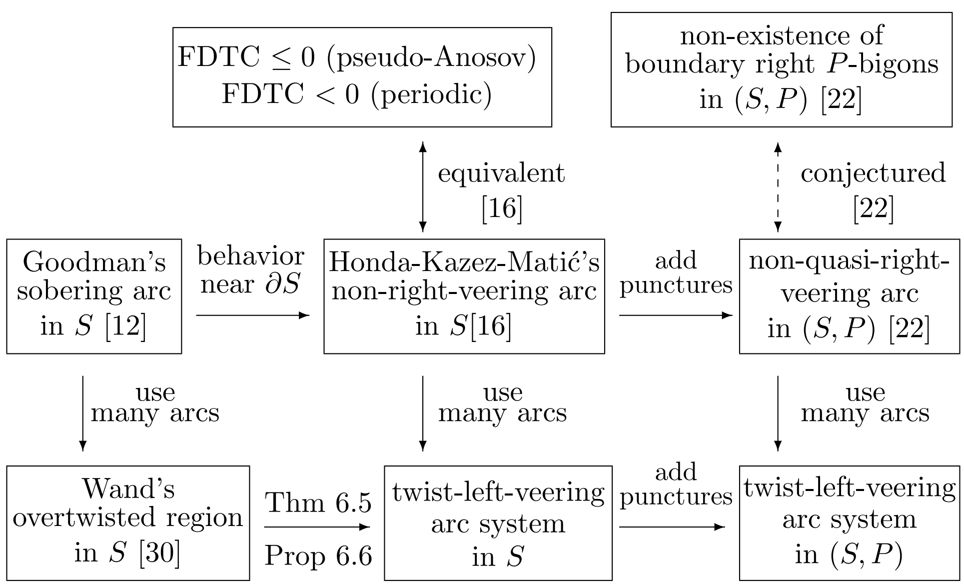

FIGURE 1. Criteria for overtwistedness and looseness.

Non-right-veering is equivalent to non-positive fractional Dehn twist coefficient, which is a numerical invariant of mapping classes [16]. On the other hand, for non-quasi-right-veering no numerical characterization has been found. Instead, a boundary right P-bigon, that is a certain punctured bigon at the boundary ∂S, plays a practically useful role to detect quasi-right-veering. In [22] it is conjectured that non-existence of boundary P-bigon is equivalent to non-quasi-right-veering.

In this paper, we extend the arc criteria in [16, 22] to arc-system criteria. More precisely, we generalize the notion of non-(quasi)-right-veering to a notion of *twist-left-veering*. Our work may be understood as an analogue of Wand's generalization of Goodman's criterion. The schematic picture in Figure 1 may be helpful.

In [22] we introduced the *right-veering ordering* $\prec_{\textsf{right}}$ and the *strong right-veering ordering* $\ll_{\textsf{right}}$ of arcs, see Definition 2.2. A mapping class $\phi \in \mathcal{MCG}(S,P)$ is called *non-right-veering* (resp. *non-quasi-right-veering*) if $\phi(\gamma) \prec_{\textsf{right}} \gamma$ (resp. $\phi(\gamma) \ll_{\textsf{right}} \gamma$) for some arc γ. Both orderings $\prec_{\textsf{right}}$ and $\ll_{\textsf{right}}$ can be naturally extended to N-arc systems.

Let $N \in \mathbb{N}$. If an N-arc system Γ and its image $\phi(\Gamma)$ form a $2N$-gon then we call it a *boundary based region*, denoted $R(\Gamma, \phi(\Gamma))$ (see Definition 3.5 and Figure 7). If a $2N$-gon is formed then we define another N-arc system $\phi^{tw}(\Gamma)$ called the *left-twist* of Γ by $\phi(\Gamma)$ (Definition 3.7). We have $\phi^{tw}(\Gamma) \prec_{\textsf{right}} \Gamma \prec_{\textsf{right}} \phi(\Gamma)$ in general. If no $2N$-gon is formed we define $\phi^{tw}(\Gamma) := \phi(\Gamma)$. We say that ϕ is (N,k)*-twist left-veering* if $\phi^{tw}(\Gamma) \ll_{\textsf{right}} \Gamma$ for some N-arc system Γ such that the boundary based region $R(\Gamma, \phi(\Gamma))$ contains k puncture points. We also say that ϕ is N*-twist left-veering* (resp. *twist left-veering*) if ϕ is (N,k)-twist left-veering for some k (resp. N and k).

When $P = \emptyset$, 1-twist left-veering and non-right-veering are equivalent. When $P \neq \emptyset$, $(1,0)$-twist left-veering and non-quasi-right-veering are equivalent. Thus, twist left-veering is a generalization of non-right-veering and non-quasi-right-veering. Although someone might want to name it non-twist right-veering, we prefer twist left-veering to avoid the prefix 'non-', and in fact, when $\phi^{tw}(\Gamma) \ll_{\textsf{right}} \Gamma$ the arc system $\phi^{tw}(\Gamma)$ is on the left of Γ near the base points.

A contact 3-manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$ is overtwisted if ϕ is non-right-veering [16]. Similarly, we showed that a transverse link represented by a braid L is loose (the complement is overtwisted) if the distinguished monodromy $\phi_L \in \mathcal{MCG}(S,P)$ is non-quasi-right-veering [22]. We generalize these results to twist left-veering.

Theorem 4.1. *Let L be a closed braid with respect to (S,ϕ) and $\phi_L \in \mathcal{MCG}(S,P)$ be its distinguished monodromy.*

(1) *If $\phi \in \mathcal{MCG}(S)$ is N-twist left-veering then $(M_{(S,\phi)}, \xi_{(S,\phi)})$ is overtwisted and there is an overtwisted disk that intersects the binding at N points.*
(2) *If $\phi_L \in \mathcal{MCG}(S, P)$ is (N,k)-twist left-veering then there is an overtwisted disk that intersects the binding at N points and intersects the closed braid L at k points. In particular, if ϕ_L is $(N,0)$-twist left-veering then L is loose.*

Recall a result of Wand [30] that if (S,ϕ) admits an overtwisted region then (S,ϕ) supports an overtwisted contact structure. The statement (1) in Theorem 4.1 suggests that the two conditions on (S,ϕ) (i) admitting an overtwisted region and (ii) twist left veering are comparable when the puncture set P is empty. However, when $P \neq \emptyset$ Wand's proof of the result may not immediately be generalized as we discuss in Remark 6.3.

For a transverse link $\mathcal{T}$ in an overtwisted contact 3-manifold (M,ξ), the *depth* of $\mathcal{T}$, $\mathsf{depth}(\mathcal{T}; M)$, is the minimum number of intersection of an overtwisted disk in M and $\mathcal{T}$. It was introduced by Baker and Onaran [1] and measures *non-looseness* of $\mathcal{T}$. As an application of Theorem 4.1 we give a diagrammatic interpretation of the depth in Corollary 5.3. In particular, we obtain the characterization of depth 1 and depth 2:

Proposition 5.7 and Theorem 5.8. *Suppose that $\xi_{(S,\phi)}$ is overtwisted. Let L be a closed braid in $M_{(S,\phi)}$ and $B_{(S,\phi)}$ be the binding.*

- $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}) = 1$ *if and only if ϕ is non-right-veering (i.e. 1-twist left-veering).*
- $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}) = 2$ *if and only if ϕ is right-veering and 2-twist left-veering.*
- $\mathsf{depth}(L \cup B_{(S,\phi)}; M_{(S,\phi)}) = \mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)} \setminus L) = 1$ *if and only if ϕ_L is non-quasi-right-veering (i.e., $(1,0)$-twist left-veering).*
- $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)} \setminus L) = 2$ *if and only if ϕ_L is quasi-right-veering and $(2,0)$-twist left-veering.*

Our boundary based region and Wand's overtwisted region (Definition 6.1) share some common properties as shown in Figure 18. In Example 6.4 and Example 8.1 we highlight their difference. The following theorem shows an overtwisted region is a special type of boundary based region.

Theorem 6.5. *Let Γ be an N-arc system with $N \geq 2$ such that the boundary based region $R(\Gamma, \phi(\Gamma))$ exists. Then $R(\Gamma, \phi(\Gamma))$ is an overtwisted region if and only if $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$, $\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma) = \emptyset$,* *and* $R(\Gamma, \phi(\Gamma))$ *is embedded.* (A parallel statement holds for $N = 1$, see Proposition 6.6.)

In the following corollary, the if-direction ($\Leftarrow$) is exactly Theorem 4.1, and the only-if-direction ($\Rightarrow$) follows from Theorem 6.5, Proposition 6.6 and Wand's significant result, the inconsistency criterion [30, Theorem 1.1].

Corollary 6.9. *An open book* (S, ϕ) *supports an overtwisted contact structure if and only if* (S, ϕ) *is twist left-veering after some stabilizations.* (*In other words,* (S, ϕ) *is twist left veering after some stabilizations if and only if* (S, ϕ) *is inconsistent.*)

In Corollary 6.9 destabilizations are not required thanks to [30, Theorem 1.1]. This makes a sharp contrast to the non-right-veering criterion of Honda, Kazez and Matić, which states that an open book (S, ϕ) is overtwisted if and only if it is *stably equivalent* to some non-right-veering open book (S', ϕ'). That is, (S, ϕ) and (S', ϕ') are related to each other by a sequence of stabilizations and destabilizations. (Every open book becomes right-veering after a number of stabilizations [16, Proposition 6.1]).

Corollary 6.9 leads us to the following question:

Question 1.1. Is it true that an open book (S, ϕ) supports an overtwisted contact structure if and only if (S, ϕ) is twist left-veering (without stabilizations)?

If the answer is "Yes" then it would imply the contrapositive of the following conjecture that is a new tightness criterion. The point of the criterion is that it does not require stabilizations or destabilizations. It is a very straightforward condition comparing to the consistency condition.

Conjecture 1.2. *If the FDTC of* ϕ *is grater than 1 for every boundary component then* (S, ϕ) *supports a tight contact structure.*

This conjecture is solved when S has genus 0 [20] and sketched in [31] for general case.

Theorem 4.1 has more application to contact cyclic branched covers. So far, we have been requiring arc systems end at ∂S. If we allow arc systems end at $\partial S \cup P$ instead, then we can define a similar ordering that we denote by $\ll^{\partial+P}_{\mathsf{right}}$. It turned out that $\phi^{tw}_L(\Gamma) \ll^{\partial+P}_{\mathsf{right}} \Gamma$ is a weaker condition than $\phi_L^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$. Using this weaker ordering $\ll^{\partial+P}_{\mathsf{right}}$ in the place of $\ll_{\mathsf{right}}$, we define a notion of *weakly* (N,k)*-twist left-veering*. We show that weakly $(N,0)$-twist left-veering guarantees virtually looseness of the closed braid L as stated in Theorem 7.8.

Theorem 7.8. *Assume that all the meridians of L are homotopically non-trivial. If ϕ_L is weakly $(N,0)$-twist left-veering; namely, there is an N-arc system $\Gamma \in \mathcal{A}_{\mathcal{B}}(S,P)$ such that $\phi_L^{tw}(\Gamma) \ll^{\partial+P}_{\mathsf{right}} \Gamma$ and the associated boundary based region $R(\Gamma,\phi(\Gamma))$ is non-punctured, then L is virtually loose.*

Figure 2 below would help us to understand four results on looseness and virtually looseness.

single arc γ	arc system Γ
[22, Theorem 4.1] $\phi_L(\gamma) \ll_{\mathsf{right}} \gamma$ (non-quasi-rightveering) $\Rightarrow L$ is loose	Theorem 4.1 $\phi_L^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$ with non-punctured $R(\Gamma,\phi_L(\Gamma))$ ($(N,0)$-twist left-veering) $\Rightarrow L$ is loose
[23, Corollary 5.7] $\phi_L(\gamma) \prec_{\mathsf{right}} \gamma$ (non-rightveering) $\Rightarrow L$ is virtually loose	Theorem 7.8 $\phi_L^{tw}(\Gamma) \ll^{\partial+P}_{\mathsf{right}} \Gamma$ with non-punctured $R(\Gamma,\phi_L(\Gamma))$ (weakly $(N,0)$-twist left-veerting) $\Rightarrow L$ is virtually loose

FIGURE 2. Four results on looseness and virtually looseness of transverse links

In Figure 3 below definitions of various left-veering type notions are summarized.

The paper is organized as follows. In Section 2 we recall basic definitions including the two orderings $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ of arcs in S and non-quasi-right-veering monodromies. In Section 3 we extend the definition of the two orderings to N-arc systems. We further define a boundary based region $R(\Gamma,\Gamma')$, left twist $\phi^{tw}(\Gamma)$ of Γ and the notion of (N,k)-twist left-veering. In Section 4 we prove the main result Theorem 4.1. In Section 5 we discuss applications of Theorem 4.1 to the depths of transverse links. In Section 6, relation between twist left veering and inconsistency is discussed. In Section 7 we introduce $\ll^{\partial+P}_{\mathsf{right}}$ and

Notion	Definition	Property
Non-right-veering	$\exists$ arc γ s.t. $\phi(\gamma) \prec_{\mathsf{right}} \gamma$ $(P=\emptyset)$ $\phi_L(\gamma) \prec_{\mathsf{right}} \gamma$ $(P\neq\emptyset)$	Overtwisted Virtually loose
Non-quasi-right-veering	$\exists$ arc γ s.t. $\phi_L(\gamma) \ll_{\mathsf{right}} \gamma$	Loose
N-twist-left-veering	$\exists$ N-arc system Γ s.t. $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$	Overtwisted
$(N,0)$-twist-left-veering	$\exists$ N-arc system Γ s.t. $\phi_L^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$, $R(\Gamma, \phi_L(\Gamma))$ contains no punctures	Loose
Weakly $(N,0)$-twist-left-veering	$\exists$ N-arc system Γ s.t. $\phi_L^{tw}(\Gamma) \ll_{\mathsf{right}}^{\partial+P} \Gamma$, $R(\Gamma, \phi_L(\Gamma))$ contains no punctures	Virtually loose

FIGURE 3. Summary of definitions of various left-veering notions and relation to overtwisted and loose properties.

prove Theorem 7.8. In Section 8 we give examples of twist left veering monodromy and weakly twist left veering monodromy.

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2. PRELIMINARIES

2.1. **Open books and closed braids.** We review the relation between open books and contact 3-manifolds, and the relation between closed braids and transverse links.

Let S be an oriented compact surface with non-empty boundary and $P=\{p_1,\dots,p_n\}$ be a (possibly empty) set of n distinct interior points of S. The mapping class group $\mathcal{MCG}(S,P)$ is the group of isotopy classes of orientation preserving diffeomorphisms of the surface S fixing ∂S pointwise and fixing P setwise. In the following, $\mathcal{MCG}(S)=\mathcal{MCG}(S,\emptyset)$.

An *(abstract) open book* is a pair (S, ϕ) of a surface S and a diffeomorphism $\phi \in \mathrm{Diff}^+(S, \partial S)$ (or a mapping class $\phi \in \mathcal{MCG}(S)$). Throughout, for simplicity a diffeomorphism and its mapping class are denoted by the same symbol if no confusions occur.

For an open book (S, ϕ) let $M_{(S,\phi)}$ be a closed 3-manifold defined by

$$M_{(S,\phi)} := S \times [0,1]/\sim \tag{2.1}$$

where $\sim$ is the equivalence relation defined by $(x, 1) \sim (\phi(x), 0)$ for all $x \in S$, $\quad (x,t) \sim (x,s)$ for all $x \in \partial S$ and $t, s \in [0,1]$. We denote the quotient map by $\Pi : S \times [0,1] \to M_{(S,\phi)} = S \times [0,1]/\sim$. The *binding* B is defined by

$$B = B_{(S,\phi)} := \Pi(\partial S \times \{t\}) \subset M_{(S,\phi)}$$

which does not depend on $t \in [0,1]$ since ϕ is identity near ∂S. The binding B is a fibered link in $M_{(S,\phi)}$ with natural fibration:

$$\begin{array}{cccc} \pi = \pi_{(S,\phi)} : & M_{(S,\phi)} \setminus B_{(S,\phi)} & \longrightarrow & S^1 = [0,1]/0 \sim 1 \\ & \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ & \Pi(x,t) & \longmapsto & t \end{array}$$

For each $t \in [0,1]$ the closure of the fiber $S_t := \overline{\pi^{-1}(t)}$ is called a *page* of the open book. Note that S_0 and S_1 represent the same page in $M_{(S,\phi)}$.

For $t \in (0,1)$ let

$$p_t = \Pi^{-1}|_{S_t} : S_t \to S \times \{t\} = S \tag{2.2}$$

be the canonical diffeomorphism arising from the definition (2.1). To define $p_0 : S_0 \to S$, for $x \in S_0$ we choose a sequence of points $\{x_n \in S_{t_n} \subset M_{(S,\phi)} \mid 0 < t_{n+1} < t_n\}$ that converges to x, and define

$$p_0(x) = \lim_{n \to \infty} p_{t_n}(x_n) \tag{2.3}$$

(this is well-defined and independent of a choice of $\{x_n\}$). The diffeomorphism $p_1 : S_1 \to S$ is defined similarly. Thus, $p_0(x) = \phi \circ p_1(x)$ if $x \in S_0 = S_1 \subset M_{(S,\phi)}$.

