Lower bound for Buchstaber invariants of real universal complexes

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Abstract. In this article, we prove that Buchstaber invariant of 4-dimensional real universal complex is no less than 24 as a follow-up to the work of Ayzenberg and Sun. Moreover, a lower bound for Buchstaber invariants of n-dimensional real universal complexes is given as an improvement of result of Erokhovets.

1 Introduction

Moment-angle complex and its real counterpart are fundamental objects in toric topology as they construct links between algebraic geometry, symplectic geometry and combinatorics (see Definition 1). These objects are equipped with certain group actions, yielding applications in both non-equivariant and equivariant categories (see [4] for more details).

For a given simplicial complex K on m vertices, the associated real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ (resp. moment-angle complex $\mathcal{Z}_K$) admits a natural $\mathbb{Z}_2^m$-action (resp. T^m-action) by coordinate-wise sign permutation (resp. rotation). However, these actions fail to be free unless K is the empty complex, leading to the definition of *real Buchstaber invariant* $s_{\mathbb{R}}(K)$ (resp. *Buchstaber invariant* $s(K)$) as the maximal rank of subgroup (resp. toric subgroup) that acts freely on $\mathbb{R}\mathcal{Z}_K$ (resp. $\mathcal{Z}_K$) (see Definition 2). These two types of invariants measure the degree of symmetry of the corresponding complexes and were first introduced in [3] for simplicial spheres with generalization in [8] for arbitrary simplicial complexes.

Buchstaber asked for a combinatorial description of $s(K)$ in [3], which turns out to be quite hard and remains open till today. As a matter of fact, calculation of $s(K)$ is not completed even for the special case where K is the boundary of a cyclic polytope. The partial results can be found in [7].

On the other hand, there exists a general bound for $s_{\mathbb{R}}(K)$ and $s(K)$:

$$m - \gamma(K) \leq s(K) \leq s_{\mathbb{R}}(K) \leq m - n \tag{1.1}$$

where K is an $(n-1)$-dimensional simplicial complex on m vertices and $\gamma(K)$ stands for ordinary chromatic number of K. This formula can be derived from relations among generalized chromatic numbers in a systematic manner, as shown in [1]. Indeed, with the help of *real universal complex* $\mathcal{K}_1^n$ and *universal complex* $\mathcal{K}_2^n$ introduced in [6], $s_{\mathbb{R}}(K)$

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and $s(K)$ can be expressed as $m - r_{\mathbb{R}}(K)$ and $m - r(K)$ respectively, where $r_{\mathbb{R}}(K)$ and $r(K)$ are minimal rank of certain colorings on K (see Section 2, also see [10]). Moreover, the upper bound of $s(K)$ and $s_{\mathbb{R}}(K)$ is controlled by the sum of rational Betti numbers since $2^{m-n} \leq \sum_i \dim H^i(\mathcal{Z}_K;\mathbb{Q}) = \sum_i \dim H^i(\mathbb{R}\mathcal{Z}_K;\mathbb{Q})$ was proved as a special case of Halperin-Carlsson conjecture (see [5, 16]).

The general inequality $s(K) \leq s_{\mathbb{R}}(K)$ follows from the fact that involutions on T^m and $\mathcal{Z}_K$ induced by complex conjugation have fixed point sets $\mathbb{Z}_2^m$ and $\mathbb{R}\mathcal{Z}_K$ respectively. Thus, any free T^r-action on $\mathcal{Z}_K$ induces a free $\mathbb{Z}_2^r$-action on $\mathbb{R}\mathcal{Z}_K$. Meanwhile, the special case $s(K) = s_{\mathbb{R}}(K)$ is closely related to *Lifting problem* (see Section 2) presented by Lü at the conference on toric topology held in Osaka in November 2011[1]. Let $\Delta(K) = s_{\mathbb{R}}(K) - s(K) = r(K) - r_{\mathbb{R}}(K)$ denote the difference, then the vanishing of $\Delta(K)$ is necessary for the validity of Lifting problem on K.

