

SOME NEW FANO VARIETIES WITH A MULTIPLICATIVE CHOW–KÜNNETH DECOMPOSITION

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ABSTRACT. Let Y be a smooth dimensionally transverse intersection of the Grassmannian $\mathrm{Gr}(2, n)$ with 3 Plücker hyperplanes. We show that Y admits a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of Y injects into cohomology.

1. INTRODUCTION

Given a smooth projective variety Y over \mathbb{C} , let $A^i(Y) := CH^i(Y)_{\mathbb{Q}}$ denote the Chow groups of Y , i.e. the groups of codimension i algebraic cycles on Y with \mathbb{Q} -coefficients, modulo rational equivalence. Let us write $A_{hom}^*(Y)$ and $A_{AJ}^*(Y)$ for the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles. Intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the *Chow ring* of Y [16]. In the case of K3 surfaces, this ring structure has a peculiar property:

Theorem 1.1 (Beauville–Voisin [3]). *Let S be a K3 surface. The \mathbb{Q} -subalgebra*

$$R^*(S) := \langle A^1(S), c_j(S) \rangle \subset A^*(S)$$

injects into cohomology under the cycle class map.

Inspired by the remarkable behaviour of K3 surfaces and of abelian varieties, Beauville [2] has famously conjectured that for certain special varieties, the Chow ring should admit a *multiplicative splitting*. To make concrete sense of Beauville’s elusive “splitting property conjecture”, Shen–Vial [42] have introduced the concept of *multiplicative Chow–Künneth decomposition*; let us abbreviate this to “MCK decomposition”.

What can one say about the class of special varieties admitting an MCK decomposition? This class is not yet well-understood. Varieties with $A_{hom}^*(Y) = 0$ (i.e. varieties with *trivial Chow groups*) admit an MCK decomposition, for trivial reasons. The question becomes interesting for varieties with $A_{AJ}^*(Y) = 0$ (conjecturally, these are exactly the varieties with Hodge level at most 1, i.e. the Hodge numbers $h^{p,q}$ are zero for $|p - q| > 1$). It is known that hyperelliptic curves have an MCK decomposition [42, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [13, Example 2.3] (for more details, cf. subsection 2.1 below). Also, there exist Fano threefolds that do not admit an MCK decomposition. On the positive side,

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here are some higher-dimensional varieties with Hodge level 1 that are known to have an MCK decomposition:

- cubic threefolds and cubic fivefolds [7], [13];
- Fano threefolds of genus 8 [28];
- complete intersections of 2 quadrics [27];
- Gushel–Mukai fivefolds [26].

The goal of the present note is to add some new varieties with Hodge level 1 to this list:

Theorem (=Theorem 3.7). *Let Y be a smooth dimensionally transverse intersection*

$$Y := \mathrm{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^{\binom{n}{2}-1},$$

where $\mathrm{Gr}(2, n)$ denotes the Grassmannian of 2-dimensional linear subspaces of a fixed n -dimensional vector space, and the H_j are Plücker hyperplanes. Then Y has an MCK decomposition.

In case n is odd, a variety Y as in Theorem 3.7 has trivial Chow groups and so the statement is vacuously true. In case n is even, there is a curve C naturally associated to Y , and one has a relation of Chow motives

$$(1) \quad h(Y) \cong h(C)((1 - \dim Y)/2) \oplus \bigoplus \mathbb{1}(\ast) \quad \text{in } \mathcal{M}_{\mathrm{rat}}$$

(cf. Theorem 3.2). The relation between Y and C has previously been studied on the level of Hodge theory in [8], and on the level of derived categories in [20], [21]. As a result of independent interest, we prove here (Theorem 3.2) that the relation (1) also holds on the level of Chow motives.

The existence of an MCK decomposition has profound intersection-theoretic consequences. This is exemplified by the following corollary, which is about a certain *tautological subring* of the Chow ring of powers of Y :

Corollary (=Corollary 4.1). *Let Y be as in Theorem 3.7, and $m \in \mathbb{N}$. Let*

$$R^*(Y^m) := \left\langle (p_i)^* \mathrm{Im}(A^*(\mathrm{Gr}(2, n)) \rightarrow A^*(Y)), (p_{ij})^*(\Delta_Y) \right\rangle \subset A^*(Y^m)$$

be the \mathbb{Q} -subalgebra generated by (pullbacks of) cycles coming from the Grassmannian and the diagonal $\Delta_Y \in A^*(Y \times Y)$. (Here p_i and p_{ij} denote the various projections from Y^m to Y resp. to $Y \times Y$). The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

Corollary 4.1 is somewhat surprising, because the corresponding statement for the associated curve C is *false*: in general it is *not* true that the \mathbb{Q} -subalgebra

$$\left\langle (p_i)^*(h), (p_{ij})^*(\Delta_C) \right\rangle \subset A^*(C^m)$$

injects into cohomology (cf. Proposition 4.3 for the precise statement). This means that the injection

$$A^*(C^m) \hookrightarrow A^*(Y^m)$$

induced by (1) does *not* send tautological cycles to tautological cycles !

Let us end this introduction with an open question. In view of Theorem 3.7, one might ask whether more generally smooth complete intersections of Grassmannians $\text{Gr}(k, n)$ with an arbitrary number of Plücker hyperplanes have an MCK decomposition. This concerns in particular the Debarre–Voisin 20folds

$$\text{Gr}(3, 10) \cap H ,$$

which are Fano varieties of K3 type [6], and also the Fano eightfolds

$$\text{Gr}(2, 8) \cap H_1 \cap \cdots \cap H_4 ,$$

which are again of K3 type [41], [10]. Such varieties (being of Hodge level > 1) are out of scope of the argument of the present note.