A contact structure $\xi = \ker \alpha$ on $M_{(S,\phi)}$ is *supported* by the open book $(B_{(S,\phi)}, \pi)$ if $d\alpha > 0$ on every page S_t and $\alpha > 0$ on $B_{(S,\phi)}$. Up to isotopy there exists a unique contact structure $\xi_{(S,\phi)}$ on $M_{(S,\phi)}$ supported by $(B_{(S,\phi)}, \pi)$ [11]. If (M, ξ) and $(M_{(S,\phi)}, \xi_{(S,\phi)})$ are contactomorphic then we say that (M, ξ) is supported by the open book (S, ϕ) or, (S, ϕ) is an open book of (M, ξ).

A link L in $M_{(S,\phi)}$ is a *closed braid* with respect to (S, ϕ) if $L \subset M_{(S,\phi)} \setminus B_{(S,\phi)}$ and L is positively transverse to every page. For a fixed (S, ϕ) two closed braids L_0 and L_1 are called *braid isotopic* if they are isotopic through a continuous family $\{L_t\}_{0\leq t\leq 1}$ of closed braids with respect to (S,ϕ).

Suppose that (M,ξ) is supported by (S,ϕ). We say that a transverse link $\mathcal{T}$ in (M,ξ) is *represented by* a closed braid L with respect to (S,ϕ) if a contactomorphism $(M_{(S,\phi)},\xi_{(S,\phi)}) \to (M,\xi)$ takes L to $\mathcal{T}$ (up to transverse isotopy). In fact, due to Bennequin [2] and Pavelescu [27, 28], given a contact manifold (M,ξ) and its open book (S,ϕ), every transverse link $\mathcal{T}$ in (M,ξ) is represented by some closed braid L with respect to (S,ϕ) and such L is unique up to braid isotopy, positive braid stabilization and positive braid destabilization.

Let $L \subset M_{(S,\phi)}$ be a closed braid with respect to (S,ϕ). Take a collar neighborhood $\nu(\partial S)$ so that $\phi|_{\nu(\partial S)} = id_{\nu(\partial S)}$, and move L by braid isotopy so that $P := p_0(L\cap S_0) \subset \nu(\partial S)$. Then ϕ is regarded as a diffeomorphism $(S,P)\to(S,P)$ hence it gives an element $j(\phi)\in \mathcal{MCG}(S,P)$. By cutting $M_{(S,\phi)}$ along the page S_0 we get a cylinder $S\times[0,1]$ and L gives rise to an element β_L of the surface braid group $B_n(S)$. We define the the *distinguished monodromy* of L by

$$\phi_L = j(\phi)\ i(\beta_L)$$

where i is the push map in the generalized Birman exact sequence [8, Theorem 9.1]

$$1 \to B_n(S) \xrightarrow{i} \mathcal{MCG}(S,P) \xrightarrow{f} \mathcal{MCG}(S) \to 1.$$

The distinguished monodromy ϕ_L is well-defined up to point-changing isomorphism [22]: If two closed braids L and L' with respect to (S,ϕ) are braid isotopic, then there is a point-changing isomorphism $\Theta: \mathcal{MCG}(S,P)\to\mathcal{MCG}(S,P')$ where $P=p_0(S_0\cap L)$, $P'=p_0(S_0\cap L')$ such that $\Theta(\phi_L)=\phi_{L'}$. Here the point-changing isomorphism Θ is an isomorphism defined by $\Theta([\psi]) = [\theta^{-1}\circ\psi\circ\theta]$ for some orientation-preserving diffeomorphism $\theta:(S,P')\to(S,P)$ such that $\theta|_{\partial S}=id_{\partial S}$ and θ is isotopic to id_S if we forget the sets of marked points P and P'. When $P=P'$, this simply means that ϕ_L and $\phi_{L'}$ are conjugate in $\mathcal{MCG}(S,P)$.

2.2. **Strong right-veering ordering and quasi-right-veering.** We review the right-veering orderings $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ of arcs and the definition of quasi-right-veering.

Take a base point $v\in\partial S$. Let $\mathcal{A}_v(S,P)$ be the set of isotopy classes of oriented properly embedded arcs $\gamma:[0,1]\to S\backslash P$ satisfying $\gamma(0)=v$ and $\gamma(1)\in\partial S\setminus\{v\}$. We call $\gamma(0)$ the base point of γ and $\gamma(1)$ the *terminal* point of γ. We allow arcs to be boundary-parallel and by an isotopy we mean isotopy fixing ∂S. Let $\overline{\gamma}$ denote the arc γ with the reversed orientation so that $\overline{\gamma}(t) = \gamma(1-t)$.

We call an element of $\mathcal{A}_v(S,P)$ an *arc* based on v. Abusing the notation, by an arc $\gamma \in \mathcal{A}_v(S,P)$ we will mean three different objects.

- A map $\gamma : [0,1] \to S \setminus P$,
- The image of the map (viewed as a submanifold) $\gamma([0,1]) \subset S \setminus P$.
- The isotopy class of the submanifold γ.

We say that arcs α and β intersect *efficiently* if they realize the geometric intersection number. Unless otherwise specified (such as Definitions 3.9 and 6.1), we will always assume that all arcs intersect pairwise efficiently. In particular, when α and β have the same isotopy class relative to the boundary we have $\mathsf{int}(\alpha) \cap \mathsf{int}(\beta) = \emptyset$.

The following orderings are important in 3-dimensional contact topology.

Definition 2.1 (Right-veering ordering $\prec_{\mathsf{right}}$)**.** For arcs $\alpha, \beta \in \mathcal{A}_v(S,P)$, we denote $\alpha \prec_{\mathsf{right}} \beta$ if $\alpha \neq \beta$ and the arc β lies on the right side of α in a small neighborhood of the base point v.

Definition 2.2 (Strong right-veering ordering $\ll_{\mathsf{right}}$ [22])**.** For arcs $\alpha, \alpha' \in \mathcal{A}_v(S,P)$, we denote $\alpha \prec_{\mathsf{disj}} \alpha'$ if $\alpha \prec_{\mathsf{right}} \alpha'$ and $\alpha \cap \alpha' = \{v\}$. We denote $\alpha \ll_{\mathsf{right}} \alpha'$ if $\alpha \neq \alpha'$ and there exists a sequence of arcs $\alpha_0, \dots, \alpha_n \in \mathcal{A}_v(S,P)$ such that

$$\alpha = \alpha_0 \prec_{\mathsf{disj}} \alpha_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \alpha_n = \alpha'. \tag{2.4}$$

The relation $\prec_{\mathsf{right}}$ is a strict total order and $\ll_{\mathsf{right}}$ is a strict partial order (strict in the sense that it is irreflexive; $a \not\prec_{\mathsf{right}} a$ for all a). The relation $\prec_{\mathsf{disj}}$ is not a strict partial order since it is not transitive; $a \prec_{\mathsf{disj}} b$ and $b \prec_{\mathsf{disj}} c \not\Rightarrow a \prec_{\mathsf{disj}} c$.

Remark 2.3. The definitions of $\ll_{\mathsf{right}}$ in Definition 2.2 and [22] are slightly different. In this paper for the sequence of arcs (2.4) we required $\alpha_i \cap \alpha_{i+1} = \{v\}$ (so α_i and α_{i+1} only shares the common base point v), whereas in [22] we required a weaker condition $\mathsf{int}(\alpha_i) \cap \mathsf{int}(\alpha_{i+1}) = \emptyset$ (so we allowed α_i and α_{i+1} have the same start point and ternimal point).

However, as for the definition of quasi-right-veering, Definition 2.7 below and the one given in [22] are equivalent. This is because when α and β share the same terminal point, we can slightly move one of the terminal points without introducing new intersections.

Remark 2.4. When $P = \emptyset$ the orderings $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ are the same, which can be proved using a work of Honda, Kazez and Matić

[16, Lemma 4.1]. However, when $P \neq \emptyset$, $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ are different as studied in [22].

The mapping class group $\mathcal{M}CG(S,P)$ acts on $\mathcal{A}_v(S,P)$ and the action preserves both $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$. Using the total ordering $\prec_{\mathsf{right}}$ the right-veering property is defined as follows.

Definition 2.5 (Right-veering [16])**.** An element $\phi \in \mathcal{M}CG(S,P)$ is *right-veering* if for any $v \in \partial S$ and any arc $\alpha \in \mathcal{A}_v(S,P)$, $\alpha \preceq_{\mathsf{right}} \phi(\alpha)$; that is, $\alpha \prec_{\mathsf{right}} \phi(\alpha)$ or $\alpha = \phi(\alpha)$.

For our purpose, however, it is much more convenient to accept *non-right-veering* as the basic concept. We view right-veering as *not non-right-veering*. Definition 2.5 is rephrased as follows:

Definition 2.6 (Non-right-veering)**.** An element $\phi \in \mathcal{M}CG(S,P)$ is *non-right-veering* if there exists $v \in \partial S$ and an arc $\alpha \in \mathcal{A}_v(S,P)$ such that $\phi(\alpha) \prec_{\mathsf{right}} \alpha$. Otherwise, we say that ϕ is *right-veering*.

Using the strong right-veering ordering $\ll_{\mathsf{right}}$ in the place of $\prec_{\mathsf{right}}$, we define quasi-right-veering.

Definition 2.7 (Non-quasi-right-veering [22])**.** An element $\phi \in \mathcal{M}CG(S,P)$ is *non-quasi-right-veering* if there exists $v \in \partial S$ and an arc $\alpha \in \mathcal{A}_v(S,P)$ such that $\phi(\alpha) \ll_{\mathsf{right}} \alpha$. Otherwise, we say that ϕ is *quasi-right-veering*.

Since $\ll_{\mathsf{right}}$ and $\prec_{\mathsf{right}}$ are the same when $P = \emptyset$, $\phi \in \mathcal{M}CG(S)$ is right-veering if and only if it is quasi-right-veering. On the other hand, when $P \neq \emptyset$, ϕ is quasi-right-veering if ϕ is right-veering [22, Proposition 3.14], but the converse is not true in general.

The following is a list of properties of non-quasi-rightveering.

Theorem 2.8. *Let (M,ξ) be a closed contact 3-manifold and $\mathcal{T}$ be a transverse link in (M,ξ).*

(1) *(M,ξ) is overtwisted if and only if there exists a non-right-veering open book (S,ϕ) that supports (M,ξ).* [16]
(2) *$\mathcal{T}$ is loose; that is, the contact structure restricted to the complement $M \setminus \mathcal{T}$ is overtwisted, if and only if there exist an open book (S,ϕ) supporting (M,ξ) and a closed braid representative L of $\mathcal{T}$ with respect to (S,ϕ) such that the distinguished monodromy ϕ_L is non-quasi-right-veering.* [22]
(3) *If $\mathcal{T}$ is represented by a closed braid L with respect to an open book (S,ϕ) whose distinguished monodromy ϕ_L is non-right-veering then $\mathcal{T}$ is virtually loose; that is, there is a finite covering of $M \setminus \mathcal{T}$ on which the lifted contact structure is overtwisted.* [23]

3. Right-veering ordering on arc systems and N-twist left-veering

3.1. **Right-veering ordering on arc systems.**

In this section we extend the strong right-veering ordering $\ll_{\mathsf{right}}$ to arc systems.

Definition 3.1 (N-arc system)**.** Let $\mathcal{B} = \{v^1, \dots, v^N\} \subset \partial S$ be an ordered set of N distinct boundary points. Let

$$\mathcal{A}_{\mathcal{B}}(S,P) = \left\{ \begin{array}{l} \Gamma = (\gamma^1, \dots, \gamma^N) \\ \quad \in \mathcal{A}_{v^1}(S,P) \times \cdots \times \mathcal{A}_{v^N}(S,P) \end{array} \middle| \gamma^i \cap \gamma^j = \emptyset \ \ (i \neq j) \right\}.$$

We call an element of $\mathcal{A}_{\mathcal{B}}(S,P)$ an N*-arc system* of (S,P) based on $\mathcal{B}$.

We may abuse the symbol Γ for different objects such as:

- A collection of maps $\{\gamma^j : [0,1] \to S \mid j = 1, \dots N\}$.
- The submanifold $\gamma^1 \cup \gamma^2 \cup \cdots \cup \gamma^N \subset S$
- The isotopy class (rel. ∂S) of the submanifold $\gamma^1 \cup \gamma^2 \cup \cdots \cup \gamma^N \subset S$.

We denote $\Gamma(1) := \{\gamma^1(1), \gamma^2(1), \dots, \gamma^N(1)\} = \partial\Gamma \setminus \mathcal{B}$ the set of terminal points.

We naturally generalize $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ of Definitions 2.1 and 2.2 as follows:

Definition 3.2 (Right-veering ordering $\prec_{\mathsf{right}}$)**.** For N-arc systems $\Gamma = (\gamma^1, \dots, \gamma^N)$ and $\Gamma' = (\gamma'^1, \dots, \gamma'^N) \in \mathcal{A}_{\mathcal{B}}(S,P)$, we denote $\Gamma \prec_{\mathsf{right}} \Gamma'$ if $\Gamma \neq \Gamma'$ and $\gamma^j \preceq_{\mathsf{right}} \gamma'^j$ for all $j = 1, \dots, N$.

Definition 3.3 (Strong right-veering ordering $\ll_{\mathsf{right}}$)**.** Let Γ and $\Gamma' \in \mathcal{A}_{\mathcal{B}}(S,P)$.

- We denote $\Gamma \prec_{\mathsf{disj}} \Gamma'$ if $\Gamma \prec_{\mathsf{right}} \Gamma'$ and $\Gamma \cap \Gamma' = \mathcal{B}$.
- We denote $\Gamma \ll_{\mathsf{right}} \Gamma'$ if $\Gamma \neq \Gamma'$ and there exists a finite sequence of N-arc systems $\Gamma_0, \dots, \Gamma_n \in \mathcal{A}_{\mathcal{B}}(S,P)$ such that $\Gamma = \Gamma_0 \prec_{\mathsf{disj}} \Gamma_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \Gamma_n = \Gamma'$.

While $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ are strict partial orderings, $\prec_{\mathsf{disj}}$ is not.

Example 3.4. See Figure 4. Sketch (i) depicts 1-arc systems α and β satisfying $\alpha \ll_{\mathsf{right}} \beta$. Sketch (ii) shows $(\alpha^1, \alpha^2) \prec_{\mathsf{disj}} (\gamma^1, \gamma^2) \prec_{\mathsf{disj}} (\beta^1, \beta^2)$; thus, $(\alpha^1, \alpha^2) \ll_{\mathsf{right}} (\beta^1, \beta^2)$. The thick arrows represent parts of the boundary ∂S with the induced orientation.

Related to Remark 2.4, if $N > 1$ the orderings $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ are not equal as we will see in Proposition 3.6.

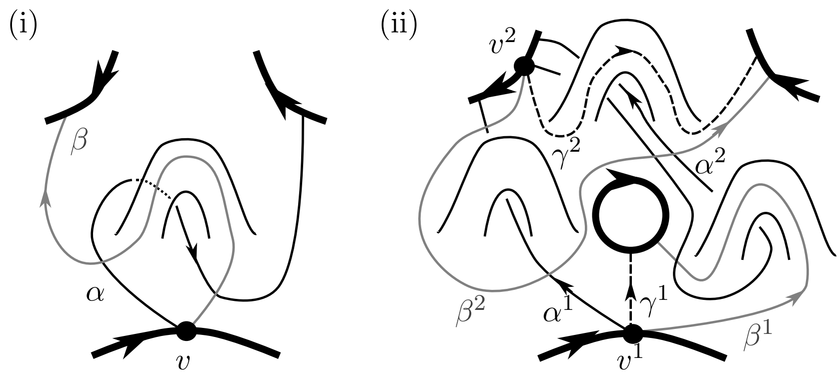

FIGURE 4. (i) $N = 1$ and $\alpha \ll_{\mathsf{right}} \beta$. (ii) $N = 2$ and $(\alpha^1, \alpha^2) \ll_{\mathsf{right}} (\beta^1, \beta^2)$.