For real universal complex $\mathcal{K}_1^n$, we have the following monotonicity and universal property:

Proposition 1 ([15, Theorem 3.3]) $\Delta(\mathcal{K}_1^n) \leq \Delta(\mathcal{K}_1^{n+1})$.

Proposition 2 ([15, Proposition 4.1]) $r_{\mathbb{R}}(K) \leq r_{\mathbb{R}}(\mathcal{K}_1^n) \Longrightarrow \Delta(K) \leq \Delta(\mathcal{K}_1^n)$.

Therefore, the value of $\Delta(\mathcal{K}_1^n)$ is significant as it gives out an upper bound for general cases. In [1] and [2], Ayzenberg showed that $\Delta(\mathcal{K}_1^n) = 0$ for $n = 1, 2, 3$ and $\Delta(\mathcal{K}_1^4) > 0$ respectively. In [15], $\Delta(\mathcal{K}_1^4) = 1$ was confirmed by Sun. Furthermore, an upper bound $\Delta(\mathcal{K}_1^n) \leq 3 \cdot 2^{n-2} - 1 - n$ for $n \geq 2$ can be viewed as a corollary of Proposition 3.3.4 in [7] (see Remark 5).

Main aim of this article is to further estimate the upper bound for $\Delta(\mathcal{K}_1^n)$, which can be deduced from construction of certain colorings. Since $s(\mathcal{K}_1^n) + \Delta(\mathcal{K}_1^n) = 2^n - 1 - n$ by definition (see Section 2), upper bound estimation of $\Delta(\mathcal{K}_1^n)$ is equivalent to lower bound estimation of $s(\mathcal{K}_1^n)$. Explicit construction of non-degenerate simplicial maps from $\mathcal{K}_1^5$ to $\mathcal{K}_2^7$ and from $\mathcal{K}_1^n$ to $\mathcal{K}_2^{2^{n-2}+1}$ for $n \geq 2$ yields the following theorems:

Theorem 1 $\Delta(\mathcal{K}_1^5) \leq 2$*, i.e.,* $s(\mathcal{K}_1^5) \geq 24$.

Theorem 2 *For* $n \geq 2$*,* $\Delta(\mathcal{K}_1^n) \leq 2^{n-2} + 1 - n$*, i.e.,* $s(\mathcal{K}_1^n) \geq 3 \cdot 2^{n-2} - 2$.

Remark 1 The construction process is equivalent to finding the solution to a system of nonlinear Diophantine equations. These equations all belong to a certain type discussed in [12], where existence and classification problem of solutions to a single equation was solved. However, existence problem of solutions to the system is much harder to deal with since the number of equations grow rapidly as n increases.

Remark 2 Vanishing of $\Delta(\mathcal{K}_1^n)$ for $n = 1, 2, 3$ follows from the fact that every matrix in $\mathrm{GL}(3, \mathbb{Z}_2)$ has integral determinant ± 1. For general n, upper bound estimation of $\Delta(\mathcal{K}_1^n)$ is also related to determinant calculation in both $\mathbb{Z}_2$ and $\mathbb{Z}$. An upper bound $\frac{(n+1)^{\frac{(n+1)}{2}}}{2^n}$

[1] http://www.sci.osaka-cu.ac.jp/masuda/toric/torictopology2011_osaka.html

for absolute value of integral determinant of matrix in $\mathrm{GL}(n, \mathbb{Z}_2)$ was given in [9]. This is a special case of *Hadamard maximum determinant problem* which aims to calculate the maximal determinant of a square matrix with elements restricted in a given set S. It should be pointed out that this problem is far from being solved even for the simplest case $S = \{0, 1\}$ since whether the bound given above is sharp or not remains unknown except for $n+1$ being power of 2. Computational results in low dimensions ($n \leq 9$) with the aid of computer were listed in [17] and recent theoretical progress can be found in [13].