Conventions. *In this note, the word variety will refer to a reduced irreducible scheme of finite type over \mathbb{C} . A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we denote by $A_j(Y)$ the Chow group of j -dimensional cycles on Y with \mathbb{Q} -coefficients; for Y smooth of dimension n the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A_{\text{hom}}^j(Y)$ and $A_{\text{AJ}}^j(Y)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.*

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [39], [34]) will be denoted \mathcal{M}_{rat} .

2. PRELIMINARIES

2.1. MCK decomposition.

Definition 2.1 (Murre [33]). Let X be a smooth projective variety of dimension n . We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in } A^n(X \times X) ,$$

such that the π_X^i are mutually orthogonal idempotents and $(\pi_X^i)_* H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 2.2. The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [33], [17].

Definition 2.3 (Shen–Vial [42]). Let X be a smooth projective variety of dimension n . Let $\Delta_X^{\text{sm}} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{\text{sm}} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X .$$

An *MCK decomposition* is a CK decomposition $\{\pi_X^i\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi_X^k \circ \Delta_X^{\text{sm}} \circ (\pi_X^i \times \pi_X^j) = 0 \quad \text{in } A^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k .$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

Remark 2.4. The small diagonal (seen as a correspondence from $X \times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{\text{sm}}: h(X) \otimes h(X) \rightarrow h(X) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Let us assume X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \rightarrow h(X) \otimes h(X) \xrightarrow{\Delta_X^{\text{sm}}} h(X) \quad \text{in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

If X has an MCK decomposition, then setting

$$A_{(j)}^i(X) := (\pi_X^{2i-j})_* A^i(X),$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A_{(j)}^i(X) \otimes A_{(j')}^{i'}(X)$ to $A_{(j+j')}^{i+i'}(X)$.

It is expected that for any X with an MCK decomposition, one has

$$A_{(j)}^i(X) \stackrel{??}{=} 0 \quad \text{for } j < 0, \quad A_{(0)}^i(X) \cap A_{\text{hom}}^i(X) \stackrel{??}{=} 0;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [33].

The property of having an MCK decomposition is restrictive, and is closely related to Beauville's "splitting property conjecture" [2]. To give an idea: hyperelliptic curves have an MCK decomposition [42, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [13, Example 2.3]. As for surfaces: a smooth quartic in \mathbb{P}^3 has an MCK decomposition, but a very general surface of degree ≥ 7 in \mathbb{P}^3 should not have an MCK decomposition [13, Proposition 3.4]. There are examples of Fano threefolds that do not admit an MCK decomposition [13, Example 1.11].

For more detailed discussion, and examples of varieties with an MCK decomposition, we refer to [42, Section 8], as well as [48], [43], [14], [22], [32], [23], [24], [27], [30], [13].

2.2. The Franchetta property.

Definition 2.5. Let $\mathcal{Y} \rightarrow B$ be a smooth projective morphism, where \mathcal{Y}, B are smooth quasi-projective varieties. We say that $\mathcal{Y} \rightarrow B$ has the *Franchetta property in codimension j* if the following holds: for every $\Gamma \in A^j(\mathcal{Y})$ such that the restriction $\Gamma|_{Y_b}$ is homologically trivial for the very general $b \in B$, the restriction $\Gamma|_b$ is zero in $A^j(Y_b)$ for all $b \in B$.

We say that $\mathcal{Y} \rightarrow B$ has the *Franchetta property* if $\mathcal{Y} \rightarrow B$ has the Franchetta property in codimension j for all j .

This property is studied in [37], [4], [11], [12].

Definition 2.6. Given a family $\mathcal{Y} \rightarrow B$ as above, with $Y := Y_b$ a fiber, we write

$$GDA_B^j(Y) := \text{Im} \left(A^j(\mathcal{Y}) \rightarrow A^j(Y) \right)$$

for the subgroup of *generically defined cycles*. In a context where it is clear to which family we are referring, the index B will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that $GDA_B^*(Y)$ injects into cohomology, under the cycle class map, for every fiber Y .

There is some flexibility with respect to the base B :

Lemma 2.7. *Let $\mathcal{Y} \rightarrow B$ be a smooth projective family, and $B_0 \subset B$ the intersection of a countable number of dense open subsets. Then $\mathcal{Y} \rightarrow B$ has the Franchetta property if and only if $\mathcal{Y} \rightarrow B_0$ has the Franchetta property.*

Proof. This follows from a well-known spread lemma [50, Lemma 3.2]. \square

2.3. A Franchetta-type result.

Proposition 2.8. *Let M be a smooth projective variety with trivial Chow groups. Let $L_1, \dots, L_r \rightarrow M$ be very ample line bundles, and let $\mathcal{Y} \rightarrow B$ be the universal family of smooth dimensionally transverse complete intersections of type*

$$Y = M \cap H_1 \cap \dots \cap H_r, \quad H_j \in |L_j|.$$

Assume the fibers $Y = Y_b$ have $H_{\text{tr}}^{\dim Y}(Y, \mathbb{Q}) \neq 0$. There is an inclusion

$$\ker\left(GDA_B^{\dim Y}(Y \times Y) \rightarrow H^{2 \dim Y}(Y \times Y, \mathbb{Q})\right) \subset \left\langle (p_1)^*GDA_B^*(Y), (p_2)^*GDA_B^*(Y) \right\rangle.$$

