Polynomial Invariants of Tribrackets in Knot Theory

Sam Nelson* Fletcher Nickerson†

Abstract

We introduce a six-variable polynomial invariant of Niebrzydowski tribrackets analogous to quandle, rack and biquandle polynomials. Using the subtribrackets of a tribracket, we additionally define subtribracket polynomials and establish a sufficient condition for isomorphic subtribrackets to have the same polynomial regardless of their embedding in the ambient tribracket. As an application, we enhance the tribracket counting invariant of knots and links using subtribracket polynomials and provide examples to demonstrate that this enhancement is proper.

1 Introduction

In [6], a two-variable polynomial invariant of finite quandles known as the *quandle polynomial* was introduced. It was generalized to the case of biquandles in [7] and racks in [3]. As an application, subquandle polynomials and their analogues were applied to define enhancements of the various counting invariants of knots and links associated to quandles, biquandles and racks.

Sets with ternary operations known as *knot-theoretic ternary quasigroups* were introduced in [10] and used to define knot and link invariants. Subsequent work in [11] presented a homology theory for these structures, which was used to enhance their associated counting invariants.

In several recent papers, the first listed author and collaborators have studied and generalized the knot-theoretic ternary quasigroup structure, also called *Niebrzydowski tribrackets*, and defined new enhanced invariants of knots and related structures. In [5], tribracket counting invariants were enhanced with module structures analogous to rack and quandle modules; in [9], *virtual tribrackets* extended tribracket coloring invariants to virtual knots and links, while in [4], *Niebrzydowski algebras* extended tribracket colorings to trivalent spatial graphs and handlebody-links. More recently, in [8], a general theory of multi-tribrackets extended tribracket colorings to various types of generalized knot theories, and in [1] tribracket-brackets analogous to biquandle brackets were defined.

In this paper we introduce a notion of tribracket polynomial analogous to the quandle polynomial from [6]. This polynomial can be conceptualized as quantifying the way in which trivial actions of pairs of elements on other elements are distributed throughout the algebraic structure, in contrast with the trivial action being concentrated in the identity element of a group. As an application, we define subtribracket polynomial enhancements of the tribracket counting invariant. The paper is organized as follows. In Section 2 we review the basics of tribracket theory. In Section 3 we define tribracket polynomials and compute some examples. In Section 4 we apply the subtribracket polynomial construction to enhance the tribracket counting invariant for oriented classical knots and links. We conclude in Section 5 with some questions and directions for future work.

*Email: Sam.Nelson@cmc.edu. Partially supported by Simons Foundation collaboration grant 702597.
†Email: nnickerson@hmc.edu. Supported by the HMC Math Department Giovanni Borrelli Fund.
[0]2020 MSC: 57K12

2 Niebrzydowski Tribrackets

Definition 1. Let X be a set. A (horizontal) *Niebrzydowski tribracket structure* on X, also called a *knot-theoretic ternary quasigroup structure* on X, is a map $[\ ,\ ,\ ] : X \times X \times X \to X$ satisfying the conditions

- For every $a, b, c \in X$ there are unique $x, y, z \in X$ such that

$$[a, b, x] = c, \quad [a, y, b] = c \quad \text{and} \quad [z, a, b] = c$$

  and

- For every $a, b, c, d \in X$ we have

$$[c, [a, b, c], [a, c, d]] = [b, [a, b, c], [a, b, d]] = [d, [a, b, d], [a, c, d]].$$

Example 1. Let G be any group. Then the operation

$$[a, b, c] = ba^{-1}c$$

defines a tribracket structure known as the *Dehn tribracket* of G.

Example 2. Let X be any module over the two-variable Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$. Then the operation

$$[a, b, c] = tb + sc - tsa$$

defines a tribracket operation known as an *Alexander tribracket*.

Example 3. Let $X = \{1, 2, \dots, n\}$. We can specify a tribracket structure on X with an *operation 3-tensor*, i.e. an ordered n-tuple of $n \times n$ matrices where the element in matrix i row j column k is $[i, j, k]$. For instance, the reader can verify that the operation 3-tensor

$$\left[\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\right]$$

satisfies the tribracket axioms and hence defines a Niebrzydowski tribracket structure on the set $X = \{1, 2, 3\}$.