This article is organized as follows. In Section 2, basic definitions and notations are listed. Section 3 is devoted to the proof of Theorem 1 via explicit construction and computer-aided verification. Section 4 includes the proof of Theorem 2 with an illustrative example.

2 Preliminaries

In the first place, we shall give out the formal definition of (real) moment-angle complex and (real) Buchstaber invariant.

Definition 1 Given a simplicial complex K on $[m] = \{1, \ldots, m\}$, we can define the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ and the *moment-angle complex* $\mathcal{Z}_K$ associated to K:

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{I \subset K} (D^1, S^0)^I \subseteq (D^1)^m; \mathcal{Z}_K = \bigcup_{I \subset K} (D^2, S^1)^I \subseteq (D^2)^m,$$

where $(X, A)^I = \{(x_1, \ldots, x_m) \in X^m, x_i \in A \textit{ if } i \notin I\}$ for $A \subseteq X$.

Definition 2 For $\mathbb{R}\mathcal{Z}_K$ and $\mathcal{Z}_K$ associated to a simplicial complex K on $[m]$:
(1) The *real Buchstaber invariant* $s_{\mathbb{R}}(K)$ is the maximal rank of a subgroup $H \subseteq \mathbb{Z}_2^m$ such that the restricted action $H \curvearrowright \mathbb{R}\mathcal{Z}_K$ is free;
(2) The *Buchstaber invariant* $s(K)$ is the maximal rank of a toric subgroup $G \subseteq T^m$ such that the restricted action $G \curvearrowright \mathcal{Z}_K$ is free.

Example 1 Let K be the boundary of a square with vertices labeled as $1, 3, 2, 4$ counter-clockwise. By definition $\mathcal{Z}_K = (D^2 \times S^1 \cup S^1 \times D^2) \times (D^2 \times S^1 \cup S^1 \times D^2) = S^3 \times S^3$ while $\mathbb{R}\mathcal{Z}_K = (D^1 \times S^0 \cup S^0 \times D^1) \times (D^1 \times S^0 \cup S^0 \times D^1) = S^1 \times S^1$. Moreover, $s(K) = s_{\mathbb{R}}(K) = 2$ follows from (1.1) and $\gamma(K) = \mathrm{dim}K + 1 = 2$, where $\gamma(K)$ is the chromatic number of K.

Secondly, we introduce the (real) universal complex and the corresponding coloring to get an equivalent expression of (real) Buchstaber invariant.

Definition 3 Let $R_d^n = \mathbb{Z}_2^n$ when $d = 1$ and $R_d^n = \mathbb{Z}^n$ when $d = 2$. The simplicial complex $\mathcal{K}_d^n$ is defined on the set of primitive vectors in R_d^n as follow:

$$[\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}] \text{ is a simplex of } \mathcal{K}_d^n \Longleftrightarrow \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\} \text{ is part of a basis of } R_d^n.$$

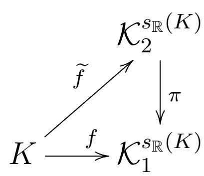
$\mathcal{K}_1^n$ is called *real universal complex* while $\mathcal{K}_2^n$ is called *universal complex*.

Definition 4 An R_d^r-*coloring* on a simpicial complex K is defined as a non-degenerate simplicial map $\lambda : K \to \mathcal{K}_d^r$. The non-degenerate condition means λ is an isomorphism on each simplex of K.

Let $r_{\mathbb{R}}(K)$ denote the minimum value of r such that there exists an R_1^r-coloring on K. Similarly, $r(K)$ is the minimum value of r such that there exists an R_2^r-coloring on K. Then $r_{\mathbb{R}}(\mathcal{K}_1^n) = r(\mathcal{K}_2^n) = n$ follows from definition. In addition, equivalent expressions $s_{\mathbb{R}}(K) = m - r_{\mathbb{R}}(K)$ and $s(K) = m - r(K)$ were first proved in [10]. Since there are $2^n - 1$ primitive vectors in $\mathcal{K}_1^n$, we have $s_{\mathbb{R}}(\mathcal{K}_1^n) = s(\mathcal{K}_1^n) + \Delta(\mathcal{K}_1^n) = 2^n - 1 - n$.