Proof. This is essentially Voisin's "spread" result [49, Proposition 1.6] (cf. also [31, Proposition 5.1] for a reformulation of Voisin's result). We give a proof which is somewhat different from [49]. Let $\bar{B} := \mathbb{P}H^0(M, L_1 \oplus \dots \oplus L_r)$ (so $B \subset \bar{B}$ is a Zariski open), and let us consider the projection

$$\pi: \mathcal{Y} \times_{\bar{B}} \mathcal{Y} \rightarrow M \times M.$$

Using the very ampleness assumption, one finds that π is a \mathbb{P}^s -bundle over $(M \times M) \setminus \Delta_M$, and a \mathbb{P}^t -bundle over Δ_M . That is, π is what is termed a *stratified projective bundle* in [11]. As such, [11, Proposition 5.2] implies the equality

$$(2) \quad GDA_B^*(Y \times Y) = \text{Im}\left(A^*(M \times M) \rightarrow A^*(Y \times Y)\right) + \Delta_*GDA_B^*(Y),$$

where $\Delta: Y \rightarrow Y \times Y$ is the inclusion along the diagonal. As M has trivial Chow groups, $A^*(M \times M)$ is generated by $A^*(M) \otimes A^*(M)$. Base-point freeness of the L_j implies that

$$GDA_B^*(Y) = \text{Im}\left(A^*(M) \rightarrow A^*(Y)\right).$$

The equality (2) thus reduces to

$$GDA_B^*(Y \times Y) = \left\langle (p_1)^*GDA_B^*(Y), (p_2)^*GDA_B^*(Y), \Delta_Y \right\rangle$$

(where p_1, p_2 denote the projection from $S \times S$ to first resp. second factor). The assumption that Y has non-zero transcendental cohomology implies that the class of Δ_Y is not decomposable in cohomology. It follows that

$$\begin{aligned} \text{Im}\left(GDA_B^{\dim Y}(Y \times Y) \rightarrow H^{2 \dim Y}(Y \times Y, \mathbb{Q})\right) = \\ \text{Im}\left(\text{Dec}^{\dim Y}(Y \times Y) \rightarrow H^{2 \dim Y}(Y \times Y, \mathbb{Q})\right) \oplus \mathbb{Q}[\Delta_Y], \end{aligned}$$

where we use the shorthand

$$\text{Dec}^j(Y \times Y) := \left\langle (p_1)^* GDA_B^*(Y), (p_2)^* GDA_B^*(Y) \right\rangle \cap A^j(Y \times Y)$$

for the *decomposable cycles*. We now see that if $\Gamma \in GDA^{\dim Y}(Y \times Y)$ is homologically trivial, then Γ does not involve the diagonal and so $\Gamma \in \text{Dec}^{\dim Y}(Y \times Y)$. This proves the proposition. \square

Corollary 2.9. *Let $\mathcal{Y} \rightarrow B$ be as in Proposition 2.8. Assume that $\mathcal{Y} \rightarrow B$ has the Franchetta property. Then for any fiber Y the cycle class map induces an injection*

$$GDA^{\dim Y}(Y \times Y) \hookrightarrow H^{2\dim Y}(Y \times Y, \mathbb{Q}).$$

Proof. This is immediate from Proposition 2.8: the Franchetta property for $\mathcal{Y} \rightarrow B$, combined with the Künneth decomposition in cohomology, implies that the right-hand side of Proposition 2.8 injects into cohomology. \square

2.4. A CK decomposition.

Lemma 2.10. *Let M be a smooth projective variety with trivial Chow groups. Let $Y \subset M$ be a smooth complete intersection of dimension $\dim Y = d$ defined by ample line bundles. The variety Y has a self-dual CK decomposition $\{\pi_Y^j\}$ with the property that*

$$h^j(Y) := (Y, \pi_Y^j, 0) = \oplus \mathbb{1}(\ast) \text{ in } \mathcal{M}_{\text{rat}} \quad \forall j \neq d.$$

Moreover, this CK decomposition is generically defined: writing $\mathcal{Y} \rightarrow B$ for the universal family (of complete intersections of the type of Y), there exist relative projectors $\pi_{\mathcal{Y}}^j \in A^d(\mathcal{Y} \times_B \mathcal{Y})$ such that $\pi_Y^j = \pi_{\mathcal{Y}}^j|_b$ (where $Y = Y_b$ for $b \in B$).

Proof. This is a standard construction, one can look for instance at [38] (in case d is odd, which will be the case in this note, the “variable motive” $h(Y)^{\text{var}}$ of [38, Theorem 4.4] coincides with $h^d(Y)$). \square

3. MAIN RESULTS

3.1. An isomorphism of motives.

Definition 3.1. Let V be a vector space of dimension n , and let

$$\text{Gr}(2, n) := \text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$$

be the Grassmannian (parametrizing 2-dimensional subspaces of V) in its Plücker embedding. Assuming n is even, let

$$\text{Pf} \subset \mathbb{P}(\wedge^2 V^\vee)$$

denote the projective dual of $\text{Gr}(2, n) \subset \mathbb{P}(\wedge^2 V)$, called the *Pfaffian*. (The Pfaffian Pf is a hypersurface of degree $n/2$ and singular locus of codimension 7.)

Assume n is even. Given a linear subspace $U \subset \wedge^2 V$ of codimension 3, one can define varieties by intersecting on the Grassmannian side and on the Pfaffian side:

$$\begin{aligned} Y = Y_U &:= \text{Gr}(2, V) \cap \mathbb{P}(U) \subset \mathbb{P}(\wedge^2 V), \\ C = C_U &:= \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\wedge^2 V^\vee). \end{aligned}$$

We say that Y and C are *dual*. For U generic, the intersections Y and C are smooth and dimensionally transverse, of dimension $2(n-2)-3$ resp. 1.