Definition 2. A *subtribracket* is a subset $S \subset X$ of a tribracket and is itself a tribracket under the restriction of $[\ ,\ ,\ ]$ to S.

We note that any subset S of a finite tribracket X which is closed under the tribracket operation is a subtribracket:

- First, the ternary quasigroup conditions are equivalent to the condition that rows, columns and sets of entries in the same position in each matrix do not contain any repeated elements, and removal of some elements cannot introduce such repeats;

- Secondly, for all $a, b, c, d \in S$ the condition

$$[c, [a, b, c], [a, c, d]] = [b, [a, b, c], [a, b, d]] = [d, [a, b, d], [a, c, d]]$$

  is already satisfied in X and removal of elements does not affect this property in S.

Example 4. In the tribracket structure on $X = \{1, 2, 3\}$ specified by the 3-tensor

$$\left[\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}\right],$$

the singleton sets $\{1\}, \{2\}$ and $\{3\}$ each form proper subtribrackets, while in the tribracket structure specified by

$$\left[\left[\begin{array}{ccc} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{array}\right], \left[\begin{array}{ccc} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{array}\right], \left[\begin{array}{ccc} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{array}\right]\right]$$

the only proper subtribracket is $\{1\}$.

Definition 3. A map $f : X \to Y$ between tribrackets is a *tribracket homomorphism* if for all $a, b, c \in X$ we have

$$[f(a), f(b), f(c)] = f([a, b, c]).$$

Definition 4. Let $f : X \to Y$ be a tribracket homomorphism. The subtribracket of Y generated by the elements $f(x)$ for all $x \in X$ is called the *image* of f, denoted $\mathrm{Im}(f)$.

The tribracket axioms are motivated by the *Reidemeister moves* of knot theory. The idea is to assign an element of X to each region in the planar complement of an oriented knot or link diagram D with the colors related as depicted in Figure 1:

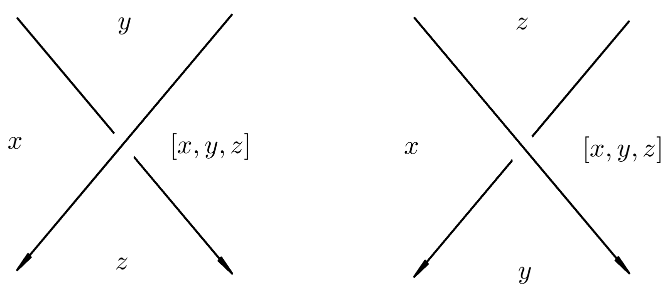


Fig. 1

Such an assignment is called an *X-coloring* of D.

Then the Niebrzydowski tribracket axioms are the conditions required to ensure that for each X-coloring on one side of a move, there is a unique corresponding X-coloring on the other side of the move which is unchanged outside the move's neighborhood. In particular, we have:

Theorem 1. *Let X be a finite Niebrzydowski tribracket and let L be an oriented link with diagram D. The cardinality of the set $\mathcal{C}(D, X)$ of X-colorings of D is an integer-valued invariant of L, known as the tribracket counting invariant, denoted $\Phi_X^{\mathbb{Z}}(L)$.*

Example 5. Let us consider the Hopf link with tribracket $X = \{1, 2, 3\}$ specified by

$$\left[\left[\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}\right], \left[\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}\right], \left[\begin{array}{ccc} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{array}\right]\right].$$

X-colorings are solutions (x, y, z, w) such that $[x, y, z] = w = [x, z, y]$ as shown in Figure 2:

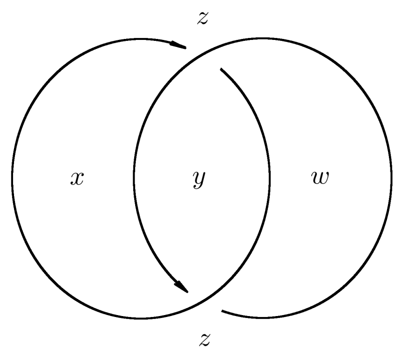


Fig. 2

The reader can verify that the Hopf link has nine such colorings. However, the unlink of two components has three non-interacting regions, which yield twenty-seven colorings. Hence, the Hopf link is distinguished from the unlink by the tribracket counting invariant for the specified tribracket X.