3.2. N**-twist left-veering.** In this section we define N-twist left-veering as a generalization of non-right-veering.

Definition 3.5 (Boundary based region)**.** Let $\Gamma, \Gamma' \in \mathcal{A}_{\mathcal{B}}(S, P)$ be N-arc systems intersecting efficiently and satisfying $\Gamma \prec_{\mathsf{right}} \Gamma'$. Suppose that there exists an embedded $2N$-gon in S with the boundary $\partial R \subset \Gamma \cup \Gamma'$ (see Figures 6, 7) such that:

- (i) The interior of R is disjoint from Γ'
- (ii) The orientation of ∂R agrees with that of Γ' and disagrees with that of Γ.
- (iii) The corners of R are read $v^1, q^1, v^2, q^2, \dots, v^N, q^N$ with respect to the orientation of ∂R where $\{v^1, \dots, v^N\} = \mathcal{B} \subset \partial S$ and $\{q^1, \dots, q^N\} \subset \Gamma \cap \Gamma' \cap \mathsf{int}(S)$.

The $2N$-gon is called a *boundary based region* for the ordered pair (Γ, Γ') and denoted by $R(\Gamma, \Gamma')$.

Here are some terminologies we use:

- Corners $v^1, \dots, v^N$ are called *base corners* and depicted by black dots $\bullet_{v^j}$.
- Corners $q^1, \dots, q^N$ are called *non-base corners* and depicted by hollow circles $\circ_{q^j}$
- If $\mathsf{int}(R(\Gamma, \Gamma')) \cap \Gamma = \emptyset$ then we say that $R(\Gamma, \Gamma')$ is *embedded*.
- If $R(\Gamma, \Gamma')$ contains k (≥ 0) punctures then we say that it has *type* (N, k).

Here are some remarks:

- If $\Gamma = (\gamma^1, \dots, \gamma^N)$ and $\Gamma' = (\gamma'^1, \dots, \gamma'^N)$, we have $v^j = \gamma^j(0) = \gamma'^j(0)$ and $q^j = \gamma'^j \cap \gamma^{j+1}$ for $j = 1, \dots, N$ (where $\gamma^{N+1} = \gamma^1$).

- All the non-base corners q_i are negative intersections of Γ and Γ'.
- ⧫ (Uniqueness Property): Given Γ, Γ', the boundary based region may not exist. If it exists $R(\Gamma, \Gamma')$ is unique due to Condition (i).
- Although Γ' and $\mathsf{int}(R(\Gamma, \Gamma'))$ are disjoint, Γ may intersect $\mathsf{int}(R(\Gamma, \Gamma'))$.
- For $N = 1$, if 1-arc systems $\Gamma = (\gamma)$ and $\Gamma' = (\gamma')$ form a boundary based region $R(\gamma, \gamma')$ then it is a punctured bigon as depicted in Figure 5. In [22] such a bigon is called a *boundary right P-bigon* from γ to γ'.

  Thus, boundary based regions can be viewed as generalization of boundary right P-bigons. However, since a boundary right P-bigon may be immersed, it is not always a boundary based region. In [22, Proposition 3.5], it is shown that a boundary right P-bigon serves as an obstruction for $\gamma \ll_{\mathsf{right}} \gamma'$. Similarly, a boundary based region serves as an obstruction for $\Gamma \ll_{\mathsf{right}} \Gamma'$.

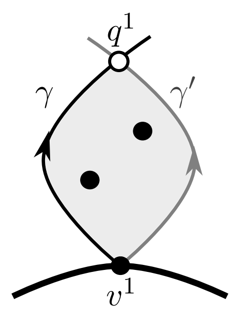

FIGURE 5. A boundary based region $R(\gamma, \gamma')$ where $N = 1$ and $k > 0$.

Proposition 3.6. *Let $N > 1$. If N-arc systems Γ and Γ' form a boundary based region $R(\Gamma, \Gamma')$ then $\Gamma \not\ll_{\mathsf{right}} \Gamma'$.*

Proof. Assume to the contrary that Γ and Γ' form a boundary based region $R(\Gamma, \Gamma')$ and $\Gamma \ll_{\mathsf{right}} \Gamma'$. Then there exists an interpolating sequence:

$$\Gamma = \Gamma_0 \prec_{\mathsf{disj}} \Gamma_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \Gamma_n = \Gamma'$$

for some $n \geq 2$. Among such ordered pairs (Γ, Γ') we choose one so that the length, n, of the interpolating sequence is minimum.

In Figure 6 (ii) the dashed arcs stand for Γ_1. Since $\Gamma \prec_{\mathsf{right}} \Gamma_1 \prec_{\mathsf{right}} \Gamma'$ there exist M-arc systems Δ_1 and Δ' for some $2 \leq M \leq N$ that are sub-systems of Γ_1 and Γ' respectively, and form a boundary based region $R(\Delta_1, \Delta')$ that is a $2M$-gon (the shaded region in Figure 6 (ii)). Taking subsequent M-arc systems $\Delta_i \subset \Gamma_i$, we obtain a sequence

$$\Delta_1 \prec_{\mathsf{disj}} \Delta_2 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \Delta_n = \Delta'$$

of length $n-1$, which contradicts the minimality of n. □

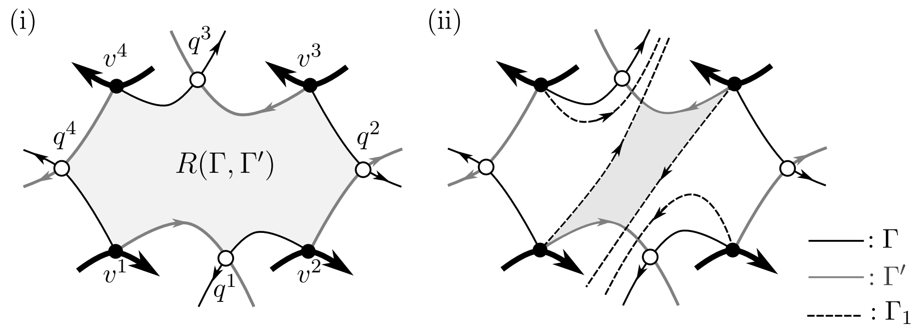

Figure 6. Proof of Proposition 3.6.

In the following, we mainly study boundary based regions $R(\Gamma, \phi(\Gamma))$ for $\phi \in \mathcal{MCG}(S, P)$.

Definition 3.7 (Left twist)**.** Let $\mathcal{B} = (v^1, \ldots, v^N)$. For $\Gamma \in \mathcal{A}_{\mathcal{B}}(S, P)$ and $\phi \in \mathcal{MCG}(S, P)$ we define $\phi^{tw}(\Gamma) \in \mathcal{A}_{\mathcal{B}}(S, P)$ the *left-twist* of Γ for ϕ as follows:

- When Γ and $\phi(\Gamma)$ do not form a boundary based region, we define $\phi^{tw}(\Gamma) := \phi(\Gamma)$.
- When Γ and $\phi(\Gamma)$ form a boundary based region $R(\Gamma, \phi(\Gamma))$, take $t^j, s^j \in [0,1]$ so that $q^j = \phi(\gamma^j(t^j)) = \gamma^{j+1}(s^j)$. We define an arc
$$\phi^{tw}(\gamma^j) := \gamma^j|_{[0,s^{j-1}]} * (\phi \circ \gamma^{j-1})|_{[t^{j-1},1]}$$
that starts at v^j and goes along γ^j until reaching q^{j-1} then turns left and switches to $\phi(\gamma^{j-1})$ to the terminal point, where $\gamma^0 = \gamma^N$. The symbol $*$ represents concatenation of paths. See Figure 7. We define
$$\phi^{tw}(\Gamma) := (\phi^{tw}(\gamma^1), \ldots, \phi^{tw}(\gamma^N)).$$
When $N = 1$ see Figure 10-(4), where $\gamma^1 = \gamma^0 = \gamma$.

Using the left-twist $\phi^{tw}(\Gamma)$ we define (N, k)-twist left-veering.

Definition 3.8 ((N,k)-twist left-veering)**.** An element $\phi \in \mathcal{MCG}(S, P)$ is (N, k)-*twist left-veering* if there exist a set $\mathcal{B}$ of N base points and an N-arc system $\Gamma \in \mathcal{A}_{\mathcal{B}}(S, P)$ such that

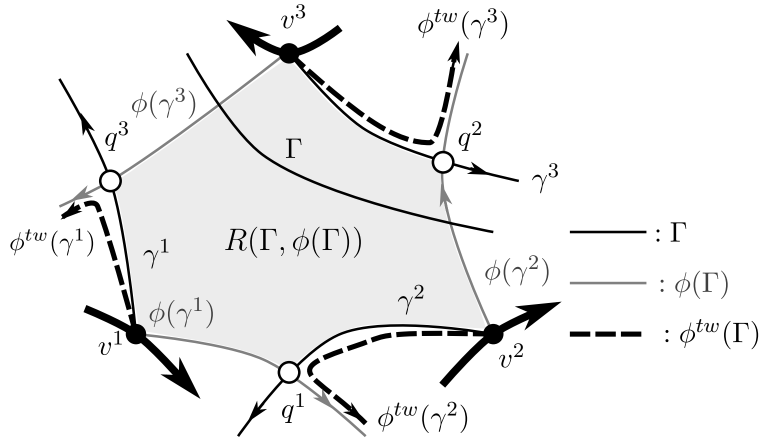

FIGURE 7. The boundary based region $R(\Gamma, \phi(\Gamma))$ and the left twist $\phi^{tw}(\Gamma)$.

- Γ and $\phi(\Gamma)$ form a boundary based region $R(\Gamma, \phi(\Gamma))$ of type (N, k), and
- $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$.

We also say that ϕ is *N-twist left-veering* if ϕ is (N, k)-twist left-veering for some k.

Finally, we introduce a type $(1, 0)$ boundary based region whose treatment is exceptional. By doing so, we can regard the (N, k)-twist left-veering as a generalization of non-right-veering, and we can state our main result Theorem 4.1 in a unified way.

Definition 3.9 (Boundary based region of type $(1, 0)$)**.** Let $\Gamma = (\gamma)$ and $\Gamma' = (\gamma') \in \mathcal{A}_v(S, P)$ be 1-arc systems with $\gamma' \prec_{\mathsf{right}} \gamma$. Since the condition $\Gamma \prec_{\mathsf{right}} \Gamma'$ is not satisfied, the ordered pair (Γ, Γ') does not form a boundary based region in the sense of Definition 3.5.

If we slightly move the arcs γ and γ' near the base point v we can create a bigon R with no punctures. See Figure 8. After the operation, γ and γ' no longer intersect efficiently. However all the conditions (i)–(iii) in Definition 3.5 are satisfied. Thus, we may call the bigon R a boundary based region of type $(1, 0)$ formed by the ordered pair (Γ, Γ') of 1-arc systems with $\Gamma' \prec_{\mathsf{right}} \Gamma$.

With this definition, we observe that Definition 3.8 of (N, k)-twist left-veering can be extended to $(N, k) = (1, 0)$.

Remark 3.10. We have the following:

- $\phi \in \mathcal{MCG}(S)$ is non-right-veering if and only if ϕ is 1-twist left-veering.

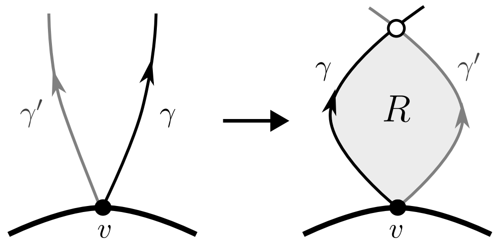

FIGURE 8. A boundary based region R of type $(1,0)$.

- $\phi \in \mathcal{MCG}(S,P)$ is non-quasi-right-veering if and only if ϕ is $(1,0)$-twist left-veering.

Thus, one can view (N,k)-twist left-veering as a generalization of non-right-veering.

4. Overtwisted disks and twist left-veering monodromies

This section is devoted to prove Theorem 4.1.

Theorem 4.1. *Let L be a closed braid with respect to (S,ϕ) and $\phi_L \in \mathcal{MCG}(S,P)$ be its distinguished monodromy.*

(1) *If $\phi \in \mathcal{MCG}(S)$ is N-twist left-veering then $(M_{(S,\phi)}, \xi_{(S,\phi)})$ is overtwisted, and there is an overtwisted disk $\mathcal{D}$ that intersects the binding B at N points.*
(2) *If $\phi_L \in \mathcal{MCG}(S,P)$ is (N,k)-twist left-veering then there is an overtwisted disk $\mathcal{D}$ that intersects the binding B at N points and intersects the closed braid L at k points. In particular, if ϕ_L is $(N,0)$-twist left-veering then L is loose.*

For the proof, we use *open book foliations* introduced by the authors in [17] and we will assume the readers are familiar with the definition and basic properties of open book foliations. See the research monograph [24] by LaFountain and Menasco for a gentle introduction to the techniques of open book foliations that is central to the new work in this paper. Open book foliations had their origins in the work of Birman and Menasco in a series of papers about *braid foliations*. See Birman and Finkelstein's article [3] for a useful guide to the work of Birman and Menasco on braid foliations, and [4] for their key paper that is relevant for us. It is the first place where braid foliations were used to solve a major then-open problem in contact topology.

In the proof, we will construct a transverse overtwisted disk D [17, Definition 4.1] which can be thought as an embedded disk D in $(M_{(S,\phi)}, \xi_{(S,\phi)})$ whose boundary ∂D is a closed braid with respect to the open book (S,ϕ) (i.e., a transverse unknot) with $sl(\partial D) = 1$. Note that the

Bennequin inequality is violated. Hence the existence of a transverse overtwisted disk is equivalent to overtwistedness of ξ. An overtwsited disk $\mathcal{D}$ can be obtained by a suitable perturbation of the transverse overtwisted disk D.

Convention 4.2. For simplicity, both a b-arc $\gamma \subset S_t$ of the open book foliation of D and its image $p_t(\gamma) \subset S$ under the canonical diffeomorphism $p_t : S_t \to S$ (defined in Section 2.1) are denoted by the same letter γ. This convention also applies in the proof of Theorem 5.8.

Proof. We prove (2) since (1) is a special case of (2) where $P = \emptyset$ and the closed braid L is empty. Assume that $\phi_L \in \mathcal{M}CG(S, P)$ is (N, k)-twist left-veering. Construction of a transverse overtwisted disk will be described in the following three cases.

Case $(N, k) = (1, 0)$.

Remark 3.10 states that ϕ_L is non-quasi-right-veering. Theorem 4.1 in [22] shows not only L is loose but also how to construct an overtwisted disk that intersects the binding at one point.

Case $N = 1$ **and** $k > 0$.

Although the construction of transverse overtwisted disk is exactly the same as in the proof of [22, Theorem 4.1], since the construction is fundamental for the case $N \geq 2$, we include it here.

(Step 1: Setting up arcs and points)

By the assumption that $\phi_L \in \mathcal{M}CG(S, P)$ is $(1, k)$-twist left-veering, there is an arc γ such that γ and $\phi_L(\gamma)$ form a boundary based region (bigon) $R(\gamma, \phi_L(\gamma))$ with k punctures and $\phi_L^{tw}(\gamma) \ll_{\mathsf{right}} \gamma$. Take an interpolating sequence

$$\phi_L^{tw}(\gamma) = \gamma_0 \prec_{\mathsf{disj}} \gamma_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \gamma_{n-1} \prec_{\mathsf{disj}} \gamma_n = \gamma.$$

Let v denote the common base point of $\gamma_0, \dots, \gamma_n$ and w_i the terminal point of the arc γ_i where $i = 0, \dots, n$. Since γ_0 and γ_n have the same terminal point, $w_0 = w_n$. In the open book foliation $\mathcal{F}_{ob}(D)$, v corresponds to a negative elliptic point and $w_0, \dots, w_{n-1}$ correspond to positive elliptic points.

Let $\alpha_i \subset S$ be a simple arc segment emanating from w_i $(i = 0, \dots, n-1)$ and contained in a small neighborhood of w_i so that α_is are pairwise disjoint.

(Step 2: Construction of a once-holed disk)

We construct via *movie presentation* a once-holed disk that is disjoint from the closed braid L. Let $0 = t_0 < t_1 < \cdots < t_n = 1$. We will construct a family

$$\{b(t) \subset S_t \mid t \in [0,1]\}$$

of b-arcs emanating from v and n families

$$\{a_0(t) \subset S_t \mid t \in [t_1, t_n]\},$$
$$\{a_i(t) \subset S_t \mid t \in [0,1] \setminus [t_i, t_{i+1}]\} \quad \text{where } i = 1, \dots, n-1$$

of a-arcs $a_i(t)$ emanating from the elliptic point w_i. The families will be building blocks of the once-holed disk.

For $t \in [0, t_1)$, using Convention 4.2, we define a b-arc $b(t) := \gamma_0 = \phi_L^{tw}(\gamma)$ and an a-arc $a_i(t) := \alpha_i$ where $i = 1, \dots, n-1$.

At $t = t_1$ the b-arc $b(t)$ and the a-arc $a_1(t)$ intersect tangentially and form a hyperbolic point.