Theorem 3.2. *Let Y be a smooth dimensionally transverse intersection*

$$Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 ,$$

where the H_j are Plücker hyperplanes.

(i) Assume n is odd. Then $A_{\text{hom}}^*(Y) = 0$.

(ii) Assume that n is even, and that Y has a smooth dual curve C . There is an isomorphism

$$h^d(Y) \cong h^1(C)((1-d)/2) \quad \text{in } \mathcal{M}_{\text{rat}} ,$$

where $d := \dim Y$ and $h^d(Y)$ is as in Lemma (2.10).

Proof. This is a special case of [29, Theorem 3.17]. Since this is crucial to the present note, let us include a (sketch of) proof.

With notation as in Definition 3.1, let us consider

$$Q := \left\{ (T, \mathbb{C}\omega) \in \text{Gr}(2, V) \times \mathbb{P}(U^\perp) \mid \omega|_T = 0 \right\} \subset \text{Gr}(2, V) \times \mathbb{P}(U^\perp) ,$$

the so-called *Cayley hypersurface*. There is a diagram

$$(3) \quad \begin{array}{ccccccc} & & Q_Y & \hookrightarrow & Q & \hookleftarrow & Q_C \\ & & \swarrow & & \swarrow^p & & \searrow^q & & \searrow \\ Y & \hookrightarrow & \text{Gr}(2, V) & & & & \mathbb{P}(U^\perp) & \hookleftarrow & C \end{array}$$

Here, C is defined to be the empty set for n is odd, and the dual curve $C \subset \text{Pf}$ in case n is even. The morphisms p and q are induced by the natural projections, and the closed subvarieties $Q_Y, Q_C \subset Q$ are defined as $p^{-1}(Y)$ resp. $q^{-1}(C)$.

The restriction of p to $Q \setminus Q_Y$ is trivial with fibre $Q_u \cong \mathbb{P}^1$, while the restriction of p to Q_Y is Zariski locally trivial with fibre $Q_{Y,y} \cong \mathbb{P}^2$. This allows us to relate the motives of Q and Y : an application of the ‘‘motivic Cayley trick’’ [18, Corollary 3.2] gives an isomorphism

$$(4) \quad \begin{aligned} h(Q) &\cong h(Y)(-2) \oplus h(\text{Gr}(2, n)) \oplus h(\text{Gr}(2, n))(-1) \\ &\cong h(Y)(-2) \oplus \bigoplus \mathbb{1}(\ast) \quad \text{in } \mathcal{M}_{\text{rat}} . \end{aligned}$$

The restriction of q to Q_C is *piecewise trivial* (in the sense of [40, Section 4.2]) with constant fiber F_1 , while the restriction of q to $Q \setminus Q_C$ is piecewise trivial with constant fiber F_2 . The fibers F_1 and F_2 are explicitly known; they have only algebraic cohomology [29, Lemma 3.5]. This allows to relate Q and C on the level of the Grothendieck ring of varieties, and hence also on the level of cohomology:

$$(5) \quad h(Q) \cong h(C)(2-n) \oplus \bigoplus \mathbb{1}(\ast) \quad \text{in } \mathcal{M}_{\text{hom}} .$$

(Here the convention is that $h(C) = 0$ in case n is odd.)

Combining (4) and (5), we find a split injection of homological motives

$$(6) \quad h^d(Y) \hookrightarrow h^1(C)((1-d)/2) \quad \text{in } \mathcal{M}_{\text{hom}} .$$

Let us now consider things family-wise. Writing $B_0 \subset \bar{B} := \mathbb{P}H^0(\mathbb{P}(\wedge^2 V), \mathcal{O}(1)^{\oplus 3})$ for the dense open parametrizing sections such that both $Y_b := \text{Gr}(2, n) \cap H_1^b \cap H_2^b \cap H_3^b$ and the dual curve $C_b \subset \text{Pf}$ are smooth and dimensionally transverse (and in addition C_b is contained in the non-singular locus $\text{Pf}^\circ \subset \text{Pf}$), we have universal families

$$\mathcal{Y} \rightarrow B_0, \quad \mathcal{C} \rightarrow B_0.$$

The above construction can be performed for every fiber $Y = Y_b$ of the family $\mathcal{Y} \rightarrow B_0$. A Hilbert schemes argument [29, Proposition 2.11] then allows to find *generically defined* correspondences (with respect to B_0) inducing the split injection (6). Then, the Franchetta-type result (Proposition 2.8) allows to lift the split injection (6) to an injection of Chow groups:

$$(7) \quad A_{hom}^*(Y) = A^*(h^d(Y)) \hookrightarrow A_{hom}^*(h^1(C)((1-d)/2)) = A_{hom}^1(C).$$

We conclude from (7) that $A_{AJ}^*(Y) = 0$ and so Y is Kimura finite-dimensional (i.e. $h(Y)$ is finite-dimensional in the sense of [19]). Combining (4) and (5), we find a numerical equality $\dim H^d(Y, \mathbb{Q}) = \dim H^1(C, \mathbb{Q})$ and so the injection (6) is actually an isomorphism of homological motives. Using Kimura finite-dimensionality of both sides, it follows that (6) is also an isomorphism of Chow motives:

$$h^d(Y) \xrightarrow{\cong} h^1(C)((1-d)/2) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

This proves the theorem. □

3.2. Some instances of the Franchetta property.

Notation 3.3. Let \bar{B} and B_0 be as in the proof of Theorem 3.2, and let $B \supset B_0$ be the set parametrizing smooth dimensionally transverse intersections $Y_b = \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3$; there is a universal family

$$\mathcal{Y} \rightarrow B.$$

Assuming n is even, let us write

$$\mathcal{C} \rightarrow B_0$$

for the universal family of smooth dual curves $C_b \subset \text{Pf}^\circ$, as in the proof of Theorem 3.2.