With notations above, we can formally state the Lifting problem as follow:

Lifting Problem ([11, Remark 6]) *For any given simplicial complex K and non-degenerate simplicial map $f : K \to \mathcal{K}_1^{s_{\mathbb{R}}(K)}$, does there exist a lifting map $\widetilde{f} : K \to \mathcal{K}_2^{s_{\mathbb{R}}(K)}$ such that the diagram below is commutative:*



where $\pi : \mathcal{K}_2^{s_{\mathbb{R}}(K)} \to \mathcal{K}_1^{s_{\mathbb{R}}(K)}$ is the natural modulo 2 projection.

3 Lower bound for $s(\mathcal{K}_1^5)$

3.1 Preparation

Let e represent the identity of $\mathbb{Z}_2$ and 1 represent the identity of $\mathbb{Z}$ to avoid confusion. As listed in the table below, we can take a partition $vt(\mathcal{K}_1^5) = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4 \sqcup V_5$ such that V_i consists of primitive vectors with $(5 - i)$ zeros and label 31 elements of $vt(\mathcal{K}_1^5)$ in lexicographic order.

V_1	V_2	V_3	V_4
$\boldsymbol{v_1} = (e, 0, 0, 0, 0)$	$\boldsymbol{v_6} = (e, e, 0, 0, 0)$	$\boldsymbol{v_{16}} = (e, e, e, 0, 0)$	$\boldsymbol{v_{26}} = (e, e, e, e, 0)$
$\boldsymbol{v_2} = (0, e, 0, 0, 0)$	$\boldsymbol{v_7} = (e, 0, e, 0, 0)$	$\boldsymbol{v_{17}} = (e, e, 0, e, 0)$	$\boldsymbol{v_{27}} = (e, e, e, 0, e)$
$\boldsymbol{v_3} = (0, 0, e, 0, 0)$	$\boldsymbol{v_8} = (e, 0, 0, e, 0)$	$\boldsymbol{v_{18}} = (e, e, 0, 0, e)$	$\boldsymbol{v_{28}} = (e, e, 0, e, e)$
$\boldsymbol{v_4} = (0, 0, 0, e, 0)$	$\boldsymbol{v_9} = (e, 0, 0, 0, e)$	$\boldsymbol{v_{19}} = (e, 0, e, e, 0)$	$\boldsymbol{v_{29}} = (e, 0, e, e, e)$
$\boldsymbol{v_5} = (0, 0, 0, 0, e)$	$\boldsymbol{v_{10}} = (0, e, e, 0, 0)$	$\boldsymbol{v_{20}} = (e, 0, e, 0, e)$	$\boldsymbol{v_{30}} = (0, e, e, e, e)$
	$\boldsymbol{v_{11}} = (0, e, 0, e, 0)$	$\boldsymbol{v_{21}} = (e, 0, 0, e, e)$	
	$\boldsymbol{v_{12}} = (0, e, 0, 0, e)$	$\boldsymbol{v_{22}} = (0, e, e, e, 0)$	
	$\boldsymbol{v_{13}} = (0, 0, e, e, 0)$	$\boldsymbol{v_{23}} = (0, e, e, 0, e)$	
	$\boldsymbol{v_{14}} = (0, 0, e, 0, e)$	$\boldsymbol{v_{24}} = (0, e, 0, e, e)$	
	$\boldsymbol{v_{15}} = (0, 0, 0, e, e)$	$\boldsymbol{v_{25}} = (0, 0, e, e, e)$	
$V_5 : \boldsymbol{v_{31}} = (e, e, e, e, e)$			