Figure 9 (iii) gives a movie presentation near $t = t_1$. Take a small $\varepsilon > 0$. On the page $S_{t_1-\varepsilon}$ we put a *describing arc* for the hyperbolic point (dotted arc joining $b(t)$ and $a_1(t)$). The describing arc is chosen to be very close to γ_1 so that after passing $t = t_1$ the isotopy type (in S) of the b-arc switches from γ_0 to γ_1, the a-arc $a_1(t)$ disappears and a new a-arc $a_0(t)$ emanating from w_0 appears. There is no change for the other a-arcs $a_2(t), \dots, a_{n-1}(t)$.

Since $\gamma_0 \prec_{\mathsf{disj}} \gamma_1$, the positive normals (dashed arrows in Figure 9 (iii)) are pointing out of the describing arc, which implies that the sign of the hyperbolic point is positive. In [20, Observation 2.5] one can see how the sign of hyperbolic point is determined by describing arc and positive normals.

Figure 9 (i) and (ii) depict part of the open book foliation $\mathcal{F}_{ob}(D)$ of the disk D intersecting with the pages S_t for $t \in [0, t_1 + \varepsilon]$. (The a-arcs $a_2(t), \dots, a_{n-1}(t)$ are not illustrated.)

For $t \in (t_1, t_2)$, we set $b(t) := \gamma_1$ and $a_i(t) := \alpha_i$ where $i = 0, 2, 3, \dots, n-1$.

At $t = t_2, \dots, t_n$ we apply the same construction as $t = t_1$. In order to create a hyperbolic point at $t = t_i$ we put a describing arc of the hyperbolic point that is very close to γ_i on the page $S_{t_i-\varepsilon}$. At $t = t_i$ the isotopy type (in S) of the b-arc $b(t)$ switches from γ_{i-1} to γ_i, the a-arc $a_i(t)$ disappears and instead a new a-arc $a_{i-1}(t)$ appears.

To summerise,

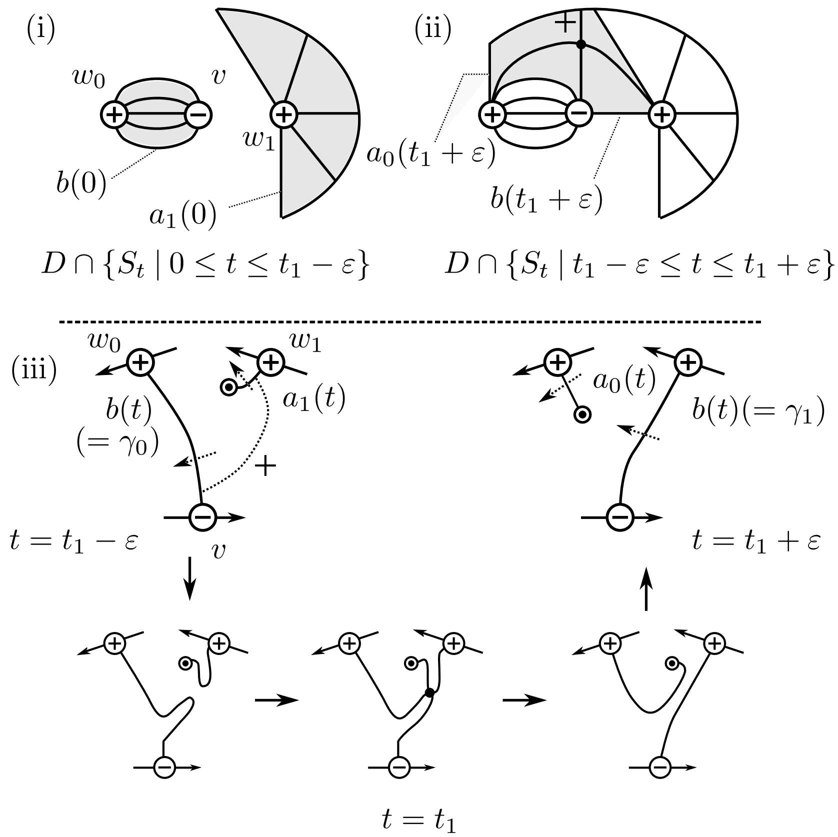

FIGURE 9. Construction of a transverse overtwisted disk D.
(i) The open book foliation on $D \cap \{S_t \mid 0 \leq t \leq t_1 - \varepsilon\}$
(ii) The open book foliation on $D \cap \{S_t | t_1 - \varepsilon \leq t \leq t_1 + \varepsilon\}$ (shaded region)
(iii) Forming a hyperbolic point at $t = t_1$. Isotopy type in S of the b-arc changes from γ_0 to γ_1.

- at $t \in [0, t_1)$, the page S_t contains the b-arc $b(t) = \gamma_0 = \phi_L^{tw}(\gamma)$ and the a-arcs $a_1(t), \ldots, a_{n-1}(t)$,
- at $t \in (t_i, t_{i+1})$ where $i = 1, \ldots, n-1$, the page S_t contains the b-arc $b(t) = \gamma_i$ and the a-arcs $a_0(t), \ldots, a_{i-1}(t), a_{i+1}(t), \ldots, a_{n-1}(t)$,
- at $t \in (t_n, 1]$ the page S_t contains the b-arc $b(t) = \gamma_n = \gamma$ and a-arcs $a_1(t), \ldots, a_{n-1}(t)$.

Since the a-arcs are very close to the boundary of the page and the monodromy ϕ_L restricted to a neighborhood of the boundary ∂S is identity, in the open book manifold $M_{(S,\phi)}$ the a-arcs $a_i(1) = \alpha_i$ and $a_i(0) = \alpha_i$ can be identified via the monodromy ϕ_L for $i = 1, \ldots, n-1$.

However, the b-arcs $b(1) = \gamma$ and $b(0) = \phi_L^{tw}(\gamma)$ cannot be identified by the monodromy ϕ_L. This is because $\phi_L(\gamma) \neq \phi_L^{tw}(\gamma)$ in $\mathcal{A}_{\mathcal{B}}(S, P)$ due to the $k(> 0)$ punctures in the bigon $R(\gamma, \phi_L(\gamma))$. Therefore, we obtain a once-holed disk (Figure 10 (1)).

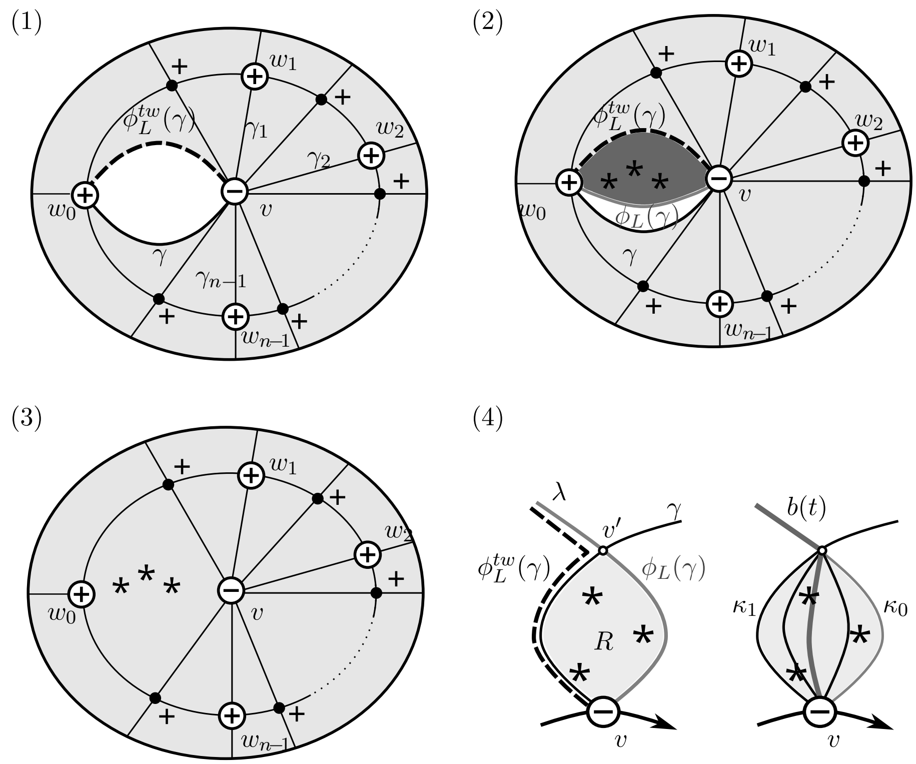

FIGURE 10. Construction of a transverse overtwisted disk for $N = 1$ and $k > 0$.

(Step 3: Filling the hole)

To fill in the hole of the disk constructed in Step 2, we modify the family of b-arcs $b(t)$ for $0 \leq t \leq t_1 - \varepsilon$. See Figure 10-(4). Let $\lambda := \phi_L(\gamma) \cap \phi_L^{tw}(\gamma)$ and $v' := \gamma \cap \phi_L(\gamma) \cap \mathsf{int}(S)$. Let $\{\kappa_s \subset R \mid s \in [0, 1]\}$ be a smooth family of arcs in the bigon $R = R(\gamma, \phi_L(\gamma))$ connecting the vertices v and v' such that

(1) $\kappa_0 = \phi_L(\gamma) \cap \partial R$,
(2) $\kappa_1 = \gamma \cap \phi_L^{tw}(\gamma)$,
(3) $\mathsf{int}(\kappa_t) \cap \mathsf{int}(\kappa_{t'}) = \emptyset$ for $t \neq t'$,
(4) $\cup_{t=0}^{t_1-\varepsilon} \kappa_t = R$ (the family gives a foliation on R),
(5) there are k arcs in the family $\{\kappa_s \mid s \in [0, 1]\}$ that contain a puncture of R.

For $0 \leq t \leq t_1 - \varepsilon$ we re-define the b-arcs as;

$$b(t) := \kappa_{\frac{t}{t_1-\varepsilon}} * \lambda$$

so that $b(0) = \phi_L(\gamma)$ and $b(t_1 - \varepsilon) = \phi_L^{tw}(\gamma)$. Figure 10-(2) depicts the open book foliation of the once-holed disk with k punctures with the modified b-arc family. The darkgray bigon stands for the modified family $\{b(t) \mid 0 \leq t \leq t_1 - \varepsilon\}$. By the condition (5), the darkgray bigon is punctured k times, where the braid L transversely intersects.

Now using the monodromy ϕ_L we can successfully identify the b-arcs $b(0) = \phi_L(\gamma)$ in S_0 and $b(1) = \gamma$ in S_1 and obtain a disk D with k punctures, see Figure 10-(3).

Since the boundary ∂D of the disk D is the trace of the terminal points of the a-arcs, ∂D transversely intersects the pages of the open book (S, ϕ_L). That is, ∂D is a transverse unknot in the contact structure $\xi_{(S,\phi_L)}$. The open book foliation $\mathcal{F}_{ob}(D)$ of D contains

- $e_+(D) = n$ positive elliptic points (from Steps 1 and 2),
- $h_+(D) = n$ positive hyperbolic points (from Step 2),
- $e_-(D) = N = 1$ negative elliptic point (from Steps 1 and 2),
- $h_-(D) = 0$ negative hyperbolic point.

Since the self-linking number satisfies

$$sl(\partial D) = -(e_+(D) - h_+(D)) + (e_-(D) - h_-(D)) = 1$$

(cf [9, p.203]) we conclude that D is a transverse overtwisted disk.

(Step 4: Modifying D into $\mathcal{D}$)

In the proof of [21, Theorem 3], we have shown that there exists a disk D' that is C^0-close to D and containing an overtwisted disk $\mathcal{D} \subset D'$ such that

(1) $|B \cap^+ (D' \setminus \mathcal{D})| = |B \cap^+ D'| = |B \cap^+ D| = e_+(D)$,
(2) $|B \cap^+ \mathcal{D}| = 0$,
(3) $|B \cap^- \mathcal{D}| = |B \cap^- D'| = |B \cap^- D| = e_-(D)$,
(4) $|L \cap \mathcal{D}| = |L \cap D'| = |L \cap D| = k$.

All the above intersections are transverse type, and $|X \cap^\pm Y|$ stands for the number of positive (resp. negative) intersection points of X and Y.

In the construction of $\mathcal{D}$ we note that *the structural stability theorem* [17, Theorem 2.21] and *the Giroux elimination lemma* [10, Lemma 3.3] play important roles. The equations (2), (3) and (4) show that $\mathcal{D}$ is a desired overtwisted disk.

Case $N \geq 2$.

(Step 1: Setting up arcs and points)

Since $\phi_L^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$ we have an interpolating sequence of N-arc systems $\in \mathcal{A}_{\mathcal{B}}(S,P)$ for some n:

$$\phi_L^{tw}(\Gamma) = \Gamma_0 \prec_{\mathsf{disj}} \Gamma_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \Gamma_n = \Gamma.$$

Since $\phi_L^{tw}(\Gamma)$ and Γ have exactly the same set of terminal points, $\phi_L^{tw}(\Gamma) \cap \Gamma \neq \mathcal{B}$ (as sets); thus $\phi_L^{tw}(\Gamma) \not\prec_{\mathsf{disj}} \Gamma$ which implies $n \geq 2$. Denote $\Gamma_i = (\gamma_i^1, \ldots, \gamma_i^N)$ for $i = 0, \ldots, n$. If $i = n$ we may also denote $\Gamma = (\gamma^1, \ldots, \gamma^N)$ so $\gamma^j = \gamma_n^j$.

For $i = 0, \ldots, n$ and $j = 1, \ldots, N$, let $w_i^j := \gamma_i^j(1)$ be the terminal point of γ_i^j, the j-th component of the i-th interpolating arc system Γ_i and $v^j := \gamma_i^j(0)$ the base point of γ_i^j. Thus, $\mathcal{B} = (v^1, \ldots, v^N)$. When $\gamma_i^j = \gamma_i^{j+1}$ we slightly move the arc γ_i^j by shifting its terminal point w_i^j so that $w_i^j \neq w_i^{j+1}$ and $\gamma_i^j \prec_{\mathsf{disj}} \gamma_i^{j+1}$ holds. This modification can be done without introducing new intersections (see Figure 11). Thus, points w_i^j are pairwise distinct, except for $w_n^{j-1} = w_0^j$ which is due to the definition of left-twist ϕ_L^{tw} and

$$w_n^{j-1} = \gamma_n^{j-1}(1) = [\phi_L(\gamma_n^{j-1})](1) = [\phi_L^{tw}(\gamma_n^j)](1) = \gamma_0^j(1) = w_0^j.$$

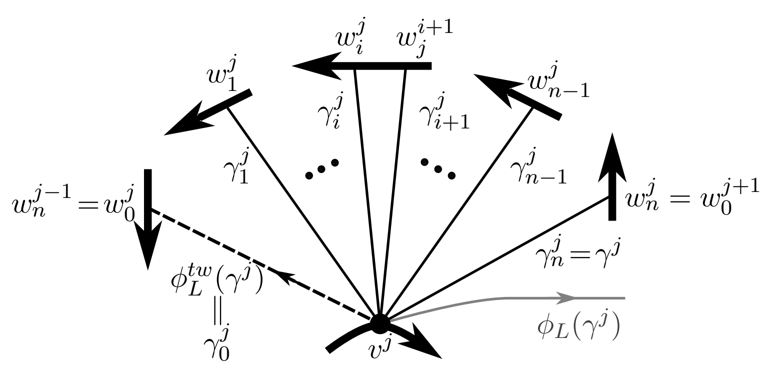

FIGURE 11. Interpolating sequence of arcs $\gamma_0^j \prec_{\mathsf{disj}} \gamma_1^j \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \gamma_n^j$ and their distinct terminal points $w_n^{j-1} = w_0^j, w_1^j, \ldots, w_{n-1}^j, w_n^j = w_0^{j+1}$.

Let $\alpha_i^j \subset S$ ($i = 0, \ldots, n-1$ and $j = 1, \ldots, N$) be a simple arc segment emanating from w_i^j and contained in a small neighborhood of w_i^j so that α_i^js are pairwise disjoint.

(Step 2: Construction of a once-holed disk)

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < 1$. We set $i = 0, \ldots, n$ and $j = 1, \ldots, N$. We will construct a family $\left\{ \mathbf{b}(t) = (b^1(t), \ldots, b^N(t)) \subset S_t \;\middle|\; t \in [0,1] \right\}$ of N-tuples of b-arcs emanating from $\mathcal{B} = (v^1, \ldots, v^N)$. With Convention 4.2, we may regard $\mathbf{b}(t)$ as an N-arc system $\in \mathcal{A}_{\mathcal{B}}(S, P)$. We also construct families of a-arcs $\left\{ a_i^j(t) \subset S_t \;\middle|\; t \in [0,1] \right\}$ emanating from the elliptic point w_i^j where $i = 0, \ldots, n-1$ and $j = 1, \ldots, N$. The once-holed disk we are construcing will intersect the page S_t in the b-arcs $\mathbf{b}(t)$ and a-arcs $a_i^j(t)$.