3 Tribracket Polynomials

We can now state our main new definition.

Definition 5. Let X be a finite Niebrzydowski tribracket. For each $i \in K$, let

- $l_1(i)$ be the number of elements $j \in X$ such that $[i, j, j] = i$,
- $c_1(i)$ be the number of elements $j \in X$ such that $[j, i, j] = i$,
- $r_1(i)$ be the number of elements $j \in X$ such that $[j, j, i] = i$,
- $l_2(i)$ be the number of elements $j \in X$ such that $[i, j, j] = j$,
- $c_2(i)$ be the number of elements $j \in X$ such that $[j, i, j] = j$, and
- $r_2(i)$ be the number of elements $j \in X$ such that $[j, j, i] = j$.

Then the *tribracket polynomial* of X is the polynomial

$$\phi(X) = \sum_{i \in X} x^{l_1(i)} y^{c_1(i)} z^{r_1(i)} u^{l_2(i)} v^{c_2(i)} w^{r_2(i)}.$$

Example 6. Let us compute the tribracket polynomial of the Niebrzydowski tribracket structure on the set $X = \{1, 2, 3\}$ specified by the operation 3-tensor

$$\left[\left[\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}\right], \left[\begin{array}{ccc} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{array}\right], \left[\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}\right]\right].$$

Starting with the element $1 \in X$, we note that

- the equation $[1, x, x] = 1$ has no solutions,
- the equation $[x, 1, x] = 1$ has one solution, namely $x = 2$,

- the equation $[x, x, 1] = 1$ has no solutions,
- the equation $[1, x, x] = x$ has one solution, namely $x = 2$,
- the equation $[x, 1, x] = x$ has one solution, namely $x = 3$, and
- the equation $[x, x, 1] = x$ has one solution, namely $x = 2$.

Thus, the element $x = 1$ contributes $x^0y^1z^0u^1v^1w^1 = yuvw$ to the polynomial ϕ. Repeating for $2, 3 \in X$, we obtain tribracket polynomial $\phi(X) = 3yuvw$.

Proposition 2. *If $f : X \to Y$ is an isomorphism of tribrackets, then $\phi(X) = \phi(Y)$.*

Proof. It suffices to show that each of the exponent variables $l_1(i)$, $c_1(i)$, $r_1(i)$, $l_2(i)$, $c_2(i)$, $r_2(i)$ used to construct the tribracket polynomial are unchanged by isomorphism f, meaning that the tribracket polynomials for the isomorphic tribrackets will be the same. Let's demonstrate that the first of these variables ($l_1(i)$) has the same value for tribrackets X and Y. Without loss of generality, this approach can be applied to the remaining five variables. Recall that in X, $l_1(i)$ counts the number of elements $j \in X$ such that $[i, j, j] = i$. Applying the definition of a tribracket homomorphism (Definition 3), our tribracket isomorphism maps this expression to $f(j) \in y$ such that $[f(i), f(j), f(j)] = f(i)$. Every element in X is replaced by its image in Y. We know there are as many elements $j \in X$ as elements $f(j) \in Y$ because f is an isomorphism. As a result, $l_1(i)$ will have the same values for the tribracket polynomials for X and Y. The same argument can be extended for the remaining tribracket polynomial exponent variables. □

Example 7. We computed the tribracket polynomials for all tribracket structures with up to $n = 5$ elements; the results are collected in the below table.