Within the rest of this article, we assume all vectors are understood as column vectors. And for a binary square matrix M, $\det_{\mathbb{Z}_2} M$ represents its determinant taken in $\mathbb{Z}_2$ while $\det M$ represents its determinant taken in $\mathbb{Z}$. The statement of Theorem 1 is equivalent to $r(\mathcal{K}_1^5) \leq 7$, i.e., there exists a vertex map $\Lambda : vt(\mathcal{K}_1^5) \to vt(\mathcal{K}_2^7)$ which induces a non-degenerate simplicial map from $\mathcal{K}_1^5$ to $\mathcal{K}_2^7$. By the restriction of non-degenerate condition, it remains to verify that:

$$\begin{aligned} &\forall \ \{\boldsymbol{v_{i_1}}, \boldsymbol{v_{i_2}}, \boldsymbol{v_{i_3}}, \boldsymbol{v_{i_4}}, \boldsymbol{v_{i_5}}\} \subseteq vt(\mathcal{K}_1^5) \textit{ satisfying } \det_{\mathbb{Z}_2}(\boldsymbol{v_{i_1}}, \boldsymbol{v_{i_2}}, \boldsymbol{v_{i_3}}, \boldsymbol{v_{i_4}}, \boldsymbol{v_{i_5}}) = e, \\ &\exists \ \boldsymbol{\alpha} = (a_1, \ldots, a_7)^{\mathrm{T}} \in \mathbb{Z}^7 \textit{ and } \boldsymbol{\beta} = (b_1, \ldots, b_7)^{\mathrm{T}} \in \mathbb{Z}^7 \textit{ with the property}: \\ &\det(\Lambda(\boldsymbol{v_{i_1}}), \Lambda(\boldsymbol{v_{i_2}}), \Lambda(\boldsymbol{v_{i_3}}), \Lambda(\boldsymbol{v_{i_4}}), \Lambda(\boldsymbol{v_{i_5}}), \boldsymbol{\alpha}, \boldsymbol{\beta}) = \pm 1. \end{aligned} \tag{$\bigstar$}$$

Indeed, it is even possible to construct Λ with additional restrictions:

$$p_j \circ \Lambda = id_j \text{ for } j \in \{1, 2, 3, 4, 5\},$$

where p_j is the projection onto the j^{th} coordinate of $\mathbb{Z}^7$ and id_j is the identity map of the j^{th} coordinate. For the 6^{th} and 7^{th} coordinate, write $\phi = p_6 \circ \Lambda$, $\psi = p_7 \circ \Lambda$ and suppose for each $i \in \{1, \ldots, 31\}$, $\phi(\boldsymbol{v_i}) = s_i \in \mathbb{Z}$, $\psi(\boldsymbol{v_i}) = t_i \in \mathbb{Z}$. Then write $\Phi(\boldsymbol{v_i}) = \binom{\boldsymbol{v_i}}{s_i} \in \mathbb{Z}^6$, $\Psi(\boldsymbol{v_i}) = \binom{\boldsymbol{v_i}}{t_i} \in \mathbb{Z}^6$ and $\boldsymbol{\alpha'} = (a_1, \ldots, a_5, a_6)^{\mathrm{T}}$, $\boldsymbol{\beta'} = (b_1, \ldots, b_5, b_7)^{\mathrm{T}}$. Furthermore, let $A = (\boldsymbol{v_{i_1}}, \boldsymbol{v_{i_2}}, \boldsymbol{v_{i_3}}, \boldsymbol{v_{i_4}}, \boldsymbol{v_{i_5}})$ and $(A)_j$ denote the matrix A with j^{th} row replaced by $(s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4}, s_{i_5})$, $(A)^j$ denote the matrix A with j^{th} row replaced by $(t_{i_1}, t_{i_2}, t_{i_3}, t_{i_4}, t_{i_5})$. Then by basic linear algebra:

$$\begin{aligned} &\det(\Phi(\boldsymbol{v_{i_1}}), \Phi(\boldsymbol{v_{i_2}}), \Phi(\boldsymbol{v_{i_3}}), \Phi(\boldsymbol{v_{i_4}}), \Phi(\boldsymbol{v_{i_5}}), \boldsymbol{\alpha'}) \\ = \ &a_6 \det A - a_1 \det(A)_1 - a_2 \det(A)_2 - a_3 \det(A)_3 - a_4 \det(A)_4 - a_5 \det(A)_5; \\ &\det(\Psi(\boldsymbol{v_{i_1}}), \Psi(\boldsymbol{v_{i_2}}), \Psi(\boldsymbol{v_{i_3}}), \Psi(\boldsymbol{v_{i_4}}), \Psi(\boldsymbol{v_{i_5}}), \boldsymbol{\beta'}) \\ = \ &b_7 \det A - b_1 \det(A)^1 - b_2 \det(A)^2 - b_3 \det(A)^3 - b_4 \det(A)^4 - b_5 \det(A)^5. \end{aligned}$$

By Chinese Remainder Theorem, the existence of $\boldsymbol{\alpha'}$ for the first determinant being ± 1 is equivalent to

$$g.c.d.(\det A, \det(A)_1, \det(A)_2, \det(A)_3, \det(A)_4, \det(A)_5) = 1. \tag{$\star 1$}$$

Similarly, the existence of $\boldsymbol{\beta'}$ for the second determinant being ± 1 is equivalent to

$$g.c.d.(\det A, \det(A)^1, \det(A)^2, \det(A)^3, \det(A)^4, \det(A)^5) = 1. \tag{$\star 2$}$$

If ($\star$1) or ($\star$2) holds, then taking $\boldsymbol{\beta} = \boldsymbol{e_7}$ or $\boldsymbol{\alpha} = \boldsymbol{e_6}$ as standard basis of $\mathbb{Z}^7$ yields the validity of ($\bigstar$). Specifically, there is nothing to verify when $|\det A| = 1$ itself. On the other hand, it follows from the upper bound given in [9] that $|\det A| \leq 5$ if $A \in \mathrm{GL}(5, \mathbb{Z}_2)$. Thus, ($\star$1) can be reformulated as:

$$\exists \ i \in \{1, \ldots, 5\} \text{ such that } \det(A)_i \not\equiv 0 \pmod{|\det A|}, \tag{$\star 1'$}$$

and ($\star$2) can be reformulated as:

$$\exists \ i \in \{1, \ldots, 5\} \text{ such that } \det(A)^i \not\equiv 0 \pmod{|\det A|}. \tag{$\star 2'$}$$

3.2 Explicit values of ϕ and ψ

Fact $\phi(\boldsymbol{v_i}) = \begin{cases} 0 & \boldsymbol{v_i} \in V_1, \\ 1 & \boldsymbol{v_i} \in V_2 \sqcup V_3 \sqcup V_4, \\ 2 & \boldsymbol{v_i} \in V_5, \end{cases}$ *and* $\psi(\boldsymbol{v_i}) = \begin{cases} 0 & \boldsymbol{v_i} \in V_2 \sqcup V_4, \\ 1 & \boldsymbol{v_i} \in V_3, \\ 2 & \boldsymbol{v_i} \in V_1 \sqcup V_5, \end{cases}$ *guarantee the validity of* $(\star 1')$ *or* $(\star 2')$. *Consequently,* $\Lambda(\boldsymbol{v_i}) = \begin{pmatrix} \boldsymbol{v_i} \\ \phi(\boldsymbol{v_i}) \\ \psi(\boldsymbol{v_i}) \end{pmatrix}$ *for* $i \in \{1, \ldots, 31\}$ *induces a non-degenerate simplicial map from* $\mathcal{K}_1^5$ *to* $\mathcal{K}_2^7$.

This fact can be verified with the help of a MATLAB program[2] that searches for all *exotic* cases ($|\det A| = 3$ or 5) and check the validity of $(\star 1')$ or $(\star 2')$ therein.

Remark 3 Due to addtional restrictions $p_j \circ \Lambda = id_j$ for $j \in \{1, \ldots, 5\}$, most cases are not exotic. Moreover, ϕ and ψ are defined to be constant on each V_i. Thus, verification can be passed down to the level of equivalence classes determined by permutations of both rows and columns. As a result, the amount of calculation is greatly reduced, making it possible to construct Λ and verify the fact by hand (see [14]).