We start with the definition of the a-arcs. For $t \in [0,1]$ we define, using Convention 4.2, $a_i^j(t) = \alpha_i^j$ an arc emanating from the positive elliptic point w_i^j for $i = 0, \ldots, n-1$ and $j = 1, \ldots, N$ with the following exceptions: For $t \in [t_k, t_{k+1}]$ the a-arc $a_k^j(t)$ is not defined (or empty) and for $t \in [t_n, 1]$ the a-arc $a_0^j(t)$ is not defined for all $j = 1, \ldots, N$.

Now we define the b-arcs. For $t \in [0, t_1)$ we define, using Convention 4.2, $b^j(t) := \phi_L^{tw}(\gamma_n^j)$ an arc connecting the negative elliptic point v^j and the positive elliptic point w_n^j. Note that

$$\mathbf{b}(t) = (b^1(t), \ldots, b^N(t)) = \phi_L^{tw}(\Gamma) = \Gamma_0.$$

The slice, $D \cap S_0$, of D by the page S_0 has

$$p_0(D \cap S_0) = \phi_L^{tw}(\Gamma) \cup \left\{ \alpha_i^j \;\middle|\; i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, N \right\}$$

where $p_0 : S_0 \to S$ is the diffeomorphism defined in Equation (2.3). The left sketch of Figure 12 depicts the a-arcs and the b-arc of the slice $D \cap S_0$ for a fixed j.

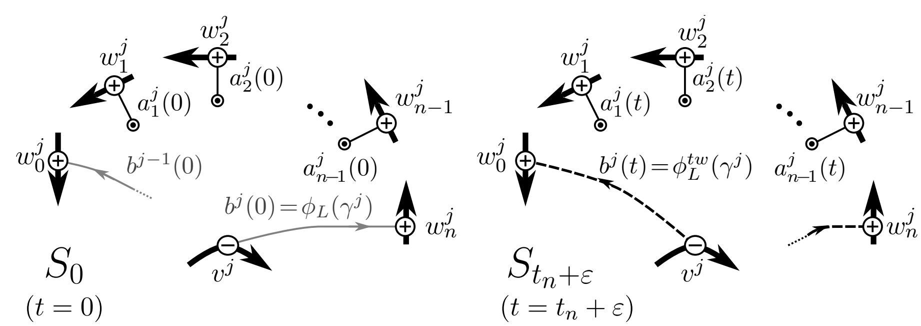

FIGURE 12. (Step 2) a-arcs and b-arcs in the pages S_0 and $S_{t_n+\epsilon}$.

For $t \in (t_i, t_{i+1})$ we define $\mathbf{b}(t) := \Gamma_i$, and for $t \in (t_n, 1]$ we define $\mathbf{b}(t) := \Gamma_n = \Gamma$.

At $t = t_i$ ($i = 1, \dots, n$) the isotopy type in S of $\mathbf{b}(t)$ changes from $\mathbf{b}(t_i - \varepsilon) = \Gamma_{i-1}$ to $\mathbf{b}(t_i + \varepsilon) = \Gamma_i$, the a-arcs $a_i^1(t), \dots, a_i^N(t)$ disappear, and new a-arcs $a_{i-1}^1(t), \dots, a_{i-1}^N(t)$ appear (no changes are made to the rest of the a-arcs). This introduces N positive hyperbolic points simultaneously whose describing arcs (see Figure 13) are parallel to γ_i^j and connecting $b^j(t_i - \varepsilon)$ and $a_i^j(t_i - \varepsilon)$ for each $j = 1, \dots, N$. This produces the fan shaped shaded region in the right sketch of Figure 16. Since $\Gamma_{i-1} \prec_{\mathsf{disj}} \Gamma_i$ the describing arcs can be pairwise disjoint, which enables us to simultaneously introduce the N hyperbolic points.

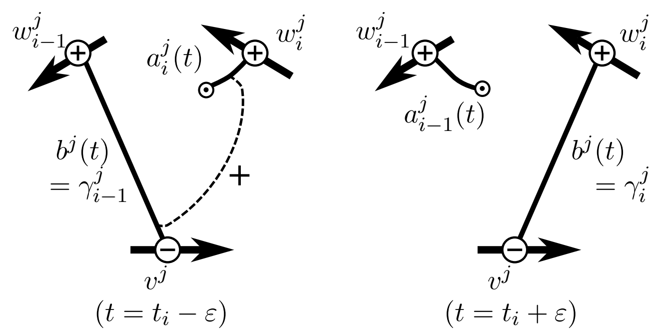

FIGURE 13. (Step 2) Movie presentation near $t = t_i$.

Similar to the $N = 1$ case, the a-arcs $a_i^j(1) \subset S_1$ and $a_i^j(0) \subset S_0$ are identified by the monodromy ϕ_L for all $i = 1, \dots, n-1$ and $j = 1, \dots, N$. However, the N-tuples $\mathbf{b}(1) = \Gamma$ and $\mathbf{b}(0) = \phi_L^{tw}(\Gamma)$ cannot be identified by ϕ_L because $\phi_L(\Gamma) \neq \phi_L^{tw}(\Gamma)$. Thus we get a disk with a hole. The once-holed disk is contained in the left sketch of Figure 16. The hole is a $2N$-gon consisting of edges $b^1(0), \dots, b^N(0)$ and $b^1(1), \dots, b^N(1)$ and vertices $v^1, \dots, v^N$ and $w_0^1, \dots, w_0^N$.

(Step 3: Filing the holes)

We fill the $2N$-gon hole by changing the family $\left\{\mathbf{b}(t) = (b^1(t), \dots, b^N(t)) \,\middle|\, 0 \leq t \leq t_1 - \varepsilon\right\}$. A similar technique is used in the case for $N = 1$. As a consequence of the modification, some *negative* hyperbolic points will be introduced. Take $0 < s_1 < s_2 < \cdots < s_{N-1} < t_1 - \varepsilon$.

Let $j = 1, \dots, N-1$. The j-th component $b^j(t)$ of $\mathbf{b}(t)$ is redefined by

$$b^j(t) := \phi_L(\gamma^j) \quad \text{for} \quad t \in [0, s_j) \text{ and}$$
$$b^j(t) := \phi_L^{tw}(\gamma^j) \quad \text{for} \quad t \in (s_j, t_1 - \varepsilon).$$

In particular, $\mathbf{b}(t) := \phi_L(\Gamma)$ for $t \in [0, s_1)$.

At each $t = s_j$ we introduce a negative hyperbolic point h_j^- in the page S_{s_j} using a describing arc (the dashed arcs in Figure 14 for $j = 1, 2$) connecting $b^j(s_j - \varepsilon)$ and $b^{j+1}(s_j - \varepsilon)$ and contained in $R(\Gamma, \phi_L(\Gamma))$. Since the positive normals are pointing into the describing arcs the signs of the hyperbolic points are both negative. Due to the hyperbolic point, the isotopy type (in S) of $b^j(t)$ changes from $\phi_L(\gamma^j)$ to $\phi_L^{tw}(\gamma^j)$. For $j' \neq j$ the isotopy type of $b^{j'}(t)$ does not change.

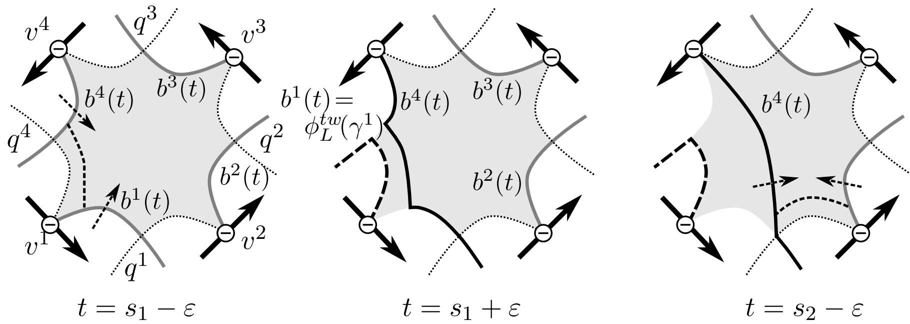

Figure 14. (Step 3) Dashed arcs are describing arcs for h_1^- and h_2^-. The shaded region is the boundary based region $R(\Gamma, \phi_L(\Gamma))$ and dashed arrows represent positive normals.

The modified family of N-tuples of b-arcs $\left\{\mathbf{b}(t) = (b^1(t), \ldots, b^N(t)) \middle| \ 0 \leq t \leq t_1 - \varepsilon\right\}$ forms a $2N$-gon (see Figure 15 where $N = 4$). Since $R = R(\Gamma, \phi(\Gamma))$ contains k punctures, the $2N$-gon and L intersect k times.

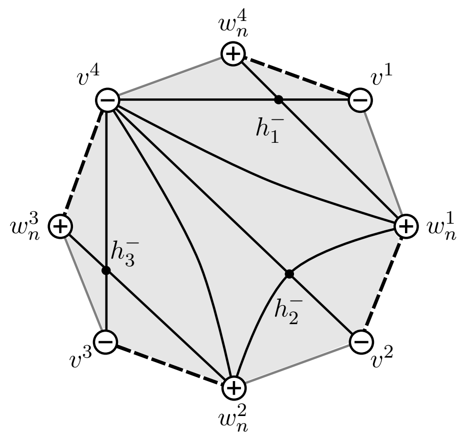

Figure 15. The open book foliation of the $2N$-gon in Step 3. If $k > 0$ then k punctures should be added to it.

Therefore, the disk with a $2N$-gon hole constructed in (Step 2) is converted to a disk with N bigon holes. More precisely, for each $j =$

$1, \ldots, N$ the pair of b-arcs $b^j(0)$ and $b^j(1)$ is forming a bigon. Since $\mathbf{b}(0) = \phi_L(\Gamma)$ and $\mathbf{b}(1) = \Gamma$, now we can identify the b-arcs $b^j(0)$ and $b^j(1)$ by the monodromy ϕ_L and all the holes are closed up. As a result, we obtain a disk D in $M_{(S,\phi)}$. The left sketch in Figure 16 depicts the open book foliation of D.

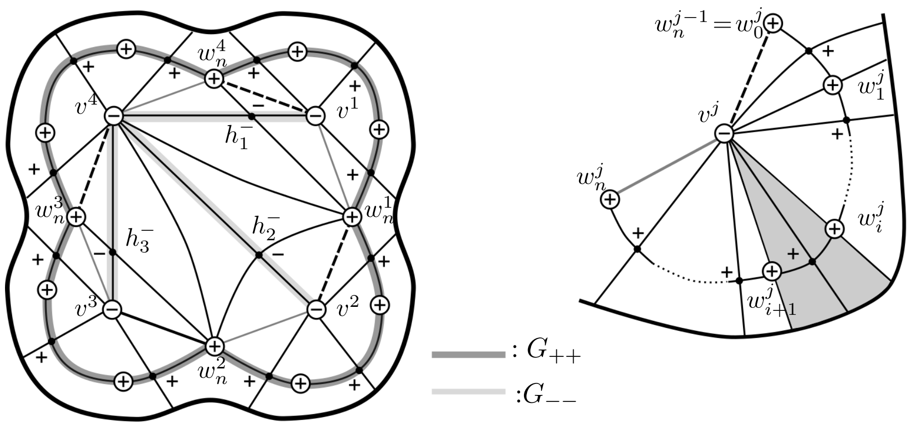

FIGURE 16. (Left) The entire open book foliation of D (where $N = 4$ and $n = 3$). (Right) Step 2 construction.

The open book foliation of the obtained disk D contains

- $e_+(D) = Nn$ positive elliptic points (from Steps 1 and 2)
- $h_+(D) = Nn$ positive hyperbolic points (from Step 2)
- $e_-(D) = N$ negative elliptic points (from Steps 1 and 2)
- $h_-(D) = (N-1)$ negative hyperbolic points (from Step 3)

Since the self-linking number satisfies

$$sl(\partial D) = -(e_+(D) - h_+(D)) + (e_-(D) - h_-(D)) = 1$$

D is a transverse overtwisted disk.

(Step 4: Modifying D into $\mathcal{D}$)

By the same argument as in Step 4 of $N = 1$ case, we can modify D into an overtwisted disk $\mathcal{D}$ that intersects the binding B (resp. braid L) in N points (resp. k points). $\square$

5. Application to depth of transverse links

Recall the depth of a transverse link defined by Baker and Onaran [1]:

Definition 5.1. [1] Let $\mathcal{T}$ be a transverse link in an overtwisted contact 3-manifold (M,ξ). The *depth* $\mathsf{depth}(\mathcal{T};M)$ of $\mathcal{T}$ is defined by

$$\mathsf{depth}(\mathcal{T};M) = \min\{\#(\mathcal{T}\cap\mathcal{D}) \mid \mathcal{D} \text{ is an overtwisted disk in } (M,\xi)\}.$$

The depth measures non-looseness of transverse links. In particular, $\mathsf{depth}(\mathcal{T};M)=0$ if and only if $\mathcal{T}$ is loose.

In [19] we have defined the *overtwisted complexity* $n(S,\phi)$ of an open book (S,ϕ).

Definition 5.2. [19]. Let

$$n(S,\phi) = \min\left\{ e_-(D) \;\middle|\; \begin{array}{l} D \text{ is a transverse overtwisted disk} \\ \text{in } (M_{(S,\phi)}, \xi_{(S,\phi)}) \end{array} \right\}$$

where $e_-(D)$ is the number of negative elliptic points in the open book foliation of D. We call $n(S,\phi)$ the *overtwisted complexity* of the open book (S,ϕ).

Recall that the binding $B_{(S,\phi)}$ is a transverse link. In [21, Theorem 3] it is shown that

$$n(S,\phi) = \mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}). \tag{5.1}$$

Theorem 4.1 (Step 4 argument) and Equation (5.1) give the following upper bounds of the depths of closed braids and the binding.

Corollary 5.3. *Let (S,ϕ) be an open book supporting an overtwisted contact structure, L be a closed braid in $M_{(S,\phi)}$, and $B_{(S,\phi)}$ be the binding.*

(a) $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}) \leq \min\{N \mid \phi \textit{ is } N\textit{-twist left-veering}\}$.
(b) $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}\setminus L) \leq \min\{N \mid \phi_L \textit{ is } (N,0)\textit{-twist left-veering}\}$
(c) $\mathsf{depth}(L; M_{(S,\phi)}) \leq \min\{k \mid \phi_L \textit{ is } (N,k)\textit{-twist left-veering for some } N\}$.
(d) $\mathsf{depth}(L\cup B_{(S,\phi)}; M_{(S,\phi)}) \leq \min\{N+k \mid \phi_L \textit{ is } (N,k)\textit{-twist left-veering}\}$.

Question 5.4. Can the above inequalities (a), ..., (d) in Corollary 5.3 be equalities?

Remark 5.5. In some cases $\mathsf{depth}(L\cup B_{(S,\phi)}; M_{(S,\phi)}) = \mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}\setminus L)$ but in general

$$\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}\setminus L) - \mathsf{depth}(L\cup B_{(S,\phi)}; M_{(S,\phi)}) \geq 0 \tag{5.2}$$

and the difference can be arbitrary large.

The next lemma concerns the equality of (5.2).

Lemma 5.6. *Suppose that $N \leq \mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)})$. We have $\mathsf{depth}(L\cup B_{(S,\phi)}; M_{(S,\phi)}) = N$ if and only if $\mathsf{depth}(B_{(S,\phi)}; M_{(S,\phi)}\setminus L) = N$.*

Proof. Let $B := B_{(S,\phi)}$ and $M := M_{(S,\phi)}$. Suppose that $N \leq \mathsf{depth}(B;M)$.

($\Leftarrow$): If $\mathsf{depth}(B;M\setminus L) = N$ then by

$$N \leq \mathsf{depth}(B;M) \leq \mathsf{depth}(L\cup B;M) \leq \mathsf{depth}(B;M\setminus L) = N$$

we get $\mathsf{depth}(L\cup B;M) = N$.

($\Rightarrow$): Assume that $\mathsf{depth}(L\cup B;M) = N$. There exists an overtwisted disk $\mathcal{D}$ that intersects $L\cup B$ at N points. By

$$N \leq \mathsf{depth}(B;M) \leq \mathsf{depth}(L\cup B;M) = N$$

all the N intersection points belong to B, and $\mathcal{D} \subset M\setminus L$. Thus $\mathsf{depth}(B;M\setminus L) \leq |B\cap\mathcal{D}| = N$. By (5.2) we get

$$N = \mathsf{depth}(L\cup B;M) \leq \mathsf{depth}(B;M\setminus L) \leq N.$$

□

Question 5.4 for (a),(b) and (d) are answered affirmatively in the case of depth 1:

Proposition 5.7. [21, Corollary 1] [22, Theorem 5.5] *Suppose that $\xi_{(S,\phi)}$ is overtwisted. Let L be a closed braid in $M_{(S,\phi)}$.*

- $\mathsf{depth}(B_{(S,\phi)};M_{(S,\phi)}) = 1$ *if and only if ϕ is non-right-veering (i.e. 1-twist left-veering).*
- $\mathsf{depth}(L\cup B_{(S,\phi)};M_{(S,\phi)}) = \mathsf{depth}(B_{(S,\phi)};M_{(S,\phi)}\setminus L) = 1$ *if and only if ϕ_L is non-quasi-right-veering (i.e., $(1,0)$-twist left-veering).*

Here is another fact supporting the affirmative answer to Question 5.4. We prove equalities for (a) and (b) in the case of depth 2.