n	$\phi(X)$
1	$uvwxyz$
2	$2uvwx^2y^2z^2,\ 2uvw$
3	$3uvwxy^3z^3,\ 3uvwx^3yz^3,\ 3uvwx^3y^3z,\ u^3v^3w^3xyz + 2xyz,\ 3uvwz,\ 3uvwy,\ 3uvwx$
4	$4uvw,\ 4uvwx^4y^4z^2,\ 4uvwx^2y^2z^2,\ 4uvwxy,\ 4uvwz^2,\ 4uvwx^2y^4z^4,\ 4uvwx^4y^2z^4,$ $4uvwxyz^4,\ 4uvwx^2y^4z^2,\ 4uvwx^4y^2z^2,\ 4uvwy^2z^2,\ 4uvwx^2z^2,\ 4uvwy^2,\ 4uvwx^2,$ $4uvwx^2y^2z^4,\ 4uvwx^2y^2,\ 4uvwx^4y^4z^4$
5	$5uvwx^5y^5z,\ uvw^5xyz + 4uvxyz,\ u^5v^5wxyz + 4wxyz,\ 5uvwz,\ 5uvwy,\ 5uvwxy,$ $5uvwxyz^5,\ 5uvwxyz,\ 5uvwx,\ 5uvwx^5yz^5,\ 5uvwxy^5z^5$

Definition 6. Let $S \subset X$ be a subtribracket. Then the *subtribracket polynomial* $\phi(S \subset X)$ of S with respect to X is the sum

$$\phi(S \subset X) = \sum_{i \in S} x^{l_1(i)} y^{c_1(i)} z^{r_1(i)} u^{l_2(i)} v^{c_2(i)} w^{r_2(i)}$$

of the contributions of the elements of S to the tribracket polynomial.

Example 8. The subtribracket $S = \{1\}$ of the tribracket structure on $X = \{1, 2, 3\}$ specified by the 3-tensor

$$\left[\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\right]$$

has subtribracket polynomial $\phi(S \subset X) = xyzu^3v^3w^3$.

The subtribracket polynomial of a subtribracket is not in general determined by the isomorphism class of the subtribracket, but carries information about the way the subtribracket is embedded in the ambient tribracket. In this way, the subtribracket polynomial is already essentially knot-theoretic. However, there is a class of tribrackets in which isomorphic subtribrackets all have the same subtribracket polynomial:

Definition 7. A tribracket is *homogeneous* if every element contributes the same monomial to the tribracket polynomial.

Proposition 3. *Any two subtribrackets of a homogeneous tribracket with the same cardinality have the same subtribracket polynomial.*

Proof. Since every element has the same contribution, the polynomial is determined by the number of contributions, i.e. by the the number of elements in the subtribracket. □

In particular we note that in Example 8, all tribrackets are homogeneous for $n = 1, 2, 4$, whereas for $n = 3, 5$ there are non-homogeneous tribrackets.

4 Subtribracket Polynomial Enhancement

As an application, we will now define a new link invariant analogous to the subquandle polynomial invariant defined in [6].

Definition 8. Let X be a finite tribracket and L an oriented classical link. The *subtribracket polynomial enhancement* of the tribracket counting invariant $\Phi_X(L)$ of L with respect to X is the multiset

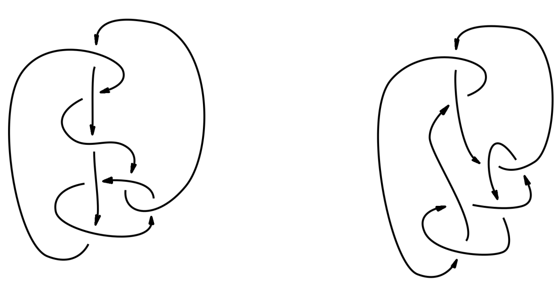
$$\Phi_{\mathrm{Im}\subset X}(L) = \{\phi(\mathrm{Im}(f) \subset X) \mid f \in \mathcal{C}(L, X)\}$$

of subtribracket polynomials of the image subtribrackets over the set of tribracket colorings of L by X.

By construction, we have our main result:

Proposition 4. *Let X be a finite tribracket. If two links L and L' are ambient isotopic, then* $\Phi_{\mathrm{Im}\subset X}(L) = \Phi_{\mathrm{Im}\subset X}(L')$.