Proposition 3.4. *The following families have the Franchetta property:*

- (i) the family $\mathcal{Y} \rightarrow B$;
- (ii) the family $\mathcal{C} \rightarrow B_0$.

Proof. For (i), let us note that the statement is vacuously true in case n is odd, because then each fiber Y_b has trivial Chow groups (Theorem 3.2(i)). Let us now assume that n is even, say $n = 2m$. We observe that the projection

$$\bar{\mathcal{Y}} \rightarrow \text{Gr}(2, n)$$

is a projective bundle, and so (reasoning with the projective bundle formula, or directly applying [11, Proposition 5.2]) one finds that for any fiber $Y := Y_b$ there is equality

$$GDA^j(Y) = \text{Im} \left(A^j(\text{Gr}(2, n)) \rightarrow A^j(Y) \right).$$

We know from Theorem 3.2(ii) that the only non-trivial Chow group of Y is

$$A^{(d+1)/2}(Y) = A^{(2(n-2)-3+1)/2}(Y) = A^{2m-3}(Y),$$

and so we only need to prove that $GDA^{2m-3}(Y)$ injects into cohomology. The Chow ring of the Grassmannian is

$$A^*(\mathrm{Gr}(2, n)) = \langle h, c \rangle,$$

where $c := c_2(Q) \in A^2(\mathrm{Gr}(2, n))$ is the second Chern class of the tautological quotient bundle [9], and so

$$A^{2m-4}(\mathrm{Gr}(2, n)) \xrightarrow{\cdot h} A^{2m-3}(\mathrm{Gr}(2, n))$$

is surjective (and hence an isomorphism, by hard Lefschetz). Let $\tau: Y \rightarrow \mathrm{Gr}(2, n)$ denote the inclusion morphism. The normal bundle formula tells us that the composition

$$A^{2m-4}(\mathrm{Gr}(2, n)) \xrightarrow{\cdot h} A^{2m-3}(\mathrm{Gr}(2, n)) \xrightarrow{\tau^*} A^{2m-3}(Y) \xrightarrow{\tau_*} A^{2m}(\mathrm{Gr}(2, n))$$

is a non-zero multiple of

$$A^{2m-4}(\mathrm{Gr}(2, n)) \xrightarrow{\cdot h^4} A^{2m}(\mathrm{Gr}(2, n)).$$

This last map is the same as

$$H^{4m-8}(\mathrm{Gr}(2, n), \mathbb{Q}) \xrightarrow{\cdot h^4} H^{4m}(\mathrm{Gr}(2, n), \mathbb{Q}),$$

which is an isomorphism thanks to hard Lefschetz for the $(4m-4)$ -dimensional variety $\mathrm{Gr}(2, n)$. This proves the required injectivity of $GDA^{2m-3}(Y)$ into cohomology.

As for (ii), one can either prove this directly, or can reduce to (i) via the generically defined isomorphism

$$A_{hom}^1(C) \xrightarrow{\cong} A_{hom}^{2m-3}(Y)$$

given by Theorem 3.2(ii). □

Proposition 3.5. *The following families have the Franchetta property:*

- (i) the family $\mathcal{C} \times_{B_0} \mathcal{C} \rightarrow B_0$;
- (ii) the family $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$.

Proof. (i) Let $\bar{\mathcal{C}} \subset \mathrm{Pf} \times \bar{B}$ denote the projective closure of \mathcal{C} , and let us consider the projection

$$\pi: \bar{\mathcal{C}} \times_{\bar{B}} \bar{\mathcal{C}} \rightarrow \mathrm{Pf} \times \mathrm{Pf}.$$

This is a *stratified projective bundle* (in the sense of [11]). As such, [11, Proposition 5.2] implies the equality

$$(8) \quad GDA_{B_0}^*(C \times C) = \mathrm{Im}\left(A^*(\mathrm{Pf}^\circ \times \mathrm{Pf}^\circ) \rightarrow A^*(C \times C)\right) + \Delta_* GDA_{B_0}^*(C),$$

where $\Delta: C \rightarrow C \times C$ is the inclusion along the diagonal, and $\mathrm{Pf}^\circ \subset \mathrm{Pf}$ denotes the non-singular locus of the Pfaffian. As Pf° has the Chow–Künneth property [29, Example 2.7], $A^*(\mathrm{Pf}^\circ \times \mathrm{Pf}^\circ)$ is generated by $A^*(\mathrm{Pf}^\circ) \otimes A^*(\mathrm{Pf}^\circ)$. The equality (8) thus simplifies to

$$(9) \quad GDA_{B_0}^*(C \times C) = \left\langle (p_j)^* \mathrm{Im}\left(A^j(\mathrm{Pf}^\circ) \rightarrow A^*(C)\right), \Delta_C \right\rangle.$$

We now proceed to check that $GDA_{B_0}^j(C \times C)$ injects into cohomology:

In case $j = 1$, we know that Δ_C is linearly independent from the decomposable classes

$$\left\langle (p_j)^* \operatorname{Im}(A^j(\operatorname{Pf}^\circ) \rightarrow A^*(C)) \right\rangle$$

in cohomology (indeed, we may assume that C has genus > 0 , for otherwise the statement is vacuously true). The required injectivity then reduces to Proposition 3.4(ii).