Theorem 5.8. *Suppose that $\xi_{(S,\phi)}$ is overtwisted. Let L be a closed braid in $M_{(S,\phi)}$.*

- $\mathsf{depth}(B_{(S,\phi)};M_{(S,\phi)}) = 2$ *if and only if ϕ is right-veering and 2-twist left-veering.*
- $\mathsf{depth}(B_{(S,\phi)};M_{(S,\phi)}\setminus L) = 2$ *if and only if ϕ_L is quasi-right-veering and $(2,0)$-twist left-veering.*

Proof. We prove the second statement as the first statement is a special case of the second. We denote $M := M_{(S,\phi)}$ and $B = B_{(S,\phi)}$.

($\Rightarrow$) Assume that $\mathsf{depth}(B;M\setminus L) = 2$. By Proposition 5.7 we know that ϕ_L is quasi-right-veering. Thus, it is enough to show that ϕ_L is $(2,0)$-twist left-veering.

Let D be an overtwisted disk in $M\setminus L$ that intersects B in two points. We recycle the argument in the proof of [21, Theorem 3]: Taking a positive transverse push-off of the Legendrian boundary ∂D, we can find a disk D' whose boundary $\partial D'$ is a transverse unknot with $sl(\partial D') = 1$.

We may assume $|D' \cap B| = |D \cap B| = 2$. Using Pavelescu's work [27, 28] we can modify D' to obtain a disk D'' such that boundaries $\partial D'$ and $\partial D''$ are transversely isotopic and $\partial D''$ is in braid position with respect to the open book (S, ϕ). We can further purtarb D'' to obtain a transverse overtwisted disk $\mathcal{D}$. During the procedure, neither intersections with L or *negative* intersections with B are introduced. That is,

$$\mathcal{D} \subset M \setminus L \quad \text{and} \quad e_-(\mathcal{D}) = |B \cap^- \mathcal{D}| \ \leq\ |B \cap D| = 2.$$

If $e_-(\mathcal{D}) = 1$ then we can construct another overtwisted disk D''' in $M \setminus L$ such that $|B \cap D'''| = e_-(\mathcal{D}) = 1$, that contradicts the assumption $\mathsf{depth}(B; M \setminus L) = 2$. Therefore,

$$e_-(\mathcal{D}) = |B \cap^- \mathcal{D}| = 2.$$

Denote the two negative elliptic points of $\mathcal{F}_{ob}(\mathcal{D})$ by v^1 and v^2, and the b-arc (if exists) in $\mathcal{D} \cap S_t$ that starts at v^i by $b^i(t)$ for $t \in [0, 1]$ and $i = 1, 2$. Since $\mathcal{D}$ is a transverse overtwisted disk (i.e., the graph G_{--} is a tree) D has exactly one negative hyperbolic point, h^- that is connected to v^1 and v^2 by a singular leaf.

Let $\{S_{t_n} \mid n = 1, \cdots, k\}$ where $0 < t_1 < t_2 < \cdots < t_k < 1$ be the set of singular pages. Assume that the unique negative hyperbolic point h^- lies in S_{t_1} and each of the other singular pages contains exactly one positive hyperbolic point.

For $t \neq t_1, \ldots, t_k$ the b-arcs define a 2-arc system

$$\Gamma_t := (b^1(t), b^2(t)) \in \mathcal{A}_{\mathcal{B}}(S, P)$$

with the base $\mathcal{B} = \{v^1, v^2\}$ (Convention 4.2 is used here). We have

$$\Gamma_0 = \phi_L(\Gamma_1).$$

Each positive hyperbolic point is formed by a pair of a-arc and b-arc. Passing a positive hyperbolic point veers the b-arc to the right near its base point and the resulting arc system is disjoint from the original one. This gives for $i = 2, \ldots, k$

$$\Gamma_{t_i - \varepsilon} \prec_{\mathsf{disj}} \Gamma_{t_i + \varepsilon}.$$

Since $\Gamma_{t_i + \varepsilon} = \Gamma_{t_{i+1} - \varepsilon}$ we have

$$\Gamma_{t_1 + \varepsilon} \prec_{\mathsf{disj}} \Gamma_{t_2 + \varepsilon} \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \Gamma_{t_k + \varepsilon} = \Gamma_1. \tag{5.3}$$

A boundary based 4-gon region $R(\Gamma_1, \phi_L(\Gamma_1))$ is formed in a neighborhood of a describing arc of the negative hyperbolic point h_- (the shaded region in Figure 17). This can be easily seen if we shift the end points of the describing arc close to v^1 and v^2 (see the left sketch in Figure 17). It follows that $\phi_L^{tw}(\Gamma_1) = \Gamma_{t_1 + \varepsilon}$. With (5.3) we obtain

$$\phi_L^{tw}(\Gamma_1) \ll_{\mathsf{right}} \Gamma_1.$$

The 4-gon $R(\Gamma_1, \phi_L(\Gamma_1))$ is not punctured since L is disjoint from the transverse overtwisted disk $\mathcal{D}$; i.e., ϕ_L is $(2,0)$-twist left-veering.

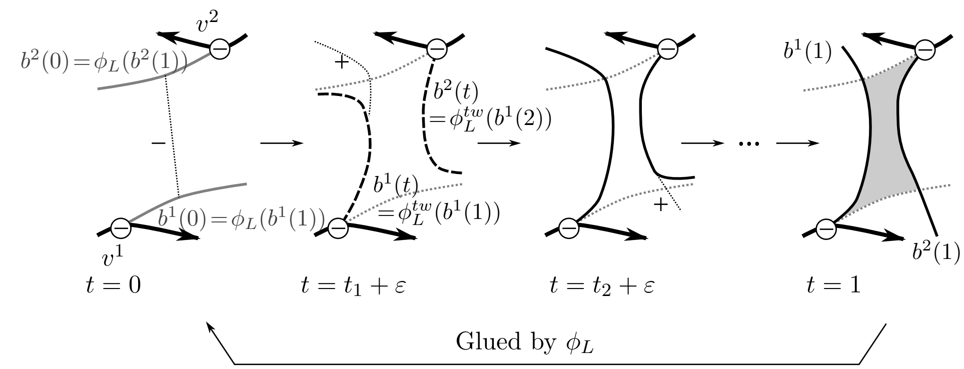

Figure 17. Movie presentation of a transverse overtwisted disk with two negative elliptic points.

($\Leftarrow$) This implication follows by Corollary 5.3 and Proposition 5.7. □

Corollary 5.9. *If ϕ is right-veering and $\xi_{(S,\phi)}$ is overtwisted then* $\mathsf{depth}(L \cup B_{(S,\phi)}; M_{(S,\phi)}) = 2$ *if and only if ϕ_L is quasi-right-veering and $(2,0)$-twist left-veering.*

Proof. By Proposition 5.7, the map ϕ is right-veering if and only if $2 \leq \mathsf{depth}(B_{(S,\phi)}, M_{(S,\phi)})$. By Lemma 5.6, $\mathsf{depth}(L \cup B_{(S,\phi)}; M_{(S,\phi)}) = 2$ if and only if $\mathsf{depth}(B_{(S,\phi)}, M_{(S,\phi)} \setminus L) = 2$, which is equivalent to ϕ_L is quasi-right-veering and $(2,0)$-twist left-veering by Theorem 5.8. □

6. Connection to Wand's inconsistency

In this section we discuss the relation between $(N,0)$-twist left-veering and Wand's inconsistency. We begin with definitions of overtwisted region and inconsistent mapping class that Wand introduced in [30], where the puncture set P is empty.

Definition 6.1 (Wand)**.** Let $\phi \in \mathcal{MCG}(S)$ and $\Gamma \in \mathcal{A}_{\mathcal{B}}(S)$ an N-arc system with $N \geq 1$. A pair (S, ϕ, Γ) consisting of open book (S,ϕ) and arc system Γ is called an *augmented open book*. An *overtwisted region* in the augmented open book (S, ϕ, Γ) is a $2N$-gon disk A embedded in S (when $N = 1$ relaxing the condition that Γ and $\phi(\Gamma)$ intersect efficiently, A can be a bigon) and $\partial A \subset (\Gamma \cup \phi(\Gamma))$ satisfying the following:

(1) The orientation of ∂A
- agrees with that of $\phi(\Gamma)$ and disagrees with that of Γ, or

- disagrees with that of $\phi(\Gamma)$ and agrees with that of Γ.

(2) Each point of $\Gamma \cap \phi(\Gamma) \cap \mathsf{int}(S)$ is a corner of A. Corners of A alternate between points in $\partial\Gamma = \mathcal{B} \cup \Gamma(1)$ and points in $\Gamma \cap \phi(\Gamma) \cap \mathsf{int}(S)$.

(3) A is the unique such disk.

As the name suggests, Wand showed that:

Proposition 6.2 (Wand)**.** *If (S, ϕ, Γ) has an overtwisted region then (S, ϕ) supports an overtwisted contact structure.*

Remark 6.3. Wand's proof of the above statement does not immediately generalize to surfaces with $P \neq \emptyset$. His construction of an overtwisted disk D uses $\Gamma \times [0, 1]$ as part of D, but in general $\Gamma \times [0, 1]$ may intersect the transverse link L that corresponds to P.

Moreover, even the definition of an overtwisted region does not immediately extend to the case where $P \neq \emptyset$: Assume that $\phi_L \in \mathcal{MCG}(S, P)$ with $f(\phi_L) = id_S \in \mathcal{MCG}(S)$ under the forgetful map $f : \mathcal{MCG}(S, P) \to \mathcal{MCG}(S)$. Note the contact structure supported by the open book (S, id_S) is tight. If $\phi_L(\gamma) \prec_{\mathsf{right}} \gamma$ and the arcs $\phi_L(\gamma)$, γ bound a punctured bigon at the base point, then the conditions (1)-(3) are satisfied, which means the contact structure is overtwisted.

In Figure 18 we compare overtwisted regions and boundary based regions. They share many common properties:

	overtwisted region	*boundary based region*
$2N$-gon	yes	yes
corners alternate between ∂S and $\mathsf{int}(S)$	yes	yes
edges alternate Γ and $\phi(\Gamma)$	yes	yes
unique	yes	yes
embedded (Definition 3.5)	yes	not required
$\mathsf{int}(\Gamma) \cap \mathsf{int}(\phi(\Gamma))$ $\stackrel{?}{=}$ non-base corners of R	yes $=$	allowed to be $\subseteq$
work with punctures (Remark 6.3)	no	yes

FIGURE 18. Comparison of overtwisted region and boundary based region.

On the other hand, Example 6.4 highlights their difference:

Example 6.4. Let S be a genus 0 surface with four boundary components, a, b, c, d. Let $e \subset S$ be a simple closed curve that separates

a and b from c and d. Let $\phi_{i,j,k} = T_a^i T_b^j T_c T_d T_e^{-k-1}$ where $i, j, k \geq 1$. As shown in Theorem 4.1 of [18], the open book $(S, \phi_{i,j,k})$ supports an overtwisted contact structure. We see a Type $(2,0)$ boundary based region $R(\Gamma, \phi(\Gamma))$ for a 2-arc system $\Gamma = (\gamma^1, \gamma^2)$ in Figure 19, and in fact $\phi_{i,j,k}$ is $(2,0)$-twist left veering. However, notice that $R(\Gamma, \phi(\Gamma))$ is not an overtwisted region as it violates Condition (2) in Definition 6.1.

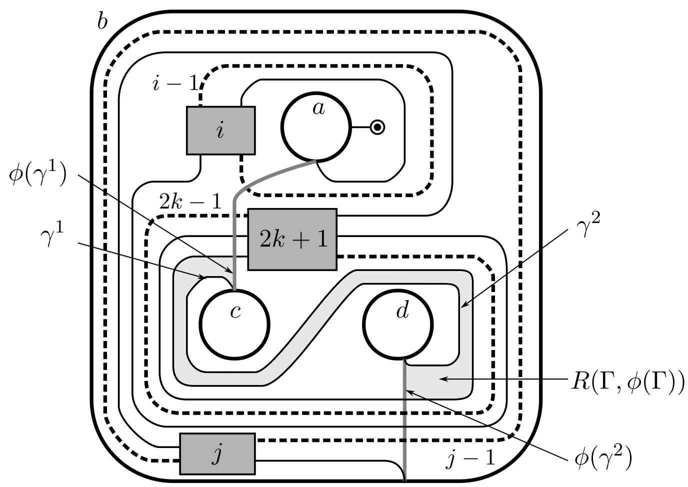

FIGURE 19. (Example 6.4). The boundary based region $R(\Gamma, \phi(\Gamma))$ is not an overtwisted region. The gray box labeled i contains i parallel strands. The dashed arc labeled $i-1$ represents $i-1$ parallel copies of the arc.

Theorem 6.5 and Proposition 6.6 show that an overtwisted region can be understood as a special type of boundary based region:

Theorem 6.5. *Let Γ be an N-arc system with $N \geq 2$ such that a boundary based region $R(\Gamma, \phi(\Gamma))$ is formed. Then $R(\Gamma, \phi(\Gamma))$ is an overtwisted region if and only if*

- $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$,
- $\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma) = \emptyset$, *and*
- $R(\Gamma, \phi(\Gamma))$ *is embedded; that is,* $\mathsf{int}(R(\Gamma, \phi(\Gamma))) \cap \Gamma = \emptyset$.

Proof. ($\Leftarrow$) Assume that $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$, $\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma) = \emptyset$, and $\mathsf{int}(R(\Gamma, \phi(\Gamma))) \cap \Gamma = \emptyset$. We will check that Conditions (1), (2), and (3) of Definition 6.1 are satisfied.

Condition (1) is clearly satisfied.

Since $\mathsf{int}(R(\Gamma, \phi(\Gamma)) \cap \Gamma = \emptyset$ we have $\partial R(\Gamma, \phi(\Gamma)) \cap \mathsf{int}(\phi(\Gamma)) \cap \mathsf{int}(\Gamma) = \{q^1, \dots, q^N\}$. Condition (2) follows by:

$$\begin{aligned} &\Gamma \cap \phi(\Gamma) \cap \mathsf{int}(S) \\ &\quad = [\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma)] \cup [\partial R(\Gamma, \phi(\Gamma)) \cap \mathsf{int}(\phi(\Gamma)) \cap \mathsf{int}(\Gamma)] \\ &\quad = \{q^1, \dots, q^N\}. \end{aligned}$$

Before we proceed to (3) we recall that, unlike boundary based regions, in Definition 6.1 of an overtwisted region A, the corners of A are not required to contain the base points $\mathcal{B}$ of Γ. Therefore, the uniqueness property (⧫) of a boundary-based region does not imply the uniqueness condition (3) of an overtwisted region.

Assume to the contrary that Condition (3) does not hold; namely, $R(\Gamma, \phi(\Gamma))$ is not the unique disk in the augmented open book (S, ϕ, Γ). Then by Condition (2), a boundary based region $R(\underline{\Gamma}, \phi(\underline{\Gamma})) \neq \emptyset$ must be formed, where $\underline{\Gamma}$ denotes the arc system Γ with the reversed orientation (so the set of base points of $\underline{\Gamma}$ is $\Gamma(1)$, not $\mathcal{B}$), see Figure 20 (a). We note that $R(\phi^{tw}(\Gamma), \Gamma)$ also exists as it contains $R(\underline{\Gamma}, \phi(\underline{\Gamma}))$, see

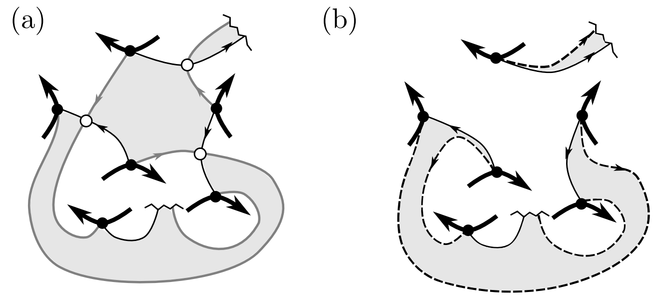

FIGURE 20. (a) Two regions satisfying Conditions (1) and (2) but (3). The black (resp. gray) oriented arcs represent Γ (resp. $\phi(\Gamma)$). (b) The boundary based region $R(\phi^{tw}(\Gamma), \Gamma)$. The oriented dashed arcs represent $\phi^{tw}(\Gamma)$.

Figure 20 (b). By Proposition 3.6 this implies $\phi^{tw}(\Gamma) \not\ll_{\mathsf{right}} \Gamma$ which contradicts the assumption.