Example 9. Our `python` computations reveal that though the links L7a3 and L7a7 shown in Figure 3



$L7a3$ $L7a7$

Fig. 3

both have $\Phi_x^{\mathbb{Z}}(L) = 64$ colorings by the tribracket structure on $X = \{1, 2, 3, 4\}$ given by the 3-tensor

$$\left[\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array}\right], \left[\begin{array}{cccc} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{array}\right], \left[\begin{array}{cccc} 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{array}\right], \left[\begin{array}{cccc} 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right]\right],$$

they are distinguished by their subtribracket polynomials with

$$\Phi_{\mathrm{Im}\subset X}(L7a3) = \{4\times uvwx^2y^2z^4,\ 48\times 4uvwx^2y^2z^4,\ 12\times 2uvwx^2y^2z^4\}$$

and

$$\Phi_{\mathrm{Im}\subset X}(L7a7) = \{4\times uvwx^2y^2z^4,\ 28\times 2uvwx^2y^2z^4,\ 32\times 4uvwx^2y^2z^4\}.$$

In particular, this example shows that $\Phi_{\mathrm{Im}\subset X}$ is not determined by $\Phi_X^{\mathbb{Z}}$ and hence is a proper enhancement.

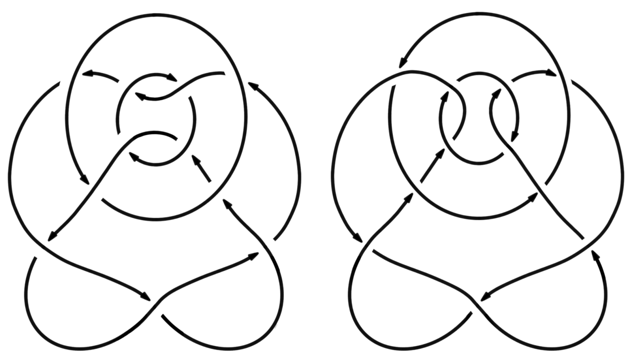
Example 10. The subtribracket polynomial can distinguish single-component knots with the same counting invariant as well. Our `python` computations show that the granny knot $3_1\#3_1$ and the 6_1 knot both have 2916 colorings by the Alexander tribracket $\mathbb{Z}_{18}$ with $t=5$ and $s=13$. However, they are distinguished by their subtribracket polynomials, as

$$\begin{aligned}\Phi_{\mathrm{Im}\subset X}(3_1\#3_1) &= \{324\times 3uvwy^2z^6,\ 972\times 6uvwx^{18}y^2z^6+12uvwy^2z^6,\ 972\times 3uvwx^{18}y^2z^6+6uvwy^2z^6,\\ &\quad 324\times 6uvwy^2z^6,\ 6\times uvwx^{18}y^2z^6,\ 156\times 3uvwx^{18}y^2z^6,\ 156\times 6uvwx^{18}y^2z^6,\\ &\quad 6\times 2uvwx^{18}y^2z^6\}\end{aligned}$$

while

$$\begin{aligned}\Phi_{\mathrm{Im}\subset X}(6_1) &= \{108\times 3uvwy^2z^6,\ 1296\times 3uvwx^{18}y^2z^6+6uvwy^2z^6,\\ &\quad 1296\times 6uvwx^{18}y^2z^6+12uvwy^2z^6,\ 108\times 6uvwy^2z^6,\ 6\times uvwx^{18}y^2z^6,\\ &\quad 48\times 3uvwx^{18}y^2z^6,\ 48\times 6uvwx^{18}y^2z^6,\ 6\times 2uvwx^{18}y^2z^6\}.\end{aligned}$$

Example 11. Links that share an Alexander polynomial can also be distinguished by their subtribracket polynomials. Consider the pair of links L11n404 and L11n406 shown in Figure 4:



$L11n404$ $L11n406$

Fig. 4

Although they both have a trivial Alexander polynomial, our subtribracket polynomial enhancement is able to differentiate them. Under the tribracket structure on $X = 1,2,3,4,5$ specified by the tensor

$$\left[\begin{bmatrix}1&3&2&5&4\\5&4&3&2&1\\4&2&5&1&3\\2&1&4&3&5\\3&5&1&4&2\end{bmatrix},\begin{bmatrix}5&4&3&2&1\\1&3&2&5&4\\1&3&2&5&4\\3&5&1&4&2\\4&2&5&1&3\end{bmatrix},\begin{bmatrix}4&2&5&1&3\\1&3&2&5&4\\3&5&1&4&2\\5&4&3&2&1\\2&1&4&3&5\end{bmatrix},\begin{bmatrix}2&1&4&3&5\\3&5&1&4&2\\5&4&3&2&1\\4&2&5&1&3\\1&3&2&5&4\end{bmatrix},\begin{bmatrix}3&5&1&4&2\\4&2&5&1&3\\2&1&4&3&5\\1&3&2&5&4\\5&4&3&2&1\end{bmatrix}\right]$$

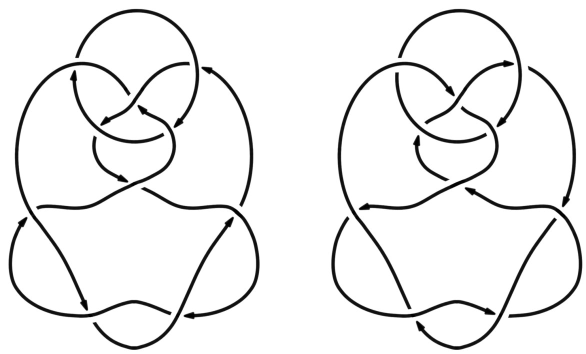
our python computations reveal that they have different subtribracket polynomials:

$$\Phi_{\mathrm{Im}\subset X}(\mathrm{L11n404}) = \{1 \times uvw^5xyz, 624 \times uvw^5xyz + 4 \times uvxyz\},$$

whereas

$$\Phi_{\mathrm{Im}\subset X}(\mathrm{L11n406}) = \{1 \times uvw^5xyz, 124 \times uvw^5xyz + 4 \times uvxyz\}.$$

This same tribracket structure also distinguishes two orientations of the same link, denoted L10n9{0} and L10n9{1}, as shown in Figure 5:



$L10n9\{0\}$ $L10n9\{1\}$

Fig. 5

Both orientations result in the same multivariable Alexander polynomial, $1 - t_1 - t_2 + t_1t_2$. Python code computed the link L10n9{0}'s subtribracket polynomials as

$$\Phi_{\mathrm{Im}\subset X}(\mathrm{L10n9}\{0\}) = \{1 \times uvw^5xyz, 124 \times uvw^5xyz + 4 \times uvxyz\},$$

while the associated polynomials for L10n9{1} are

$$\Phi_{\mathrm{Im}\subset X}(\mathrm{L10n9}\{1\}) = \{1 \times uvw^5xyz, 24 \times uvw^5xyz + 4 \times uvxyz\}.$$

In particular, these examples show that the subtribracket polynomial invariant (and indeed the tribracket counting invariant) are not determined by the multivariable Alexander polynomial and are sensitive to orientation.

Example 12. Using our python code, we computed the subtribracket polynomial invariant for a choice of orientation for each of the prime links with up to seven crossings in the Thistlethwaite link table at the Knot Atlas [2]. These invariant values were calculated with respect to the Alexander tribracket structure on

$X = \mathbb{Z}_8$ with parameters $t = 3$ and $s = 5$. The results are collected in the below table.