In case $j = 2$, we know that $A^1(\operatorname{Pf}^\circ)$ is 1-dimensional, generated by a hyperplane class H (cf. Lemma 3.6 below). Since $C \subset \mathbb{P}^2$ is a plane curve, clearly we have an equality

$$\Delta_C \cdot (p_i)^*(H) = \sum_{r=0}^2 \frac{1}{\deg C} (p_1)^*(H^r) \cdot (p_2)^*(H^{2-r}) \quad \text{in } A^2(C \times C),$$

and so

$$\begin{aligned} GDA_{B_0}^2(C \times C) &= \left\langle (p_j)^* \operatorname{Im}(A^j(\operatorname{Pf}^\circ) \rightarrow A^*(C)), \Delta_C \right\rangle \cap A^2(C \times C) \\ &= \left\langle (p_j)^* \operatorname{Im}(A^j(\operatorname{Pf}^\circ) \rightarrow A^*(C)) \right\rangle \cap A^2(C \times C). \end{aligned}$$

The required injectivity then reduces to Proposition 3.4(ii).

In the above, we have used the following lemma:

Lemma 3.6. *Let $\operatorname{Pf}^\circ \subset \operatorname{Pf}$ denote (as above) the non-singular locus of the Pfaffian. We have*

$$A^1(\operatorname{Pf}^\circ) \cong \mathbb{Q}[H].$$

Proof. (of the lemma.) We consider

$$\widetilde{\operatorname{Pf}} := \left\{ (\omega, K) \in \operatorname{Pf} \times \operatorname{Gr}(2, n) \mid K \subset \ker \omega \right\} \subset \operatorname{Pf} \times \operatorname{Gr}(2, n).$$

The projection $\widetilde{\operatorname{Pf}} \rightarrow \operatorname{Gr}(2, n)$ is a projective bundle (and so $\widetilde{\operatorname{Pf}}$ is smooth), and the projection $\widetilde{\operatorname{Pf}} \rightarrow \operatorname{Pf}$ is an isomorphism over the non-singular locus (and so $\widetilde{\operatorname{Pf}} \rightarrow \operatorname{Pf}$ is a resolution of singularities).

Being a projective bundle over a Grassmannian, $\widetilde{\operatorname{Pf}}$ has Picard number 2:

$$A^1(\widetilde{\operatorname{Pf}}) = \mathbb{Q}^2.$$

The complement of (the isomorphic pre-image of) Pf° inside $\widetilde{\operatorname{Pf}}$ is an irreducible divisor D (it is a partial flag variety). The localization sequence

$$A_*(D) \rightarrow A^1(\widetilde{\operatorname{Pf}}) \rightarrow A^1(\operatorname{Pf}^\circ) \rightarrow 0$$

then gives the result. \square

(ii) Again, we may assume that n is even (for otherwise the statement is vacuously fulfilled). In view of Lemma 2.7, it will suffice to prove the Franchetta property for $\mathcal{Y} \times_{B_0} \mathcal{Y} \rightarrow B_0$. Thanks to Theorem 3.2(ii), for any fiber $Y = Y_b$ with $b \in B_0$ we have split injections

$$A^j(Y \times Y) \hookrightarrow A^{j+1-d}(C \times C) \oplus \bigoplus A^*(C) \oplus \mathbb{Q}^s.$$

The isomorphism of Theorem 3.2 being generically defined, there are also split injections

$$GDA^j(Y \times Y) \hookrightarrow GDA^{j+1-d}(C \times C) \oplus \bigoplus GDA^*(C) \oplus \mathbb{Q}^s.$$

The required injectivity now follows from (i) and Proposition 3.4(ii). \square

3.3. MCK.

Theorem 3.7. *Let Y be a smooth dimensionally transverse intersection*

$$Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 ,$$

where the H_j are Plücker hyperplanes. Then Y has an MCK decomposition.

Proof. In case n is odd, Y has trivial Chow groups (Theorem 3.2(i)) and so the statement is vacuously true. In case $n = 4$, Y is a rational curve and again the statement is vacuously true. We may thus suppose that n is even and ≥ 6 . We have the following general result:

Proposition 3.8. *Let $\mathcal{Y} \rightarrow B$ be a family of smooth projective varieties, verifying*

(a1) *the fibers Y_b are of odd dimension $d \geq 5$ and*

$$A_{\text{hom}}^j(Y_b) = 0 \quad \forall j > (d+1)/2 \quad \forall b \in B ;$$

(a2) *the fibers Y_b have a generically defined Künneth decomposition, i.e. there exist $\{\pi_{\mathcal{Y}}^j\} \in A^d(\mathcal{Y} \times_B \mathcal{Y})$ such that the fiberwise restriction $\pi_{Y_b}^j := \pi_{\mathcal{Y}}^j|_b \in A^d(Y_b \times Y_b)$ is a Künneth decomposition for all $b \in B$;*

(a3) *the family $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$ has the Franchetta property.*

Then $\{\pi_{Y_b}^j\}$ is an MCK decomposition for any $b \in B$.