Remark 4 The value of ϕ and ψ can be taken modulo 15 since only coprimeness to 3 and 5 is concerned. As a matter of fact, the value of ψ can be taken modulo 3 and it can be arbitrary on set $V_1 \sqcup V_5$ since ϕ guarantees the validity of $(\star 1')$ for most of the exotic cases, resulting in only a few restrictions on ψ (see [14]).

4 Lower bound for $s(\mathcal{K}_1^n)$

By Proposition 2, $\Delta(\mathcal{K}_1^n)$ can be viewed as an upper bound for general cases. However, it remains open whether or not $\Delta(\mathcal{K}_1^n)$ is bounded when n goes to $+\infty$. On the other hand, by mapping $2^n - 1$ primitive vectors in $vt(\mathcal{K}_1^n)$ to standard basis of $\mathbb{Z}^{2^n-1}$, one can easily verify that $\Delta(\mathcal{K}_1^n) \leq 2^n - 1 - n$. With some symmetric modifications, this upper bound can be improved to $2^{n-2} + 1 - n$ for $n \geq 2$, as stated in Theorem 2.

Proof. Since $n \geq 2$, one can choose two arbitrary primitive vectors $\boldsymbol{x}, \boldsymbol{y} \in vt(\mathcal{K}_1^n)$, then a partition of $vt(\mathcal{K}_1^n)$ is given by $A_0 = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\}$ and $A_i = \{\boldsymbol{a}_i, \boldsymbol{a}_i+\boldsymbol{x}, \boldsymbol{a}_i+\boldsymbol{y}, \boldsymbol{a}_i+\boldsymbol{x}+\boldsymbol{y}\}$ for $i=1,\ldots,2^{n-2}-1$ with addition taken in $\mathbb{Z}_2$. Define a vertex map $\Lambda : vt(\mathcal{K}_1^n) \to vt(\mathcal{K}_2^{2^{n-2}+1})$ with the following assignment:

$$\boldsymbol{x} \mapsto \boldsymbol{e_1} \qquad \boldsymbol{y} \mapsto \boldsymbol{e_2} \qquad \boldsymbol{a_i} \mapsto \boldsymbol{e_{i+2}}$$
$$\boldsymbol{x} + \boldsymbol{y} \mapsto \boldsymbol{e_1} + \boldsymbol{e_2}$$
$$\boldsymbol{a_i} + \boldsymbol{x} \mapsto \boldsymbol{e_1} + \boldsymbol{e_{i+2}}$$
$$\boldsymbol{a_i} + \boldsymbol{y} \mapsto \boldsymbol{e_2} + \boldsymbol{e_{i+2}}$$
$$\boldsymbol{a_i} + \boldsymbol{x} + \boldsymbol{y} \mapsto \boldsymbol{e_1} + \boldsymbol{e_2} + \boldsymbol{e_{i+2}}$$

[2]The script for computer-aided verification is available at https://github.com/QiFanSHEN/Buchstaber-invariant-Verification/blob/main/Buchstaber%20invariant%20verification.m.

where $\{\boldsymbol{e_j}\}_{j=1}^{2^{n-2}+1}$ is the standard basis. It suffices to verify that Λ induces a non-degenerate simplicial map $\widetilde{\Lambda}$ from $\mathcal{K}_1^n$ to $\mathcal{K}_2^{2^{n-2}+1}$. Apparently, for each simplex $\sigma \in \mathcal{K}_1^n$, $A_0 \nsubseteq vt(\sigma)$. Let $\sharp(X)$ denote the number of elements in set X, then there are three different cases:

Case 1 $\sharp(A_0 \cap vt(\sigma))$=2.
By linear dependency, $\sharp(A_i \cap vt(\sigma)) \leq 1$ for any $i \geq 1$. Thus, images of $vt(\sigma)$ are parts of columns in the matrix equivalent to

$$P = \begin{pmatrix} P_2 & * \\ 0 & \mathrm{I}_{2^{n-2}-1} \end{pmatrix},$$

where P_2=$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathrm{I}_{2^{n-2}-1}$ stands for identity matrix of dimension $2^{n-2}-1$.