($\Rightarrow$) Assume that $R(\Gamma, \phi(\Gamma))$ is an overtwisted region satisfying (1)–(3) of Definition 6.1. We observe that Condition(2) implies $\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma) = \emptyset$ and $\mathsf{int}(R(\Gamma, \phi(\Gamma))) \cap \Gamma = \emptyset$. Thus, we are left to show $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$.

We want to point out that $\mathsf{int}(\phi^{tw}(\Gamma)) \cap \mathsf{int}(\Gamma) = \emptyset$ does *not* immediately imply $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$. This is because the arc systems $\phi^{tw}(\Gamma)$ and Γ share not only the same set of base points $\mathcal{B}$ but also the terminal points $\Gamma(1)$, which means the condition $\phi^{tw}(\Gamma) \cap \Gamma = \mathcal{B}$ in Definition 3.3 of $\prec_{\mathsf{disj}}$ is *not* satisfied. In other words, we have $\phi^{tw}(\Gamma) \not\prec_{\mathsf{disj}} \Gamma$.

Denote

$$(\gamma^1, \ldots, \gamma^N) := \Gamma,$$
$$(\gamma_0^1, \ldots, \gamma_0^N) := \Gamma_0 := \phi^{tw}(\Gamma),$$

and $\mathcal{B} = (v^1, \ldots, v^N)$ the base point set for Γ and Γ_0. We will find a sequence such that

$$\phi^{tw}(\Gamma) = \Gamma_0 \prec_{\mathsf{disj}} \Gamma_1 \prec_{\mathsf{disj}} \Gamma_2 \prec_{\mathsf{disj}} \Gamma \tag{6.1}$$

which concludes $\phi^{tw}(\Gamma) \ll_{\mathsf{right}} \Gamma$.

For $j = 1, \ldots, N$ let R^j be the connected component of $S \setminus (\Gamma_0 \cup \Gamma)$ which lies between γ_0^j and γ^j. We note that R^j cannot be a boundary based region due to the uniqueness Condition (3). It is possible that $R^j = R^k$ for some $j \neq k$, and $R^1 = \cdots = R^N$ is also possible.

(Case 1): If $\partial R^j \cap \partial S$ has positive measure (or positive length), we can find simple arcs $\gamma_1^j \prec_{\mathsf{disj}} \gamma_2^j$ in R^j connecting v^j and the boundary ∂S so that $\Gamma_1 = (\gamma_1^1, \ldots, \gamma_1^N)$ and $\Gamma_2 = (\gamma_2^1, \ldots, \gamma_2^N)$ form N-arc systems $\in \mathcal{A}_{\mathcal{B}}(S)$ and satisfy (6.1).

(Case 2): If $\partial R^j \cap \partial S$ is a set of discrete points then $R^1 = \cdots = R^N$ (call it $R^\star$) whose boundary is

$$\partial R^\star = \underline{\gamma_0^1} \cup \gamma^1 \cup \underline{\gamma_0^2} \cup \gamma^2 \cup \cdots \cup \underline{\gamma_0^N} \cup \gamma^N$$

where $\underline{\gamma}$ is the arc γ with the reversed orientation. We claim that $R^\star$ has non-trivial genus: If $R^\star$ was a disk with $2N$ sides, that would contradict Condition (3) that $R(\Gamma, \phi(\Gamma))$ is the unique overtwisted region in the augmented open book (S, ϕ, Γ).

We can always find arc systems $\Gamma_1 = (\gamma_1^1, \ldots, \gamma_1^N)$ and $\Gamma_2 = (\gamma_2^1, \ldots, \gamma_2^N) \in \mathcal{A}_{\mathcal{B}}(S)$ satisfying (6.1) as follows.

Take an arc in $R^\star$ starting at v^1 and terminating at v^N so that it does not separate $R^\star$. Then move the terminal point slightly to the right of v^N and call it γ_1^1. To define γ_1^j for $j = 2, \ldots, N$, consider the arc $\gamma_0^j * \underline{\gamma^{j-1}} \subset \partial R^\star$ starting at v^j and terminating at v^{j-1}. Move the terminal point to the right side of v^{i-1} then call it γ_1^j. It satisfies $\gamma^j \prec_{\mathsf{disj}} \gamma_1^j$ and $\Gamma_0 \prec_{\mathsf{disj}} \Gamma_1$. See the gray arcs in Figure 21.

Next take γ_2^1 be an arc starting at v^1 and terminating on the left-hand side of v^1 so that it is disjoint from $\Gamma_1 = (\gamma_1^1, \ldots, \gamma_1^N)$ and that $\gamma_1^1 \prec_{\mathsf{disj}} \gamma_1^2 \prec_{\mathsf{disj}} \gamma^1$. To define γ_2^j for $j = 2, \ldots, N$, consider the arc

$$\gamma_0^j * \underline{\gamma^{j-1}} * \gamma_0^{j-1} * \cdots * \gamma_0^2 * \underline{\gamma^1} * \gamma_2^1$$

that starts at v^j and terminates at v^1. Move the terminal point to the left side of v^1 so that it becomes a simple arc which we call γ_2^j. It satisfies $\gamma_1^j \prec_{\mathsf{disj}} \gamma_2^j \prec_{\mathsf{disj}} \gamma^j$. We make $\gamma_2^1, \ldots, \gamma_2^N$ pairwise disjoint (see the dotted arcs in Figure 21) to achieve (6.1). $\square$

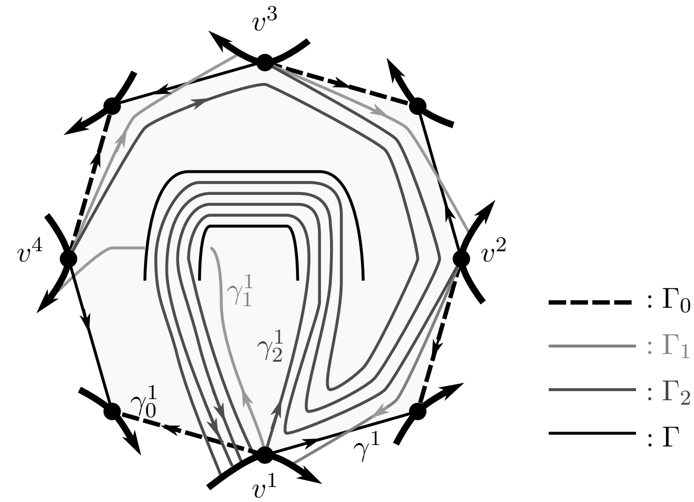

FIGURE 21. **(Case 2):** Construction of N-arc systems $\phi^{tw}(\Gamma) = \Gamma_0 \prec_{\mathsf{disj}} \Gamma_1 \prec_{\mathsf{disj}} \Gamma_2 \prec_{\mathsf{disj}} \Gamma$ where $N = 4$.

When $N = 1$ a parallel statement to Theorem 6.5 holds:

Proposition 6.6. *Let γ be an arc (i.e., a 1-arc system) in S. The augmented open book (S, ϕ, γ) has an overtwisted region if and only if $\phi^{tw}(\gamma) \ll_{\mathsf{right}} \gamma$ and that $\mathsf{int}(\phi^{tw}(\gamma)) \cap \mathsf{int}(\gamma) = \emptyset$.*

Proof. Note that $\phi(\gamma) = \phi^{tw}(\gamma)$ since γ is a 1-arc system.

($\Rightarrow$) Suppose that (S, ϕ, γ) has a bigon overtwisted region. Reversing the orientation of γ if necessary, we may further assume that the bigon is at the base point $\gamma(0)$. By the uniqueness property (3) of Definition 6.1 we know that $\phi \neq id$. After removing the bigon formed by γ and $\phi(\gamma)$ we see that $\phi(\gamma) \prec_{\mathsf{right}} \gamma$ and $\phi(\gamma) \cap \gamma = \partial\gamma$; that means γ and $\phi(\gamma)$ are almost disjoint. Since $\phi \neq id$ the component of $S \setminus (\phi(\gamma) \cup \gamma)$ that lies between $\phi(\gamma)$ and γ at the base point $\gamma(0)$ is not a disk. Applying Honda, Kazez and Matić's algorithm [16] we can find a sequence of arcs with

$$\phi(\gamma) \prec_{\mathsf{disj}} \gamma_1 \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \gamma_l \prec_{\mathsf{disj}} \gamma,$$

thus $\phi^{tw}(\gamma) = \phi(\gamma) \ll_{\mathsf{right}} \gamma$.

($\Leftarrow$) Assume that $\phi^{tw}(\gamma) \ll_{\mathsf{right}} \gamma$ and that $\mathsf{int}(\phi^{tw}(\gamma)) \cap \mathsf{int}(\gamma) = \emptyset$. Since $\phi^{tw}(\gamma) = \phi(\gamma)$ we have $\mathsf{int}(\phi(\gamma)) \cap \mathsf{int}(\gamma) = \emptyset$ so by isotopy, $\phi(\gamma)$

and γ form a bigon at the base point which is an overtwisted region for (S, ϕ, γ). $\square$

Now we recall the definition of inconsistency.

Definition 6.7 (Wand)**.** A class $\phi \in \mathcal{MCG}(S)$ is *inconsistent* if there is some arc system Γ in S and a stabilization (S', ϕ') of (S, ϕ) such that $(S', \phi', \iota(\Gamma))$ has an overtwisted region, where $\iota : S \to S'$ is the inclusion map (for simplicity $\iota(\Gamma)$ is denoted by Γ in the following). Otherwise, ϕ is *consistent*.

Inconsistency is a central concept in Wand's work due to the following result:

Theorem 6.8. [30, Theorem 1.1] *(S, ϕ) supports an overtwisted contact structure if and only if ϕ is inconsistent.*

We obtain the following corollary.

Corollary 6.9. *An open book (S, ϕ) supports an overtwisted contact structure if and only if a stabilization of (S, ϕ) is N-twist left-veering for some N.*

Proof. ($\Leftarrow$) is exactly Theorems 4.1. ($\Rightarrow$) follows from Theorem 6.5, Proposition 6.6 and Wand's Theorem 6.8. $\square$

7. Variation of twist-left-veering and virtual looseness

Non-right-veering closed braids are not necessarily loose but they are *virtually loose*; that is, some finite cover of its complement is overtwisted [23, Corollary 5.7]. In this section we generalize the result from arcs to N-arc systems. We do this by introducing $\ll^{\partial+P}_{\mathsf{right}}$, a variation of the ordering $\ll_{\mathsf{right}}$.

We begin with reviewing the standard branched cyclic coverings studied in [23], and then discuss how twist-left-veering can be related to virtual looseness.

Let $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_m$ be an m-component transverse link in a contact 3-manifold (M, ξ). Let $\mu_i \in \pi_1(M \setminus \mathcal{T})$ be represented by a meridian of $\mathcal{T}_i$. In the following, we assume that all μ_i are non-trivial. Note that if M contains no $S^1 \times S^2$ summands (i.e., M is irreducible) every link in M satisfies this property (cf. the proof of Proposition 1.4 in [14]).

Since $\pi_1(M \setminus \mathcal{T})$ is residually finite [15, Theorem 1.1], there exists a finite group G_i and a homomorphism

$$f_i : \pi_1(M \setminus \mathcal{T}) \to G_i$$

such that $f_i(\mu_i) \neq 1$. Let $f : \pi_1(M \setminus \mathcal{T}) \to G_1 \times G_2 \times \cdots \times G_m$ be a homomorphism defined by $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$. Let

$G := \text{image}(f)$. Then G is a non-trivial subgroup of $G_1 \times G_2 \times \cdots \times G_m$, in particular G is a non-trivial finite group, and

$$f : \pi_1(M \setminus \mathcal{T}) \to G$$

is a surjective homomorphism such that $f(\mu_i) \neq 1$ for all i. Let

$$\pi : (\widetilde{M}, \widetilde{\xi}) \to (M, \xi)$$

be the covering branched along $\mathcal{T}$ [9, 26] such that the restriction $\pi : \widetilde{M} \setminus \widetilde{\mathcal{T}} \to M \setminus \mathcal{T}$ is a normal covering that corresponds to $\ker(f)$.

Assume that (M, ξ) is supported by an open book (S, ϕ) and $\mathcal{T}$ is represented by a closed n-braid L in (S, ϕ) with the distinguished monodromy $\phi_L \in \mathcal{MCG}(S, P)$ where P is a set of n interior points of S.

Assume that S has $k+1$ boundary components, $C_0, \ldots, C_k$. Take a base point v on C_0. Let σ_j $(j = 1, \ldots, k)$ be a path in $S \setminus P$ connecting v and a point on C_j. As shown in [6]

$$\pi_1(M \setminus \mathcal{T}) = \left\langle s \in \pi_1(S \setminus P) \,\middle|\, s\phi_{L*}(s^{-1}),\ \sigma_j \phi_{L*}(\underline{\sigma_j}) \quad (j = 1, \ldots, k) \right\rangle.$$

Here $\underline{\sigma_j}$ is the path σ_j with the reversed orientation, hence $\sigma_j \phi_{L*}(\underline{\sigma_j})$ represents a loop in $S \setminus P$. Note that $\pi_1(M \setminus \mathcal{T})$ is a quotient group of $\pi_1(S \setminus P)$. Let

$$q : \pi_1(S \setminus P) \to \pi_1(M \setminus \mathcal{T})$$

denote the quotient map. Define a surjective homomorphism

$$f' := f \circ q : \pi_1(S \setminus P) \to G.$$

Let

$$\pi_S : (\widetilde{S}, \widetilde{P}) \to (S, P) \tag{7.1}$$

be the branched covering that corresponds to $\ker(f')$. Since π_S is a normal covering, the property that $f(\mu_i) \neq 1 \in G$ for all $i = 1, \ldots, m$ implies that the branched covering $\pi_S : (\widetilde{S}, \widetilde{P}) \to (S, P)$ is *fully ramified* with branch set P. That is, in a neighborhood of each $\tilde{p} \in \pi_S^{-1}(p) \subset \widetilde{P}$ the map π_S is a non-trivial branch covering.

The presentation of $\pi_1(M \setminus \mathcal{T})$ implies that $q \circ \phi_{L*} = q$. Therefore, $f' \circ \phi_{L*} = f \circ q \circ \phi_{L*} = f \circ q = f'$. By the *lifting criterion* [13, Proposition 1.33] there is a diffeomorphism

$$\widetilde{\phi_L} : (\widetilde{S}, \widetilde{P}) \to (\widetilde{S}, \widetilde{P}) \tag{7.2}$$

such that $\phi_L \circ \pi_S = \pi_S \circ \widetilde{\phi_L}$.

Lemma 7.1. *If the lift* $\widetilde{\phi_L} : (\widetilde{S}, \widetilde{P}) \to (\widetilde{S}, \widetilde{P})$ *is* $(N, 0)$*-twist-left-veering then* $\mathcal{T}$ *is virtually loose.*

Proof. The preimage $\widetilde{L} := \pi^{-1}(L)$ is a closed braid representating $\widetilde{\mathcal{T}} = \pi^{-1}(\mathcal{T})$ and its distinguished monodromy is $\widetilde{\phi_L}$. If $\widetilde{\phi_L}$ is $(N,0)$-twist-left-veering then by theorem 4.1 (2) $\widetilde{L}$ is loose. Thus $\mathcal{T}$ is virtually loose. □

The above lemma motivates us to ask when the lift $\widetilde{\phi_L}$ becomes twist-left-veering. To this end, we extend the right-veering orderings to a slightly bigger set.

Definition 7.2. We introduce the following:

- A $(\partial + P)$-*arc* γ is an oriented properly embedded arc in $S \setminus P$ with the starting point $\gamma(0) \in \partial S$ and the terminal point $\gamma(1) \in \partial S \cup P$.
- A $(\partial + P)$-*arc system* $\Gamma = (\gamma^1, \ldots, \gamma^N)$ is defined similarly with $\Gamma(1) \subset \partial S \cup P$.
- Let $\mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$ denote the set of $(\partial + P)$-arc systems that start at $\mathcal{B} \subset \partial S$.
- The orderings $\prec_{\textsf{right}}$ and $\ll_{\textsf{right}}$, and the relation $\prec_{\textsf{disj}}$ on $\mathcal{A}_{\mathcal{B}}(S,P)$ can be extended to $\mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$ and denoted by $\prec^{\partial+P}_{\textsf{right}}$, $\ll^{\partial+P}_{\textsf{right}}$, and $\prec^{\partial+P}_{\textsf{disj}}$ respectively.

Remark 7.3. By the definition $\mathcal{A}_{\mathcal{B}}(S,P) \subset \mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$.

For $\Gamma, \Gamma' \in \mathcal{A}_{\mathcal{B}}(S,P) \subset \mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$, we have $\Gamma \prec_{\textsf{right}} \Gamma'$ if and only if $\Gamma \prec^{\partial+P}_{\textsf{right}} \Gamma'$. However, in general $\Gamma \ll^{\partial+P}_{\textsf{right}} \Gamma'$ does not imply $\Gamma \ll_{\textsf{right}} \Gamma'$ since $\Gamma \prec^{\partial+P}_{\textsf{disj}} \Gamma'$ does not imply $\Gamma \prec_{\textsf{disj}} \Gamma'$.