Proof. (of Proposition 3.8.) Condition (a1) implies (via the Bloch–Srinivas argument, cf. [5]) that for every fiber Y_b there exists a curve C_b and a split injection of motives

$$(10) \quad h(Y_b) \hookrightarrow h(C_b)((1-d)/2) \oplus \bigoplus \mathbb{1}(*). \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Condition (a3) implies that the Künneth decomposition $\{\pi_{Y_b}^j\}$ of (a2) is a self-dual CK decomposition. Let $h(Y_b) = \bigoplus_j h^j(Y_b)$ denote the corresponding decomposition of the motive of X . Using the injection (10), one finds that $h^j(Y_b) = \bigoplus \mathbb{1}(*)$ for all $j \neq d$, while for $j = d$ one finds a split injection

$$(11) \quad h^d(Y_b) \hookrightarrow h^1(C_b)((1-d)/2) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Let us now establish that the CK decomposition $\{\pi_{Y_b}^j\}$ is MCK. By definition, what we need to check is that the cycle

$$\Gamma_{ijk} := \pi_{Y_b}^k \circ \Delta_{Y_b}^{\text{sm}} \circ (\pi_{Y_b}^i \times \pi_{Y_b}^j) \in A^{2d}(Y_b \times Y_b \times Y_b)$$

is zero for all $i + j \neq k$.

Let us assume at least one of the integers i, j, k is different from d . In this case, there is an injection

$$\Gamma_{ijk} \in (\pi_{Y_b}^{2d-i} \times \pi_{Y_b}^{2d-j} \times \pi_{Y_b}^k)_* A^{2d}(Y_b \times Y_b \times Y_b) \hookrightarrow \bigoplus A^*(Y_b \times Y_b) ,$$

and this injection sends generically defined cycles to generically defined cycles. But Γ_{ijk} is generically defined and homologically trivial, and so the Franchetta property for $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$ gives the required vanishing $\Gamma_{ijk} = 0$.

Next, let us assume $i = j = k = d$. In this case, the injection of motives (11) induces an injection of Chow groups

$$\Gamma_{ijk} \in (\pi_{Y_b}^d \times \pi_{Y_b}^d \times \pi_{Y_b}^d)_* A^{2d}(Y_b \times Y_b \times Y_b) \hookrightarrow A^{(d+3)/2}(C_b \times C_b \times C_b).$$

But the right-hand side vanishes for dimension reasons for any $d \geq 5$, and so $\Gamma_{ijk} = 0$. \square

Let us now consider the family $\mathcal{Y} \rightarrow B$ of all smooth complete intersections $\text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3$, where $n \geq 6$ is even. Each fiber Y_b has a generically defined CK decomposition $\{\pi_{Y_b}^j\}$ (Lemma 2.10). To check that $\{\pi_{Y_b}^j\}$ is MCK, it suffices to do this over a dense open of B ; for instance we may take $B_0 \subset B$ the locus as before where Y_b has a smooth dual curve C_b contained in Pf° . Let us check that $\mathcal{Y} \rightarrow B_0$ verifies the conditions of Proposition 3.8. Condition (a1) is immediate from Theorem 3.2(ii). Condition (a2) is fulfilled by the $\{\pi_{Y_b}^j\}$. As for condition (a3), this is Proposition 3.5(ii). This ends the proof. \square

4. THE TAUTOLOGICAL RING

4.1. A positive result.

Corollary 4.1. *Let Y be as in Theorem 3.7, and $m \in \mathbb{N}$. Let*

$$R^*(Y^m) := \left\langle (p_i)^* \text{Im}(A^*(\text{Gr}(2, n)) \rightarrow A^*(Y)), (p_{ij})^*(\Delta_Y) \right\rangle \subset A^*(Y^m)$$

be the \mathbb{Q} -subalgebra generated by (pullbacks of) cycles coming from $\text{Gr}(2, n)$ and (pullbacks of) the diagonal $\Delta_Y \in A^d(Y \times Y)$. (Here p_i and p_{ij} denote the various projections from Y^m to Y resp. to $Y \times Y$). The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \text{ for all } m \in \mathbb{N}.$$

Proof. This is inspired by the analogous result for cubic hypersurfaces [12, Section 2.3], which in turn is inspired by analogous results for hyperelliptic curves [44], [45] (cf. Remark 4.2 below) and for K3 surfaces [51].

The Chow ring $A^*(\text{Gr}(2, n))$ is generated by the Plücker polarization $h \in A^1(\text{Gr}(2, n))$ and the Chern class $c_2(Q) \in A^2(\text{Gr}(2, n))$, where $Q \rightarrow \text{Gr}(2, n)$ is the universal quotient bundle [9]. As in [12, Section 2.3], let us write

$$o := \frac{1}{\deg(h^d)} h^d \in A^d(Y), \quad c := c_2(Q)|_Y \in A^2(Y),$$

and

$$\tau := \pi_Y^d = \Delta_Y - \sum_{j \neq d} \pi_Y^j \in A^d(Y \times Y),$$

where the π_Y^j are as above, and $d := \dim Y$.

Moreover, for any $1 \leq i < j \leq m$ let us write

$$\begin{aligned} o_i &:= (p_i)^*(o) \in A^d(Y^m), \\ h_i &:= (p_i)^*(h) \in A^1(Y^m), \\ c_i &:= (p_i)^*(c) \in A^2(Y^m), \\ \tau_{ij} &:= (p_{ij})^*(\tau) \in A^d(Y^m). \end{aligned}$$

Note that (by definition) we have

$$R^*(Y^m) = \langle o_i, h_i, c_i, \tau_{ij} \rangle \subset A^*(Y^m).$$

Let us now define the \mathbb{Q} -subalgebra

$$\bar{R}^*(Y^m) := \langle o_i, h_i, c_i, \tau_{ij} \rangle \subset H^*(Y^m, \mathbb{Q})$$