Case 2 $\sharp(A_0 \cap vt(\sigma))$=1.
By linear dependency, there exists at most one index $i_0 \geq 1$ such that $\sharp(A_{i_0} \cap vt(\sigma))$=2 while $\sharp(A_i \cap vt(\sigma)) \leq 1$ is valid for any other $i \geq 1$. Take subtraction between columns if such i_0 do exist, then images of $vt(\sigma)$ are parts of columns in the matrix equivalent to

$$Q = \begin{pmatrix} Q_2 & * \\ 0 & \mathrm{I}_{2^{n-2}-1} \end{pmatrix},$$

where Q_2=$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$.

Case 3 $\sharp(A_0 \cap vt(\sigma))$=0.
Similar to *Case 2*, either there exists at most one index $j_0 \geq 1$ such that $\sharp(A_{j_0} \cap vt(\sigma))$=3 while $\sharp(A_j \cap vt(\sigma)) \leq 1$ for any other $j \geq 1$, or there are at most two indices $j_1, j_2 \geq 1$ such that $\sharp(A_{j_1} \cap vt(\sigma))$=$\sharp(A_{j_2} \cap vt(\sigma))$=2 while $\sharp(A_j \cap vt(\sigma)) \leq 1$ for any other $j \geq 1$. Take subtraction between columns if such j_0 or j_1, j_2 do exist, then images of $vt(\sigma)$ can also be viewed as parts of columns in the matrix equivalent to Q.
Since $\det P = \det Q = 1$, the induced map $\widetilde{\Lambda}$ is non-degenerate as desired. □

Remark 5 Erokhovets gave an upper bound of $r(K)$ for general simplicial complex K on $[m]$ in terms of minimal non-simplices: $r(K) \leq \sum_{i=0}^{l} \dim\omega_i$ if there exists a collection of minimal non-simplices $\{\omega_i\}_{i=0}^{l}$ such that $\cup_{i=0}^{l}\omega_i = [m]$ (see Proposition 3.3.4 in [7]). Take $K = \mathcal{K}_1^n$ and $\omega_i = A_i$, then an upper bound $3 \cdot 2^{n-2} - 1 - n$ is obtained for $\Delta(\mathcal{K}_1^n)$. Theorem 2 can be regarded as an improvement of this result and it gives out sharp upper bound when $n \leq 4$.

Remark 6 Choosing one primitive vector in $\mathcal{K}_1^n$, an upper bound $2^{n-1} - n$ can be obtained for any $n \geq 1$ by similar argument. However, similar construction can not give out better results. If three linearly independent primitive vectors in $\mathcal{K}_1^n$ are chosen at the beginning, then for any simplex $\sigma \in \mathcal{K}_1^n$, the images of $vt(\sigma)$ can be viewed as parts of columns in the matrix equivalent to

$$R = \begin{pmatrix} R_3 & * \\ 0 & \mathrm{I}_{2^{n-3}-1} \end{pmatrix}.$$

Here R_3 may be equal to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ due to necessary column subtractions, leading to $\det R = -3$ instead of ± 1. Starting from choosing more linearly independent primitive vectors causes more problems like this.

Example 2 For n=4, take primitive vectors $\boldsymbol{x}, \boldsymbol{y}$ as $(1,0,0,0)^{\mathrm{T}}$ and $(0,1,0,0)^{\mathrm{T}}$ respectively, then $\Lambda : vt(\mathcal{K}_1^4) \to vt(\mathcal{K}_2^5)$ is defined as follow:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
$$\mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since $p_j \circ \Lambda \neq id_j$ for $j = 3, 4$, this map is different from the construction given in [15].

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