As stated in Proposition 3.6, when the boundary based region $R(\Gamma, \Gamma')$ is nonempty then $\Gamma \not\ll_{\textsf{right}} \Gamma'$. The next lemma states that this is not the case for $\ll^{\partial+P}_{\textsf{right}}$.

Lemma 7.4. *If a boundary based region $R(\Gamma, \Gamma')$ is embedded (i.e., its interior does not intersect Γ) and contains a puncture point, then $\Gamma \ll^{\partial+P}_{\textsf{right}} \Gamma'$ (see Figure 22).*

Let $\pi_S : (\widetilde{S}, \widetilde{P}) \to (S,P)$ be a fully ramified branched covering with branched set P. For example, the above branched covering (7.1) satisfies this property. For each base point $v^j \in \mathcal{B}$ we choose a lift $\widetilde{v}^j \in \pi_S^{-1}(v^j) \subset \partial\widetilde{S}$ and define a set of base points $\widetilde{\mathcal{B}} = \{\widetilde{v}^1, \ldots, \widetilde{v}^N\}$ for $\widetilde{S}$. For an N-arc system $\Gamma = \{\gamma^1, \ldots, \gamma^N\} \in \mathcal{A}_{\mathcal{B}}(S,P)$, we denote by $\widetilde{\Gamma} = \{\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^N\} \subset \mathcal{A}_{\widetilde{\mathcal{B}}}(\widetilde{S}, \widetilde{P})$ the N-arc system such that $\widetilde{\gamma}^j$ is the lift of γ^j with $\widetilde{\gamma}^j(0) = \widetilde{v}^j$.

Proposition 7.5. *For $\Gamma, \Gamma' \in \mathcal{A}_{\mathcal{B}}(S,P)$, if $\Gamma \ll^{\partial+P}_{\textsf{right}} \Gamma'$ then $\widetilde{\Gamma} \ll_{\textsf{right}} \widetilde{\Gamma}'$ in $\mathcal{A}_{\widetilde{\mathcal{B}}}(\widetilde{S}, \widetilde{P})$.*

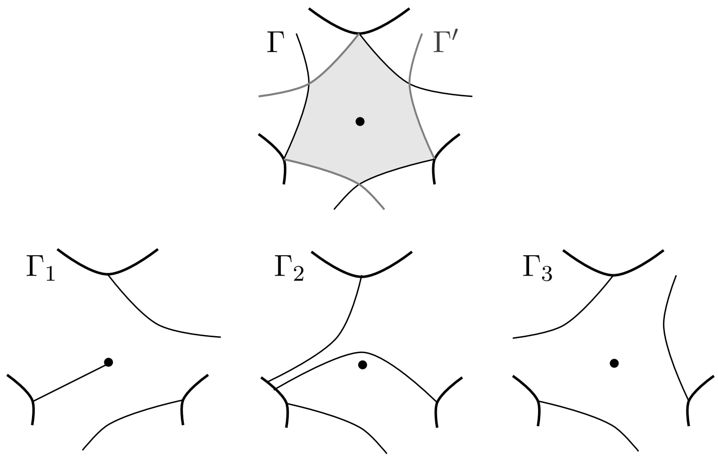

FIGURE 22. Embedded boundary based region with puncture gives $\Gamma \ll^{\partial+P}_{\mathsf{right}} \Gamma'$ by $\Gamma \prec^{\partial+P}_{\mathsf{disj}} \Gamma_1 \prec^{\partial+P}_{\mathsf{disj}} \Gamma_2 \prec^{\partial+P}_{\mathsf{disj}} \Gamma_3 = \Gamma'$

Proof. Assume that $\Gamma \ll^{\partial+P}_{\mathsf{right}} \Gamma'$. Then there is a sequence of $(\partial+P)$-arc systems $\Gamma_1, \ldots, \Gamma_{k-1}$ such that

$$\Gamma =: \Gamma_0 \prec^{\partial+P}_{\mathsf{disj}} \Gamma_1 \prec^{\partial+P}_{\mathsf{disj}} \cdots \prec^{\partial+P}_{\mathsf{disj}} \Gamma_{k-1} \prec^{\partial+P}_{\mathsf{disj}} \Gamma_k := \Gamma'.$$

For each $\Gamma_i = (\gamma_i^1, \ldots, \gamma_i^N) \in \mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$ we construct $\Gamma_i^* = (\gamma_i^{*1}, \ldots, \gamma_i^{*N}) \in \mathcal{A}_{\mathcal{B}}(S,P)$ as follows: For $j = 1, \ldots, N$, if the terminal point of the j-th arc γ_i^j lies on ∂S, then we define $\gamma_i^{*j} := \gamma_i^j$. If the terminal point of γ_i^j is a puncture point $p \in P$ then we define (see Figure 23)

$$\gamma_i^{*j} := \gamma_i^j * c_p * \underline{\gamma_i^j}.$$

where c_p is a small loop around the point p clockwise. We slightly move the terminal point of γ_i^{*j} to the right of v_i along the boundary.

By the construction, γ_i^{*j} is disjoint from $\gamma_{i\pm1}^{*j'}$ if $j' \neq j$. When $j' = j$ we have $\gamma_i^{*j} \cap \gamma_{i\pm1}^{*j} \neq \emptyset$ (see the hollowed point near v_j in Figure 23). Since we assume that the covering π_S is fully ramified, the loop c_p cannot lift to a loop. This means that the intersection point can be removed in the covering space. Thus, we have

$$\widetilde{\Gamma} \prec_{\mathsf{disj}} \widetilde{\Gamma_1^*} \prec_{\mathsf{disj}} \cdots \prec_{\mathsf{disj}} \widetilde{\Gamma_{k-1}^*} \prec_{\mathsf{disj}} \widetilde{\Gamma}',$$

which means $\widetilde{\Gamma} \ll_{\mathsf{right}} \widetilde{\Gamma}'$ in $\mathcal{A}_{\widetilde{\mathcal{B}}}(\widetilde{S}, \widetilde{P})$. □

Definition 7.6. Let L be a closed braid with respect to an open book (S, ϕ). If there is an N-arc system $\Gamma \in \mathcal{A}_{\mathcal{B}}(S,P)$ such that

$$\phi_L^{tw}(\Gamma) \ll^{\partial+P}_{\mathsf{right}} \Gamma \text{ in } \mathcal{A}^{\partial+P}_{\mathcal{B}}(S,P)$$

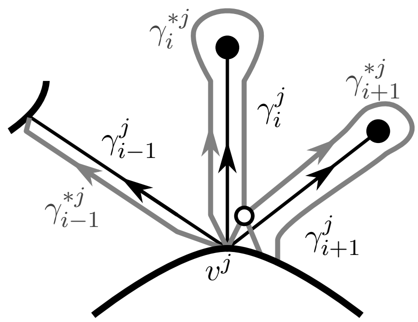

FIGURE 23. Construction of γ_i^{*j}. The intersection point of γ_i^{*j} and γ_{i+1}^{*j} disappears when we take lifts.

and the associated boundary based region $R(\Gamma, \phi_L(\Gamma))$ has type (N, k) then we say that ϕ_L is *weakly* (N, k)-twist left veering.

Remark 7.7. When $N = 1$, [22, Corollary 7.6] implies that $\gamma \ll^{\partial+P}_{\textsf{right}} \gamma'$ if and only if $\gamma \prec_{\textsf{right}} \gamma'$. Thus weakly $(1, 0)$-twist left-veering is nothing but non-right-veering.

Theorem 4.1 states that if ϕ_L is $(N, 0)$-twist left veering then L is loose. In the next theorem, with a weaker condition we show that L is virtually loose.

Theorem 7.8. *Assume that all the meridians of L are homotopically non-trivial. If ϕ_L is weakly $(N, 0)$-twist left-veering then L is virtually loose.*

Proof. Let $N = 1$. By Remark 7.7, ϕ_L is non-right-veering. Then by [23, Corollary 5.7], L is virtually loose.

Let $N > 1$. Let $\widetilde{\phi_L} : (\widetilde{S}, \widetilde{P}) \to (\widetilde{S}, \widetilde{P})$ be a lift of $\phi_L : (S, P) \to (S, P)$ as in (7.2). With a suitable lift of the base $\mathcal{B}$, $\widetilde{\phi_L}(\widetilde{\Gamma})$ and $\widetilde{\Gamma}$ can form a non-punctured boundary based region which is a lift of the non-punctured $R(\Gamma, \phi_L(\Gamma))$ and we have $\widetilde{\phi_L^{tw}(\Gamma)} = \widetilde{\phi_L}^{tw}(\widetilde{\Gamma})$. Since $\phi_L^{tw}(\Gamma) \ll^{\partial+P}_{\textsf{right}} \Gamma$ by Proposition 7.5 we obtain $\widetilde{\phi_L}^{tw}(\widetilde{\Gamma}) = \widetilde{\phi_L^{tw}(\Gamma)} \ll_{\textsf{right}} \widetilde{\Gamma}$. Thus, $\widetilde{\phi_L}$ is $(N, 0)$-twist left-veering. By Lemma 7.1 L is virtually loose. ☐

8. EXAMPLES

Example 8.1. (3-twist left veering) Let S be a sphere with 9 boundary components. Let a_i, b_i, c_i with $i = 1, 2, 3$ be simple closed curves as depicted in Figure 24. Let

$$\phi = T_{c_1} T_{c_2} T_{c_3} T_{b_1} T_{b_2} T_{b_3} T_{a_1}^{-1} T_{a_2}^{-1} T_{a_3}^{-1} \in \mathcal{MCG}(S).$$

Let Γ be an arc system as shown in Sketch (1). Then ϕ is 3-twist left veering with the boundary based region $R(\Gamma, \phi(\Gamma))$ in Sketch (3). However, $R(\Gamma, \phi(\Gamma))$ is not an overtwisted region as it violates the Condition (2) in Definition 6.1.

This construction can be generalized to N-twist left veering for any $N \geq 2$. The construction works under additional genera and boundary components.

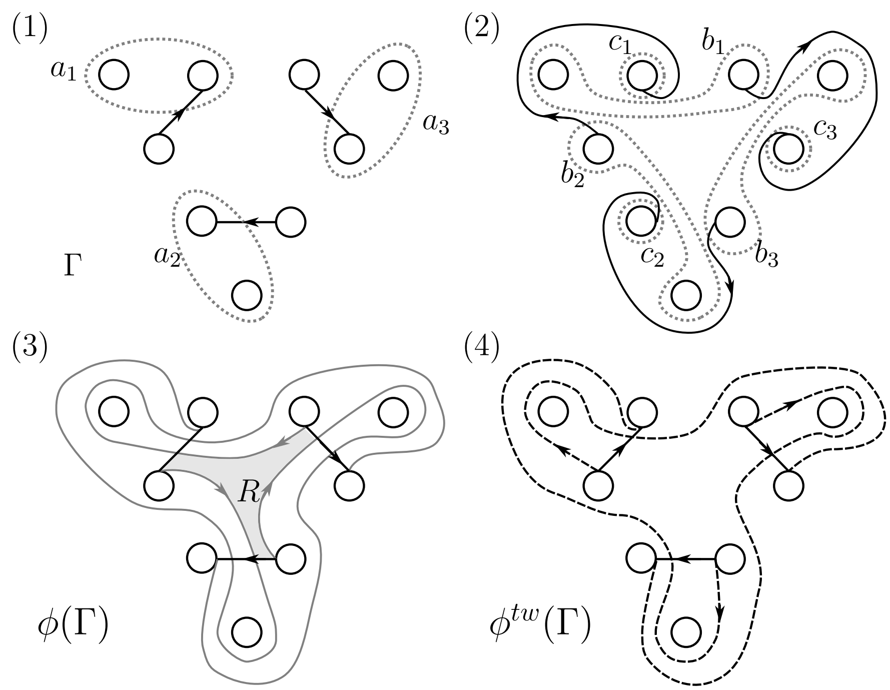

FIGURE 24. (Example 8.1). (1) The circles a_1, a_2 and a_3 and the 3-arc system Γ. (2) The arc system $T_{a_1}^{-1}T_{a_2}^{-1}T_{a_3}^{-1}(\Gamma)$ and the circles $b_1, b_2, b_3, c_1, c_2, c_3$. (3) The arc system $\phi(\Gamma)$ where $\phi = T_{c_1}T_{c_2}T_{c_3}T_{b_1}T_{b_2}T_{b_3}T_{a_1}^{-1}T_{a_2}^{-1}T_{a_3}^{-1}$ and the boundary based region $R(\Gamma, \phi(\Gamma))$. (4) The arc system $\phi^{tw}(\Gamma)$.

Example 8.2. (Weakly $(3,0)$-twist left veering but *not* twist left veering) Let S be a sphere with six boundary components and six punctures. Let a_i, b_i, c_i with $i = 1, 2, 3$ be simple closed curves as depicted in Figure 25. Let

$$\phi_L = T_{c_1}T_{c_2}T_{c_3}T_{b_1}T_{b_2}T_{b_3}T_{a_1}^{-1}T_{a_2}^{-1}T_{a_3}^{-1} \in \mathcal{MCG}(S, P).$$

Sketch (4) shows that ϕ_L is weakly $(3,0)$-twist left veering with the boundary based region $R(\Gamma, \phi_L(\Gamma))$ in Sketch (3); thus, by Theorem 7.8 the link L is virtually overtwisted.

Under the forgetful map $f : \mathcal{M}CG(S,P) \to \mathcal{M}CG(S)$ the distinguished monodromy ϕ_L becomes $\phi = f(\phi_L)$ that is isotopic to a product of positive Dehn twists (note the negative Dehn twist about a_i and the positive Dehn twist about c_i cancel each other after forgetting the punctures). That is, (S, ϕ) supports a tight contact structure and L is non-loose. By Theorem 4.1-(2) we can conclude that ϕ_L is *not* twist left veering.

This construction can be generalized to weakly $(N,0)$-twist left veering for any $N \geq 2$. The construction may work under additional genera, boundary components and punctures.

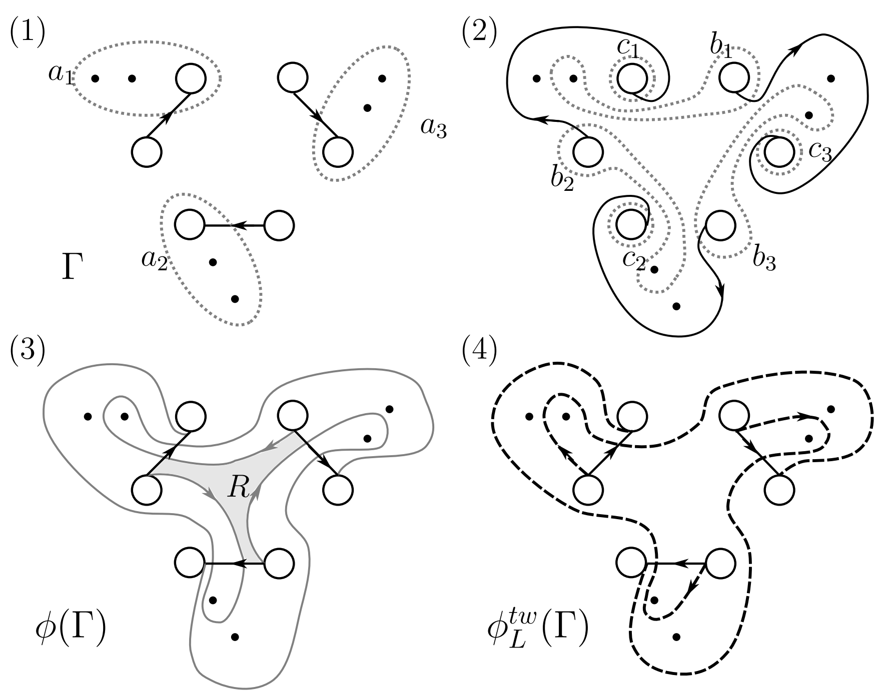

FIGURE 25. (Example 8.2). (1) The circles a_1, a_2 and a_3, and the 3-arc system Γ. (2) The arc system $T_{a_1}^{-1}T_{a_2}^{-1}T_{a_3}^{-1}(\Gamma)$ and the circles $b_1, b_2, b_3, c_1, c_2, c_3$. (3) The arc system $\phi_L(\Gamma)$ where $\phi_L = T_{c_1}T_{c_2}T_{c_3}T_{b_1}T_{b_2}T_{b_3}T_{a_1}^{-1}T_{a_2}^{-1}T_{a_3}^{-1}$ and the boundary based region $R(\Gamma, \phi(\Gamma))$. (4) The arc system $\phi_L^{tw}(\Gamma)$.

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