(where i ranges over $1 \leq i \leq m$, and $1 \leq i < j \leq m$); this is the image of $R^*(Y^m)$ in cohomology. One can prove (just as [12, Lemma 2.11] and [51, Lemma 2.3]) that the \mathbb{Q} -algebra $\bar{R}^*(Y^m)$ is isomorphic to the free graded \mathbb{Q} -algebra generated by o_i, h_i, c_i, τ_{ij} , modulo the following relations:

$$(12) \quad h_i \cdot o_i = c_i \cdot o_i = 0, \quad c_i^{(d+1)/2} = 0, \quad c_i^{(d-1)/2} = \lambda h_i^{d-1}, \quad \dots, \quad h_i^d = \deg(h^d) o_i;$$

$$(13) \quad \tau_{ij} \cdot o_i = \tau_{ij} \cdot h_i = \tau_{ij} \cdot c_i = 0, \quad \tau_{ij} \cdot \tau_{ij} = -b_d o_i \cdot o_j;$$

$$(14) \quad \tau_{ij} \cdot \tau_{ik} = \tau_{jk} \cdot o_i;$$

$$(15) \quad \sum_{\sigma \in \mathfrak{S}_{b_d+2}} \prod_{i=1}^{b_d/2+1} \tau_{\sigma(2i-1), \sigma(2i)} = 0.$$

Here $\lambda \in \mathbb{Q}$, and the dots “...” in (12) indicate certain relations of type $c_i^{m_j} h_i^{n_j} = \lambda_j h_i^{2m_j+n_j}$. By definition, $b_d := \dim H^d(Y, \mathbb{Q})$ and \mathfrak{S}_r denotes the symmetric group on r elements.

To prove Corollary 4.1, it suffices to check that all these relations are verified modulo rational equivalence. The relations (12) take place in $R^*(Y)$ and so they follow from the Franchetta property for Y (Proposition 3.4). The relations (13) take place in $R^*(Y^2)$. The last relation is trivially verified, because (Y being Fano) $A^{2d}(Y^2) = \mathbb{Q}$. As for the other relations of (13), these follow from the Franchetta property for $Y \times Y$ (Proposition 3.5).

Relation (14) takes place in $R^*(Y^3)$ and follows from the MCK decomposition. Indeed, we have

$$\Delta_Y^{\text{sm}} \circ (\pi_Y^d \times \pi_Y^d) = \pi_Y^{2d} \circ \Delta_Y^{\text{sm}} \circ (\pi_Y^d \times \pi_Y^d) \quad \text{in } A^{2d}(Y^3),$$

which (using Lieberman’s lemma) translates into

$$(\pi_Y^d \times \pi_Y^d \times \Delta_Y)_* \Delta_Y^{\text{sm}} = (\pi_Y^d \times \pi_Y^d \times \pi_Y^{2d})_* \Delta_Y^{\text{sm}} \quad \text{in } A^{2d}(Y^3),$$

which means that

$$\tau_{13} \cdot \tau_{23} = \tau_{12} \cdot o_3 \quad \text{in } A^{2d}(Y^3).$$

Finally, relation (15), which takes place in $R^*(Y^{b_d+2})$, is related to the Kimura finite-dimensionality relation [19]: relation (15) expresses the vanishing

$$\mathrm{Sym}^{b_d+2} H^d(Y, \mathbb{Q}) = 0,$$

where $H^d(Y, \mathbb{Q})$ is seen as a super vector space. This relation is also verified modulo rational equivalence, (i.e., relation (15) is also true in $A^{d(b_d+2)}(Y^{b_d+2})$): relation (15) involves a cycle in

$$A^*(\mathrm{Sym}^{b_d+2} h^d(Y)),$$

and $\mathrm{Sym}^{b_d+2} h^d(Y)$ is 0 because Y has Kimura finite-dimensional motive (Theorem 3.2).

This ends the proof. \square

Remark 4.2. Given any curve C and an integer $m \in \mathbb{N}$, one can define the *tautological ring*

$$R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m)$$

(where p_i, p_{ij} denote the various projections from C^m to C resp. $C \times C$). Tavakol has proven [45, Corollary 6.4] that if C is a hyperelliptic curve, the cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

On the other hand, there exist curves for which the tautological ring $R^*(C^3)$ does *not* inject into cohomology, cf. Proposition 4.3 below.

4.2. A negative result.

Proposition 4.3. *Let*

$$Y := \mathrm{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3$$

be a very general intersection of the Grassmannian with 3 Plücker hyperplanes, where n is even and $8 \leq n \leq 2000$. Let $C \subset \mathrm{Pf}$ be the curve dual to Y (Definition 3.1). The \mathbb{Q} -subalgebra

$$R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m)$$

does not inject into cohomology for $m = 3$.

Proof. The point is that C is a plane curve of degree $n/2$, and that the general plane curve of degree $n/2$ arises in this way [1]. Using the spread lemma (Lemma 2.7), it follows that the assumption that $R^*(C^m)$ injects into cohomology for the very general C as in Proposition 4.3 would imply that $R^*(C^m)$ injects into cohomology for every plane curve of degree $n/2$. Taking $m = 3$, this would mean that every plane curve of degree $n/2$ has a self-dual MCK decomposition. As explained in [15, Proposition 7.1] and [13, Remark 2.4], this would imply that for every plane curve C of degree $n/2$ the Ceresa cycle

$$C - [-1]_*(C) \in A_1(\mathrm{Jac}(C))$$

is algebraically trivial. But this is known to be false for the Fermat curve of degree between 4 and 1000, cf. [36]. \square

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