L	$\Phi_{\mathrm{Im}\subset X}(L)$
$L2a1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 32 \times 4uvwx^8y^2z^4,\ 64 \times 8uvwx^8y^2z^4\}$
$L4a1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L5a1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 384 \times 8uvwx^8y^2z^4\}$
$L6a1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L6a2$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 32 \times 4uvwx^8y^2z^4,\ 64 \times 8uvwx^8y^2z^4\}$
$L6a3$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 32 \times 4uvwx^8y^2z^4,\ 64 \times 8uvwx^8y^2z^4\}$
$L6a4$	$\{8 \times uvwx^8y^2z^4,\ 56 \times 2uvwx^8y^2z^4,\ 448 \times 4uvwx^8y^2z^4,\ 512 \times 8uvwx^8y^2z^4\}$
$L6a5$	$\{8 \times uvwx^8y^2z^4,\ 56 \times 2uvwx^8y^2z^4,\ 64 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L6n1$	$\{8 \times uvwx^8y^2z^4,\ 56 \times 2uvwx^8y^2z^4,\ 64 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L7a1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 384 \times 8uvwx^8y^2z^4\}$
$L7a2$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L7a3$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 384 \times 8uvwx^8y^2z^4\}$
$L7a4$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 384 \times 8uvwx^8y^2z^4\}$
$L7a5$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 32 \times 4uvwx^8y^2z^4,\ 64 \times 8uvwx^8y^2z^4\}$
$L7a6$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 32 \times 4uvwx^8y^2z^4,\ 64 \times 8uvwx^8y^2z^4\}$
$L7a7$	$\{8 \times uvwx^8y^2z^4,\ 56 \times 2uvwx^8y^2z^4,\ 64 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L7n1$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 128 \times 8uvwx^8y^2z^4\}$
$L7n2$	$\{8 \times uvwx^8y^2z^4,\ 24 \times 2uvwx^8y^2z^4,\ 96 \times 4uvwx^8y^2z^4,\ 384 \times 8uvwx^8y^2z^4\}$

5 Questions

We conclude with some open questions for future research.

- To what extent is the algebraic structure of a tribracket determined by its tribracket polynomial?
- What conditions on a polynomial make it a tribracket polynomial? That is, what conditions on a six-variable polynomial p are individually necessary and jointly sufficient for the existence of a tribracket X such that $p = \phi(x)$?
- Does the geometry of the algebraic variety in $\mathbb{C}^6$ determined by a tribracket polynomial or subtribracket polynomial carry any special significance?
- Compared with the case of quandles with their 2-dimensional operation tables, the 3-dimensional operation tensors of tribrackets present a greater variety of options for defining tribracket polynomials. We have made one such choice here, but others are possible and might be of interest to explore.

References

[1] L. Aggarwal, S. Nelson, and P. Rivera. Quantum enhancements via tribracket brackets. *arXiv:1907.03011*, 2019.

[2] D. Bar-Natan. The knot atlas http://katlas.org/wiki/Main_Page.

[3] T. Carrell and S. Nelson. On rack polynomials. *J. Algebra Appl.*, 10(6):1221–1232, 2011.

[4] P. Graves, S. Nelson, and S. Tamagawa. Niebrzydowski algebras and trivalent spatial graphs. *Internat. J. Math.*, 29(14):1850102, 16, 2018.

[5] D. Needell, S. Nelson, and Y. Shi. Tribracket Modules. *Internat. J. Math.*, 31(4):2050028, 13, 2020.

[6] S. Nelson. A polynomial invariant of finite quandles. *J. Algebra Appl.*, 7(2):263–273, 2008.

[7] S. Nelson. Generalized quandle polynomials. *Canad. Math. Bull.*, 54(1):147–158, 2011.

[8] S. Nelson and E. Pauletich. Multi-tribrackets. *J. Knot Theory Ramifications*, 28(12):1950075, 16, 2019.

[9] S. Nelson and S. Pico. Virtual tribrackets. *J. Knot Theory Ramifications*, 28(4):1950026, 12, 2019.

[10] M. Niebrzydowski. On some ternary operations in knot theory. *Fund. Math.*, 225(1):259–276, 2014.

[11] M. Niebrzydowski, A. Pilitowska, and A. Zamojska-Dzienio. Knot-theoretic ternary groups. *Fund. Math.*, 247(3):299–320, 2019.

Department of Mathematical Sciences
Claremont McKenna College
850 Columbia Ave.
Claremont, CA 91711

Department of Mathematics
Harvey Mudd College
301 Platt Boulevard
Claremont, CA 91711